Introduction to Directed Algebraic Topology with a view towards modelling Concurrency II

Mathematical Structures of Computations - Lyon 2014

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The 29th of January

Summary

Diagram in **Top**

from a precubical set K

- $\partial_i^- \cong (x \cdots x \underbrace{0}_{i^{th}} x \cdots x)$ and $\partial_i^+ \cong (x \cdots x \underbrace{1}_{i^{th}} x \cdots x)$

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- for all $n \in \mathbb{N}$ for all $x \in K_n$ for all $i \in \{0,\ldots,\text{n-1}\}$
  and for $\varepsilon \in \{0,1\}$ we have the inclusion map

$$\begin{array}{rccc} \phi^{\varepsilon}_{i,n,x} : & \{\partial^{\varepsilon}_i(x)\} \times [0,1]^{n-1} & \rightarrow & \{x\} \times [0,1]^n \\ & (t_1,\ldots,t_{n-1}) & \mapsto & (t_1,\ldots,t_{i-1},\varepsilon,t_i,\ldots,t_{n-1}) \end{array}$$

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- $|K|$: the geometric realization of K is the colimit of this diagram in **Top**

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- $|K|$: the geometric realization of K is the colimit of this diagram in **Top**
- for all K, K' precubical sets, $|K \otimes K'| \cong |K| \times |K'|$

Geometric realization in **Top**

a calculation

- $K_0 = \{a, b\}$ and $K_1 = \{\alpha, \beta\}$
  $\partial^- \alpha = \partial^- \beta = a$ and $\partial^+ \alpha = \partial^+ \beta = b$

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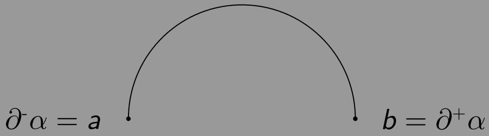

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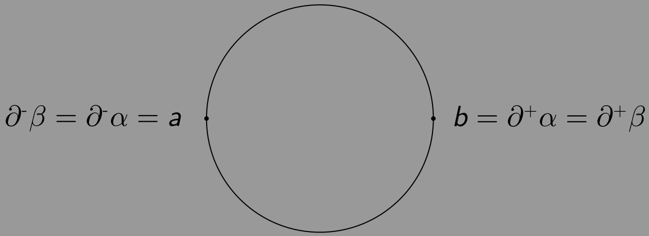

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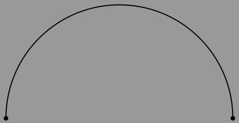
$\partial^- \alpha = a$  $b = \partial^+ \alpha$

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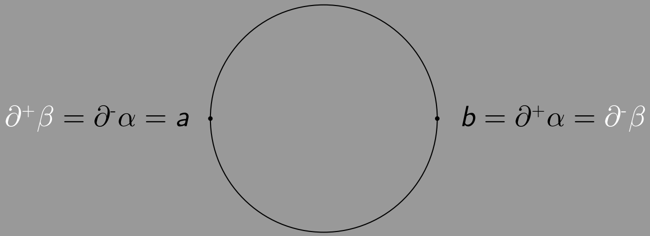

Partially Ordered Spaces - pospaces

Eilenberg 41 / Nachbin 48

- A topological space X together with a closed partial order

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- A topological space X together with a closed partial order
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- e.g. $\mathbb{R}$ with its standard topology and order
- Potop is complete and cocomplete
  but its colimits do not preserve the topology

Directed geometric realization in **PoTop**

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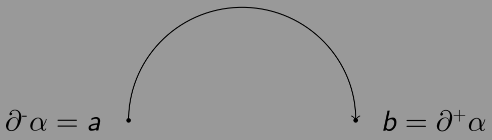

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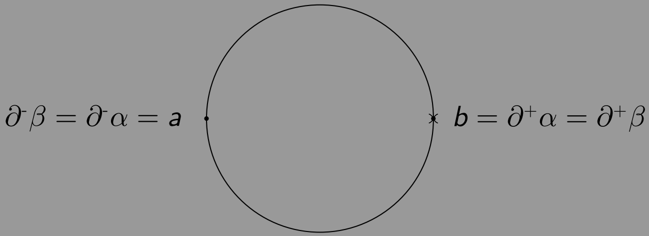

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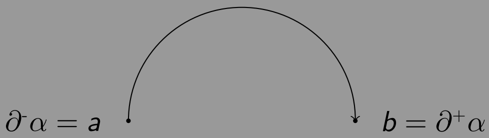

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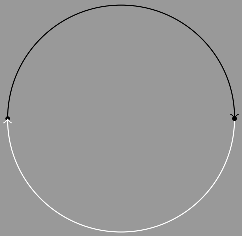
$\partial^+ \beta = \partial^- \alpha = a$  $b = \partial^+ \alpha = \partial^- \beta$

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just one point remains

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- ordered chart on X: pospace over some open subset of X
- ordered atlas on X: collection $\mathcal{U}$ of ordered charts s.t.
  i) for all $U, U' \in \mathcal{U}$ and $x \in U \cap U'$ there exists $U'' \in \mathcal{U}$ s.t. $x \in U'' \subseteq U \cap U'$ and the order on U'' matches both orders on U and U'
  ii) $\mathcal{U}$ induces a basis of topology of X

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- morphism of atlases $f : (X, \mathcal{U}) \to (Y, \mathcal{V})$: a continuous map $f : X \to Y$ such that for all $x \in X$ there exists $U \in \mathcal{U}$, $V \in \mathcal{V}$ neighborhoods of x and $f(x)$ such that f induces a morphism of pospaces from U to V

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- If $\mathcal{U} \sim \mathcal{U}'$, $\mathcal{V} \sim \mathcal{V}'$, and $f : \mathcal{U} \to \mathcal{V}$ morphism of atlases then $f : \mathcal{U}' \to \mathcal{V}'$ morphism of atlases

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- e.g. the exponential map $t \in \mathbb{R} \mapsto \mathrm{e}^{it} \in \mathbb{S}^1$
- Lpotop is finitely complete but
  misses some infinite products
  its cocompleteness is an open question
  its colimits do not preserve the topology

Directed geometric realization in **LpoTop**

a claim

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$$U(\upharpoonleft K \downharpoonright_{\mathbf{LpoTop}}) = |K|$$

- therefore

$$\upharpoonleft K \otimes K' \downharpoonright_{\mathbf{LpoTop}} \cong \upharpoonleft K \downharpoonright_{\mathbf{LpoTop}} \times \upharpoonleft K' \downharpoonright_{\mathbf{LpoTop}}$$

Realization of graphs

as local pospaces

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  for all $\alpha \in A$ such that $\partial^-\alpha = v$ and $0 < \varepsilon < 1$
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- directed atlas
  $\{\alpha\}\times]a,b[$ with $\alpha \in A$ and $0 \leqslant a < b \leqslant 1$, and
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  with obvious partial order
- denoted by $\upharpoonleft G \downharpoonright$

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virtual machine

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- for any execution trace s, there exists a dipath γ such that $[\![\gamma]\!] = s$

Continuous parallel
dynamics

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- if the process is conservative then for any δ, $[\![\gamma]\!] \cdot \delta$ only depends on $\partial^{-}\gamma$ and $\partial^{+}\gamma$
- therefore we have a potential function $F : \lceil G \rfloor \times \mathcal{R} \to \mathbb{N}$

Areas

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- $(G_1, \ldots, G_d)$-areas: finite union of blocks
- The collection of $(G_1, \ldots, G_d)$-areas forms a boolean subalgebra of $2^{\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_d \downharpoonright}$

Race conditions

conflicts in variable access

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- Race conditions is the subset of $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_d \downharpoonleft$ s.t. there is $1 \leqslant i < j \leqslant d$ such that $\lambda(v_i)$ and $\lambda(v_j)$ are actions sharing some variable

Forbidden area

via potential function

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- $F : \upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_d \downharpoonleft \times \mathcal{R} \to \mathbb{N}$ the potential function

$$F(v_1, \ldots, v_d, x) = \sum_{k=1}^{d} F_k(v_k, x)$$

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- Forbidden area is the subset of $\rceil G_1 \lfloor \times \cdots \times \rceil G_d \lfloor$

$$\{(v_1, \ldots, v_d) \mid \exists x \in \mathcal{R},\ F(v_1, \ldots, v_d, x) \geqslant \text{arity}(x)\}$$

Walls

and geometric model

- Walls is the subset of $(v_1, \ldots, v_d) \in \upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_d \downharpoonright$ s.t.
  there exists a synchronization x s.t.
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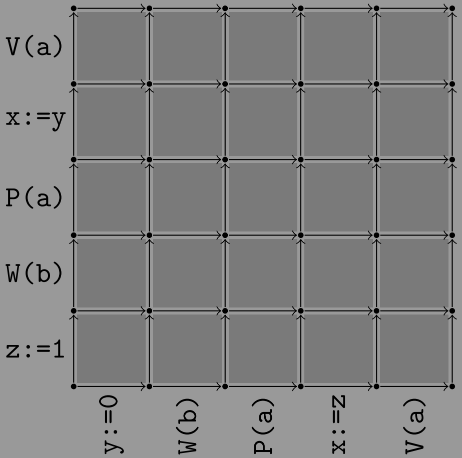
  there exists a synchronization x s.t.
  the cardinal of $\{k \in \{1, \ldots, d\} \mid \lambda(v_k) = W(x)\}$
  is neither 0 nor the arity of x
- The geometric model is then defined as
  $\upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_d \downharpoonleft \setminus (\text{Race} \cup \text{Forbidden} \cup \text{Walls})$

Geometric model: an example

y:=0.W(b).P(a).x:=z.V(a)|z:=0.W(b).P(a).x:=y.V(a)



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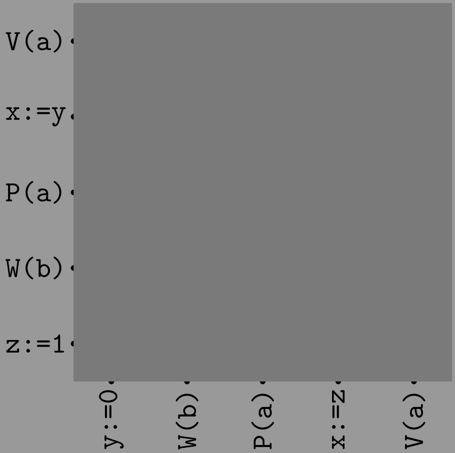

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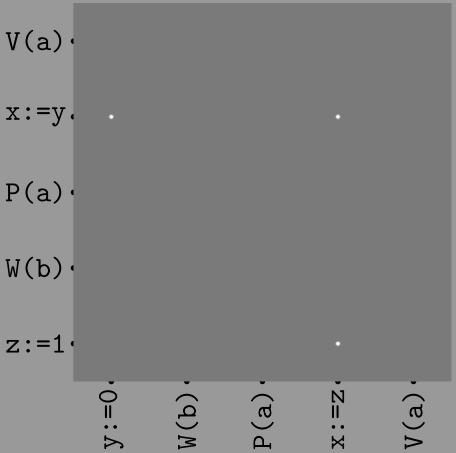

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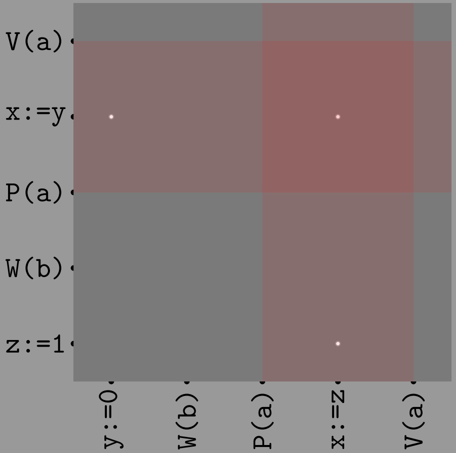

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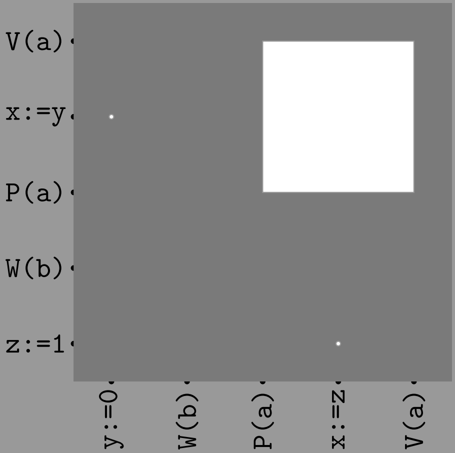

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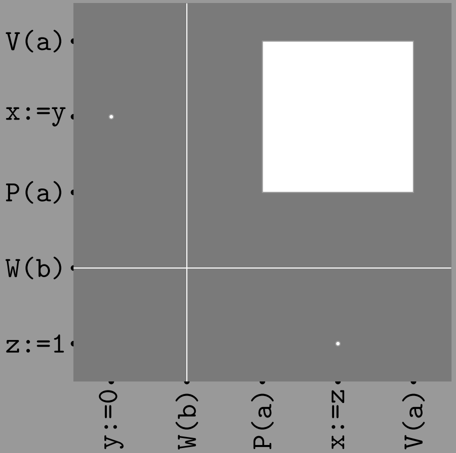

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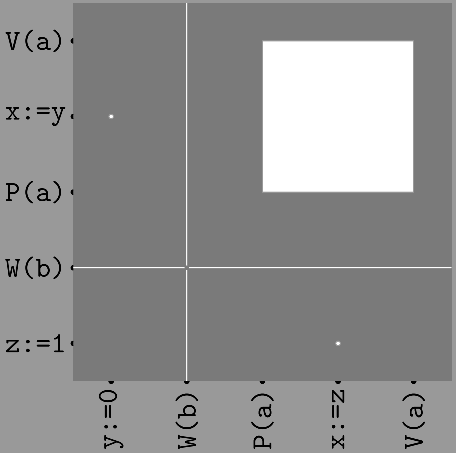
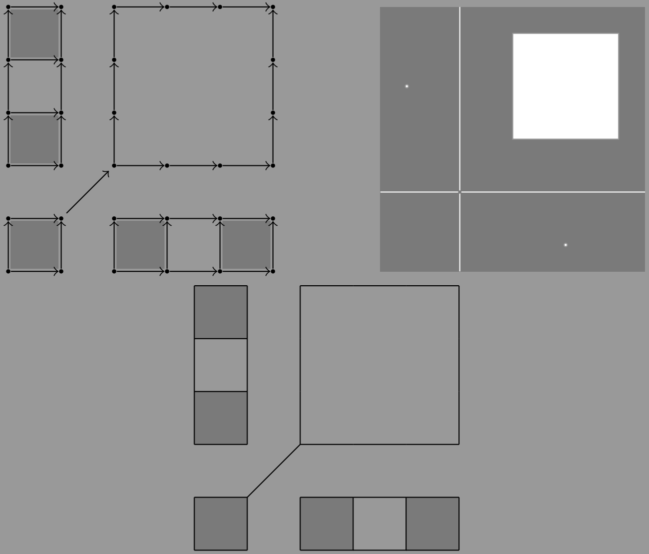


Comparing

Discrete vs Continuous



Fundamental category

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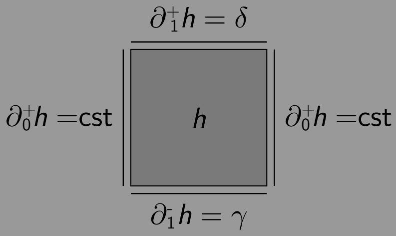

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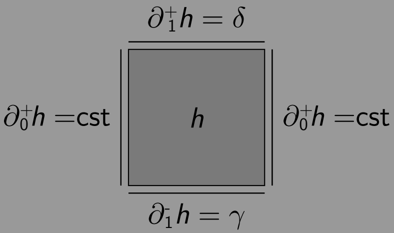

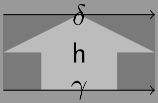

...and even more.

Dihomotopy

2-category

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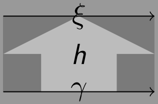

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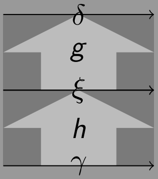

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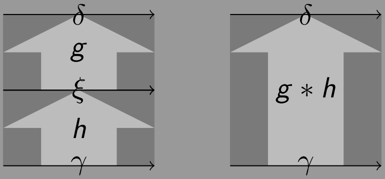

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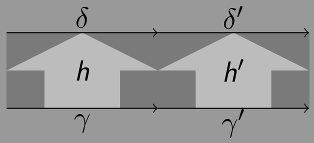

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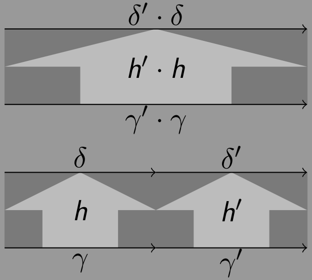
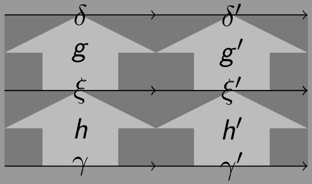


Godement

Exchange property



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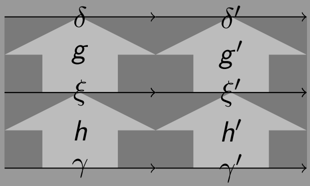

$$(g' * h') \cdot (g * h) = (g' \cdot g) * (h' \cdot h)$$

Directed topology vs Category

2-category

Directed topology	Category
point	category
dipath	functor
dihomotopy	natural transformation
path concatenation	composition of functors
‘piled up’ homotopies	composition of natural transformations
‘side-by-side’ homotopies	juxtaposition of natural transformations

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The dihomotopy relation

γ and δ dipaths defined over $[0, r]$ and $[0, r']$

- Write $\gamma \sim \delta$ when $\partial^{-}\gamma = \partial^{-}\delta$, $\partial^{+}\gamma = \partial^{+}\delta$ and there is an elementary homotopy between $c \cdot \gamma$ and $d \cdot \delta$ where c (resp. d) is constant over $[0, (r \vee r') - r]$ (resp. $[0, (r \vee r') - r']$)

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- it fails if $\overrightarrow{\pi_1} X$ contains loops