Concurrency

and

Directed Algebraic Topology

- MPRI -

Examination 2012

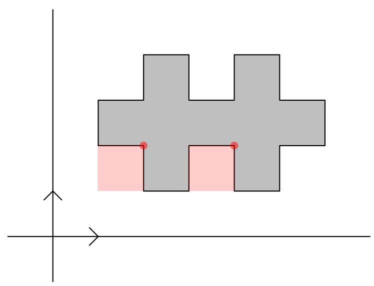
Answers

Exercise 1: PV programs and Geometric Models

1)

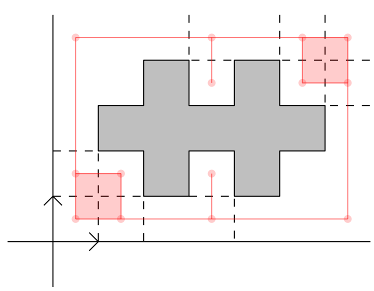
```
#sem a,b 2
procs:
p = P(a).P(b).V(b).P(b).V(b).V(a)
q = P(b).P(a).V(a).V(b)
init:
p|q
```

2)



Deadlocks and their attractors

3)



4a) and 4b) $F_{a,b} = [1,4]^2 \times \mathbb{R}^2 \ \cup\ \mathbb{R}^2 \times [1,4]^2$ and $F_c = \mathbb{R} \times [2,3]^3 \ \cup\ [2,3] \times \mathbb{R} \times [2,3]^2 \ \cup\ [2,3]^2 \times \mathbb{R} \times [2,3] \ \cup\ [2,3]^3 \times \mathbb{R}$

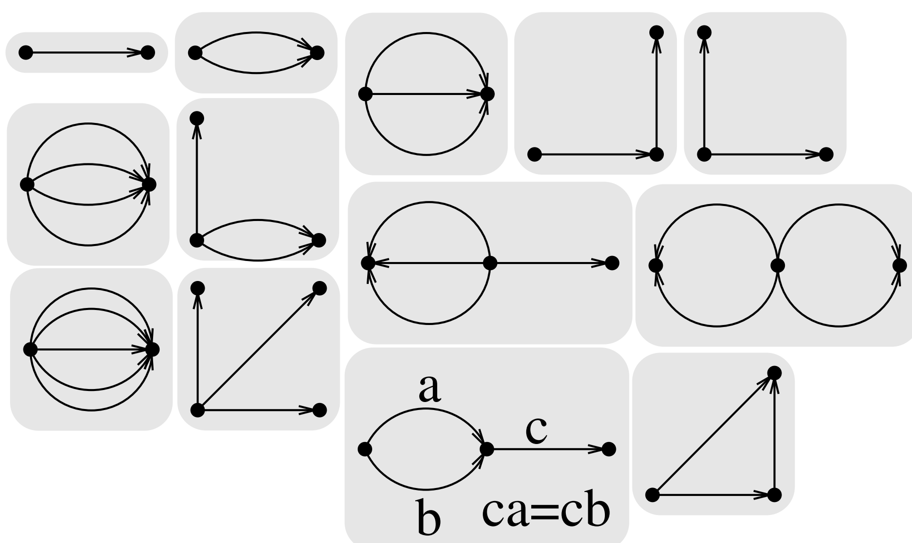
4c) We have $F_c \subset F_{a,b}$ therefore we can actually "drop" the semaphore c to obtain an "equivalent" program

```
#sem a,b 2
process definition:
p = P(a).V(a)
q = P(b).V(b)
program:
p | p | q | q
```

Exercise 2:

1) The only element of $\mathbb{M}$ of size 1 is the neutral element of M. There is no element of size 2 in $\mathbb{M}$ since such an element should contain at least 2 identities and therefore would not be connected. Then $\cdot \to \cdot$ is the only element of $\mathbb{M}$ of size 3. In order to minimize the size of a product of non trivial elements of $\mathbb{M}$, one has to minimize both the number of terms and theirs sizes. Hence the least non prime element of $\mathbb{M}$ is $(\cdot \to \cdot) \times (\cdot \to \cdot)$.

2) In addition to the neutral element of $\mathbb{M}$ there are 13 elements of size at most 7. The least non free element appears in the following list.



3) Define the n-star as the element S_n of $\mathbb{M}$ with n arrows outgoing from the same object. In addition of these arrows, S_n has $n+1$ identities. In particular for all $N \geqslant 3$ the size of the $\frac{n-1}{2}$ -star is n. Moreover any n -star is a free category therefore it is prime. We can also consider the category with 2 objects, say a and b, and $n-2$ arrows from a to b.

4) In $\mathbb{N}_0[X]$ we have $(1+X^3)\cdot(1+X+X^2) = (1+X)\cdot(1+X^2+X^4)$ while $1+X$

is irreducible and does not divide (in $\mathbb{N}_0[X]$) $1+X^3$ nor $1+X+X^2$, hence is not prime. According to the caracterization of free commutative monoids, $\mathbb{N}_0[X]$ is not free.

5) The uniqueness derives from the fact that X generates $\mathbb{N}[X]$ as a semiring. The morphism $\mathsf{eval}_{\mathcal{C}}$ is defined as follows
$\alpha_0 + \alpha_1 X + \cdots + \alpha_n X^n \in \mathbb{N}[X] \mapsto \beta_0 \sqcup \beta_1 \mathcal{C} \sqcup \cdots \sqcup \beta_n \mathcal{C}^n \in \mathbb{S}$ where $\beta_k = 1_{\mathbb{M}}$ if $\alpha_k = 1$, the empty category otherwise and $\sqcup$ is the disjoint union of categories.

6) Let $\mathcal{C}$ be some prime element of $\mathbb{M}$. Considering the size of categories we have $\mathcal{C}^n = \mathcal{C}^m$ if and only if $n = m$. Therefore $\mathsf{eval}_{\mathcal{C}}$ is one-to-one. Suppose $\mathcal{A} \times \mathcal{B}$ belongs to the image of $\mathsf{eval}_{\mathcal{C}}$. Since $\mathbb{M}$ is commutative free, each connected component of $\mathcal{A}$ and $\mathcal{B}$ is a power of $\mathcal{C}$. As a consequence given nonzero polynomials $P, P' \in \mathbb{N}[X]$, P divides P' in $\mathbb{N}[X]$ if and only if $\mathsf{eval}_{\mathcal{C}}(P)$ divides $\mathsf{eval}_{\mathcal{C}}(P')$ in $\mathbb{S}_0$. Thus $\mathbb{S}_0$ contains some nonprime irreducible element - because so does $\mathbb{N}_0[X]$, see question 4 - and therefore it is not free.

Exercise 3:

1) Given a directed loop $\gamma : [0, r] \to X$ one has $\gamma(0) = \gamma(r)$ and $\gamma(0) \sqsubseteq \gamma(t) \sqsubseteq \gamma(r)$ for all $0 \leqslant t \leqslant r$. Since $\sqsubseteq$ is antisymetric γ is a constant map.

2) Any constant map is continuous and monotonic therefore dX contains all of them. The concatenation of two directed paths is still directed because the partial relation on X is transitive. The directed paths are also stable under subpaths because the composition of two continuous monotonic map is continuous monotonic. We have defined the object part of the functor. Given a morphism of pospaces $f : X \to Y$ and a directed path γ on X, $f \circ \gamma$ is a directed path on Y as composite of monotonic continuous maps. Thus we have the morphism part of the functor.

3) Let $(F_i)_{i \in I}$ be a family of future stable subsets of X. Let γ be a dipath on X starting in some F_i. The path γ is contained in F_i hence in the union of the family. If γ starts in the intersection of the family, γ is contained in F_i for all i, in other words in the intersection of the family.

4) Given any points x and y on the circle, there is a directed path from x to y since the mapping $t \mapsto e^{it}$ covers the circle. Therefore, a stable subset of the directe circle is either empty or the whole circle. The lattice is thus isomorphic to $\{0 < 1\}$.

5) Given any complex numbers z and z', there is a directed path from z to z' if and only if $|z| \leqslant |z'|$. If follows the future stable subsets of the directed complex plane are the complement of the disks centered at 0 i.e. $\{z \in \mathbb{C};\ |z| < r\}$ (open) or $\{z \in \mathbb{C};\ |z| \leqslant r\}$ (closed) with $r \in [0, +\infty]$. Given two such disks D and D' denote by r and r' the radius of D and the radius of D'. One has $D \subseteq D'$ when $r < r'$ or ($r = r'$ and (D is open or D' is closed)). In other words the lattice of future subsets is isomorphic to (the opposite of) the lexicographic order on $(\mathbb{R}_+ \times \{0 < 1\}) \cup \{(\infty, *)\}$.