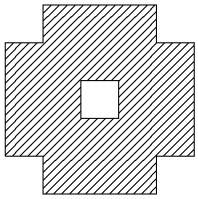
2.3.1: Concurrency

tuesday, the 1^{st} of march 2022
duration: 3h

All the programs under consideration are supposed to be conservative.

Exercise 1: Denote the following (hatched) isothetic region by X:



1) Write a program whose forbidden region is X.

2) Draw the deadlock attractor of this program.

3) What are the maximal blocks of the forbidden region.

4) Give the prime factorization of the geometric model
(i.e. the complement of the hatched region).

5) Draw the category of components of the geometric model.

Exercise 2:

1) Draw the geometric models of the following programs (`a` and `b` are mutices, i.e. semaphores of arity 1):

```
Pa.Va | Pa.Va
Pa.Pb.Va.Vb | Pa.Pb.Va.Vb
```

2) Give their categories of components.

3) Find a family Π_n of programs such that

a. the geometric models of all the programs Π_n have the same category of components, and
b. the number of maximal blocks of $[\![\Pi_n]\!]$ is *strictly less* than the number of maximal blocks of $[\![\Pi_{n+1}]\!]$.

Exercise 3: The 'jump' instruction `J(_)` interrupts the execution of the current process to start the execution of the process given in argument. For example the process `p = x:=x+1 ; J(p)` is a loop that endlessly increments the variable `x`. Consider the following program Π :

```
sync 1 a b
proc:
  p = x:=x+1 ; W(a) ; W(b) ; J(p)
  c = W(a) ; x:=x-1 ; W(b) ; J(c)
init: p c
```

1) Draw the geometric model of the program Π (the torus can be represented as a square whose opposite edges are identified, more precisely we have $(0,t) \sim (1,t)$ and $(t,0) \sim (t,1)$ for all $t \in [0,1]$).

2) Describe all the possible sequences of multi-instructions after finitely many execution steps of the program Π.

3) What are all the values of the variable `x` during an execution of the program Π.

4) What if we remove the instruction `W(b)` from the program Π ?

Exercise 4:
1) Give the prime decomposition of the following homogeneous languages (their underlying alphabets are $\{\mathtt{0},\mathtt{1}\}$ and $\{\mathtt{A},\mathtt{B},\mathtt{C},\mathtt{I}\}$ respectively).

01 10	010 100 001	0010 1000 0001 0100	ABIC IBCA CABI AIBC

Denote by $\mathcal{H}\Sigma$ the commutative monoid of homogeneous finite language on the alphabet Σ.

Let L be an element of $\mathcal{H}\{\mathtt{0},\mathtt{1}\}$ of dimension n, and X be a representative of L, i.e. L is the equivalence class of X. The language X is said to be a *permutation* when i) every word of X contains exactly one occurrence of `'1'`, and ii) for every $k \in \{1,\ldots,n\}$ there is a word in X whose k^{th} letter is `'1'`. (Equivalently, if you stack the words of X to form a matrix M, then M is the matrix of a permutation).

2a) Suppose that X' is another representative of L, i.e. $X \sim X'$ up to the action of the n^{th} symmetric group. Prove that if X is a permutation, then so is X'.

Then we say that L is a permutation when so is any of its representative.

2b) Prove that for all $n \geqslant 1$, there is a unique permutation $L \in \mathcal{H}\{\mathtt{0},\mathtt{1}\}$ of dimension n.

2c) Prove that for $n \geqslant 1$, the permutation of dimension n is prime.

3a) Prove that any map $f : \Sigma \to \Sigma'$ induces a morphism $\mathcal{H}f$ from the commutative monoid $\mathcal{H}\Sigma$ of homogeneous language to $\mathcal{H}\Sigma'$.

3b) Given $L \in \mathcal{H}\Sigma$, prove that if $\mathcal{H}f(L)$ is prime, then so is L.

3c) Give an example proving that the converse of 3b) is false.

Suppose that $\Sigma = \Sigma_0 \cup \Sigma_1$ with $\Sigma_0 \cap \Sigma_1 = \emptyset$. Let $L \in \mathcal{H}\Sigma$ with a representative X such that i) every word of X contains exactly one letter in Σ_1, and ii) for every $k \in \{1,\ldots,n\}$ there is a word in X whose k^{th} letter belongs to Σ_1.

4) Prove that L is prime.

Exercise 5: Let X be the geometric model of a program, and suppose that $X^{[0,r]}(p,q)$, the set of directed paths $\gamma : [0,r] \to X$ from p to q is equipped with the compact-open topology. Suppose that the set of states of the system is equipped with the discrete topology (i.e. singletons are open).

1) Prove that the map sending $\gamma \in X^{[0,r]}(p,q)$ to the state $\gamma \cdot \sigma$, i.e. the state resulting from the action of γ on the state σ, is continuous.

2) Deduce from 1) that if γ and γ' are weakly dihomotopic on X, then for any state σ we have $\gamma \cdot \sigma = \gamma' \cdot \sigma$.

Exercise 6: Find a length metric space E admitting a ball B with two points p, $q \in B$ such that there is $\varepsilon > 0$ and a path γ from p to q in E such that for any path δ from p to q in B, we have $\mathrm{length}(\delta) \geqslant \mathrm{length}(\gamma) + \varepsilon$. (Intuitively, the shortest paths from p to q have to leave B)