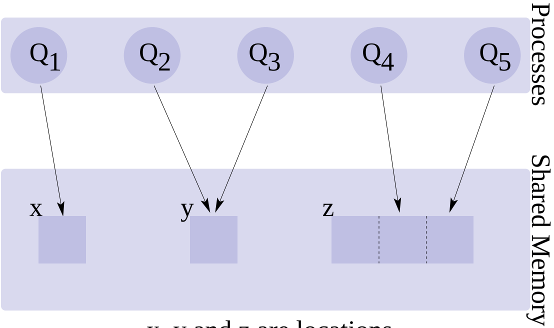
Directed Algebraic Topology and Concurrency

Eric Goubault and Emmanuel Haucourt

GEOCAL 2006 Marseille

Concurrency and Geometry ?

shared memory style

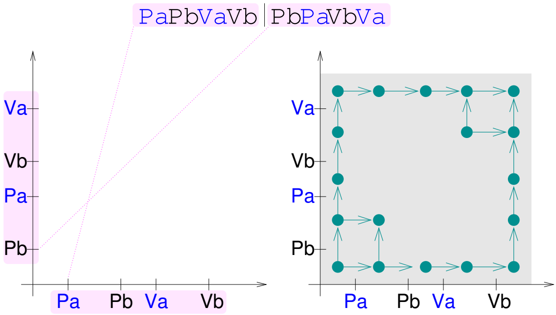


x, y and z are locations

Not sequential programs, bad states, chaotic behavior
$\Longrightarrow$ Need for synchronizations $\Longrightarrow$ Need for locks
$\Longrightarrow$ deadlocks might appear.

First Model

directed graphs of actions

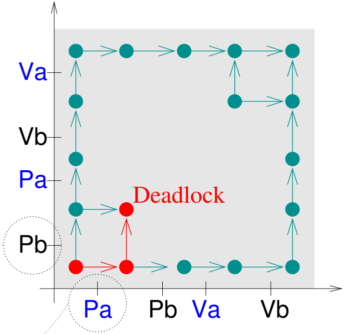


18 states and 20 arrows

A potential execution

program T_1 =PaPbVaVb| T_2 =PbPaVbVa

T_1	T_2
Pa	–
–	Pb

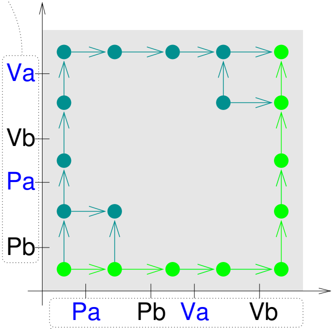


Deadlock

Anoter potential execution

program T_1 =PaPbVaVb| T_2 =PbPaVbVa

T_1	T_2
Pa	—
Pb	—
Va	—
Vb	—
—	Pb
—	Pa
—	Vb
—	Va



Termination

Notice that...

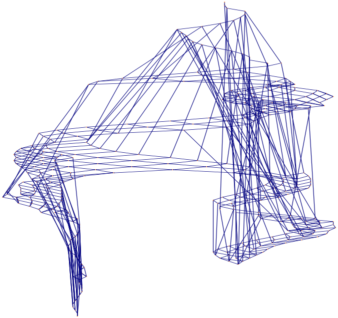
... there are very few "interesting" paths

Suppose $T_1 = Pa(a = a + 1)Pb(b = b + 1)VaVb$, $T_2 = Pb(b = b - 1)Pa(b = 2 * b)VbVa$ and in the beginning $a = 1$ and $b = 2$, we have:

- 1 path "T_2 then T_1" which computes $\underline{b = 3}$ (2*(2-1)+1) and $\underline{a = 2}$.
- 1 path "T_1 then T_2" which computes $\underline{b = 4}$ (2*((2+1)-1)) and $\underline{a = 2}$.
- 2 "equivalent" paths near the diagonal: they do not "terminate" with $a = 2$ and $b = 1$.

Size explosion problem

Dekker's algorithm

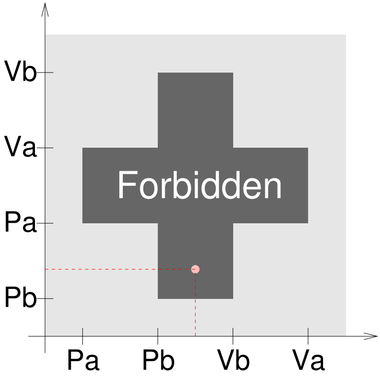


Few lines of C on 2 processes lead to few hundreds of paths, only 2 of which are interesting!

Geometry

"progress graphs" E.W.Dijkstra'68 (later V.Pratt'91)

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb

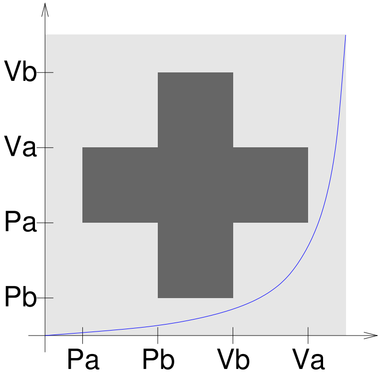


"Continuous model": x_i = local time; dark grey region=forbidden!

Execution paths

are continuous

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb

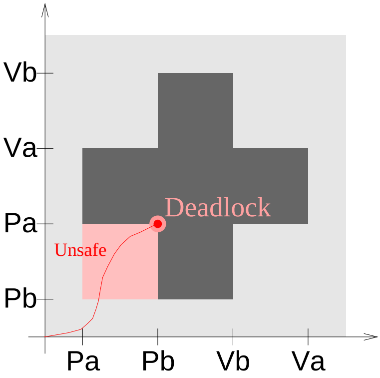


Traces are continuous paths increasing in each coordinate: dipaths.

Deadlocks and Unsafe regions

Swiss flag example

T1=Pa.Pb.Vb.Va in parallel with T2=Pb.Pa.Va.Vb



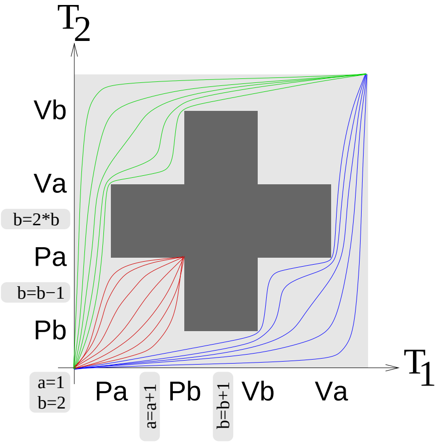
Entering the unsafe region $\Longrightarrow$ finishing to its deadlock.

Classes of equivalent dipaths
up to dihomotopy

T1 gets a and b before T2 => a=2 and b=4

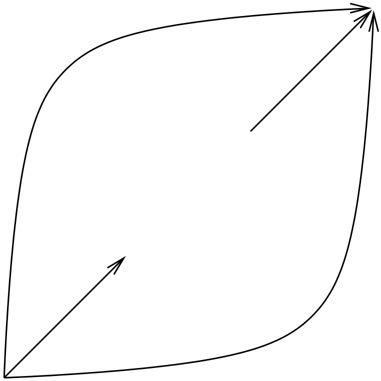
T2 gets b and a before T1 => a=2 and b=3

Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1



Ideally...

not quite true though

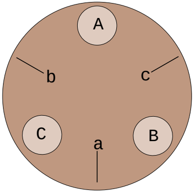


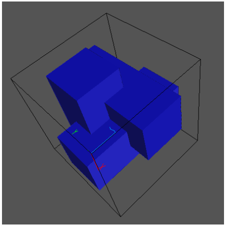
We will get back to this later.

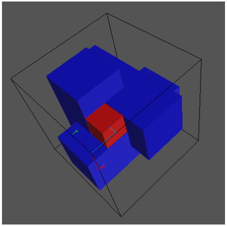
In higher-dimension

philosophers and chopsticks

A=Pb.Pc.Vb.Vc
B=Pc.Pa.Vc.Va
C=Pa.Pb.Va.Vb

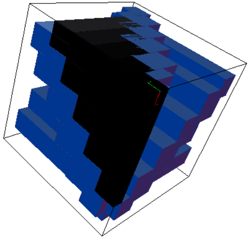




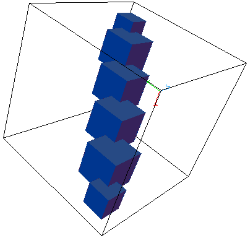


Effect of the level of sharing

```
A=Pa.Pb.Va.Pc.Vb.Pd.Vc.Pe.Vd.Pf.Ve.Vf
B=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va
C=Pf.Pe.Vf.Pd.Ve.Pc.Vd.Pb.Vc.Pa.Vb.Va
```



$a, \ldots$ binary sem.



$a, \ldots$ counting sem.

Correspondences
almost

Model [discrete]	combinatorial complex
Model [continuous]	topological space
Relation discrete/continuous	geometric realisation
Parallel composition	product
Action refinement	subdivision
Compositionality	Seifert/van Kampen
Deadlocks/reachability	connected components
Scheduling properties	fundamental group
Observational equivalence	homotopy equivalence (weak/strong)
Computable properties	topological invariants (homology etc.)

Other types of related subjects
and their applications

- Rewriting invariants (Squier like - see talks by Y. Lafont for instance)
- Fault-tolerant distributed systems (realizability and complexity, see M. Herlihy, S. Rajsbaum, N. Shavit etc.)

Models

- Po-spaces, local po-spaces, (pre-)cubical sets (see MFPS'98, with L. Fajstrup and M. Raussen)
- Globular CW-complexes: with P. Gaucher, "Topological Deformation of Higher-Dimensional Automata", *HHA 2003*
- Ω-categories, Category "Flow" (Philippe Gaucher)
- *d*-spaces (Marco Grandis)
- Higher-Dimensional Transition Systems (Vladimiro Sassone and Gian Luca Cattani, LICS'96)
- ECHIDNA (Richard Buckland and Michael Johnson, AMAST'96)
- Sanjeevi Krishnan's spaces
- *et cetera*

Partially Ordered Spaces

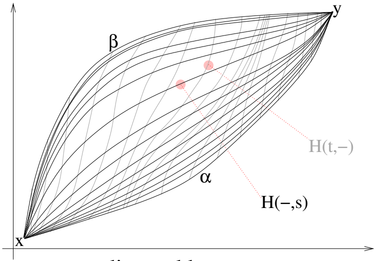
framework for "progress graphs" (one only needs *MFPS'98*)

A topological space X with a (global) closed partial order $\sqsubseteq$

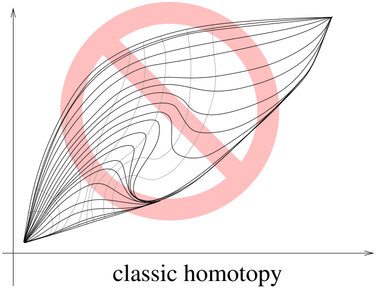
- Morphisms are increasing and continuous maps: dimaps
- (Finite) Traces on $(X, \sqsubseteq)$ are dimaps from $\vec{I} = ([0,1], \leq)$ to $(X, \sqsubseteq)$: dipaths
- Dihomotopies between dipaths α and β with fixed extremities x and y are dimaps $H : \overrightarrow{I} \times \overrightarrow{I} \rightarrow X$ such that for all $s \in \overrightarrow{I}$, $t \in \overrightarrow{I}$,
  - $H(t,0) = \alpha(t)$ and $H(t,1) = \beta(t)$
  - $H(0,s) = x$ and $H(1,s) = y$

Deformation of execution paths

dihomotopy *vs* homotopy



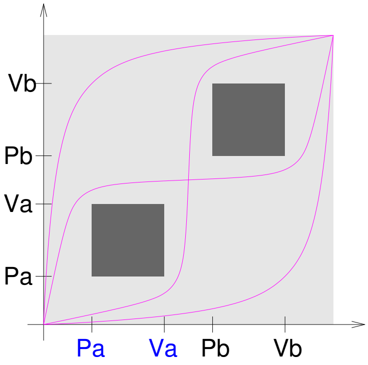
directed homotopy

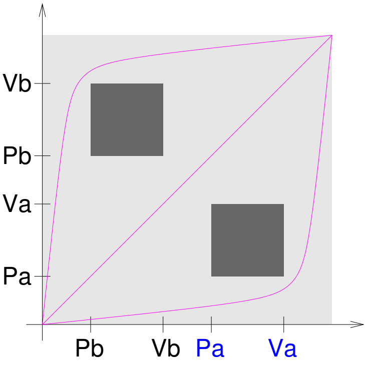


classic homotopy

First subtlety

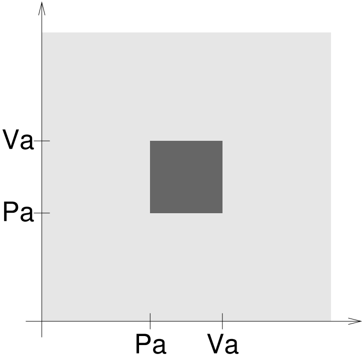
directed homotopy is not classic homotopy

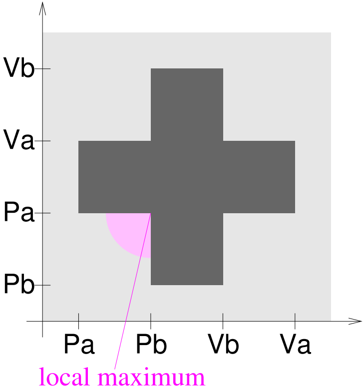




Second subtlety

classic homotopy cannot “see” local extrema

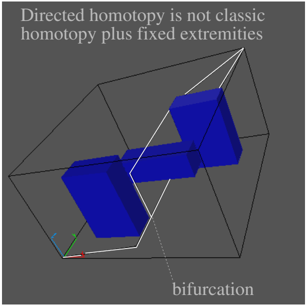


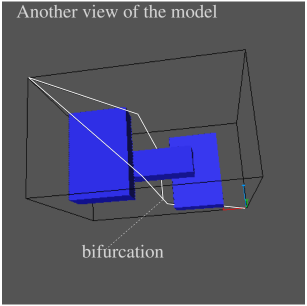


Third subtlety

Floating cube between two pillars

A=Pb.Pc.Vb.Vc
B=Pc.Pa.Vc.Va
C=Pa.Pb.Va.Vb





A typical object of study

fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$ of a pospace $\overrightarrow{X}$

- its objects are the points of X,
- its morphisms are the classes of dipaths up to dihomotopy: a morphism from x to y is a dihomotopy class $[\alpha]$ of a dipath α going from x to y.

A detailed example (1)

square with centered hole

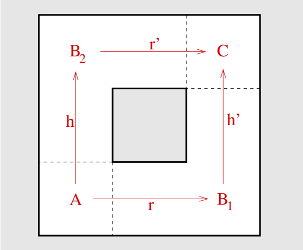
$x \in$	$y \in$	$\overrightarrow{\pi_1}(\overrightarrow{X})[x,y]$
A	A	$\{\sigma_{x,y}\}$
B_1	B_1	$\{\sigma_{x,y}\}$
B_2	B_2	$\{\sigma_{x,y}\}$
C	C	$\{\sigma_{x,y}\}$
A	B_1	$\{r_{x,y}\}$
A	B_2	$\{h_{x,y}\}$
B_1	C	$\{h'_{x,y}\}$
B_2	C	$\{r'_{x,y}\}$
B_1	B_2	$\emptyset$
B_2	B_1	$\emptyset$
A	C	$\{u_{x,y}\ ,\ d_{x,y}\}$

With

$r'_{y,z} \circ h_{x,y} = u_{x,z}$, $h'_{y,z} \circ r_{x,y} = d_{x,z}$

and 3 points x, y, z of the square such that $x \sqsubseteq y \sqsubseteq z$;

if $x \not\sqsubseteq y$ then $\overrightarrow{\pi_1}(\overrightarrow{X}) = \emptyset$.



A detailed example (2)

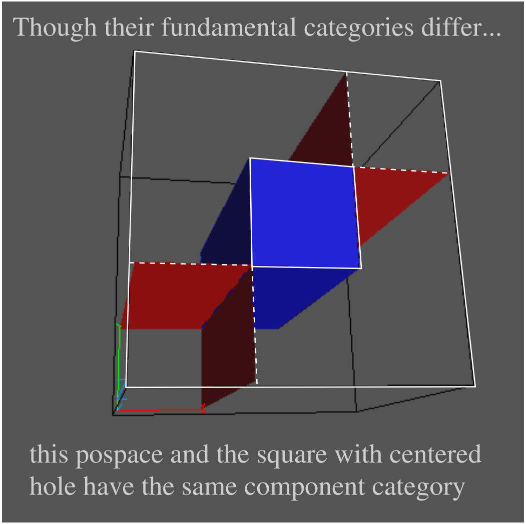
the previous calculation suggests that

- we have a partition A, B_1, B_2, C of the objects of $\overrightarrow{\pi_1}(\overrightarrow{X})$,
- any arrow of $\overrightarrow{\pi_1}(\overrightarrow{X})$ can be given a "type" (σ, h, h', r, r', u or d) according to the components its extremities x and y belong to,
- the type σ is "neutral" in the sense that $\sigma_{y,z} \circ \sigma_{x,y} = \sigma_{x,z}$
- the map which sends
  - any object x of $\overrightarrow{\pi_1}(\overrightarrow{X})$ to its component (A, B_1, B_2 or C)
  - any morphism α to its "type" (σ, h, h', r, r', u or d)

is both an equivalence and a fibration and its codomain is, by definition, the category of components of $\overrightarrow{X}$.

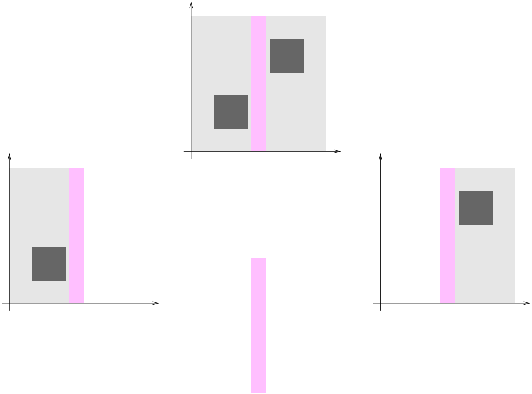
Example of product

parallel “independent” composition



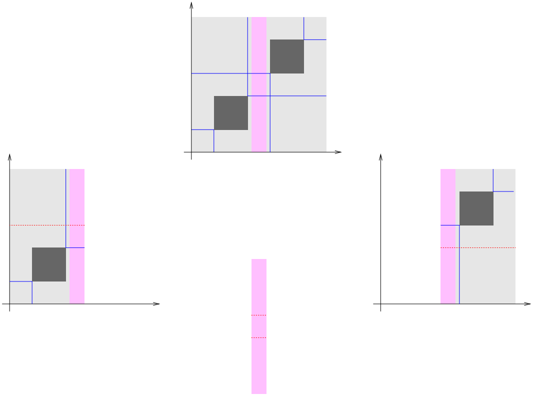
The Seifert/Van Kamen theorem for fundamental category

compositionality



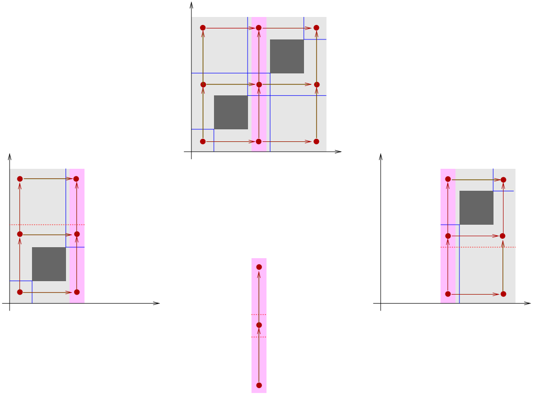
A Seifert/Van Kamen theorem for components category (1)

subdivisions are necessary



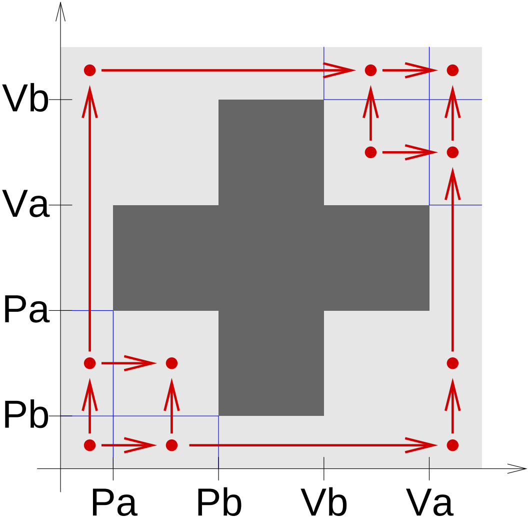
A Seifert/Van Kamen theorem for components category (2)

the resulting category of components



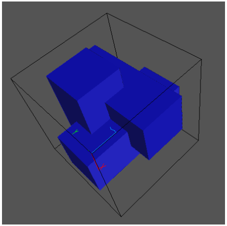
The bounding diagrams of the grey squares do not commute.

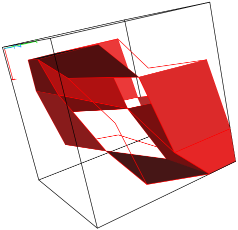
The category of components of the swiss flag



The components category of the 3 philosophers

non-orthogonal representation

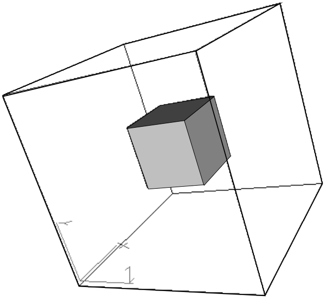




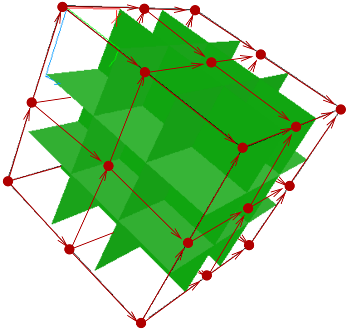
the pospace

its category of components

The components category of a 2-semaphore



the pospace



its category of components

Computations: some theoretical and practical tools

for handling concrete cases

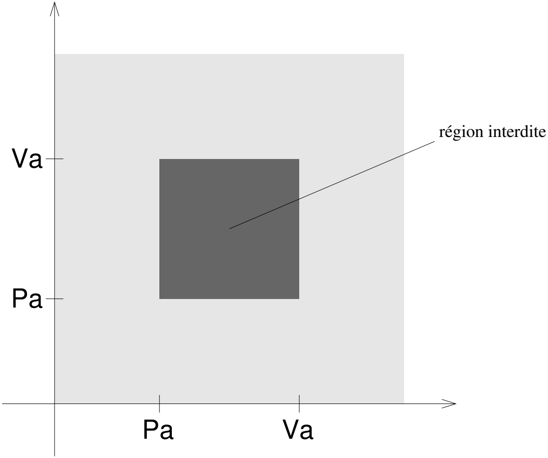
- We have a Seifert/van Kampen for local po-spaces (last ATMCS - or M. Grandis' proof)
- We also have a form of Seifert/van Kampen for components categories, "up to subdivision" (Emmanuel Haucourt), which is of value for practical computations.
- Also, some specific algorithms for mutual exclusion models (M. Raussen in dimension 2, and sub-optimal algorithm by E. Goubault in all dimensions).

Some figures
[Eric Goubault's algorithm]

- new3phil.pv: (0.05s) Objects: 27, Morphisms: 48, Relations: 18
- new4phil.pv: (0.07s) Objects: 85, Morphisms: 200, Relations: 132
- new7phil.pv: 147.36s; 81 Mo; (about one million transitions in a standard model) Objects: 2467, Morphisms: 10094, Relations: 15484
- new8phil.pv: 320.02s; 121Mo; (about 10 million transitions in a standard interleaving model) Objects: 3214, Morphisms: 14282, Relations: 24396

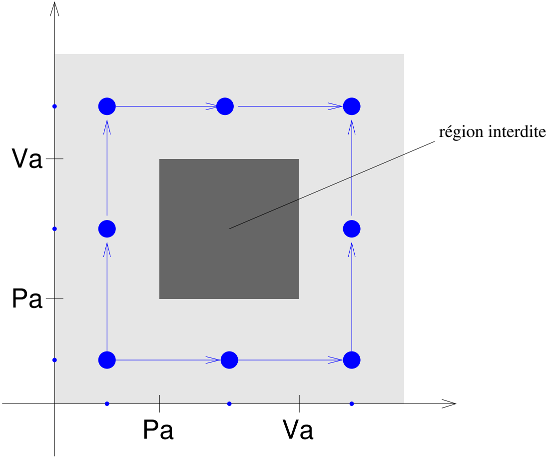
PaVa|PaVa

Dijkstra 68, *Pratt/van Glabbeek* 91, *Goubault* 92



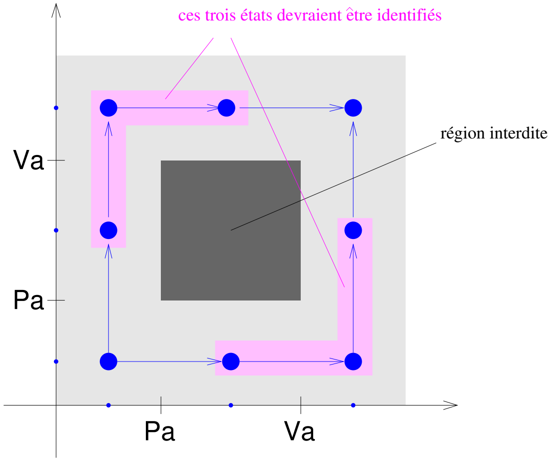
PaVa|PaVa

modèle discret classique



PaVa|PaVa

réduction



Pospace $\overrightarrow{X}$

Eilenberg 41, *Nachbin* 48 65, *Johnstone* 82

1. a topological space X,
2. a partial order $\sqsubseteq$ over $|X|$ whose graph is closed in $X \times X$.

Lemma: for any $x \in \overrightarrow{X}$, $\{y \in X | x \sqsubseteq y\}$ (denoted $\uparrow x$) is closed in X.

Morphisms of pospaces from $\overrightarrow{X}$ to $\overrightarrow{Y}$

A map $f : |X| \longrightarrow |Y|$ inducing:

1. a continuous map from X to Y and
2. an increasing map from $(|X|, \sqsubseteq_X)$ to $(|Y|, \sqsubseteq_Y)$.

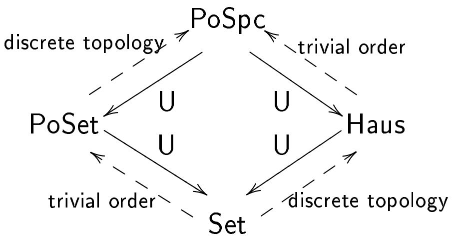
Hence the category of pospaces denoted: PoSpc.

Usual Pospaces

some common examples

1. directed real line $\mathbb{R}$ with its classical topology and order ($\overrightarrow{\mathbb{R}}$),
2. directed unit segment $[0,1]$ with the structure induced by $\overrightarrow{\mathbb{R}}$ ($\overrightarrow{[0,1]}$),
3. any morphism of PoSpc from $\overrightarrow{[0,1]}$ to $\overrightarrow{X}$ is called a directed path on $\overrightarrow{X}$. Formelly, the set of directed paths on $\overrightarrow{X}$ is $\mathrm{PoSpc}\left[\overrightarrow{[0,1]}, \overrightarrow{X}\right]$, also denoted $d\overrightarrow{X}$.

Forgetful functors PoSpc



Categorical properties of PoSpc

analogy between Top and PoSpc

1. complete and cocomplete,
2. symmetric monoidal closed,
3. compact pospaces is complete, cocomplete and admits $\overrightarrow{[0,1]}$ as a cogenerator,
4. the full sub-category of compactly generated pospaces is reflective in PoSpc and cartesian closed.

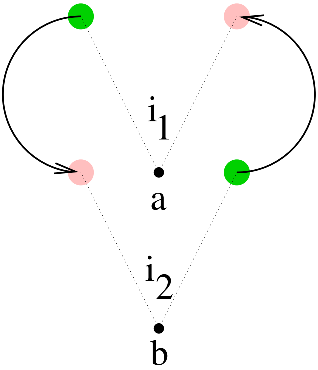
Cocompleteness of PoSpc

sketch of proof

1. Prove that the category RSpc admits quotients.
2. Use quotients of RSpc to prove its cocompletness.
3. Use quotients of RSpc to construct the reflect of any object of RSpc in PoSpc.
4. It is a general fact that any reflective subcategory of a cocomplete category is cocomplete, hence PoSpc is cocomplete.

A pushout in PoSpc (1)

the directed circle in PoSpc squashed to a point



a and b are not ordered

A pushout in PoSpc (2)

the directed circle in PoSpc squashed to a point

$i_1 : \{a, b\} \to \overrightarrow{[0,1]}$; $i_1(a) = 0$; $i_1(b) = 1$
$i_1 : \{a, b\} \to \overrightarrow{[0,1]}$; $i_2(a) = 1$; $i_2(b) = 0$
suppose $f, g : \{a, b\} \to \overrightarrow{[0,1]}$ with $f \circ i_1 = g \circ i_2$,
we have $f(i_1(a)) = g(i_2(a))$ i.e. $f(0) = g(1)$ and the same way $g(0) = f(1)$.
Hence $f(0) \sqsubseteq f(1) = g(0) \sqsubseteq g(1) = f(0) \Longrightarrow$
$f(0) = f(1) = g(0) = g(1)$
and then f and g are constant and equal.

Directed homotopy on $\overrightarrow{X}$ from α to β

Grandis 01, *Fajstrup/Raussen/Goubault* 98...

A morphism h defined on $\overrightarrow{[0,1]} \times \overrightarrow{[0,1]}$ with values in $\overrightarrow{X}$ such that $U(h)$ be a classic homotopy from $U(\alpha)$ to $U(\beta)$.
We denote $\sim_{\overrightarrow{X}}$ the symmetric and transitive closure of

$$\left\{(\alpha,\beta) \in d\overrightarrow{X} \times d\overrightarrow{X} \middle| \text{il existe une homotopie dirigée de } \alpha \text{ vers } \beta\right\}.$$

Two dipaths α and β are said dihomotopic when $\alpha \sim_{\overrightarrow{X}} \beta$.

Directed Homotopy *vs* classic homotopy

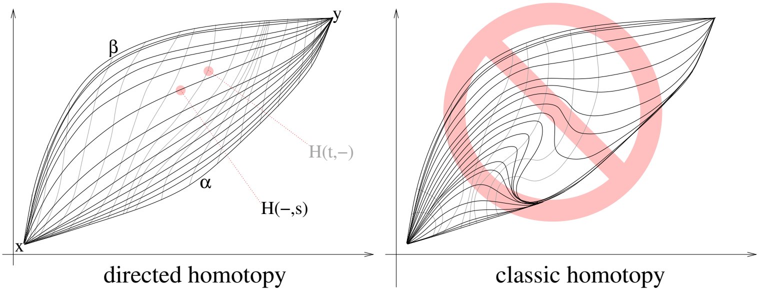


Image of a dipath

Singular facts about pospaces

1. The image of a dipath α on a pospace $\overrightarrow{X}$ is either isomorphic (in PoSpc) to $\overrightarrow{[0,1]}$ or $\{\bullet\}$.
2. Two dipaths sharing the same image are dihomotopic.
3. There is no directed *Peano* curve.

Fundamental category of a pospace $\overrightarrow{X}$

denoted $\overrightarrow{\pi_1}(\overrightarrow{X})$

1. objects: the elements de $|X|$,
2. morphisms from x to y: the set of $\sim_{\overrightarrow{X}}$-equivalence classes of

$$\left\{\alpha \in d\overrightarrow{X} \middle| \alpha(0) = x \text{ et } \alpha(1) = y\right\}$$

Loop-free categories

play the role of the groupoids

A (small) category $\mathcal{C}$ such that for any objects x and y of $\mathcal{C}$, if $\mathcal{C}[x,y] \neq \emptyset$ and $\mathcal{C}[y,x] \neq \emptyset$ then $x = y$ and $\mathcal{C}[x,x] = \{id_x\}$. We denote LfCat the full subcategory of Cat whose objects are the small loop-free category.

1. LfCat is cartesian closed and reflective in Cat.
2. The fundamental category of a pospace is loop-free, hence the functor

$$\mathrm{PoSpc} \xrightarrow{\vec{\pi}_1} \mathrm{LfCat}$$

$\overrightarrow{\pi_1}(\overrightarrow{X})$ is loop-free

proof

A morphism of $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the $\sim_{\overrightarrow{X}}$-equivalence class of some dipath α from x to y, hence $x \sqsubseteq y$; suppose that $\overrightarrow{\pi_1}(\overrightarrow{X})[y,x] \neq \emptyset$, then we also have $y \sqsubseteq x$ and then $x = y$. Further, if α is a dipath from x to x, then for any $t \in [0,1]$, we have
$x = \alpha(0) \sqsubseteq \alpha(t) \sqsubseteq \alpha(1) = x$ i.e. α is constant.

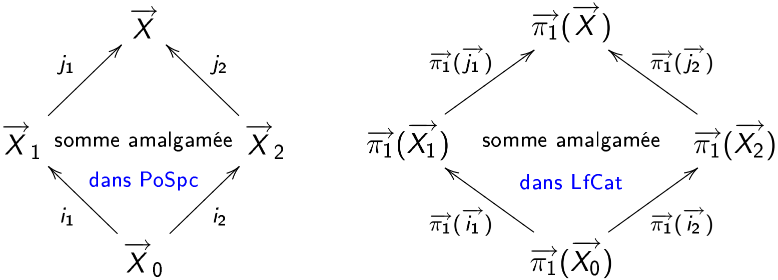
Sections and retractions

of a loop-free category $\mathcal{C}$

Suppose that $f_2 \circ f_1 = id_x$, the source and the target of f_1 and f_2 is x and then $f_1 = f_2 = id_x$. Hence the only isomorphisms of $\mathcal{C}$ are its identities and the collection of identities of $\mathcal{C}$ is pure in $\mathcal{C}$. In particular, the limits and colimits in a loop-free category are strictly unique and not only up to isomorphism.

Directed *Van Kampen* Theorem

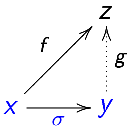
Grandis 01, *Goubault* 01



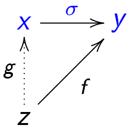
Yoneda morphism
axiomatizing the preservation of the future and the past (1)

Let $\mathcal{C}$ be a small category. A *Yoneda* morphism σ is an element of $\mathcal{C}[x,y]$ such that for all object z of $\mathcal{C}$,

future if $\mathcal{C}[y,z] \neq \emptyset$ then for all $f \in \mathcal{C}[x,z]$, there is a unique $g \in \mathcal{C}[y,z]$ such that



past if $\mathcal{C}[z,x] \neq \emptyset$ then for all $f \in \mathcal{C}[z,y]$, there is a unique $g \in \mathcal{C}[z,x]$ such that



Some properties of *Yoneda* morphisms

statements

- *Yoneda* morphisms compose
- if $\mathcal{C}$ is loop-free and $\sigma \in \mathcal{C}[x,y]$ is a *Yoneda* morphism, then $\mathcal{C}[x,y] = \{\sigma\}$
- any *Yoneda* morphism is a monomorphism and an epimorphism

Some properties of *Yoneda* morphisms

proofs

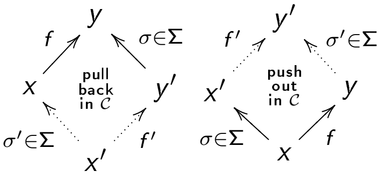
- *Yoneda* morphisms compose since injective maps compose as well as surjective ones.
- If σ is a *Yoneda* morphism, then the map $\gamma \in \mathcal{C}[y,y] \mapsto \gamma \circ \sigma \in \mathcal{C}[x,y]$ is a bijection; since $\mathcal{C}$ is loop-free, $\mathcal{C}[y,y] = \{id_y\}$, hence the result.
- A *Yoneda* morphism σ is an epimorphism since $\gamma \in \mathcal{C}[y,z] \mapsto \gamma \circ \sigma \in \mathcal{C}[x,z]$ is a bijection as soon as $\mathcal{C}[y,z] \neq \emptyset$, the same we prove that σ is a monomorphism.

Yoneda system of a small category $\mathcal{C}$

axiomatizing the preservation of the future and the past (2)

A collection Σ of morphisms of $\mathcal{C}$ such that:

1. Σ is stable under composition,
2. Σ contains all the isomorphisms of $\mathcal{C}$,
3. all the elements of Σ are *Yoneda* morphisms and
4. Σ is stable under change and cochange of base.



Pureness of *Yoneda* system Σ

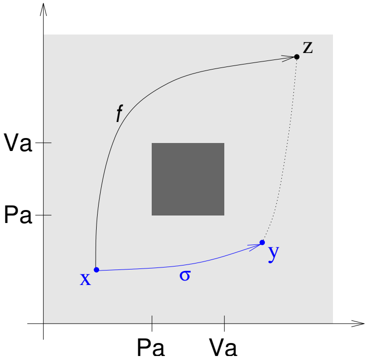
of a loop-free category $\mathcal{C}$

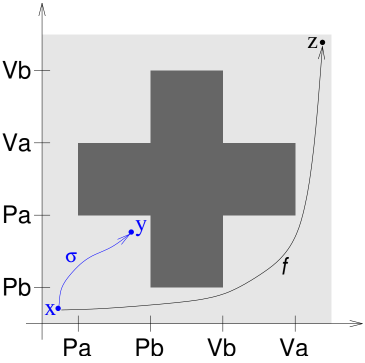
Suppose $f_2 \circ f_1 = \sigma \in \Sigma$, by cochange of base, we have the left hand side pushout below.

Still, f_1 in an epimorphism since so is σ; it follows that the right hand square above is also a pushout. By the strict uniqueness of the colimits of a loop-free category, f_1' is an identity and $f_2 = \sigma' \in \Sigma$. Using change of base, we prove that $f_1 \in \Sigma$ too.

Examples

of morphisms which do not belong to a *Yoneda* system





Locale of *Yoneda* systems

pointless topology on a small loop-free category

The collection of *Yoneda* systems of a small loop-free category, ordered by inclusion, forms a locale whose greatest and least elements are respectively denoted $\Sigma_\top$ and $\Sigma_\bot$. Besides $\Sigma_\bot$ is the collection of identities of $\mathcal{C}$.

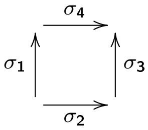
Σ-zigzags and Σ-components
of a loop-free category $\mathcal{C}$

A Σ-zigzag between two objects x and y of $\mathcal{C}$ is a finite sequence $(\sigma_n, \ldots, \sigma_0)$ $(n \in \mathbb{N})$ of morphisms of Σ such that there is a finite sequence $(z_0, \ldots, z_{n+1})$ of objects of $\mathcal{C}$ such that $z_0 = x$, $z_{n+1} = y$ and for all $k \in \{0, \ldots, n\}$, $\sigma_k \in \mathcal{C}[x_k, x_{k+1}] \cup \mathcal{C}[x_{k+1}, x_k]$.
Then we say that x and y are Σ related: thus we have an equivalence relation since Σ contains identities and is stable under composition.
The equivalence classes of this relation are called the Σ-components.

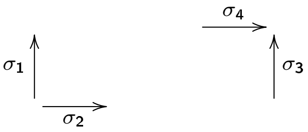
Fundamental theorem of the Σ-components

$\mathcal{C}$ loop-free and Σ *Yoneda* system of $\mathcal{C}$

Any Σ-component X of $\mathcal{C}$ ordered by $x \sqsubseteq y$ when $\mathcal{C}[x,y] \neq \emptyset$ is a lattice. Further given x, $y \in X$, $\mathcal{C}[x,y]$ is a singleton whose only element belongs to Σ. Finally, any square of arrows of Σ



is both at once the pushout of the left hand side diagram and the pullback of the right hand side diagram below



lattice = the l.u.b. and the g.l.b. of any pair of elements of X exists

Components of compact pospaces

statement

- If $\overrightarrow{K}$ is a compact pospace such that any pair of element of K has an upper/lower bound ($\vee$-lattice/$\wedge$-lattice), then $\overrightarrow{K}$ has a greatest/least element.
- If $\overrightarrow{K}$ is a compact pospace, then any component of $\overrightarrow{\pi_1}(\overrightarrow{K})$ has both a greatest lower bound and an least upper bound in $(|K|, \sqsubseteq)$.

Components of compact pospaces

proof

- Suppose $\overrightarrow{K}$ does not have a greatest element, then $K = \bigcup_{x \in K} (\uparrow x)^c$. Still, for K is compact and $(\uparrow x)^c$ is open, we have $K = \bigcup_{x \in F} (\uparrow x)^c$ for some finite $F \subseteq K$, but F has an upper bound $\top$ in K and thus $K = (\uparrow \top)^c$, which is a contradiction.
- C is a lattice, then, for K is compact we know [*Nachbin*] that the topological closure of $\downarrow D$ is a $\vee$-lattice and a compact subset of K so we can apply the first point.

Category of components

directed counterpart of the collection of arcwise connected components

The category of components of a small loop-free category $\mathcal{C}$ is then quotient category $\mathcal{C}/_{\Sigma_\top}$.

Fundamental theorem

fractions *vs* quotients

Let $\mathcal{C}$ be a small loop-free category and Σ a *Yoneda* system of $\mathcal{C}$, then

1. the collection Σ is pure in $\mathcal{C}$,
2. the small category $\mathcal{C}/_\Sigma$ is loop-free,
3. the small categories $\mathcal{C}[\Sigma^{-1}]$ and $\mathcal{C}/_\Sigma$ are equivalent and
4. the category $\mathcal{C}[\Sigma^{-1}]$ is fibered over $\mathcal{C}/_\Sigma$.

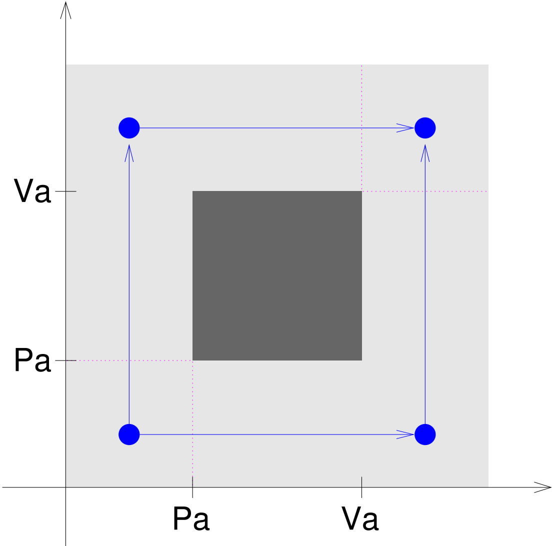
extension and improvement of *Components of the Fundamental Category* - APCS 04

Pureness

of a collection of morphisms

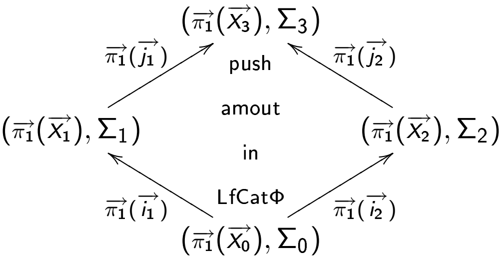
A collection Σ of morphisms of a category $\mathcal{C}$ is said pure in $\mathcal{C}$ when for all morphisms f_2, f_1 of $\mathcal{C}$, if $f_2 \circ f_1 \in \Sigma$ then f_2, $f_1 \in \Sigma$.

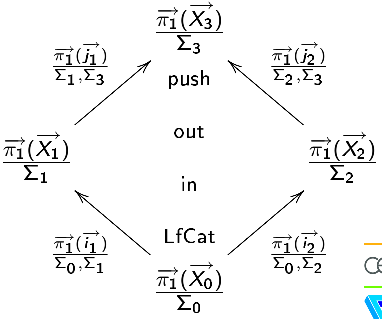
Example



Van Kampen theorem
for categories of components (1)

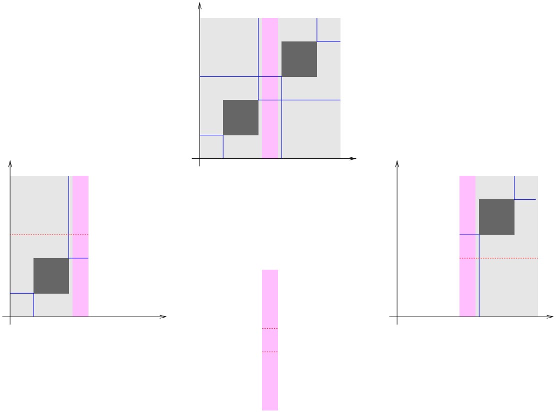
Let Σ_1 and Σ_2 be two *Yoneda* systems of $\vec{\pi}_1(\vec{X_1})$ and $\vec{\pi}_1(\vec{X_2})$. Suppose that $\Sigma_3 := \vec{\pi}_1(\vec{j_1})(\Sigma_1) \uplus \vec{\pi}_1(\vec{j_2})(\Sigma_2)$ is a *Yoneda* system of $\vec{\pi}_1(\vec{X_3})$ and that $\vec{\pi}_1(\vec{i_1})(\Sigma_0) \subseteq \Sigma_1$ et $\vec{\pi}_1(\vec{i_2})(\Sigma_0) \subseteq \Sigma_2$, then



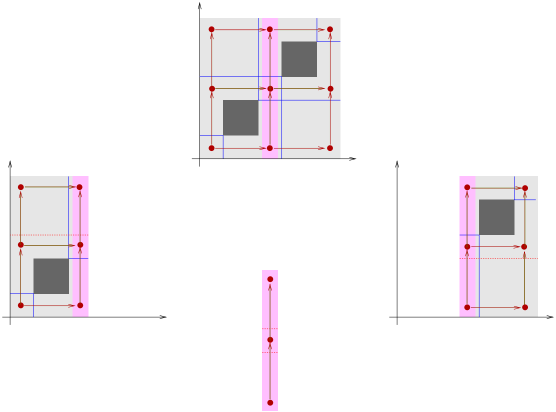


Van Kampen theorem

for categories of components: subdivisions (2)



Van Kampen theorem
for categories of components: subdivisions (3)



Generic segment of C

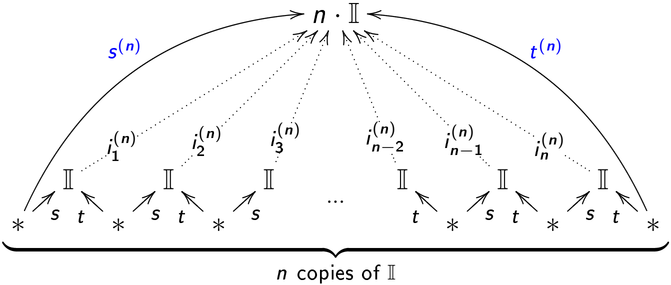
axiomatizing the notion of *Moore* paths (1)

A generic segment of C is a triple $(\mathbb{I}, s, t)$ where $\mathbb{I}$ is an object of C and s, t two points of $\mathbb{I}$ such that:

1. for any automorphism ϕ of $\mathbb{I}$ we have

$$\{\phi \circ s, \phi \circ t\} = \{s, t\}$$

2. and for any $n \in \mathbb{N}$ we have the colimit



Directed generic segment

axiomatization of the notion of direction

- A generic segment $(\mathbb{I}, s, t)$ is said directed when for any automorphism ϕ of $\mathbb{I}$, we have $\phi \circ s = s$ and $\phi \circ t = t$.
- Any automorphism ϕ of $\mathbb{I}$ such that $\phi \circ s = t$ and $\phi \circ t = s$ is called an inversion of (the) time (flow)
- In PoSpc, the generic segment $\overrightarrow{[0,1]}$ is directed while the generic segment $([0,1], =)$ does not.

the map $t \mapsto 1 - t$ is an inversion of time

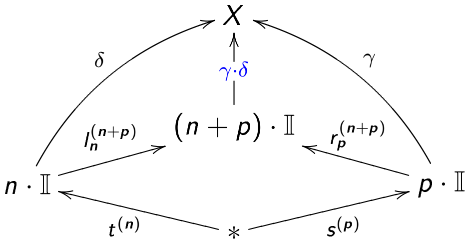
Category of paths on an object X of C

axiomatization of the notion of *Moore* path (2)

The objects of this category, denoted $\Gamma(X)$, are the points of X and its morphisms, called the paths on X, are the elements of

$$\bigcup_{n\in\mathbb{N}} \mathsf{C}[n\cdot\mathbb{I}, X],$$

the source and the target of $\gamma \in \mathsf{C}[n\cdot\mathbb{I}, X]$ are $\gamma\circ s^{(n)}$ and $\gamma\circ t^{(n)}$; the concatenation being given by the push-out:



Homotopic congruence over C

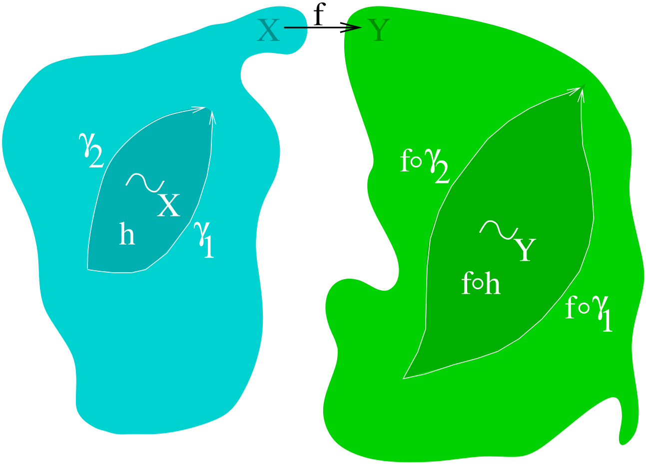
axiomatization of the notion of (di)homotopic (di)paths

A path $\gamma \in \mathcal{C}\left[n \cdot \mathbb{I}, X\right]$ is said constant when it can be written $\gamma = p \circ \mu$ where p is a point of X, it is the value of γ.
A homotopic congruence on C is defined by, for each object X of C, a congruence $\sim_X$ on the category of paths on X, such that for all paths γ_1 and γ_2 on X,

1. if γ_1 and γ_2 are constant with the same value, then $\gamma_1 \sim_X \gamma_2$,
2. if $\gamma_1 \sim_X \gamma_2$, then
    1. γ_1 and γ_2 share the same extremities and
    2. for all morphism f of C from X to Y we have $f \circ \gamma_1 \sim_Y f \circ \gamma_2$.

Homotopic congruence

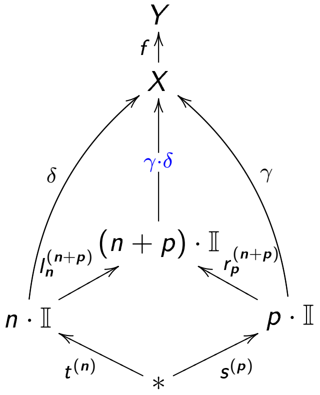
in picture



Think of $\sim_X$ as "there exists a classic homotopy h from the paths γ_1 to γ_2"

Generalized fundamental category

We set $\overrightarrow{\pi_1}(\overrightarrow{X}) := \Gamma(X)/\sim_X$ and we have a functor $\overrightarrow{\pi_1} : \mathsf{C} \longrightarrow \mathsf{Cat}$.



Since $\gamma_1 \sim_X \gamma_2$ implies $f \circ \gamma_1 \sim_Y f \circ \gamma_2$, we can define $\overrightarrow{\pi_1}(\overrightarrow{f})[\gamma]_{\sim_X} := [f \circ \gamma]_{\sim_Y}$, moreover, the left hand side diagram shows that we have $f \circ (\gamma \cdot \delta) = (f \circ \gamma) \cdot (f \circ \delta)$ whence the functoriality of $\overrightarrow{\pi_1}(\overrightarrow{f})$ from $\overrightarrow{\pi_1}(\overrightarrow{X})$ to $\overrightarrow{\pi_1}(\overrightarrow{Y})$.

directed *vs* undirected generic segment

in the framework of PoSpc

- With the generic segment $([0,1],=)$ over PoSpc, for any pospace $\overrightarrow{X}$, $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the fundamental groupoid of X.
- With the generic segment $([0,1],\leq)$ over PoSpc, for any pospace $\overrightarrow{X}$, $\overrightarrow{\pi_1}(\overrightarrow{X})$ is the fundamental category of $\overrightarrow{X}$.