Introduction to Directed Algebraic Topology with a view towards modelling Concurrency III

Mathematical Structures of Computations - Lyon 2014

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Summary

Category of components

- The loop-free case
- Beyond loop-freeness

Unique factorization theorems

- Free commutative monoid
- Finite connected loop-free categories
- Homogeneous sets of words

Components

Motivations

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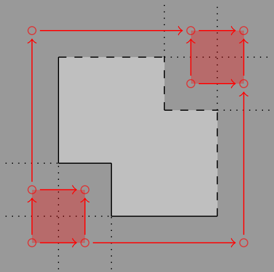
- Yet $\overrightarrow{\pi_1}[\![P]\!]$ has uncountably many objects
- Still we expect a finite description of $\overrightarrow{\pi_1}[\![P]\!]$



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- the only isomorphisms of $\overrightarrow{\pi_1}[\![P]\!]$ are its identities
  therefore $\overrightarrow{\pi_1}[\![P]\!]$ is its own skeleton
- find a nontrivial collection of morphisms enjoying
  properties similar to those of the class of isomorphisms

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Loop-free categories

introduced by André Haefliger as "small categories without loops"

- A category $\mathcal{C}$ such that for all objects x and y
  if both $\mathcal{C}[x, y]$ and $\mathcal{C}[y, x]$ are nonempty
  then $x = y$ and $\mathcal{C}[x, x] = \{\mathrm{id}_x\}$

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- The fundamental category of a pospace is loop-free

Weak isomorphism

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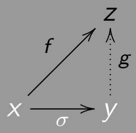

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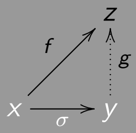

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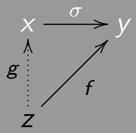

Weak isomorphisms

when loops occurs

If $\sigma : x \to y$ is a weak isomorphism and $\mathcal{C}[y, x] \neq \emptyset$ then σ is an isomorphism.

System of weak isomorphisms

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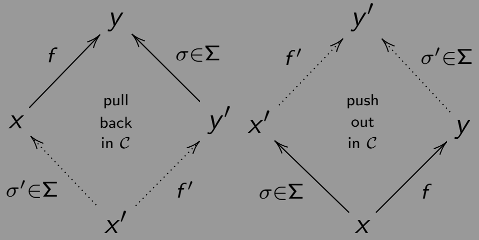

Structure of Σ-components

Σ system of weak isomorphisms over $\mathcal{C}$ loop-free

1. the relation $x \sim y \equiv \exists z \in |\mathcal{C}|\ \Sigma[x,z] \neq \emptyset$ and $\Sigma[y,z] \neq \emptyset$ is an equivalence relation

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3. If $a \sim b$ then

$$\begin{array}{ccc} a & \longrightarrow & a \vee b \\ \uparrow & & \uparrow \\ a \wedge b & \longrightarrow & b \end{array}$$

is both a pullback and a pushout in $\mathcal{C}$

Locale of systems of weak isomorphisms

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The poset ({systems of weak isomorphisms}, $\subseteq$) is a locale. Let $\overline{\Sigma}$ be its greatest element.

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- $\overrightarrow{\pi_0}(\mathcal{A} \times \mathcal{B}) \cong \overrightarrow{\pi_0}\mathcal{A} \times \overrightarrow{\pi_0}\mathcal{B}$

Fundamental theorem

$\mathcal{C}$ loop-free category and Σ system of weak isomorphisms over $\mathcal{C}$

1. Σ is pure in $\mathcal{C}$ i.e. $\beta \circ \alpha \in \Sigma \Rightarrow \beta, \alpha \in \Sigma$,

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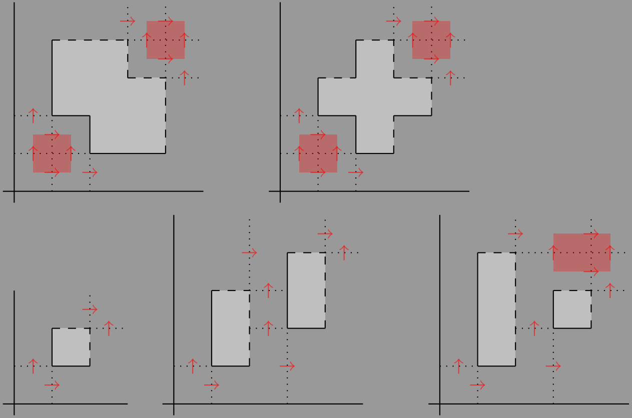
4. $\mathcal{C}[\Sigma^{-1}]$ is fibered over the base $\mathcal{C}/_\Sigma$.

Examples

in dimension 2



Commutative monoids

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- For all set X the collection MX of multisets over X
  i.e. maps $\phi : X \to \mathbb{N}$ s.t. $\{x \in X \mid \phi(x) \neq 0\}$ is finite
  forms a commutative monoid

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- Functor $M : \mathbf{Set} \to \mathbf{CMon}$

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- p prime: p nonunit and $p|ab$ implies $p|a$ or $p|b$

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monoid	irreducibles	primes	units

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$\mathbb{R}_+, \vee, 0$	$\emptyset$	$\mathbb{R}_+ \setminus \{0\}$	$\{0\}$
$\mathbb{Z}_6, \times, 1$	$\emptyset$	$\{0, 2, 3, 4\}$	$\{1, 5\}$

Graded commutative monoid

- $(M, *, \varepsilon)$ graded: there is a one-to-one morphism from $(M, *, \varepsilon)$ to $(\mathbb{N}, +, 0)$

Graded commutative monoid

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- $(M, *, \varepsilon)$ graded: there is a one-to-one morphism from $(M, *, \varepsilon)$ to $(\mathbb{N}, +, 0)$
- If M is graded then
  $\{$irreducibles of $M\}$ generates M
  $\{$primes of $M\} \subseteq \{$irreducibles of $M\}$

Irreducible that are not prime

$M = (\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}; a \neq 0 \text{ or } b \neq 0\}, \times, 1)$

- $N : M \to (\mathbb{Z} \setminus \{0\}, \times, 1)$; $N(a + b\sqrt{10}) = a^2 - 10b^2$

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$4 \pm \sqrt{10}$	6	$\pm 1, \pm 2, \pm 3, \pm 6$

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  since $2 \cdot 3 = (4 + \sqrt{10}) \cdot (4 - \sqrt{10})$
- $\{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \setminus \{0\}$ is graded by the number of prime factors of $N(u)$

$\mathbb{N}[X]$ polynomials with coefficients in $\mathbb{N}$

Junji Hashimoto 51

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- therefore $X+1$, X^2+X+1, X^3+1, and X^4+X^2+1 are irreducible but not prime
- $\mathbb{N}[X] \setminus \{0\}$ is graded by the degree

Characterization
of the free commutative monoids

The following are equivalent:
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- M is graded and $\{\text{irreducibles of } M\} \subseteq \{\text{primes of } M\}$

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- $(\mathcal{A} \times \mathcal{B}) \times \mathcal{C} \cong \mathcal{A} \times (\mathcal{B} \times \mathcal{C})$
- $1 \times \mathcal{A} \cong \mathcal{A} \cong \mathcal{A} \times 1$

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- $\mathcal{H}(\mathbb{R})$ subsets of $\mathbb{R}^n$ for n ranging through $\mathbb{N}$

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