Directed Algebraic Topology and Concurrency

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CEA LIST,
Modeling and Analysing Interaction between Systems Laboratory

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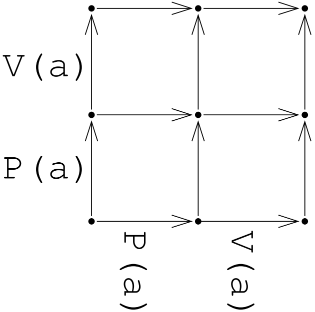
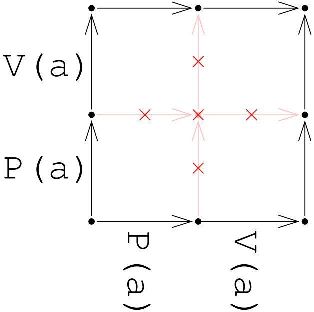
$$\begin{array}{c} A \\ \mathsf{s}\,\downdownarrows\,\mathsf{t} \\ V \end{array}
\qquad
\begin{array}{ccc} A & \xrightarrow{\phi_1} & A' \\ \mathsf{s}\,\downdownarrows\,\mathsf{t} & & \mathsf{s}'\,\downdownarrows\,\mathsf{t}' \\ V & \xrightarrow[\phi_0]{} & V' \end{array}
\qquad
\begin{array}{ccccc} A & \xrightarrow{\phi_1} & A' & \xrightarrow{\psi_1} & A'' \\ \mathsf{s}\,\downdownarrows\,\mathsf{t} & & \mathsf{s}'\,\downdownarrows\,\mathsf{t}' & & \mathsf{s}''\,\downdownarrows\,\mathsf{t}'' \\ V & \xrightarrow[\phi_0]{} & V' & \xrightarrow[\psi_0]{} & V'' \end{array}$$

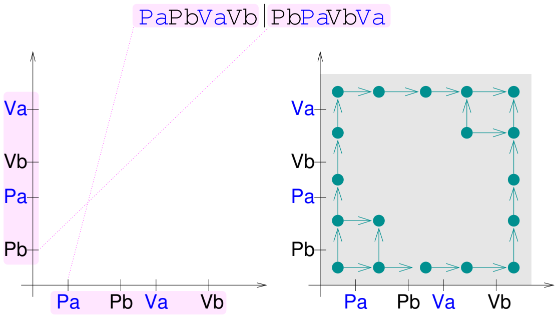
An example of model of a multi-task program
from *Edsger Wybe Dijkstra* "Pakken/Vrijlaten" language





Another example

from *Edsger Wybe Dijkstra* “Pakken/Vrijlaten” language



18 states and 20 arrows

Underlying graph and Category of paths II

adjunction between Cat and Grph

Path : morphism of graph from $\mathbb{I}_n$ to Γ

$$\underset{0}{\bullet} \longrightarrow \underset{1}{\bullet} \longrightarrow \underset{2}{\bullet} \longrightarrow \underset{3}{\bullet} \longrightarrow \cdots \underset{n}{\bullet} \longrightarrow \underset{n+1}{\bullet} \longrightarrow \cdots$$

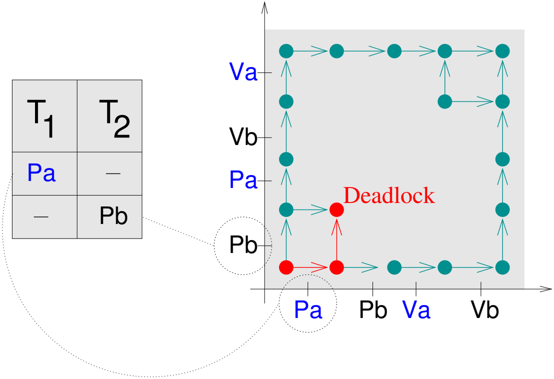
Forgetful functor $U : \mathsf{Cat} \longrightarrow \mathsf{Grph}$

"Category of paths" functor $F : \mathsf{Grph} \longrightarrow \mathsf{Cat}$

$$F \dashv U$$

A potential execution

program T_1 =PaPbVaVb | T_2 =PbPaVbVa

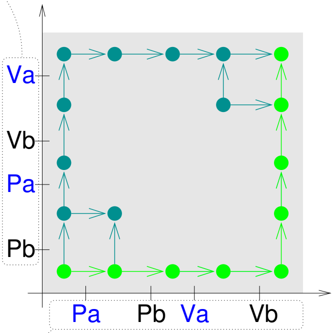


Deadlock

Anoter potential execution

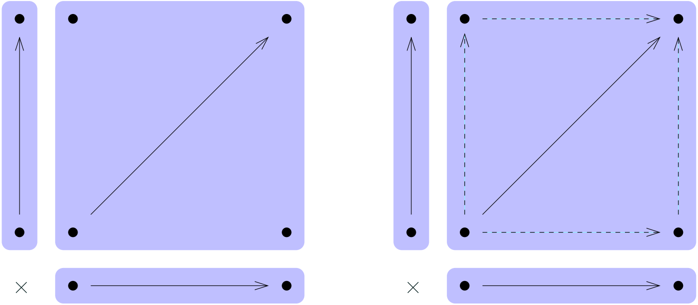
program T_1 =PaPbVaVb | T_2 =PbPaVbVa

T_1	T_2
Pa	—
Pb	—
Va	—
Vb	—
—	Pb
—	Pa
—	Vb
—	Va



Termination

$$F(\Gamma \times \Gamma') \ncong F(\Gamma) \times F(\Gamma')$$



Transitions Systems, CCS/π-calculus, *Mazurkiewicz* Traces ...

Partially ordered spaces
The category PoTop

Pospace $\overrightarrow{X}$: $\begin{cases} X & \text{topological space} \\ \sqsubseteq & \text{closed in } X \times X \end{cases}$

morphism f from $\overrightarrow{X}$ to $\overrightarrow{X'}$: continuous and order preserving maps

$$\begin{array}{ccc} \text{PoTop} & \longrightarrow & \text{PoSet} \\ \downarrow & & \downarrow \\ \text{Haus} & \longrightarrow & \text{Set} \end{array}$$

Categorical properties of PoTop

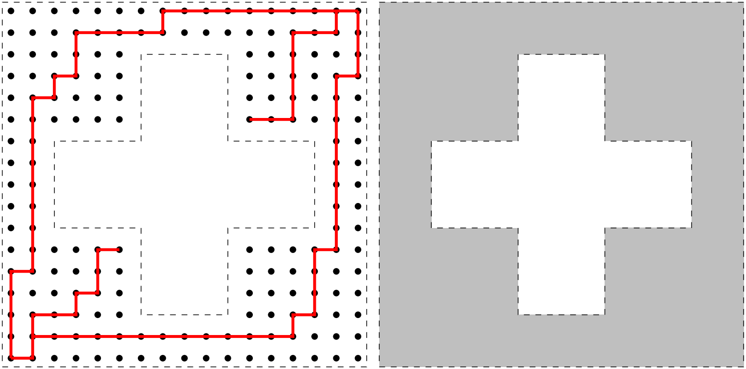
Theorem

- The directed compact unit segment is exponentiable in PoTop
- PoTop is complete and cocomplete
- PoTop is symmetric monoidal closed
- CGPoTop is a Cartesian closed reflective subcategory of PoTop
- CPoTop is a (complete and cocomplete) Cartesian closed reflective subcategory of CGPoTop cogenerated by the directed compact unit segment
- PoTop has no loop

Partially ordered spaces
examples

- Real line with standard order and topology : $\overrightarrow{\mathbb{R}}$
- Subset of a pospace (in particular $\overrightarrow{[0,1]}$)
- Geometric realization of a graph
- Cartesian Product
- Closed subsets of a metric space together with inclusion

Size reduction



Graph $\Gamma_{\overrightarrow{X}}$

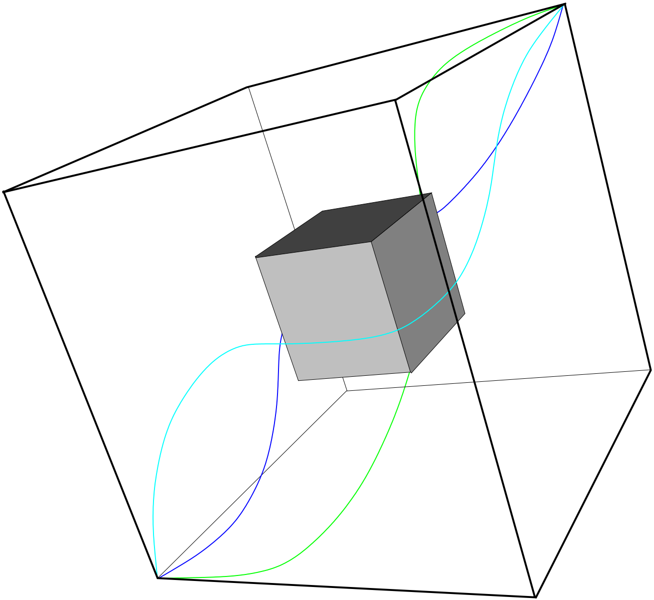
of paths on a pospace $\overrightarrow{X}$

- paths on $\overrightarrow{X}$: morphisms from $\overrightarrow{[0,1]}$ to $\overrightarrow{X}$
- arrows of $\Gamma_{\overrightarrow{X}}$: paths on $\overrightarrow{X}$
- source and target of a path γ on, $\overrightarrow{X}$: $\gamma(0)$ and $\gamma(1)$

1. The image of a dipath α on a pospace $\overrightarrow{X}$ is either isomorphic (in PoSpc) to $\overrightarrow{[0,1]}$ or $\{*\}$ (hence no directed *Peano* curve).
2. Two dipaths sharing the same image are dihomotopic.

Some paths around a cubic hole

P(a).V(a) | P(a).V(a) | P(a).V(a) with $\alpha_a = 3$



- Composition on $F(\Gamma_{\overrightarrow{X}})$ denoted by $\circ$

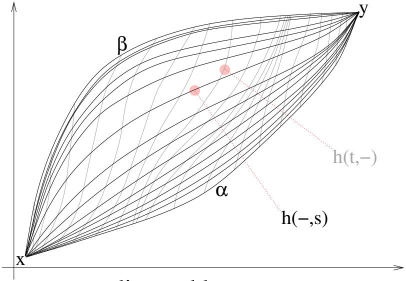
- Given $\gamma = (\gamma_n, \ldots, \gamma_1)$ a path on $\Gamma_{\overrightarrow{X}}$, we define the following path on $\overrightarrow{X}$

$$(\nu(\gamma))(t) = \begin{cases} \gamma_k(nt-k) & \text{si } t \in [\frac{k}{n}, \frac{k+1}{n}[\text{ et } k < n-1 \\ \gamma_n(nt-n+1) & \text{si } t \in [\frac{n-1}{n}, 1] \end{cases}$$

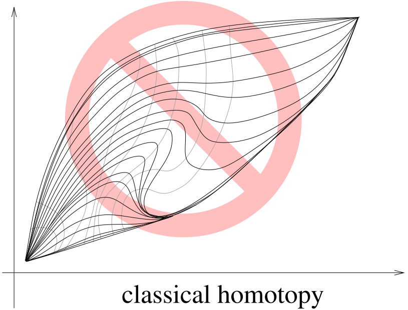
Directed homotopy

what it is and looks like

Morphism h from $\overrightarrow{[0,1]}^2$ to $\overrightarrow{X}$ such that $U(h)$ is a homotopy from γ to δ



directed homotopy



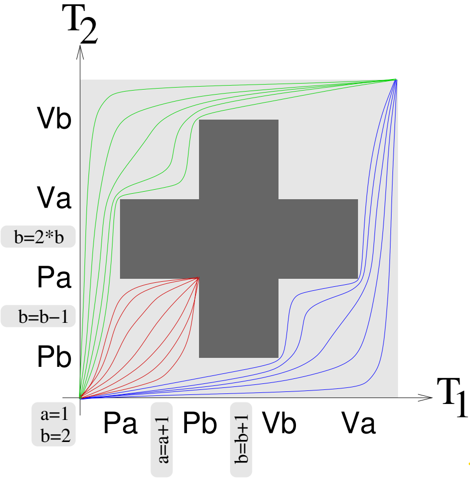
classical homotopy

Directed homotopy
an example

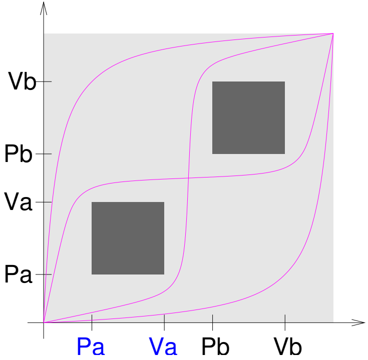
T1 gets a and b before T2 => a=2 and b=4

T2 gets b and a before T1 => a=2 and b=3

Each of T1 and T2 gets a ressource
=> Deadlock with a=2 and b=1



A subtlety
directed homotopy is not classical homotopy



Loop-Free and One-Way categories

André Haefliger (scwol) / *Dmitry Kozlov* (acyclic) and *Colin McLarty*

Loop-Free category or small categories without loops (LfCat) :
$\mathcal{C}[x,x] = \{\mathrm{id}_x\}$ and $(\mathcal{C}[x,y] \times \mathcal{C}[y,x] \neq \emptyset \implies x = y)$

One-Way category (OwCat) :
$\mathcal{C}[x,x] = \{\mathrm{id}_x\}$

$\mathcal{C}$ one-way $\iff$ sk($\mathcal{C}$) is loop-free

$$\mathrm{LfCat} \hookrightarrow \mathrm{OwCat} \hookrightarrow \mathrm{Cat}$$

Let $\sim$ be the congruence over $F(\Gamma_{\overrightarrow{X}})$ generated by

$$\Big\{ \big((\gamma_n, \ldots, \gamma_1), (\delta_p, \ldots, \delta_1)\big) \;\Big|\; \text{there is a dihomotopy from } \nu(\gamma) \text{ to } \nu(\delta) \Big\}$$

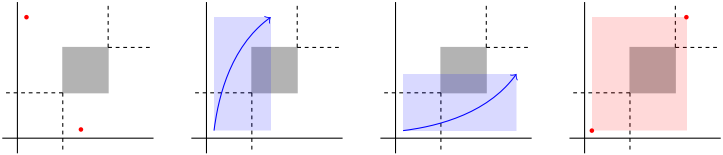
The fundamental category $\overrightarrow{\pi_1}(\overrightarrow{X})$ is $F(\Gamma_{\overrightarrow{X}})/\sim$ and we have

$$\overrightarrow{\pi_1}(\overrightarrow{X} \times \overrightarrow{Y}) \cong \overrightarrow{\pi_1}(\overrightarrow{X}) \times \overrightarrow{\pi_1}(\overrightarrow{Y})$$

$\overrightarrow{\pi_1}(\overrightarrow{X})$ is loop-free

van Kampen theorem

A detailed example

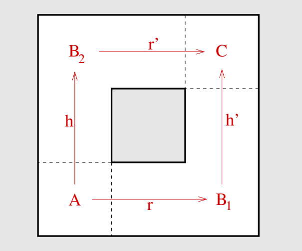


A detailed example

square with centered hole

$x \in$	$y \in$	$\overrightarrow{\pi_1}(\overrightarrow{X})[x,y]$
A	A	$\{\sigma_{x,y}\}$
B_1	B_1	$\{\sigma_{x,y}\}$
B_2	B_2	$\{\sigma_{x,y}\}$
C	C	$\{\sigma_{x,y}\}$
A	B_1	$\{r_{x,y}\}$
A	B_2	$\{h_{x,y}\}$
B_1	C	$\{h'_{x,y}\}$
B_2	C	$\{r'_{x,y}\}$
B_1	B_2	$\emptyset$
B_2	B_1	$\emptyset$
A	C	$\{u_{x,y}\ ,\ d_{x,y}\}$

With $r'_{y,z} \circ h_{x,y} = u_{x,z}$, $h'_{y,z} \circ r_{x,y} = d_{x,z}$ and 3 points x, y, z of the square such that $x \sqsubseteq y \sqsubseteq z$; if $x \not\sqsubseteq y$ then $\overrightarrow{\pi_1}(\overrightarrow{X})[x,y] = \emptyset$.



Yoneda morphism $\sigma \in \mathcal{C}[x, y]$

preserving the past and the future I

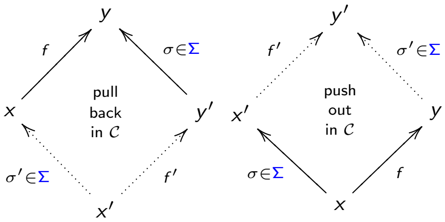
future if $\mathcal{C}[y, z] \neq \emptyset$, then $\mathcal{C}[y, z] \longrightarrow \mathcal{C}[x, z]$ is a bijection and

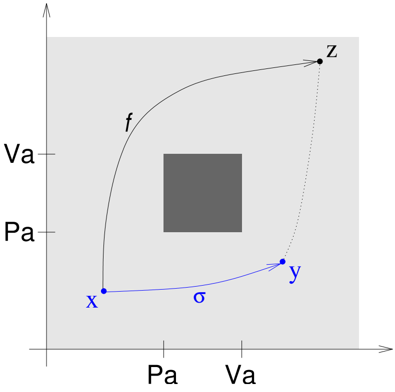
$$\gamma \longmapsto \gamma \circ \sigma$$

past if $\mathcal{C}[z, x] \neq \emptyset$, then $\mathcal{C}[z, x] \longrightarrow \mathcal{C}[z, y]$ is a bijection

$$\delta \longmapsto \sigma \circ \delta$$

1. Σ is stable under composition,
2. Σ contains all the isomorphisms of $\mathcal{C}$,
3. all the elements of Σ are *Yoneda* morphisms and
4. Σ is stable under change and cochange of base.





z
f
Va
Pa
y
x
σ
Pa
Va
z
Vb
Va
Pa
y
σ
f
Pb
x
Pa
Pb
Vb
Va

Structure of Σ-components

$\mathcal{C}$ loop-free category and Σ *Yoneda* system over $\mathcal{C}$

Theorem

1. $\exists z\ \Sigma[x,z] \times \Sigma[y,z] \neq \emptyset$ *iff* $\exists z\ \Sigma[z,x] \times \Sigma[z,y] \neq \emptyset$
2. *"* $\exists z\ \Sigma[x,z] \times \Sigma[y,z] \neq \emptyset$ *" defines an equivalence relation* $x \sim y$
3. *Given any* $\sim$*-equivalence class* K*, the full subcategory of* $\mathcal{C}$ *whose set of objects is* K *is a non empty lattice*
4. *If* $a \sim b$*, then the following square is both a* *pullback* *and a* *pushout* *in* $\mathcal{C}$*.*

$$\begin{array}{ccc} x & & \\ {\scriptstyle\Sigma}\big\uparrow & & \\ z & \xrightarrow[\Sigma]{} & y \end{array} \qquad \begin{array}{ccc} x & \xrightarrow{\Sigma} & z \\ & & \big\uparrow{\scriptstyle\Sigma} \\ & & y \end{array} \qquad \begin{array}{ccc} a & \longrightarrow & a \vee b \\ \big\uparrow & & \big\uparrow \\ a \wedge b & \longrightarrow & b \end{array}$$

Locale of *Yoneda* systems
topology without point over a one-way category

Theorem

The collection, ordered by inclusion, of the Yoneda *systems of a one-way category, forms a locale whose maximum is denoted $\overline{\Sigma}$. Beside, its minimum is the collection of all isomorphisms of $\mathcal{C}$.*

Category of components
generalizing the set of arcwise components

The category of components of a loop-free category $\mathcal{C}$ is the quotient $\mathcal{C}/\overline{\Sigma}$

Theorem

A loop-free category $\mathcal{C}$ is a non empty lattice iff its category of components is $\{*\}$

Fundamental theorem

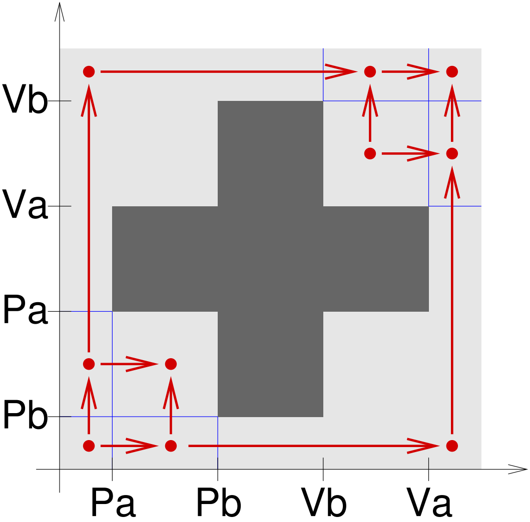
$\mathcal{C}$ one-way category and Σ *Yoneda* system over $\mathcal{C}$

Theorem

1. *the collection* Σ *is pure in* $\mathcal{C}$ ($\beta \circ \alpha \in \Sigma \Rightarrow \beta, \alpha \in \Sigma$),
2. *the category* $\mathcal{C}/_{\Sigma}$ *is loop-free and the category* $\mathcal{C}[\Sigma^{-1}]$ *is one-way*
3. *the categories* $\mathcal{C}[\Sigma^{-1}]$ *and* $\mathcal{C}/_{\Sigma}$ *are equivalent and*
4. *the category* $\mathcal{C}[\Sigma^{-1}]$ *is fibered over the base* $\mathcal{C}/_{\Sigma}$.

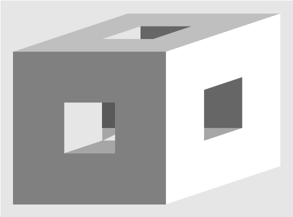
The category of components

of the swiss flag

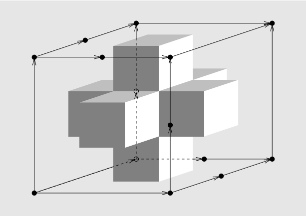


The category of components

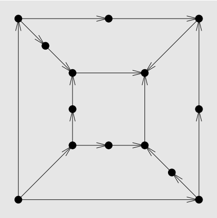
Menger sponge first iteration : `P(a).V(a) | P(a).V(a) | P(a).V(a)` with $\alpha_{\mathtt{a}} = 2$



Interior of the pospace



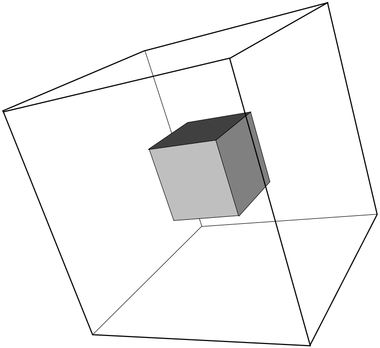
Category of components



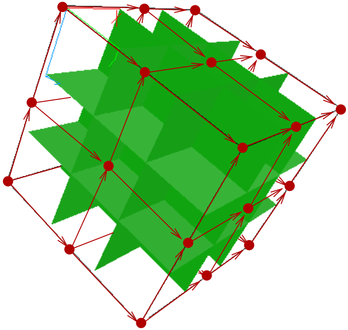
Flattened

The components category

of a 2-semaphore : `P(a).V(a) | P(a).V(a) | P(a).V(a)` avec $\alpha_{\mathtt{a}} = 3$



the pospace



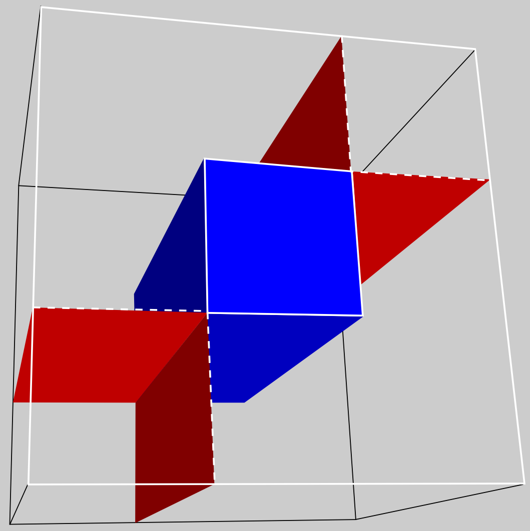
its category of components

Factorization

Theorem (J. Hashimoto -T. Balabonski)

The monoid of (isomorphism classes of) non empty, connected, finite, loop-free categories is countable and free

Example of product
parallel "independent" composition



The Directed Circle

Obects : S^1 Morphisms : $S^1 \times \mathbb{N} \times S^1$ Identities : (x,0,x) for x$\in S^1$