Introduction to Directed Algebraic Topology with a view towards modelling Concurrency II

Mathematical Structures of Computations - Lyon 2014

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Summary

Geometric realization

Directed Topology Local pospaces Realization of graphs Continuous interpretation Geometric model

Fundamental category

Precubical sets Local pospaces Some calculations

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Diagram in **Top** from a precubical set *K*

$$-\partial_i \cong (x \cdots x \underbrace{0}_{ith} x \cdots x) \text{ and } \partial_i^+ \cong (x \cdots x \underbrace{1}_{ith} x \cdots x)$$

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Diagram in **Top**

-
$$\partial_i^{\pm} \cong (x \cdots x \underbrace{0}_{i^{th}} x \cdots x)$$
 and $\partial_i^{\pm} \cong (x \cdots x \underbrace{1}_{i^{th}} x \cdots x)$
- for all $n \in \mathbb{N}$ for all $x \in K_n$ for all $i \in \{0, \dots, n-1\}$
and for $\varepsilon \in \{0, 1\}$ we have the inclusion map

$$\begin{array}{rcl} \phi_{i,n,x}^{\varepsilon} & : & \{\partial_i^{\varepsilon}(x)\} \times [0,1]^{n-1} & \to & \{x\} \times [0,1]^n \\ & & (t_1,\ldots,t_{n-1}) & \mapsto & (t_1,\ldots,t_{i-1},\varepsilon,t_i,\ldots,t_{n-1}) \end{array}$$

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- |K|: the geometric realization of K is the colimit of this diagram in **Top**

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- |K|: the geometric realization of K is the colimit of this diagram in **Top**
- for all K, K' precubical sets, $|K \otimes K'| \cong |K| imes |K'|$

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a calculation

-
$$K_0 = \{a, b\}$$
 and $K_1 = \{\alpha, \beta\}$
 $\partial^{+} \alpha = \partial^{-} \beta = a$ and $\partial^{+} \alpha = \partial^{+} \beta = b$

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a calculation

$$\mathcal{K}_0 = \{a, b\} \text{ and } \mathcal{K}_1 = \{\alpha, \beta\}$$

 $\partial^+ \alpha = \partial^+ \beta = a \text{ and } \partial^+ \alpha = \partial^+ \beta = b$
 $\partial^+ \alpha = a$
 $b = \partial^+ \alpha$

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a calculation

$$\mathcal{K}_{0} = \{a, b\} \text{ and } \mathcal{K}_{1} = \{\alpha, \beta\}$$

$$\partial^{+} \alpha = \partial^{+} \beta = a \text{ and } \partial^{+} \alpha = \partial^{+} \beta = b$$

$$\partial^{-} \beta = \partial^{-} \alpha = a$$

$$b = \partial^{+} \alpha = \partial^{+} \beta$$

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another calculation

-
$$K_0 = \{a, b\}$$
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another calculation

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$$K_0 = \{a, b\}$$
 and $K_1 = \{\alpha, \beta\}$
 $\partial \alpha = \partial^+ \beta = a$ and $\partial^+ \alpha = \partial^- \beta = b$

$$\partial^2 \alpha = a$$
 $b = \partial^+ \alpha$

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Geometric realization

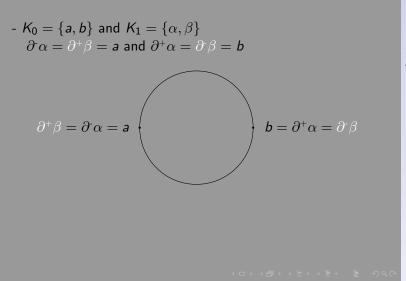
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- A topological space X together with a closed partial order

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- A topological space X together with a closed partial order - morphisms: increasing continuous maps

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- A topological space X together with a closed partial order
- morphisms: increasing continuous maps
- e.g. ${\mathbb R}$ with its standard topology and order

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- A topological space X together with a closed partial order
- morphisms: increasing continuous maps
- e.g. ${\mathbb R}$ with its standard topology and order
- Potop is complete and cocomplete but its colimits do not preserve the topology

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-
$$K_0 = \{a, b\}$$
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$$a_{0} = \{a, b\} \text{ and } K_{1} = \{\alpha, \beta\}$$

 $\alpha = \partial^{+}\beta = a \text{ and } \partial^{+}\alpha = \partial^{+}\beta = b$
 $\partial^{-}\alpha = a$
 $b = \partial^{+}\alpha$

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Directed geometric realization in **PoTop** a calculation

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$$\mathcal{K}_0 = \{a, b\} \text{ and } \mathcal{K}_1 = \{\alpha, \beta\}$$

 $\partial \alpha = \partial \beta = a \text{ and } \partial^+ \alpha = \partial^+ \beta = b$
 $\partial \beta = \partial \alpha = a$
 $b = \partial^+ \alpha = \partial^+ \beta$

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$$K_0 = \{a, b\}$$
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$$K_0 = \{a, b\}$$
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 $\partial^{-}\alpha = \partial^{+}\beta = a$ and $\partial^{+}\alpha = \partial^{-}\beta = b$

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$$\mathcal{K}_0 = \{a, b\} \text{ and } \mathcal{K}_1 = \{\alpha, \beta\}$$

 $\mathcal{F}\alpha = \partial^+\beta = a \text{ and } \partial^+\alpha = \partial^-\beta = b$
 $\partial^+\alpha = a$
 $b = \partial^+\alpha$

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$$\mathcal{K}_{0} = \{a, b\} \text{ and } \mathcal{K}_{1} = \{\alpha, \beta\}$$

$$\partial^{+}\beta = \partial^{+}\beta = a \text{ and } \partial^{+}\alpha = \partial^{-}\beta = b$$

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just one point remains

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- X underlying topological space

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- X underlying topological space
- ordered chart on X: pospace over some open subset of X

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Locally Partially Ordered Spaces - local pospaces

Fajstrup, Goubault, and Raussen 98 (original version)

- X underlying topological space
- ordered chart on X: pospace over some open subset of X
- ordered atlas on X: collection \mathcal{U} of ordered charts s.t.
 - i) for all $U, U' \in \mathcal{U}$ and $x \in U \cap U'$ there exists $U'' \in \mathcal{U}$ s.t. $x \in U'' \subseteq U \cap U'$ and the order on U''matches both orders on U and U'
 - ii) \mathcal{U} induces a basis of topology of X

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 - ii) \mathcal{U} induces a basis of topology of X
- morphism of atlases f : (X, U) → (Y, V):
 a continuous map f : X → Y such that for all x ∈ X
 there exists U ∈ U, V ∈ V neighborhoods of x and f(x)
 such that f induces a morphism of pospaces from U to V

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- Atlases \mathcal{U} and \mathcal{U}' on X are equivalent when their union is still an atlas

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- Atlases \mathcal{U} and \mathcal{U}' on X are equivalent when their union is still an atlas
- The union of all atlases equivalent to $\ensuremath{\mathcal{U}}$ is an atlas

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- Local pospace: equivalence class of atlases

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- Atlases \mathcal{U} and \mathcal{U}' on X are equivalent when their union is still an atlas
- The union of all atlases equivalent to $\ensuremath{\mathcal{U}}$ is an atlas
- Local pospace: equivalence class of atlases
- If $\mathcal{U} \sim \mathcal{U}'$, $\mathcal{V} \sim \mathcal{V}'$, and $f : \mathcal{U} \to \mathcal{V}$ morphism of atlases then $f : \mathcal{U}' \to \mathcal{V}'$ morphism of atlases

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- e.g. the exponential map $t \in \mathbb{R} \mapsto \mathsf{e}^{it} \in \mathbb{S}^1$

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- Lpotop is finitely complete but misses some infinite products its cocompleteness is an open question its colimits do not preserve the topology

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Directed geometric realization in **LpoTop**

 For all finite precubical sets K, the directed geometric realization 1K|LpoTop exists

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Directed geometric realization in **LpoTop** a claim

- For all finite precubical sets K, the directed geometric realization |K|_{LpoTop} exists
- and preserves the topology

 $U(|K|_{LpoTop}) = |K|$

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Directed geometric realization in **LpoTop** a claim

- For all finite precubical sets K, the directed geometric realization |K|_{LpoTop} exists
- and preserves the topology

 $U(|K|_{LpoTop}) = |K|$

- therefore

 $|K \otimes K'|_{LpoTop} \cong |K|_{LpoTop} \times |K'|_{LpoTop}$

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as local pospaces

$$G: A \xrightarrow[\partial^+]{\partial^+} V$$

- underlying set $V \sqcup A \times]0, 1[$

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as local pospaces

$$G: A \xrightarrow[\partial^+]{\partial^+} V$$

- underlying set $V \sqcup A \times]0, 1[$
- v_{ε}^+ union of $\{\alpha\} \times]0, \varepsilon[$ for all $\alpha \in A$ such that $\partial^{\cdot} \alpha = v$ and $0 < \varepsilon < 1$
- v_{ε}^{-} union of $\{\alpha\} \times]1 \varepsilon, 1[$

for all $\alpha \in {\cal A}$ such that $\partial^+ \alpha = {\it v}$ and $0 < \varepsilon < 1$

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as local pospaces

$$G: A \xrightarrow{\partial^-} V$$

- underlying set $V \sqcup A \times]0, 1[$
- v_{ε}^+ union of $\{\alpha\} \times]0, \varepsilon[$ for all $\alpha \in A$ such that $\partial^- \alpha = v$ and $0 < \varepsilon < 1$
- v_{ε}^{-} union of $\{\alpha\} \times]1 \varepsilon, 1[$ for all $\alpha \in A$ such that $\partial^{+}\alpha = v$ and $0 < \varepsilon < 1$
- directed atlas

 $\{\alpha\} \times]a, b[$ with $\alpha \in A$ and $0 \leq a < b \leq 1$, and $\{v\} \cup v_{\varepsilon}^+ \cup v_{\varepsilon}^-$ with $v \in V$ and $0 < \varepsilon < 1$ with obvious partial order

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as local pospaces

$$G: A \xrightarrow{\partial^-} V$$

- underlying set $V \sqcup A \times]0, 1[$
- v_{ε}^+ union of $\{\alpha\} \times]0, \varepsilon[$ for all $\alpha \in A$ such that $\partial^- \alpha = v$ and $0 < \varepsilon < 1$
- v_{ε}^{-} union of $\{\alpha\} \times]1 \varepsilon, 1[$ for all $\alpha \in A$ such that $\partial^{+}\alpha = v$ and $0 < \varepsilon < 1$
- directed atlas

 $\{\alpha\} \times]a, b[$ with $\alpha \in A$ and $0 \leq a < b \leq 1$, and $\{v\} \cup v_{\varepsilon}^+ \cup v_{\varepsilon}^-$ with $v \in V$ and $0 < \varepsilon < 1$ with obvious partial order

- denoted by |G|

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virtual machine

- The labelling $\lambda: D \to A$, with $D = \{(\alpha, \frac{1}{2}) \mid \alpha \in A\}$

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virtual machine

- The labelling $\lambda : D \to A$, with $D = \{(\alpha, \frac{1}{2}) \mid \alpha \in A\}$
- for $\gamma : [0, r] \rightarrow |G|$ the set $\gamma^{-1}(D)$ is a finite union of disjoint compact intervals $[a_1, b_1] \cup \cdots \cup [a_n, b_n]$

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virtual machine

- The labelling $\lambda: D \to A$, with $D = \{(\alpha, \frac{1}{2}) \mid \alpha \in A\}$
- for $\gamma : [0, r] \to |G|$ the set $\gamma^{-1}(D)$ is a finite union of disjoint compact intervals $[a_1, b_1] \cup \cdots \cup [a_n, b_n]$
- Instructions are performed when they are touched so [[γ]] = γ(a_n),..., γ(a₁) is associated with γ therefore the action of γ upon a distribution δ is [[γ]] · δ

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virtual machine

- The labelling $\lambda: D \to A$, with $D = \{(\alpha, \frac{1}{2}) \mid \alpha \in A\}$
- for $\gamma : [0, r] \rightarrow |G|$ the set $\gamma^{-1}(D)$ is a finite union of disjoint compact intervals $[a_1, b_1] \cup \cdots \cup [a_n, b_n]$
- Instructions are performed when they are touched so [[γ]] = γ(a_n),..., γ(a₁) is associated with γ therefore the action of γ upon a distribution δ is [[γ]] · δ
- for any execution trace s, there exists a dipath γ such that $[\![\gamma]\!]=s$

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Continuous parallel

dynamics

- if the process is conservative then for any δ , $[\![\gamma]\!]\cdot\delta$ only depends on $\partial^{\scriptscriptstyle -}\gamma$ and $\partial^{\scriptscriptstyle +}\gamma$

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Continuous parallel

dynamics

- if the process is conservative then for any δ , $[\![\gamma]\!] \cdot \delta$ only depends on $\partial^* \gamma$ and $\partial^+ \gamma$ - therefore we have a potential function

 $F:|G|\times \mathcal{R} \to \mathbb{N}$

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Areas

- G_1, \ldots, G_d finite graphs

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- G_1, \ldots, G_d finite graphs - (G_1, \ldots, G_d) -block: $B_1 \times \cdots \times B_n$ with B_k connected subset of $|G_k|$

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- G_1, \ldots, G_d finite graphs
- (G_1, \ldots, G_d) -block: $B_1 \times \cdots \times B_n$ with B_k connected subset of $|G_k|$
- (G_1, \ldots, G_d) -areas: finite union of blocks

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- G_1, \ldots, G_d finite graphs
- (G_1, \ldots, G_d) -block: $B_1 \times \cdots \times B_n$ with B_k connected subset of $|G_k|$
- (G_1, \ldots, G_d) -areas: finite union of blocks
- The collection of (G₁,..., G_d)-areas forms a boolean subalgebra of 2¹G₁↓×···×↑G_d↓

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Race conditions

conflicts in variable access

- G_1, \ldots, G_d the control flow graphs of each process

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Race conditions

conflicts in variable access

- G_1, \ldots, G_d the control flow graphs of each process
- Race conditions is the subset of $|G_1| \times \cdots \times |G_d|$ s.t. there is $1 \leq i < j \leq d$ such that $\lambda(v_i)$ and $\lambda(v_j)$ are actions sharing some variable

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Fundamental category

Forbidden area

via potential function

- F_1, \ldots, F_d the associated potential functions

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Forbidden area

via potential function

- F_1, \ldots, F_d the associated potential functions - $F : |G_1| \times \cdots \times |G_d| \times \mathcal{R} \to \mathbb{N}$ the potential function

$$F(v_1,\ldots,v_d,x)=\sum_{k=1}^d F_k(v_k,x)$$

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Forbidden area

via potential function

- F_1, \ldots, F_d the associated potential functions - $F : |G_1| \times \cdots \times |G_d| \times \mathcal{R} \to \mathbb{N}$ the potential function

$$F(v_1,\ldots,v_d,x)=\sum_{k=1}^d F_k(v_k,x)$$

- Forbidden area is the subset of $|G_1| \times \cdots \times |G_d|$

$$\{(v_1,\ldots,v_d) \mid \exists x \in \mathcal{R}, F(v_1,\ldots,v_d,x) \ge \operatorname{arity}(x)\}$$

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Walls

and geometric model

- Walls is the subset of $(v_1, \ldots, v_d) \in |G_1| \times \cdots \times |G_d|$ s.t. there exists a synchronization x s.t. the cardinal of $\{k \in \{1, \ldots, d\} \mid \lambda(v_k) = W(x)\}$ is neither 0 nor the arity of x

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Walls

and geometric model

- Walls is the subset of $(v_1, \ldots, v_d) \in |G_1| \times \cdots \times |G_d|$ s.t. there exists a synchronization x s.t. the cardinal of $\{k \in \{1, \ldots, d\} \mid \lambda(v_k) = W(x)\}$ is neither 0 nor the arity of x
- The geometric model is then defined as $|G_1| \times \cdots \times |G_d| \setminus (\text{Race} \cup \text{Forbidden} \cup \text{Walls})$

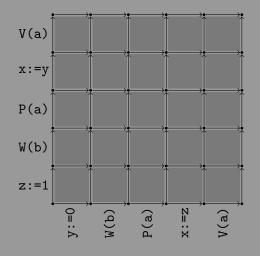
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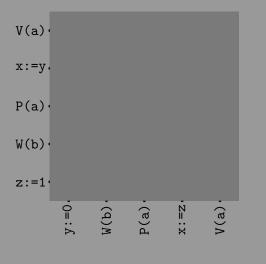
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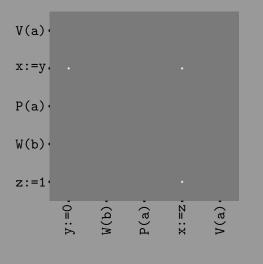
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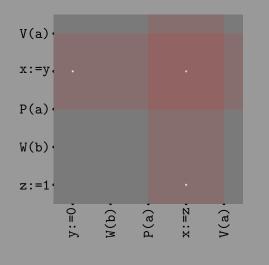
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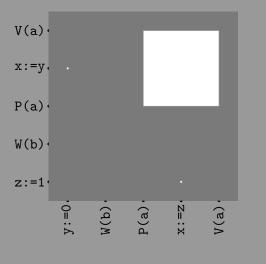
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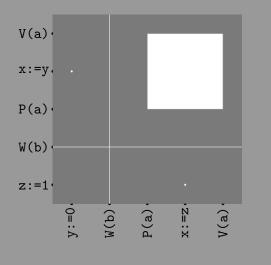
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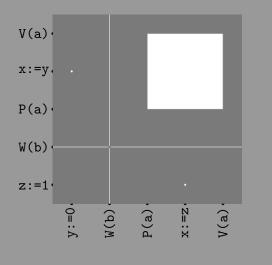
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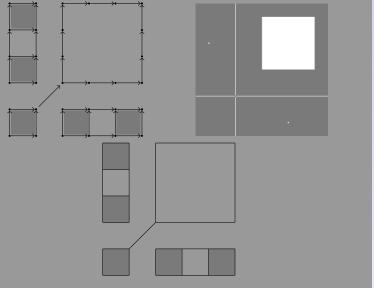
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Comparing

Discrete vs Continuous



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Fundamental category

of a precubical set K

 F(trunc₁(K)) the category of paths on the underlying graph of K

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Fundamental category

Fundamental category

of a precubical set K

- F(trunc₁(K)) the category of paths on the underlying graph of K
- the congruence \sim over $F(trunc_1(K))$ generated by $\gamma \sim \delta$ when γ and δ start and finish at the lower and upper corners of the same *n*-cube

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Fundamental category

Fundamental category

of a precubical set K

- F(trunc₁(K)) the category of paths on the underlying graph of K
- the congruence ~ over F(trunc₁(K)) generated by γ ~ δ when γ and δ start and finish at the lower and upper corners of the same *n*-cube - define π₁[→]K = F(trunc₁(K))/ ~

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Fundamental category

Fundamental category

of a precubical set K

- F(trunc₁(K)) the category of paths on the underlying graph of K
- the congruence ~ over F(trunc₁(K)) generated by γ ~ δ when γ and δ start and finish at the lower and upper corners of the same *n*-cube - define π₁[→]K = F(trunc₁(K))/ ~
- $-\overrightarrow{\pi_1}K = \overrightarrow{\pi_1}(trunc_2(K))$

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- Dipath: morphism $\gamma : [0, r] \to X$ with $r \ge 0$ $\partial^{\cdot} \gamma = \gamma(0)$ and $\partial^{+} \gamma = \gamma(r)$

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- Dipath: morphism
$$\gamma : [0, r] \to X$$
 with $r \ge 0$
 $\partial^{\cdot} \gamma = \gamma(0)$ and $\partial^{+} \gamma = \gamma(r)$
- Concatenation $\gamma \cdot \delta : [0, r + r'] \to X$ when $\partial^{\cdot} \gamma = \partial^{+} \delta$;
 $\gamma \cdot \delta(t) = \begin{cases} \delta(t) & \text{if } t \leqslant r \\ \gamma(t) & \text{if } r \leqslant t \end{cases}$

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Fundamental category

- Dipath: morphism
$$\gamma : [0, r] \to X$$
 with $r \ge 0$
 $\partial^{2}\gamma = \gamma(0)$ and $\partial^{+}\gamma = \gamma(r)$
- Concatenation $\gamma \cdot \delta : [0, r + r'] \to X$ when $\partial^{2}\gamma = \partial^{+}\delta$;
 $\gamma \cdot \delta(t) = \begin{cases} \delta(t) & \text{if } t \le r \\ \gamma(t) & \text{if } r \le t \end{cases}$

- Dipath functor P : **LpoTop** \rightarrow **Cat**

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- Dipath functor P : **LpoTop** \rightarrow **Cat**
- If X is the model of a program then the dipaths on X is an overapproximation of the execution traces

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- Dipath: morphism
$$\gamma : [0, r] \to X$$
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 $\partial^{-}\gamma = \gamma(0)$ and $\partial^{+}\gamma = \gamma(r)$
- Concatenation $\gamma \cdot \delta : [0, r + r'] \to X$ when $\partial^{-}\gamma = \partial^{+}\delta$;
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- Dipath functor $P : \mathbf{LpoTop} \to \mathbf{Cat}$
- If X is the model of a program then the dipaths on X is an overapproximation of the execution traces
- Infinitely many paths between two points

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Dihomotopy between dipaths on X

- morphism $h: [0, r] \times [0, \rho] \rightarrow X$ s.t. $h(0, _)$ and $h(r, _)$ are both constant

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Dihomotopy between dipaths on X

 morphism h: [0, r] × [0, ρ] → X s.t. h(0, _) and h(r, _) are both constant
 2-dimensional precubical set...

$$\partial_0^+ h = \operatorname{cst} \left| \begin{array}{c} \partial_1^+ h = \delta \\ \hline h \\ \hline \partial_0^- h = \operatorname{cst} \\ \hline \partial_1^- h = \gamma \end{array} \right| \partial_0^+ h = \operatorname{cst}$$

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Dihomotopy between dipaths on X

 morphism h : [0, r] × [0, ρ] → X s.t. h(0, _) and h(r, _) are both constant
 2-dimensional precubical set...

$$\partial_0^+ h = \operatorname{cst} \left| \begin{array}{c} \partial_1^+ h = \delta \\ \\ h \\ \hline \\ \partial_0^+ h = - \operatorname{cst} \end{array} \right| \left| \partial_0^+ h = \operatorname{cst} \right|$$



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...and even more.

-
$$h: [0, r] \times [0, \rho] \rightarrow X$$
 and $g: [0, r] \times [0, \rho'] \rightarrow X$
with $h(-, \rho) = g(-, 0)$

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-
$$h: [0, r] \times [0, \rho] \rightarrow X$$
 and $g: [0, r] \times [0, \rho'] \rightarrow X$
with $h(_{-}, \rho) = g(_{-}, 0)$
 $g * h: [0, r] \times [0, \rho + \rho'] \rightarrow X$ defined by
 $g * h(t, x) = \begin{cases} h(t, x) & \text{if } x \leqslant \rho \\ g(t, x - \rho) & \text{if } \rho \leqslant x \end{cases}$

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-
$$h: [0, r] \times [0, \rho] \rightarrow X$$
 and $g: [0, r] \times [0, \rho'] \rightarrow X$
with $h(\neg, \rho) = g(\neg, 0)$
 $g * h: [0, r] \times [0, \rho + \rho'] \rightarrow X$ defined by
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$$\delta$$

 g
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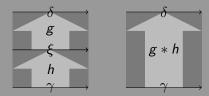
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-
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 and $g: [0, r] \times [0, \rho'] \rightarrow X$
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Fundamental category

2-category

- $h: [0, r] \times [0, \rho] \rightarrow X$ and $h': [0, r'] \times [0, \rho] \rightarrow X$ with $\partial^+_0 h = \partial^-_0 h'$ i.e. $h(r, _) = h'(0, _)$

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2-category

-
$$h: [0, r] \times [0, \rho] \to X$$
 and $h': [0, r'] \times [0, \rho] \to X$
with $\partial^+{}_0 h = \partial_0 h'$ i.e. $h(r, _) = h'(0, _)$
 $h' \cdot h: [0, r] \times [0, \rho + \rho'] \to X$ defined by
 $h' \cdot h(t, x) = \begin{cases} h(t, x) & \text{if } t \leqslant r \\ h'(t - r, x) & \text{if } r \leqslant t \end{cases}$

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2-category

-
$$h: [0, r] \times [0, \rho] \to X$$
 and $h': [0, r'] \times [0, \rho] \to X$
with $\partial^+{}_0 h = \partial_0 h'$ i.e. $h(r, _) = h'(0, _)$
 $h' \cdot h: [0, r] \times [0, \rho + \rho'] \to X$ defined by
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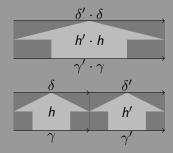
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Fundamental category

2-category

-
$$h: [0, r] \times [0, \rho] \to X$$
 and $h': [0, r'] \times [0, \rho] \to X$
with $\partial^+{}_0 h = \partial^-{}_0 h'$ i.e. $h(r, _) = h'(0, _)$
 $h' \cdot h: [0, r] \times [0, \rho + \rho'] \to X$ defined by
 $h' \cdot h(t, x) = \begin{cases} h(t, x) & \text{if } t \leqslant r \\ h'(t - r, x) & \text{if } r \leqslant t \end{cases}$



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Exchange property

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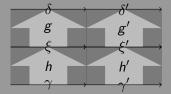
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Godement

Exchange property



$$(g'*h')\cdot(g*h)=(g'\cdot g)*(h'\cdot h)$$

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Directed topology vs Category 2-category

Directed topology	Category
point	category
dipath	functor
dihomotopy	natural transformation
path concatenation	composition of functors
'piled up' homotopies	composition of natural transformations
'side-by-side' homotopies	juxtaposition of natural transformations

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Elementary homotopy

- anti-dihomotopy $h:[0,r] \times [0,\rho] \to X$ such that $(t,x) \mapsto h(t,-x)$ is a dihomotopy

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Elementary homotopy

- anti-dihomotopy $h: [0, r] \times [0, \rho] \to X$ such that $(t, x) \mapsto h(t, -x)$ is a dihomotopy

- elementary homotopy $h_n * \cdots * h_1$ where each h_k is either a dihomotopy or an antidihomotopy

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Elementary homotopy

- anti-dihomotopy $h: [0, r] \times [0, \rho] \to X$ such that $(t, x) \mapsto h(t, -x)$ is a dihomotopy
- elementary homotopy $h_n * \cdots * h_1$ where each h_k is either a dihomotopy or an antidihomotopy
- a finite juxtaposition of dihomotopies and anti-dihomotopies can be 'replaced' by an elementary homotopy

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The dihomotopy relation γ and δ dipaths defined over [0, r] and [0, r']

- Write $\gamma \sim \delta$ when $\partial^{-}\gamma = \partial^{-}\delta$, $\partial^{+}\gamma = \partial^{+}\delta$ and there is an elementary homotopy between $c \cdot \gamma$ and $d \cdot \delta$ where c (resp. d) is constant over $[0, (r \vee r') - r]$ (resp. $[0, (r \vee r') - r']$)

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The dihomotopy relation γ and δ dipaths defined over [0, r] and [0, r']

- Write $\gamma \sim \delta$ when $\partial^{-}\gamma = \partial^{-}\delta$, $\partial^{+}\gamma = \partial^{+}\delta$ and there is an elementary homotopy between $c \cdot \gamma$ and $d \cdot \delta$ where c (resp. d) is constant over $[0, (r \vee r') - r]$ (resp. $[0, (r \vee r') - r']$)

- The relation \sim is a congruence over PX

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- By definition $\overrightarrow{\pi_1}X = PX/\sim$

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- By definition $\overrightarrow{\pi_1}X = PX/\sim$
- $f \circ (h_n * \cdots * h_1) = (f \circ h_n) * \cdots * (f \circ h_1)$ therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$

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- By definition $\overrightarrow{\pi_1}X = PX/\sim$
- $f \circ (h_n * \cdots * h_1) = (f \circ h_n) * \cdots * (f \circ h_1)$ therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$
- Hence a functor $\overrightarrow{\pi_1}$: LpoTop \rightarrow Cat

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- $f \circ (h_n * \cdots * h_1) = (f \circ h_n) * \cdots * (f \circ h_1)$ therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$
- Hence a functor $\overrightarrow{\pi_1}: \mathbf{LpoTop} \to \mathbf{Cat}$
- $-\overrightarrow{\pi_1}(A\times B)\cong\overrightarrow{\pi_1}A\times\overrightarrow{\pi_1}B$

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- By definition $\overrightarrow{\pi_1}X=PX/\sim$
- $f \circ (h_n * \cdots * h_1) = (f \circ h_n) * \cdots * (f \circ h_1)$ therefore $\gamma \sim \delta$ implies $f \circ \gamma \sim f \circ \delta$
- Hence a functor $\overrightarrow{\pi_1}: \mathbf{LpoTop} \to \mathbf{Cat}$
- $-\overrightarrow{\pi_1}(A\times B)\cong\overrightarrow{\pi_1}A\times\overrightarrow{\pi_1}B$
- for all dipaths $\gamma : [0, r] \to X$ for all θ morphisms from [0, r'] onto [0, r], $\gamma \sim \gamma \circ \theta$

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of the *n*-cube

-
$$\mathsf{Obj}(\overrightarrow{\pi_1}[0,1]) = [0,1]$$

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of the *n*-cube

$$\begin{array}{l} - \ \operatorname{Obj}(\overrightarrow{\pi_1}[0,1]) = [0,1] \\ - \ (\overrightarrow{\pi_1}[0,1])[a,b] = \left\{ \begin{array}{l} \{(a,b)\} & \text{if } a \leqslant b \\ \emptyset & \text{otherwise} \end{array} \right. \end{array}$$

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of the *n*-cube

-
$$\operatorname{Obj}(\overrightarrow{\pi_1}[0,1]) = [0,1]$$

$$- (\overrightarrow{\pi_1}[0,1])[a,b] = \begin{cases} (a,b) & \text{if } a \leq b \\ \emptyset & \text{otherwise} \end{cases}$$

$$-\overrightarrow{\pi_1}[0,1]^n=([0,1],\leqslant)^n=([0,1]^n,\leqslant^n)$$

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The fundamental category of the realization of a graph G as a local pospace

- A presentation is given by the graph

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The fundamental category of the realization of a graph G as a local pospace

 A presentation is given by the graph vertex: V ⊔ A×]0,1[

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The fundamental category of the realization of a graph *G* as a local pospace

- A presentation is given by the graph vertex: $V \sqcup A \times]0,1[$ arrows: (t, α, t') with α arrow of G and $t < t' \in [0,1]$ $\partial^{-}(t, \alpha, t') = (\alpha, t)$ if t > 0; $\partial^{-}\alpha$ otherwise $\partial^{+}(t, \alpha, t') = (\alpha, t')$ if t < 1; $\partial^{+}\alpha$ otherwise MSC - Lyon 2014

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of the realization of a graph G as a local pospace

A presentation is given by the graph vertex: V ⊔ A×]0,1[arrows: (t, α, t') with α arrow of G and t < t' ∈ [0,1] ∂(t, α, t') = (α, t) if t > 0; ∂⁻α otherwise ∂⁺(t, α, t') = (α, t') if t < 1; ∂⁺α otherwise
with the relations (t', α, t") ∘ (t, α, t') = (t", α, t) for α arrow of G and t < t' < t" ∈ [0,1]

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of the directed circle

- $\mathbb{S}^1 = \{z \in \mathbb{C} \mid z \text{ of magnitude } 1\}$

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of the directed circle

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$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid z \text{ of magnitude } 1\}$$

- $\mathsf{Obj}(\overrightarrow{\pi_1} \mathbb{S}^1) = \mathbb{S}^1$

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$$\mathbb{S}^1 = \{z \in \mathbb{C} \mid z \text{ of magnitude } 1\}$$

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$$\overrightarrow{\pi_1}\mathbb{S}^1[a,b]\cong\{a\} imes\mathbb{N} imes\{b\}$$

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of the directed circle

$$\begin{array}{l} - \mathbb{S}^1 = \{z \in \mathbb{C} \mid z \text{ of magnitude } 1\} \\ - \operatorname{Obj}(\overrightarrow{\pi_1} \mathbb{S}^1) = \mathbb{S}^1 \\ - \overrightarrow{\pi_1} \mathbb{S}^1[a, b] \cong \{a\} \times \mathbb{N} \times \{b\} \end{array}$$

$$(b,m,c)\circ(a,n,b)=\left\{egin{array}{cc} (a,n+m,c) & ext{if } ab\cup bc
eq\mathbb{S}^1\\ (a,n+m+1,c) & ext{otherwise} \end{array}
ight.$$

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of the directed complex plane

- The directed complex plane is not a local pospace yet it contains the directed circle

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$$\begin{array}{l} - \operatorname{Obj}(\overrightarrow{\pi_1}\mathbb{C}) = \mathbb{C} \\ - \overrightarrow{\pi_1}\mathbb{C}[a,b] \cong \begin{cases} \{a\} \times \mathbb{N} \times \{b\} & \text{if } a \neq 0 \text{ and } |a| \leqslant |b| \\ \{(0,b)\} & \text{if } a = 0 \\ \emptyset & \text{otherwise} \end{cases}$$

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- The fundamental category of the directed Riemann sphere is analoguous

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of cubical areas - a conjecture

- Cubical area X: finite union of *n*-cubes

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Fundamental categories

of cubical areas - a conjecture

- Cubical area X: finite union of *n*-cubes
- There exists a finite family K of sub-cubical areas of X such that ∀γ, δ dipaths on X sharing their extremities, γ ~ δ iff ∀K ∈ K s.t. img(γ) ⊆ K ⇔ img(δ) ⊆ K

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- it fails if $\overrightarrow{\pi_1}X$ contains loops

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