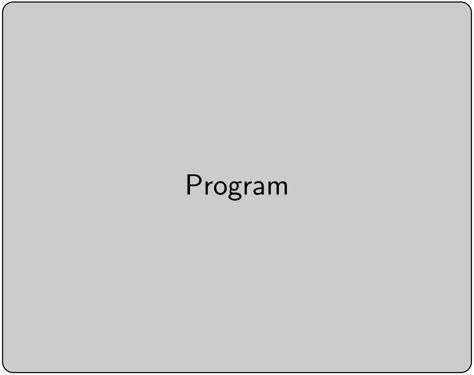
Control Flow Structures of Concurrent Programs are Higher Dimensional Mathematical Objects

Emmanuel Haucourt

Wednesday 13^{th} April 2016

A Toy Language

from Dijkstra's "Cooperating Sequential Processes" paper



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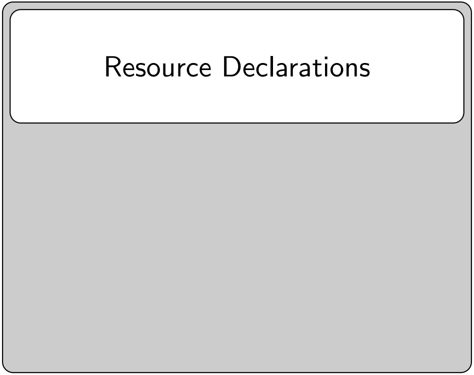

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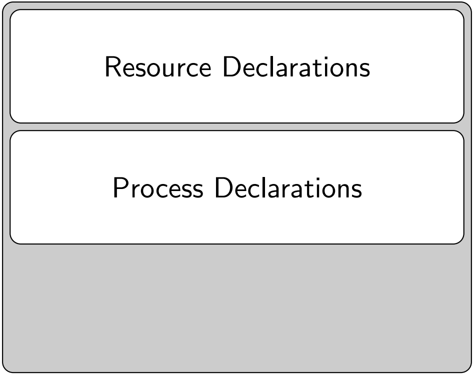

A Toy Language

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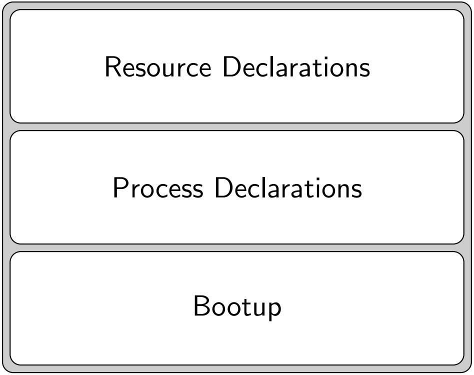

A Toy Language

Resource Declaration

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- sem: *<int>* *<set of identifiers>*

A Toy Language

Resource Declaration

- sem: *<int>* *<set of identifiers>*
- sync: *<int>* *<set of identifiers>*

A Toy Language

Resource Declaration

- sem: *<int>* *<set of identifiers>*
- sync: *<int>* *<set of identifiers>*
- var: *<identifier>* = *<constant>*

```
var:  x = 7
```

```
proc:
  p = ()+[x=1]+C(q)

proc:
  q = (x:=x/2  ; C(p))+[x % 2 = 0]+
      (x:=3*x+1; C(p))
```

```
init:  p
```

Building the Control Flow Graph

of the Hasse-Syracuse algorithm

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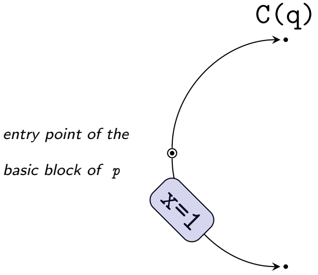
of the Hasse-Syracuse algorithm



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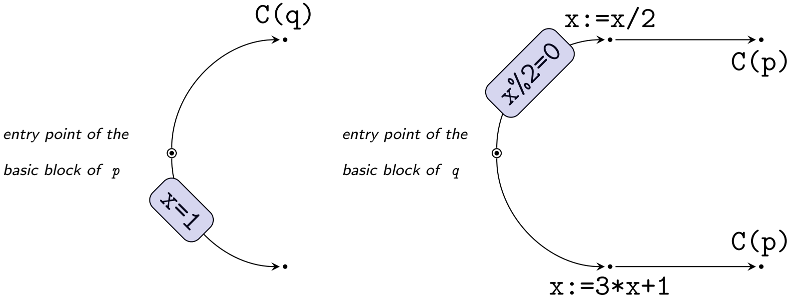
of the Hasse-Syracuse algorithm



Building the Control Flow Graph

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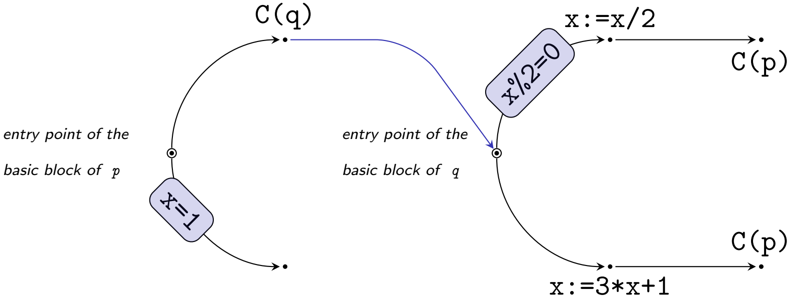

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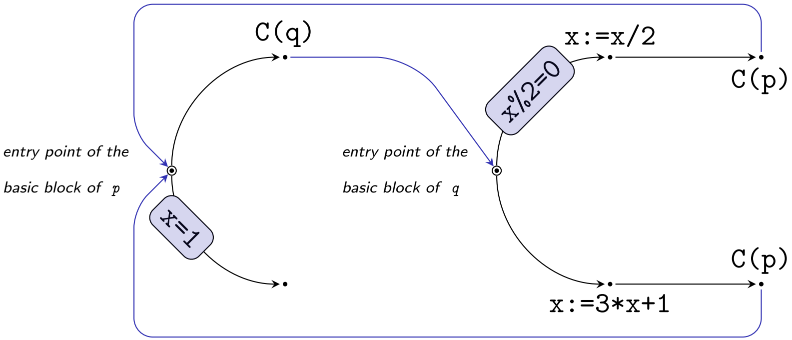

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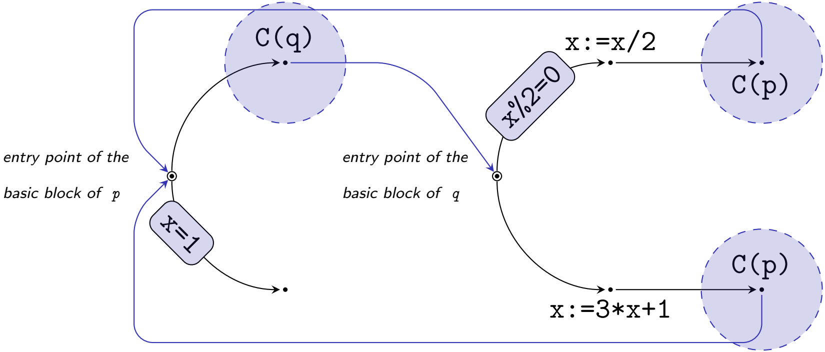

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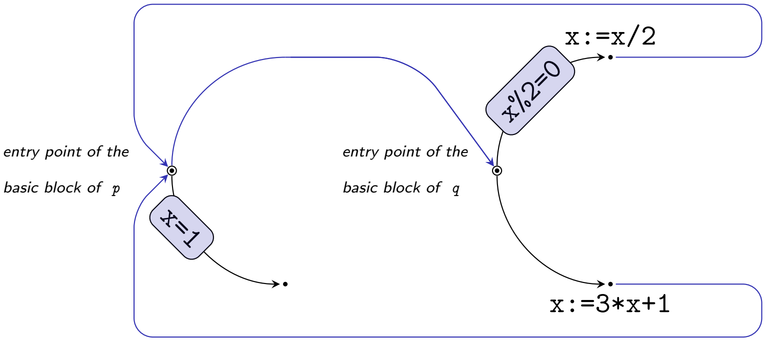
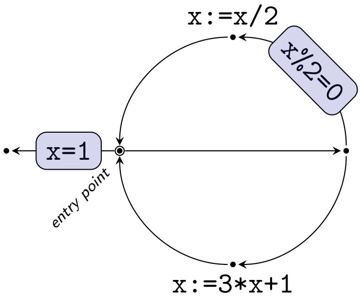


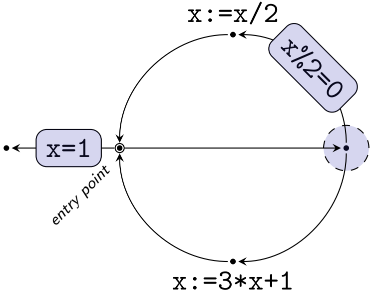
Reducing the Control Flow Graph

of the Hasse-Syracuse algorithm



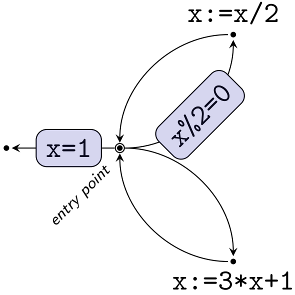
Reducing the Control Flow Graph

of the Hasse-Syracuse algorithm



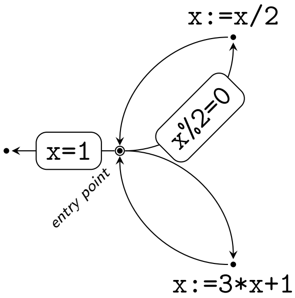
Reducing the Control Flow Graph

of the Hasse-Syracuse algorithm



An Execution Trace

on a control flow graph



An Execution Trace

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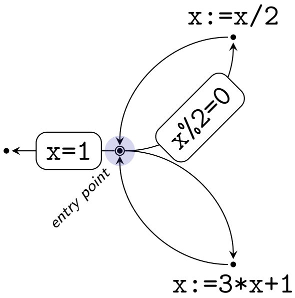
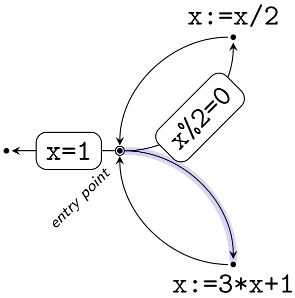


the current value of x is 7

An Execution Trace

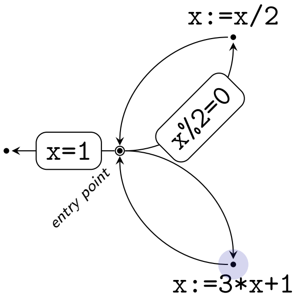
on a control flow graph



the current value of x is 7

An Execution Trace

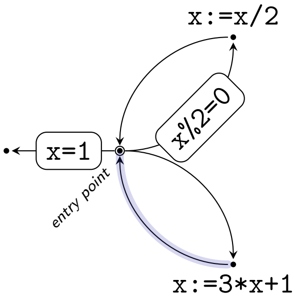
on a control flow graph



the current value of x is 22

An Execution Trace

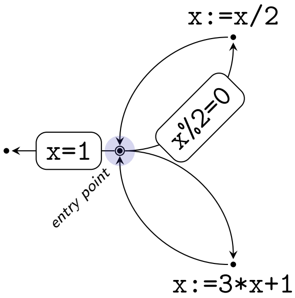
on a control flow graph



the current value of x is 22

An Execution Trace

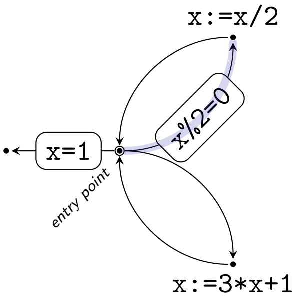
on a control flow graph



the current value of x is 22

An Execution Trace

on a control flow graph



the current value of x is 22

An Execution Trace

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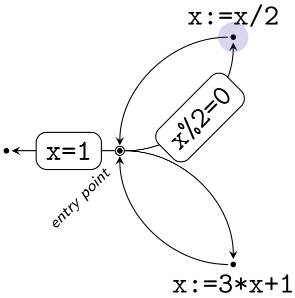
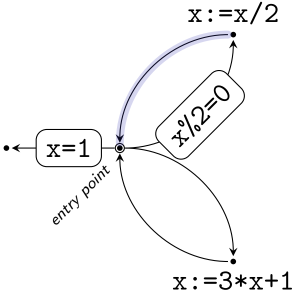


the current value of x is 11

An Execution Trace

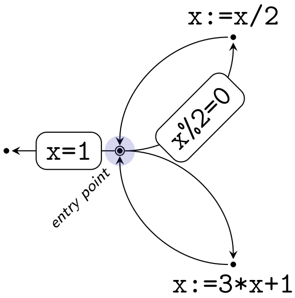
on a control flow graph



the current value of x is 11

An Execution Trace

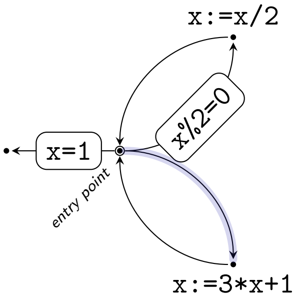
on a control flow graph



the current value of x is 11

An Execution Trace

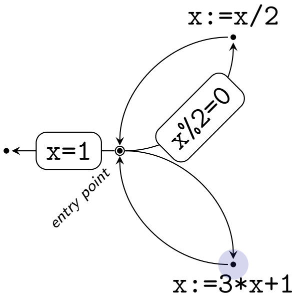
on a control flow graph



the current value of `x` is 11

An Execution Trace

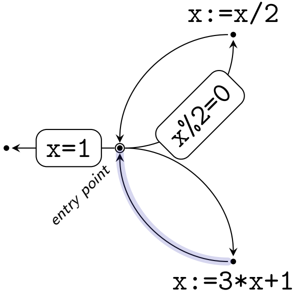
on a control flow graph



the current value of x is 34

An Execution Trace

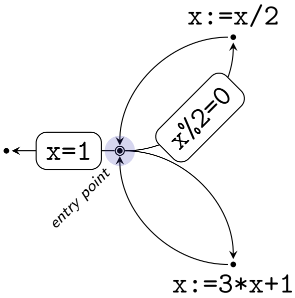
on a control flow graph



the current value of `x` is 34

An Execution Trace

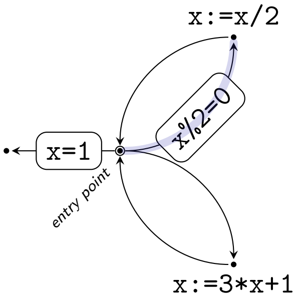
on a control flow graph



the current value of x is 34

An Execution Trace

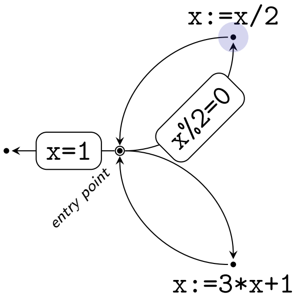
on a control flow graph



the current value of x is 34

An Execution Trace

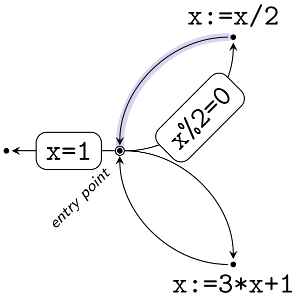
on a control flow graph



the current value of x is 17

An Execution Trace

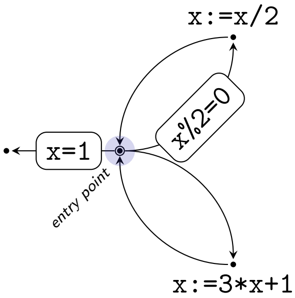
on a control flow graph



the current value of x is 17

An Execution Trace

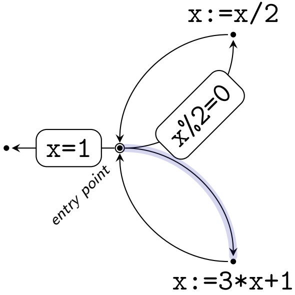
on a control flow graph



the current value of x is 17

An Execution Trace

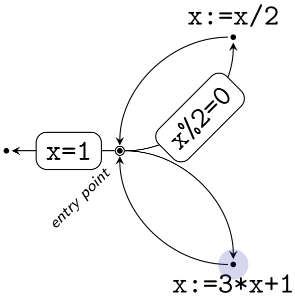
on a control flow graph



the current value of x is 17

An Execution Trace

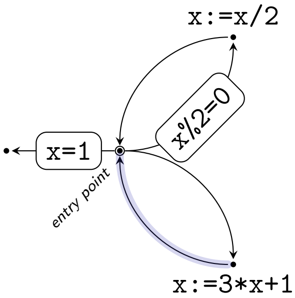
on a control flow graph



the current value of x is 52

An Execution Trace

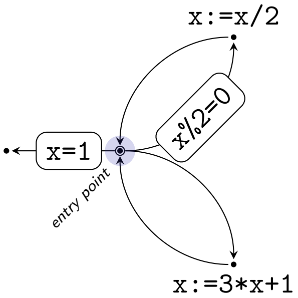
on a control flow graph



the current value of x is 52

An Execution Trace

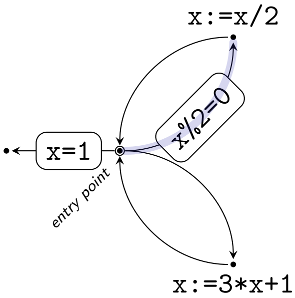
on a control flow graph



the current value of x is 52

An Execution Trace

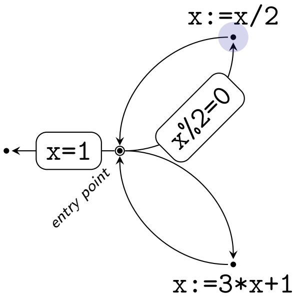
on a control flow graph



the current value of x is 52

An Execution Trace

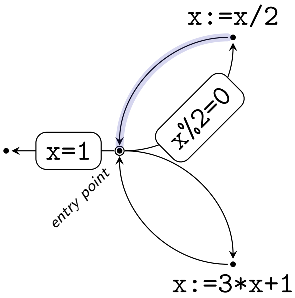
on a control flow graph



the current value of x is 26

An Execution Trace

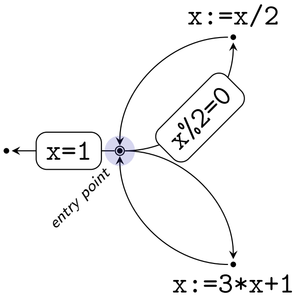
on a control flow graph



the current value of $\mathtt{x}$ is 26

An Execution Trace

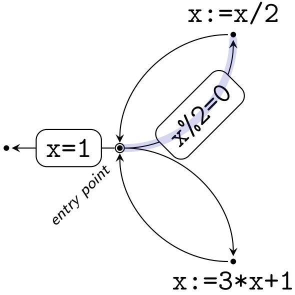
on a control flow graph



the current value of x is 26

An Execution Trace

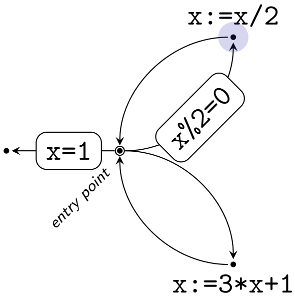
on a control flow graph



the current value of x is 26

An Execution Trace

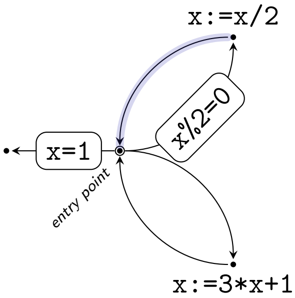
on a control flow graph



the current value of $\texttt{x}$ is 13

An Execution Trace

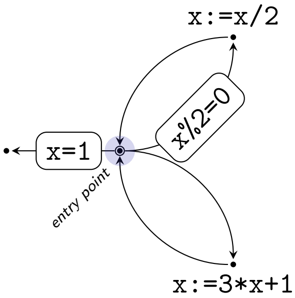
on a control flow graph



the current value of x is 13

An Execution Trace

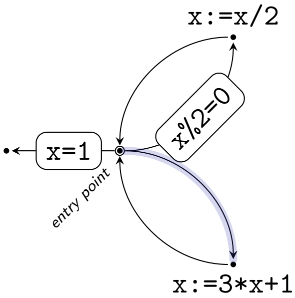
on a control flow graph



the current value of x is 13

An Execution Trace

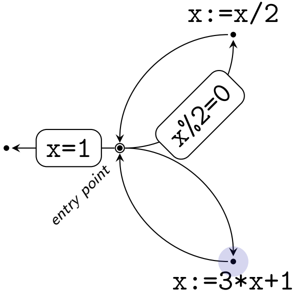
on a control flow graph



the current value of x is 13

An Execution Trace

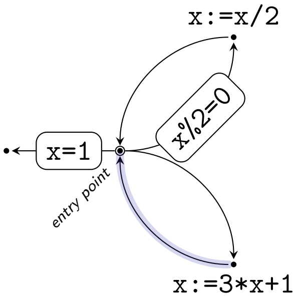
on a control flow graph



the current value of x is 40

An Execution Trace

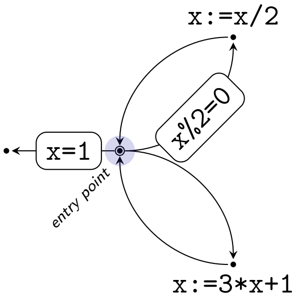
on a control flow graph



the current value of `x` is 40

An Execution Trace

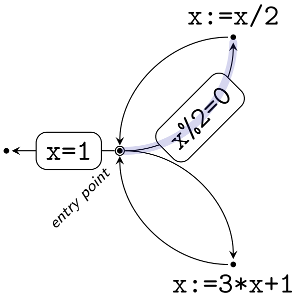
on a control flow graph



the current value of `x` is 40

An Execution Trace

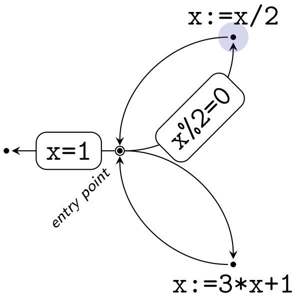
on a control flow graph



the current value of x is 40

An Execution Trace

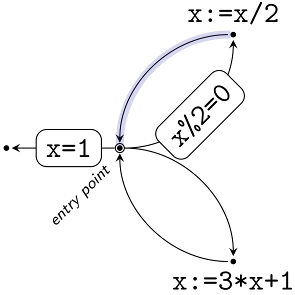
on a control flow graph



the current value of x is 20

An Execution Trace

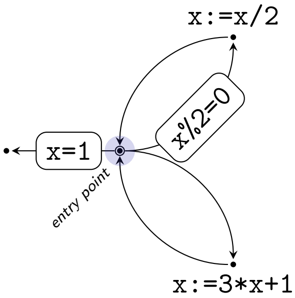
on a control flow graph



the current value of x is 20

An Execution Trace

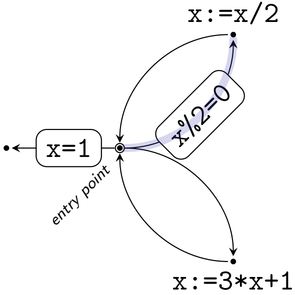
on a control flow graph



the current value of x is 20

An Execution Trace

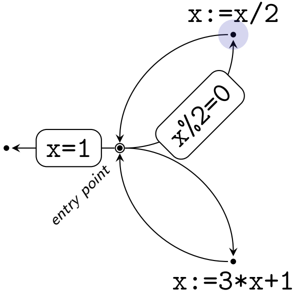
on a control flow graph



the current value of x is 20

An Execution Trace

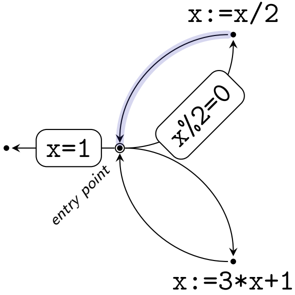
on a control flow graph



the current value of x is 10

An Execution Trace

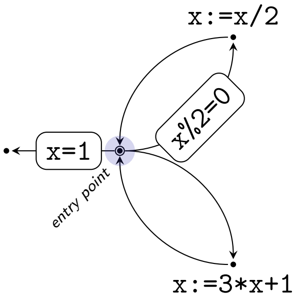
on a control flow graph



the current value of x is 10

An Execution Trace

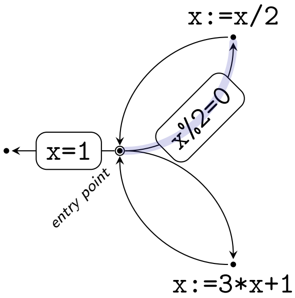
on a control flow graph



the current value of x is 10

An Execution Trace

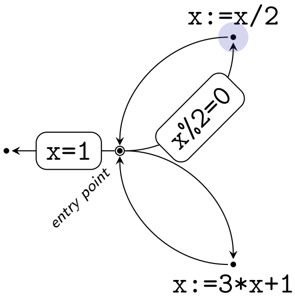
on a control flow graph



the current value of x is 10

An Execution Trace

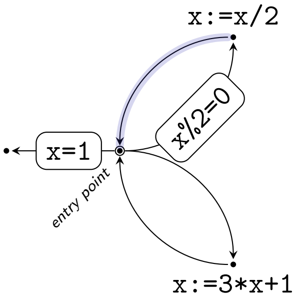
on a control flow graph



the current value of x is 5

An Execution Trace

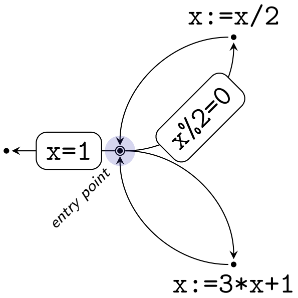
on a control flow graph



the current value of x is 5

An Execution Trace

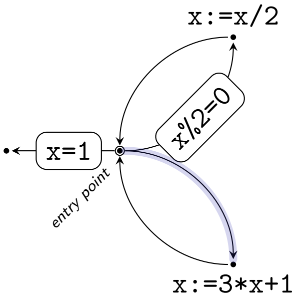
on a control flow graph



the current value of x is 5

An Execution Trace

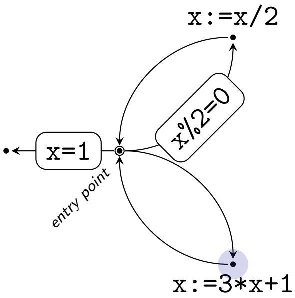
on a control flow graph



the current value of x is 5

An Execution Trace

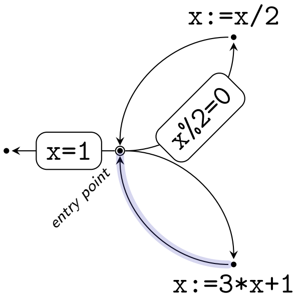
on a control flow graph



the current value of x is 16

An Execution Trace

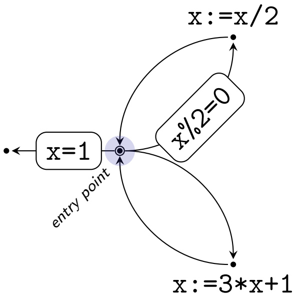
on a control flow graph



the current value of x is 16

An Execution Trace

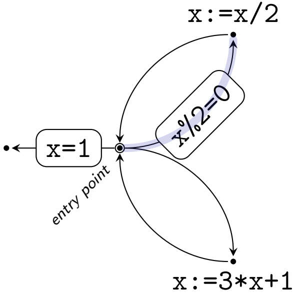
on a control flow graph



the current value of x is 16

An Execution Trace

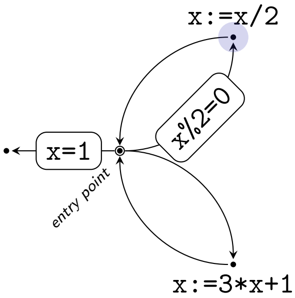
on a control flow graph



the current value of x is 16

An Execution Trace

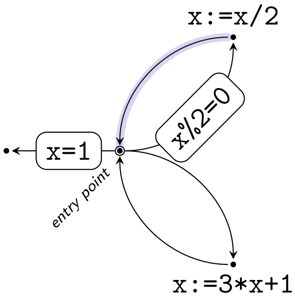
on a control flow graph



the current value of x is 8

An Execution Trace

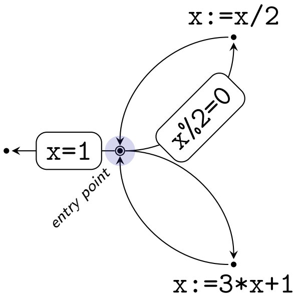
on a control flow graph



the current value of x is 8

An Execution Trace

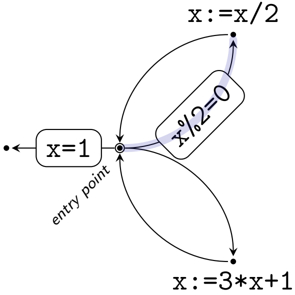
on a control flow graph



the current value of x is 8

An Execution Trace

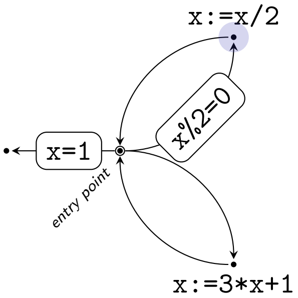
on a control flow graph



the current value of x is 8

An Execution Trace

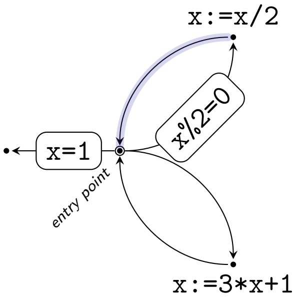
on a control flow graph



the current value of x is 4

An Execution Trace

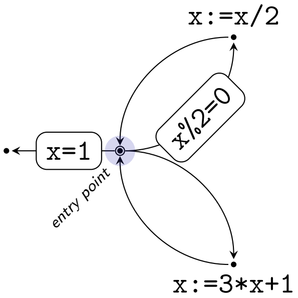
on a control flow graph



the current value of x is 4

An Execution Trace

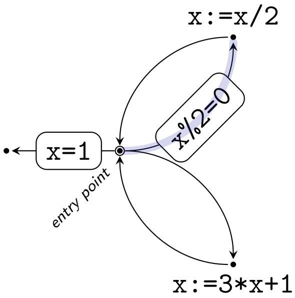
on a control flow graph



the current value of x is 4

An Execution Trace

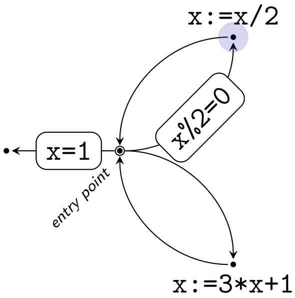
on a control flow graph



the current value of x is 4

An Execution Trace

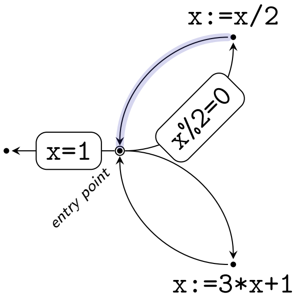
on a control flow graph



the current value of $\texttt{x}$ is 2

An Execution Trace

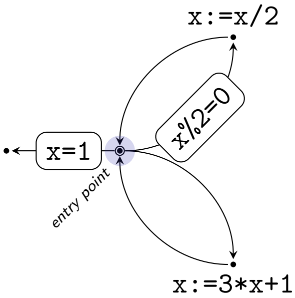
on a control flow graph



the current value of x is 2

An Execution Trace

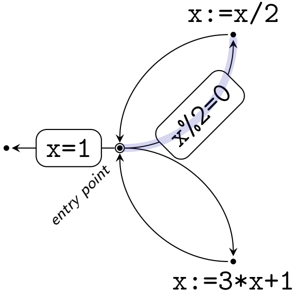
on a control flow graph



the current value of x is 2

An Execution Trace

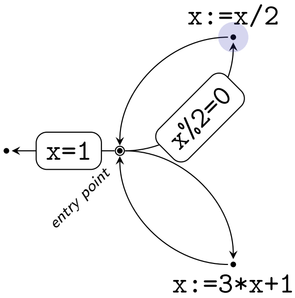
on a control flow graph



the current value of x is 2

An Execution Trace

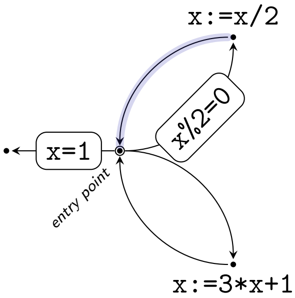
on a control flow graph



the current value of x is 1

An Execution Trace

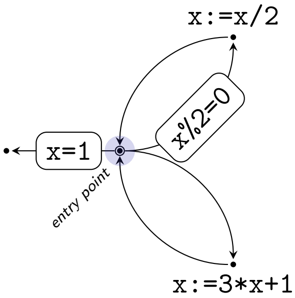
on a control flow graph



the current value of x is 1

An Execution Trace

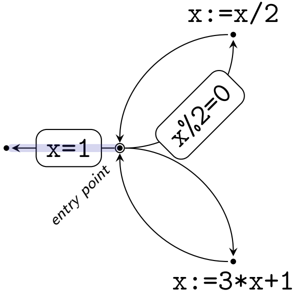
on a control flow graph



the current value of x is 1

An Execution Trace

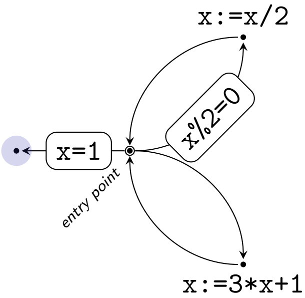
on a control flow graph



the current value of x is 1

An Execution Trace

on a control flow graph



the current value of x is 1

Precubical sets

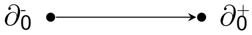
higher dimensional graphs

Precubical sets

higher dimensional graphs

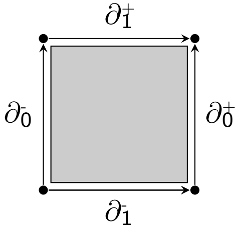
Precubical sets

higher dimensional graphs



Precubical sets

higher dimensional graphs



Precubical sets

higher dimensional graphs

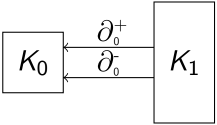
Precubical sets

higher dimensional graphs

K_0

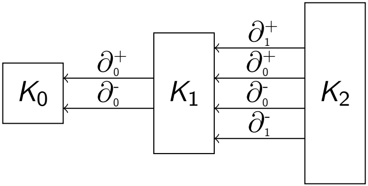
Precubical sets

higher dimensional graphs



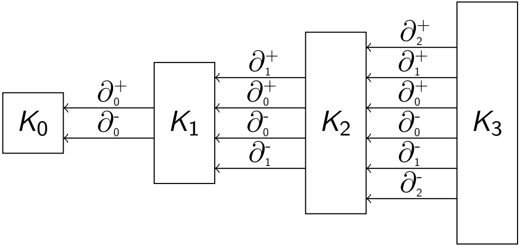
Precubical sets

higher dimensional graphs



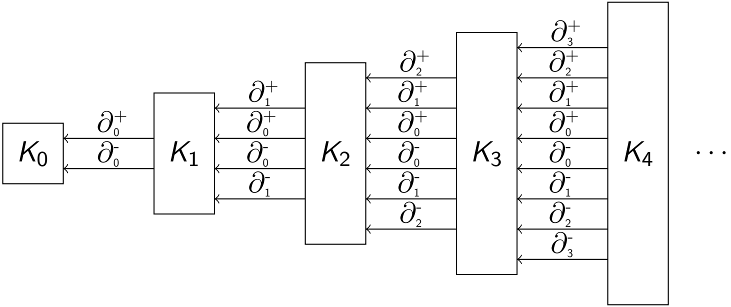
Precubical sets

higher dimensional graphs



Precubical sets

higher dimensional graphs



Tensor product

of precubical sets

Given precubical sets K and K' of dimension p and q, the set of d-cubes for $0 \leqslant d \leqslant p+q$

$$(K \otimes K')_d = \bigsqcup_{i+j=d} K_i \times K'_j$$

For $x \otimes y \in K_i \times K'_j$ with $i+j=d$ the k^{th} face map, with $0 \leqslant k < d$, is given by

$$\partial_k^{\pm}(x \otimes y) = \begin{cases} \partial_k^{\pm}(x) \otimes y & \text{if } 0 \leqslant k < i \\ x \otimes \partial_{k-p}^{\pm} y) & \text{if } i \leqslant k < d \end{cases}$$

A Toy Language

Synchronization: the W(_) instruction

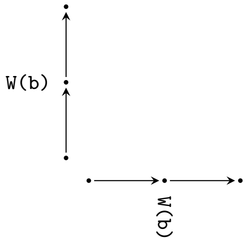
```
sync:  1 b
```

```
proc:  p = W(b)
```

```
init:  2p
```

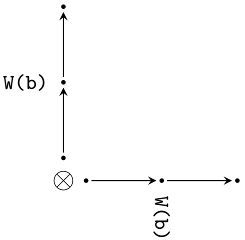
Tensor product

of control flow graphs



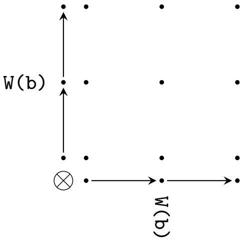
Tensor product

of control flow graphs



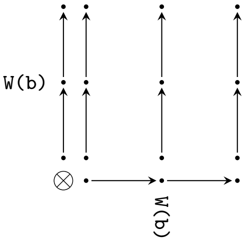
Tensor product

of control flow graphs



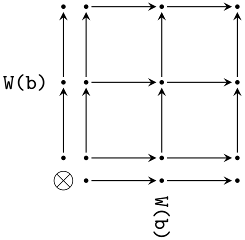
Tensor product

of control flow graphs



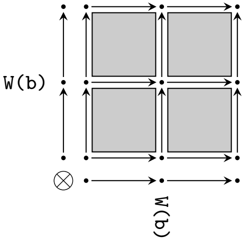
Tensor product

of control flow graphs



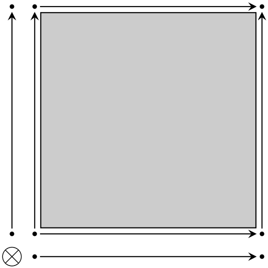
Tensor product

of control flow graphs



Discrete paths

are "continuous"



Discrete paths

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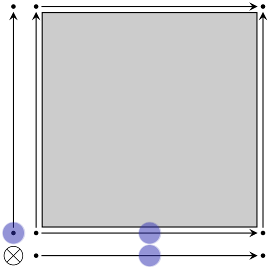

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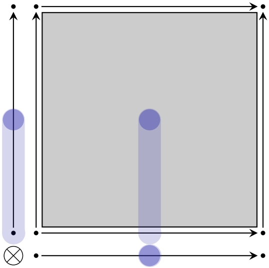

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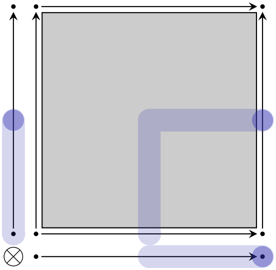

Discrete paths

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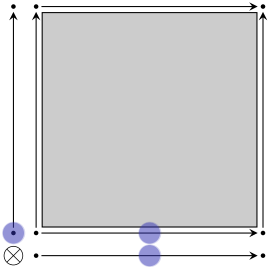
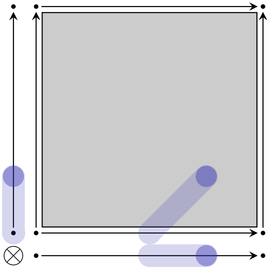


Discrete paths

are “continuous”



Discrete paths

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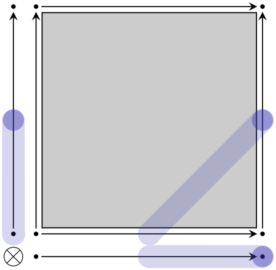

Discrete paths

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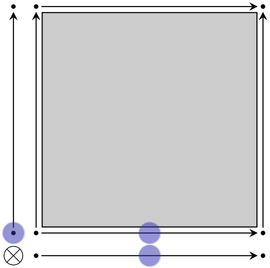

Discrete paths

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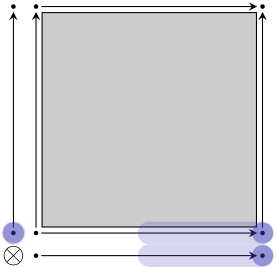
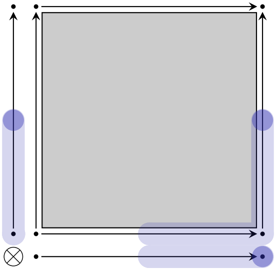


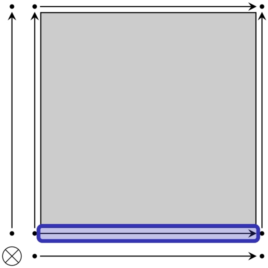
Discrete paths

are "continuous"



Discrete paths

are "continuous"



Discrete paths

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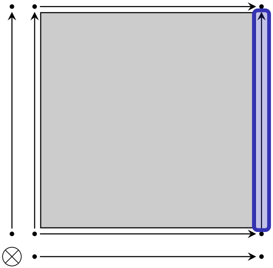
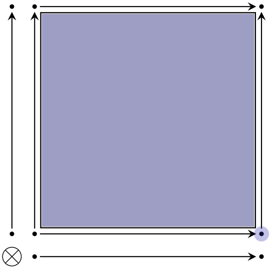


Discrete paths

are “continuous”



Discrete path on a model of dimension N

A sequence of points $p_0, \ldots, p_K$ s.t. for all $k \in \{1, \ldots, K\}$ one has

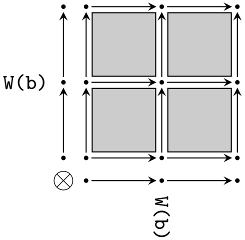
$$\text{for all } n \in \{1, \ldots, N\} \ \partial^+ p_n(k-1) = p_n(k) \text{ or } p_n(k) = p_n(k-1)$$

or

$$\text{for all } n \in \{1, \ldots, N\} \ p_n(k-1) = \partial^- p_n(k) \text{ or } p_n(k) = p_n(k-1)$$

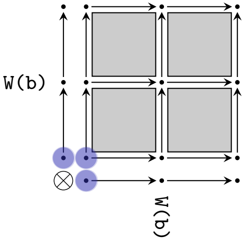
Concurrent execution trace

sync: 1 b



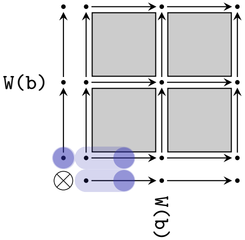
Concurrent execution trace

`sync:  1 b`



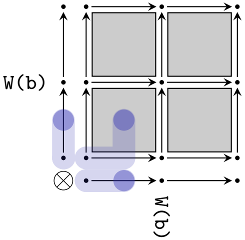
Concurrent execution trace

`sync: 1 b`



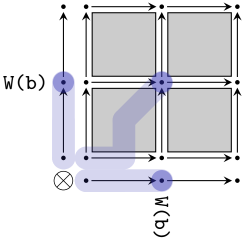
Concurrent execution trace

`sync:  1 b`



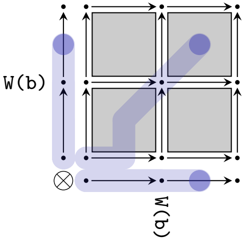
Concurrent execution trace

`sync:  1 b`



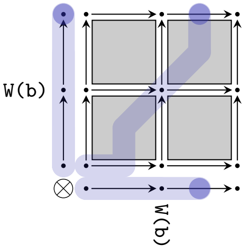
Concurrent execution trace

`sync: 1 b`



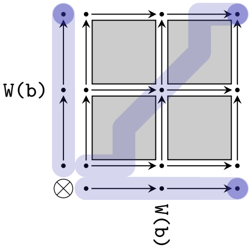
Concurrent execution trace

`sync:  1 b`



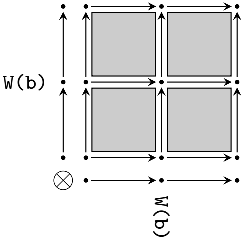
Concurrent execution trace

`sync: 1 b`



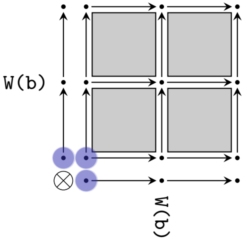
Not admissible concurrent execution trace

```
sync:  1 b
```



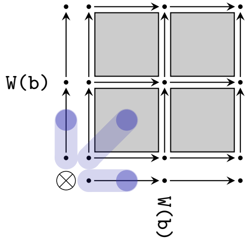
Not admissible concurrent execution trace

sync: 1 b



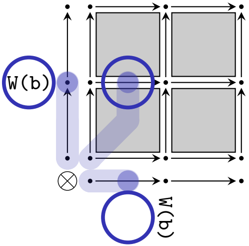
Not admissible concurrent execution trace

sync: 1 b



Not admissible concurrent execution trace

sync: 1 b



Forbidden points

due to synchronization

Each point $p = (p_1, \ldots, p_d)$ such that

$$0 \quad < \quad \mathrm{card}\Big\{ k \in \{1, \ldots, d\} \;\Big|\; \mathrm{label}(p_k) = \mathtt{W(b)} \Big\} \quad \leqslant \quad \mathrm{arity}(\mathtt{b})$$

is forbidden.

A Toy Language

conflicting assignments

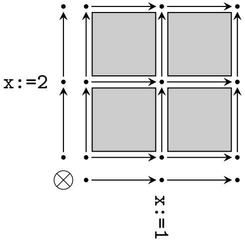
```
var:  x = 0
```

```
proc:  p = (x := 1)
proc:  q = (x := 2)
```

```
init:  p q
```

Not admissible execution trace

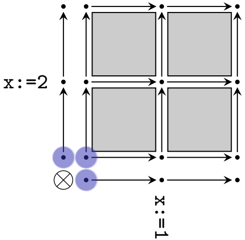
due to race condition



the value of $\mathtt{x}$ is 0

Not admissible execution trace

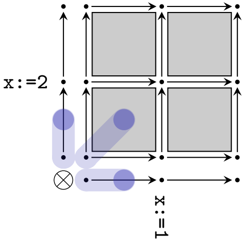
due to race condition



the value of $\mathtt{x}$ is 0

Not admissible execution trace

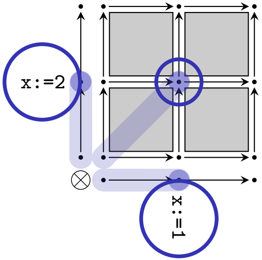
due to race condition



the value of $\mathtt{x}$ is 0

Not admissible execution trace

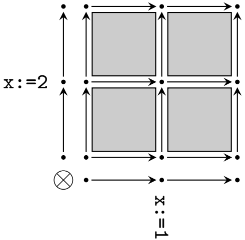
due to race condition



the value of x is ?

Admissible execution trace

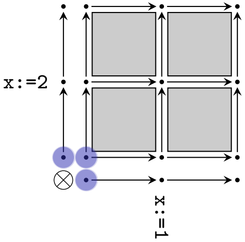
that however meets a forbidden point



the value of $\mathtt{x}$ is 0

Admissible execution trace

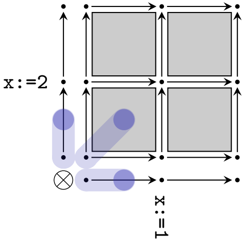
that however meets a forbidden point



the value of x is 0

Admissible execution trace

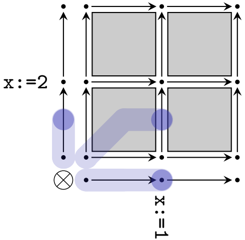
that however meets a forbidden point



the value of x is 0

Admissible execution trace

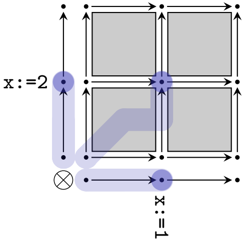
that however meets a forbidden point



the value of x is 1

Admissible execution trace

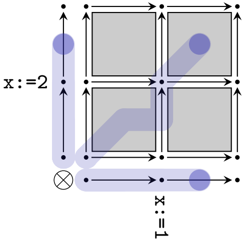
that however meets a forbidden point



the value of x is 2

Admissible execution trace

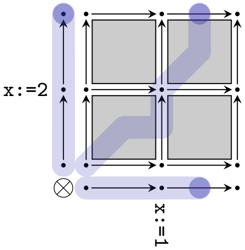
that however meets a forbidden point



the value of $\mathtt{x}$ is 2

Admissible execution trace

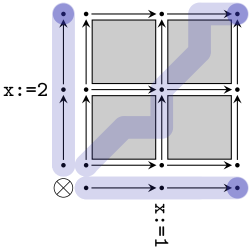
that however meets a forbidden point



the value of $\mathtt{x}$ is 2

Admissible execution trace

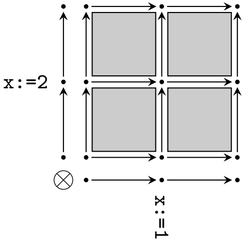
that however meets a forbidden point



the value of $\mathtt{x}$ is 2

Admissible execution trace

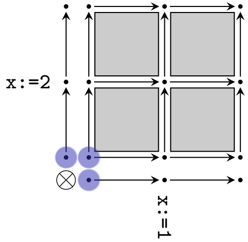
avoiding forbidden points



the value of $\mathtt{x}$ is 0

Admissible execution trace

avoiding forbidden points



the value of $\texttt{x}$ is 0

Admissible execution trace

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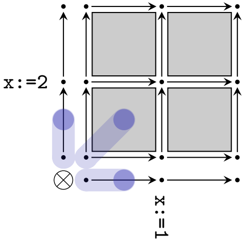

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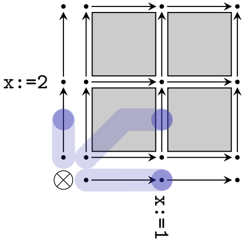

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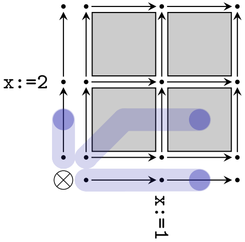

the value of x is 1

Admissible execution trace

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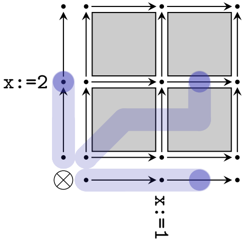

the value of $\texttt{x}$ is 2

Admissible execution trace

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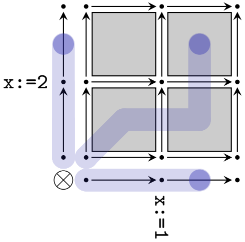

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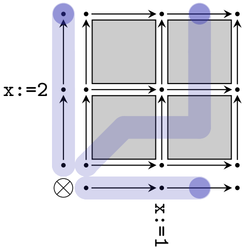

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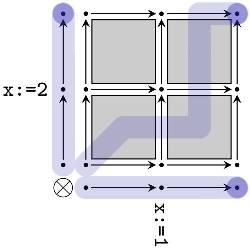

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Forbidden points

due to race conditions

A point $p = (p_1, \ldots, p_d)$ is a race condition when there exist $i \neq j$ such that
- both $\lambda_i(p_i)$ and $\lambda_j(p_j)$ are assignments trying to alter the same variable or
- $\lambda_i(p_i)$ tries to alter a free variable of $\lambda_j(p_j)$ or $\lambda_j(\alpha)$ for some arrow α such that $\partial^{-}\alpha = p_j$.

In that case the point p is forbidden.

The replacement property

for admissible execution traces

Replacement

Any admissible execution trace that meets a race condition is "equivalent" to an admissible execution trace which avoids all of them.

A Toy Language

Desynchronization: the P(_) and V(_) instructions

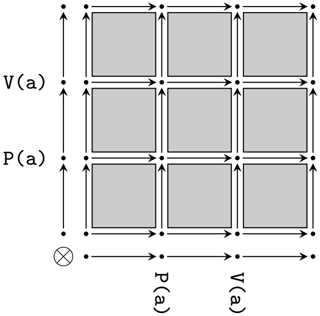
```
sem:  1 a
```

```
proc:  p = P(a);V(a)
```

```
init:  2p
```

Admissible concurrent execution trace

`sem: 1 a`



Admissible concurrent execution trace

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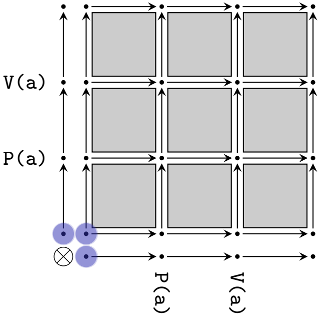

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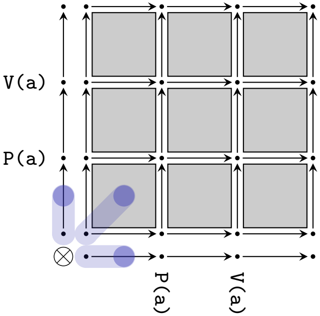

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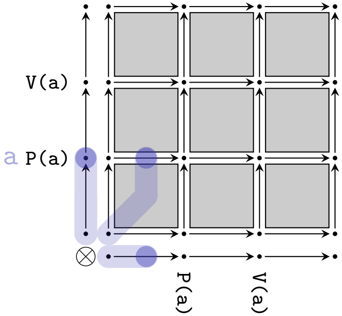

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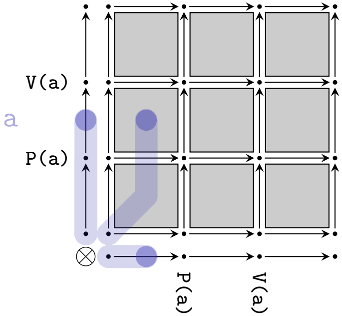

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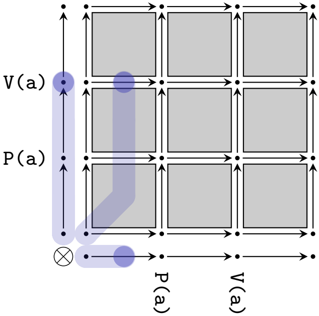

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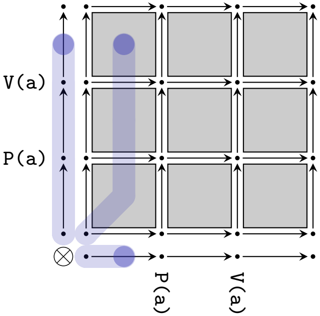

Admissible concurrent execution trace

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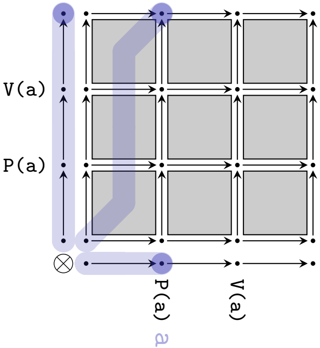

Admissible concurrent execution trace

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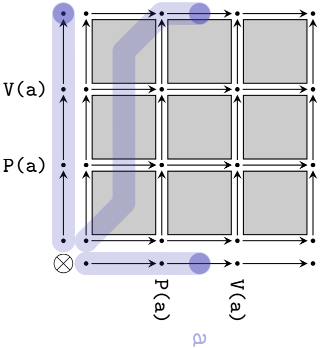

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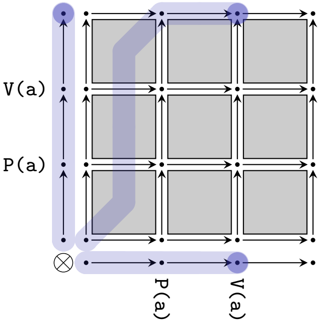

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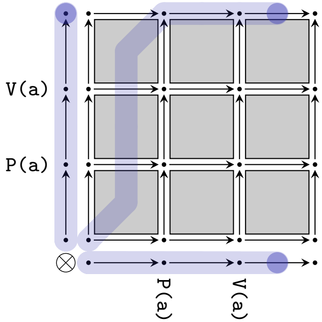

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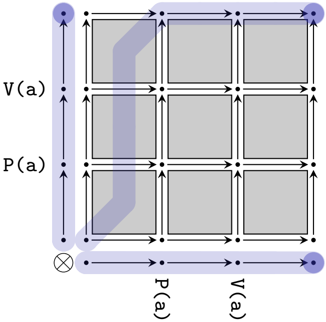

Not admissible concurrent execution trace

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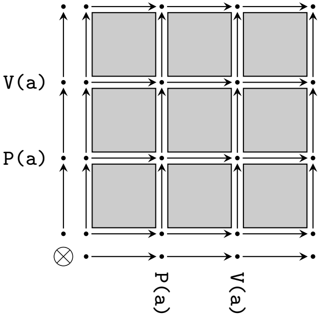

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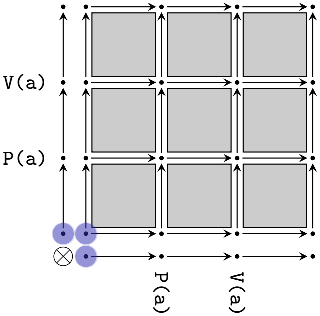

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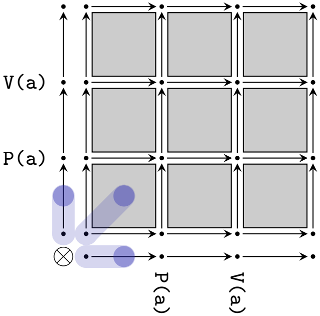

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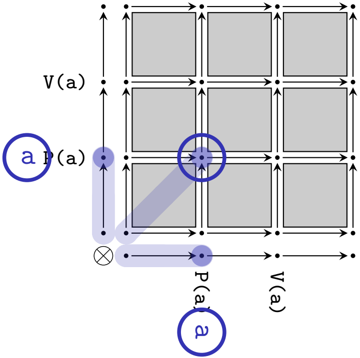

The potential functions

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A process π is conservative when for all paths and all semaphores s, the amount of tokens of type s held by the process at the end of the execution trace only depends on its arrival point.

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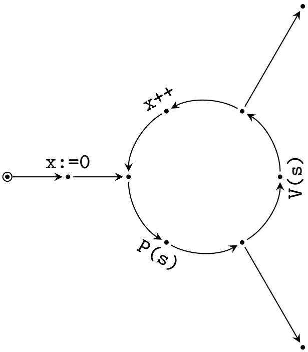
If $F_\Pi(s, p) > \text{arity}(s)$ for some semaphore s, then p is forbidden.

Conservative process

example



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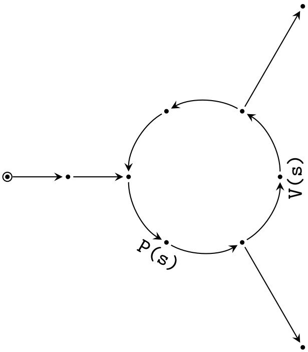

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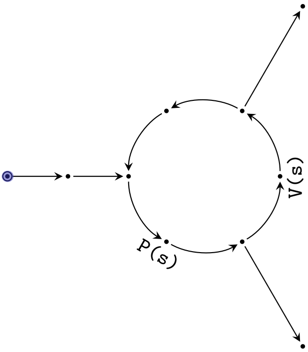

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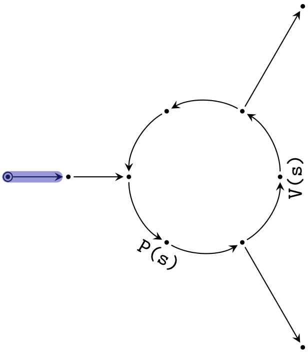

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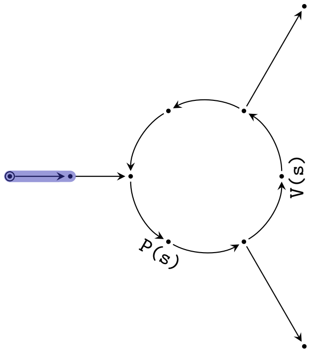

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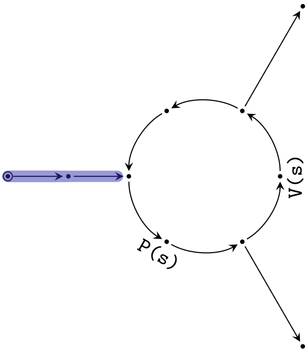

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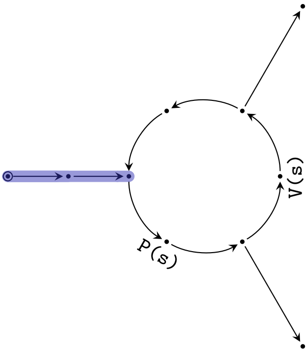

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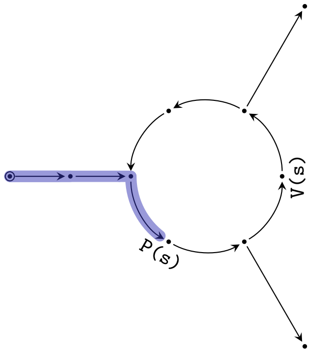

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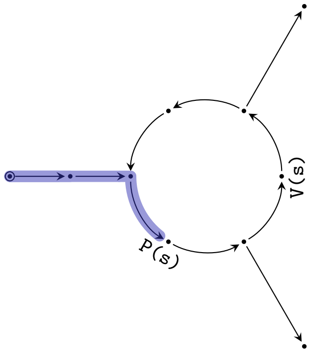

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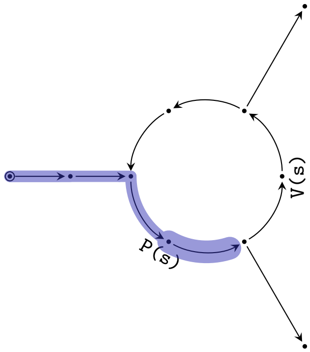

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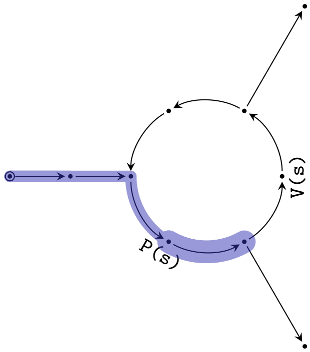

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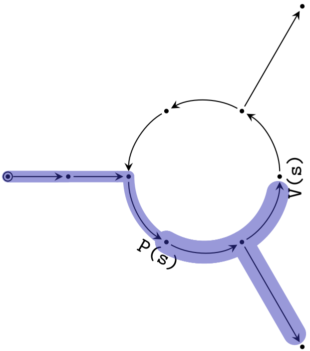

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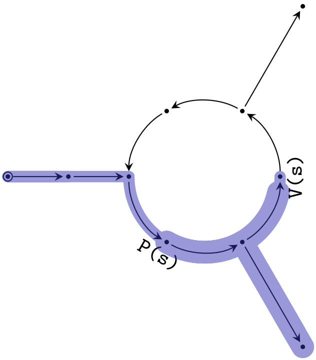

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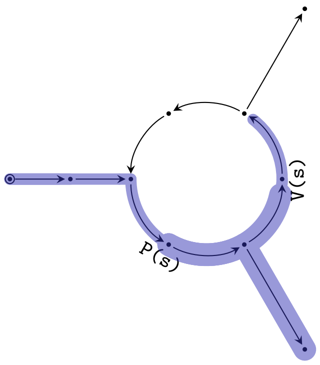

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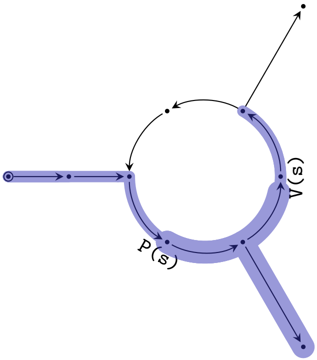

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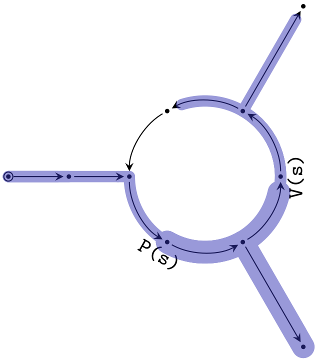

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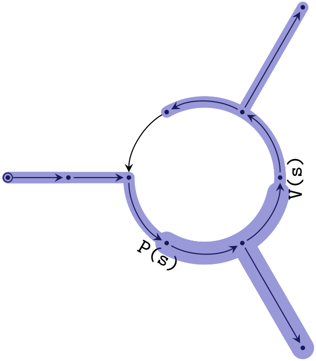

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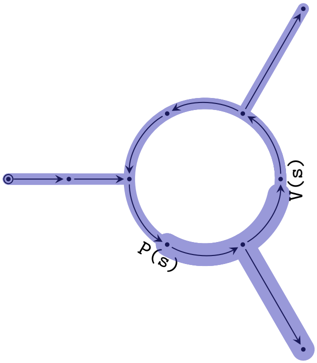

Not conservative process

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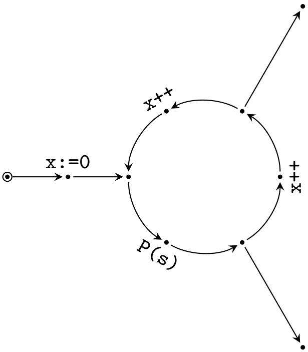

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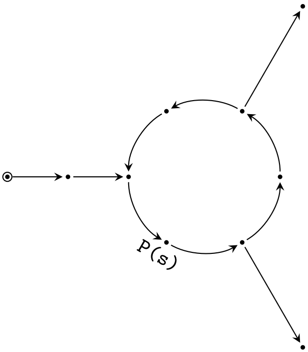

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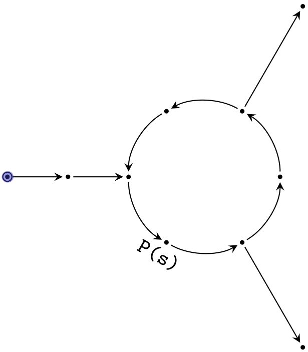

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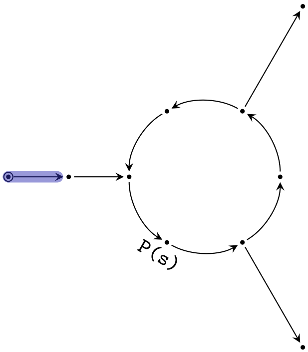

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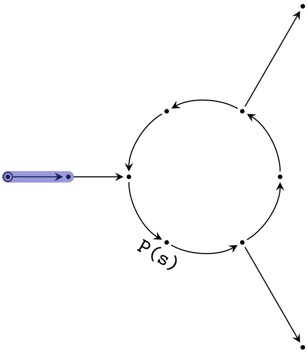

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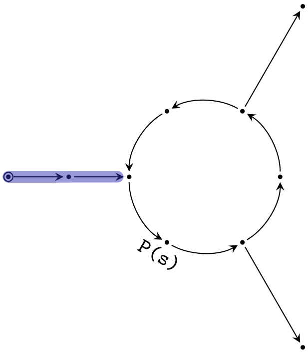

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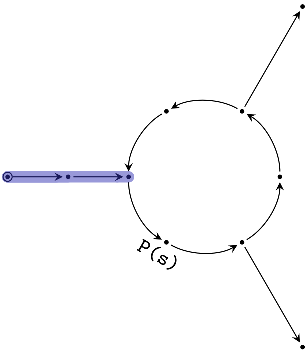

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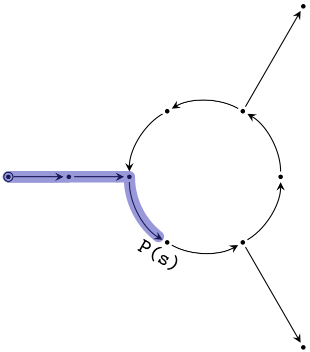

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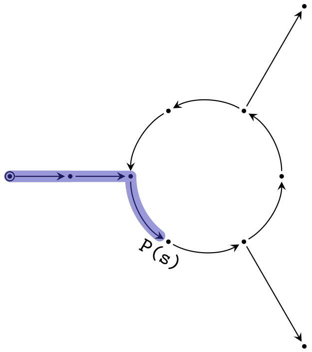

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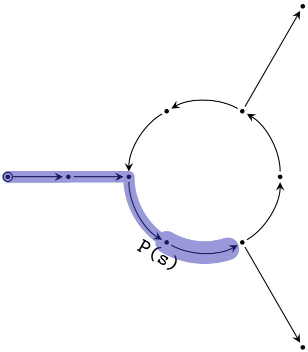

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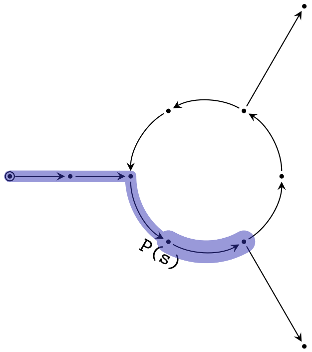

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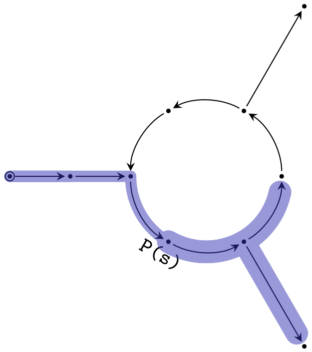

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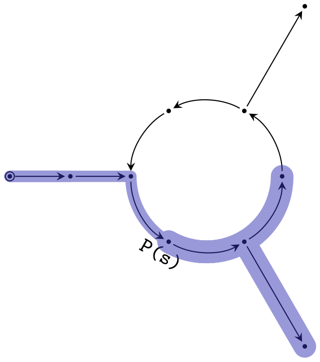

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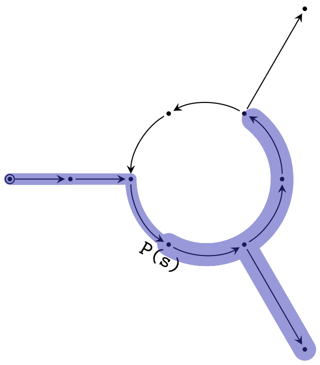

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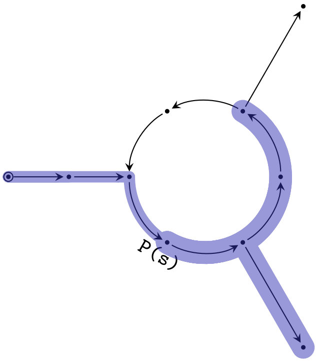

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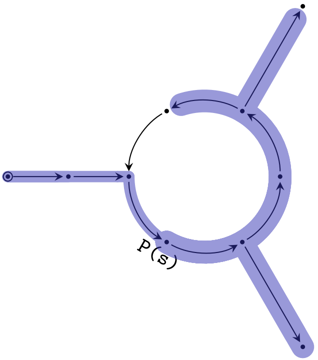

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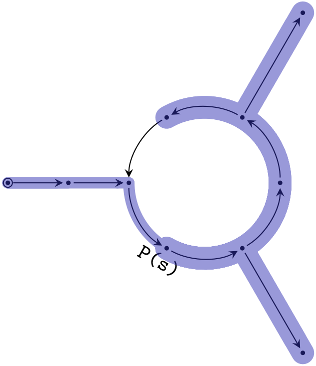

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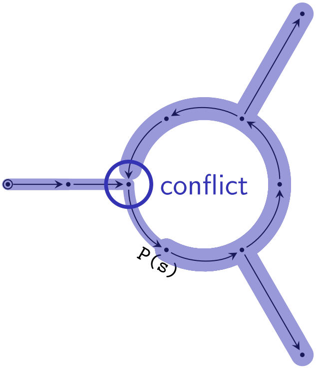
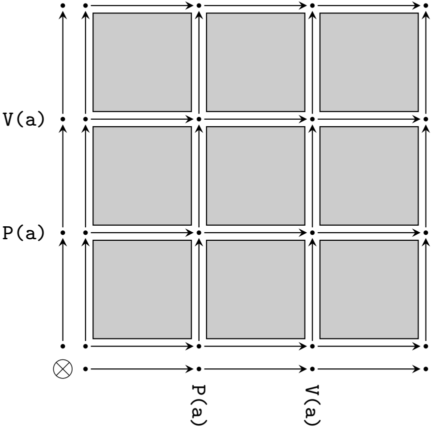


Discrete model

`sem: 1 a`



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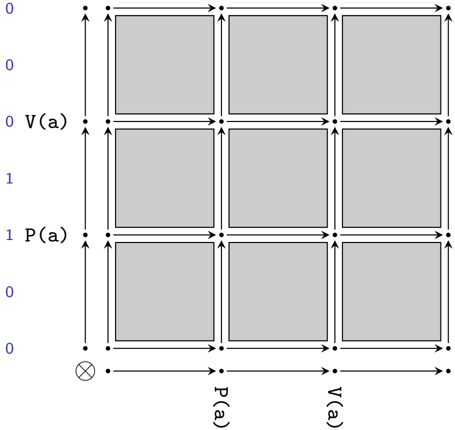

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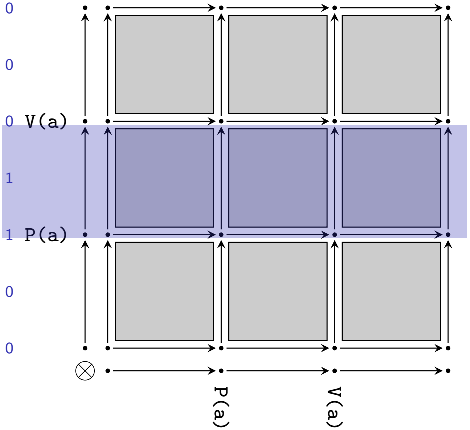

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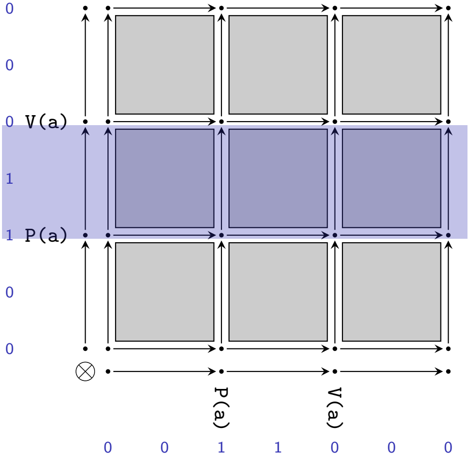

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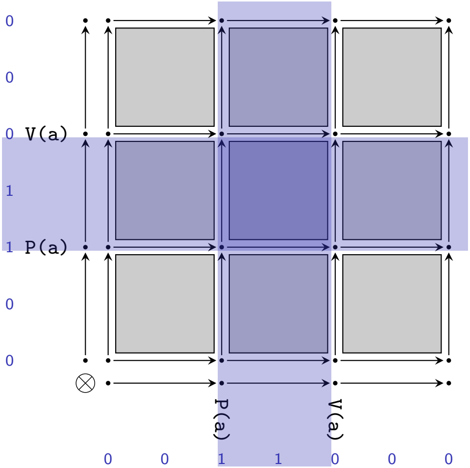

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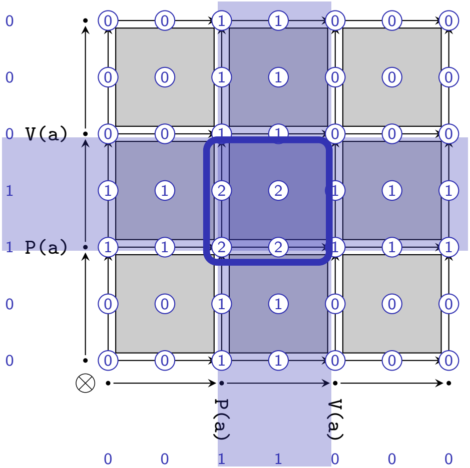

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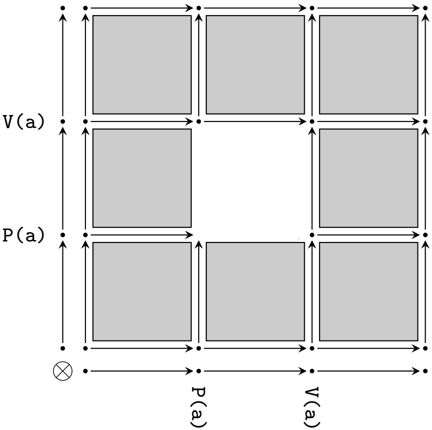

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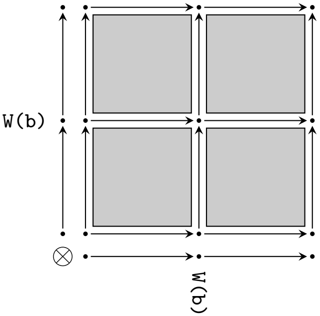
sync: 1 b



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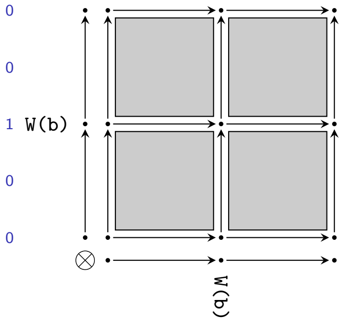

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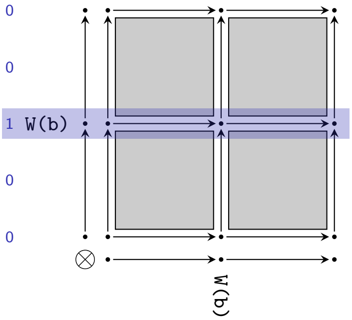

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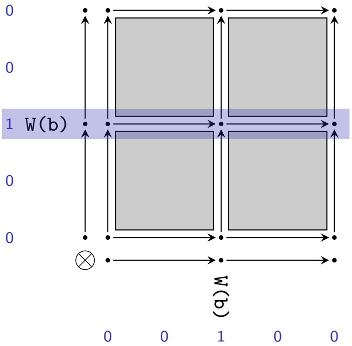

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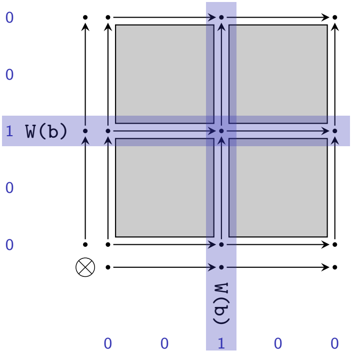

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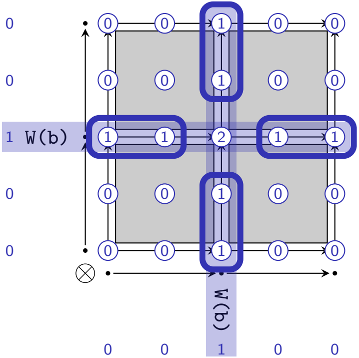

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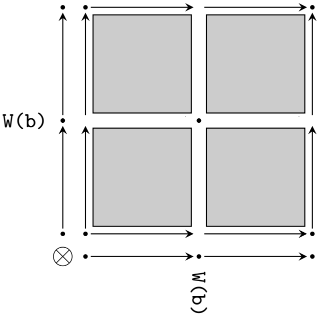

Locally ordered spaces

Directed atlas $\mathcal{U}$

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For all points p,

Locally ordered spaces

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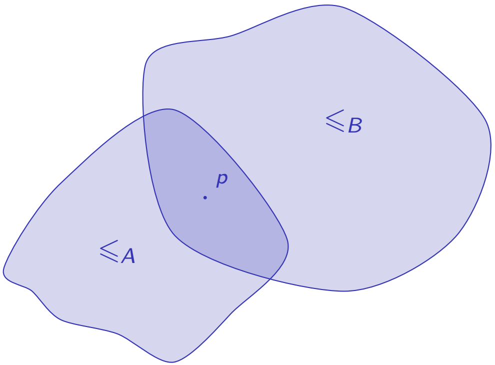

Locally ordered spaces

Directed atlas $\mathcal{U}$

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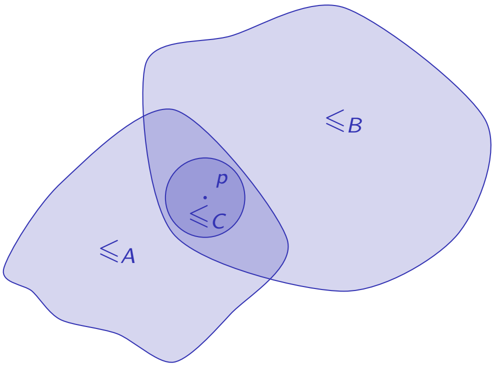
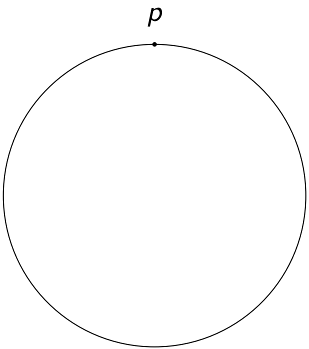


The directed circle

as a local pospace



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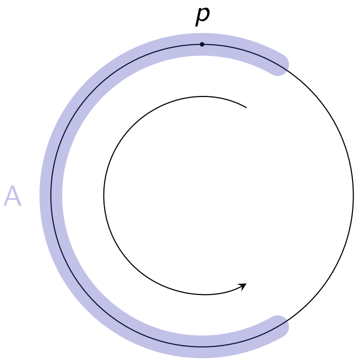

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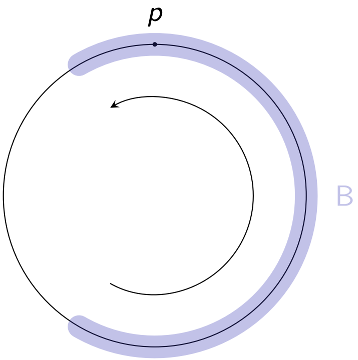

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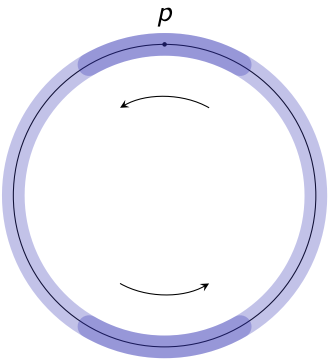

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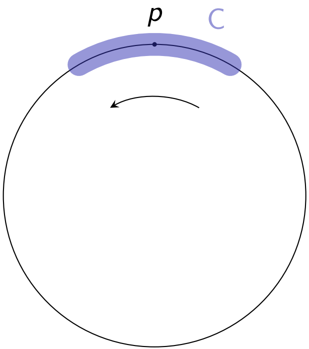

Directed geometric realization

Main property

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The main property

$$\upharpoonleft K^{(1)} \otimes \cdots \otimes K^{(n)} \downharpoonleft \quad \cong \quad \upharpoonleft K^{(1)} \downharpoonleft \times \cdots \times \upharpoonleft K^{(n)} \downharpoonleft$$

The continuous model

of a conservative program

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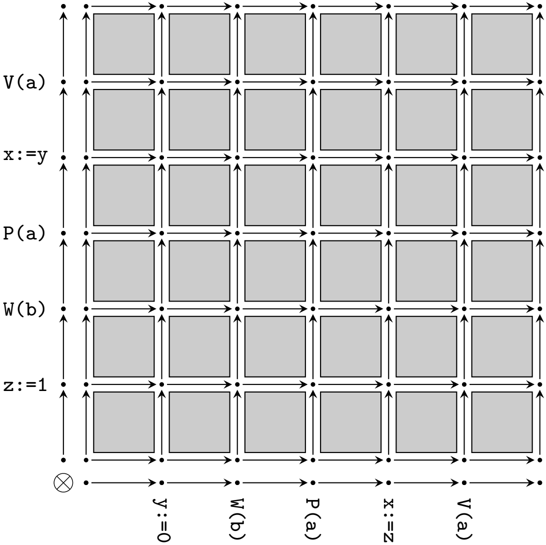
The continuous model

$$\bigsqcup_{d \in \mathbb{N}} (K_d \setminus F_d) \times]0,1[^d$$

From discrete to continuous

sem: 1 a   sync: 1 b



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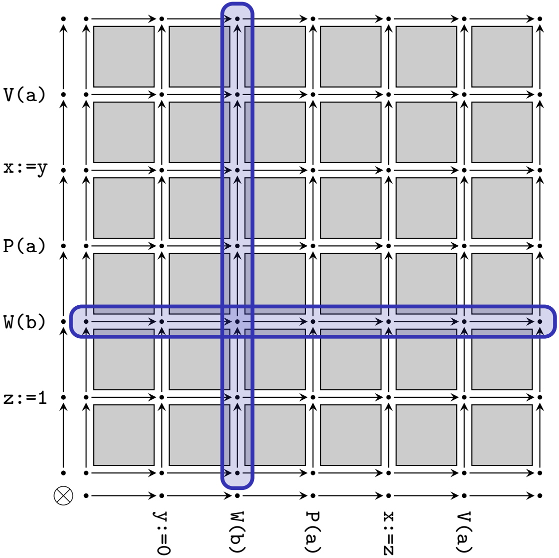

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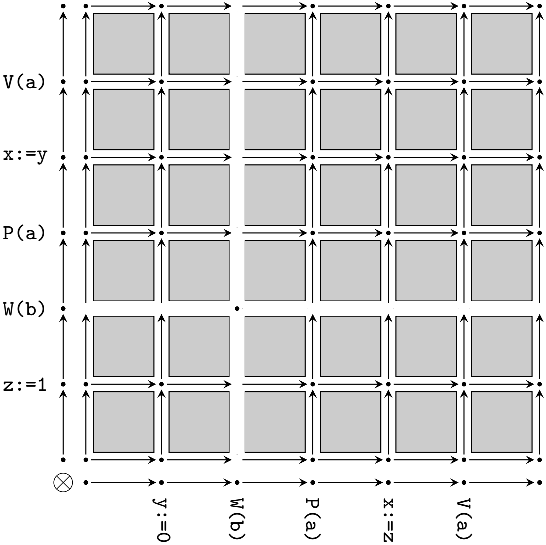

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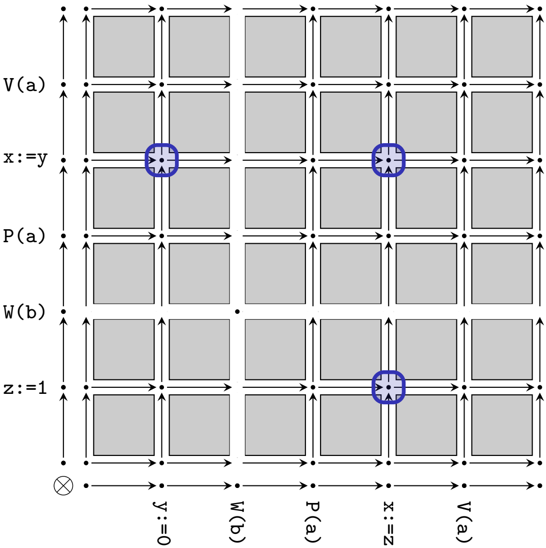

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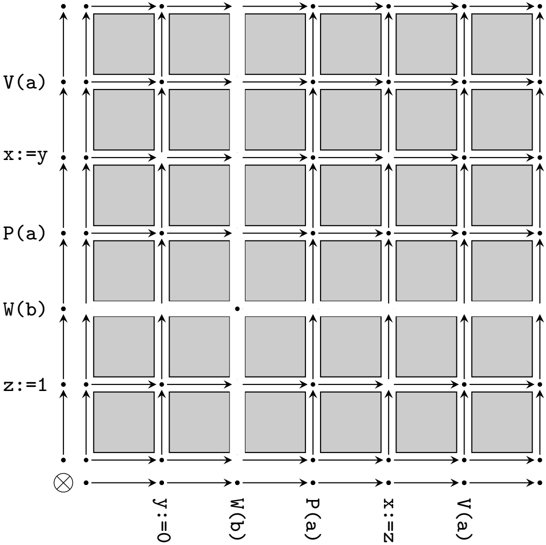

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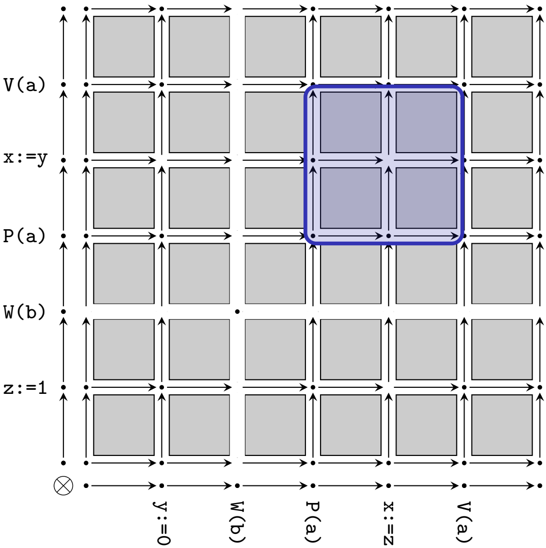

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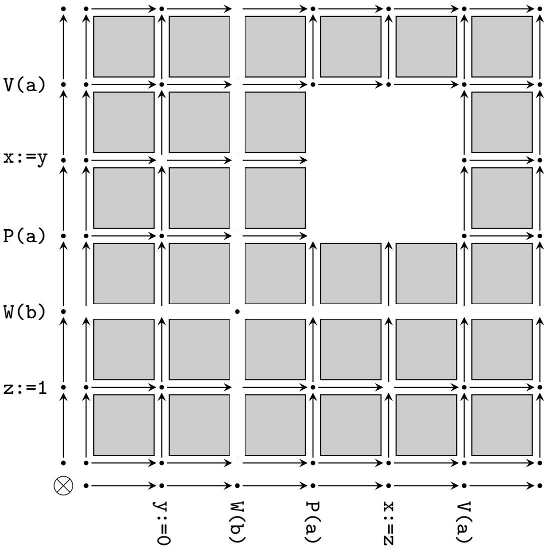

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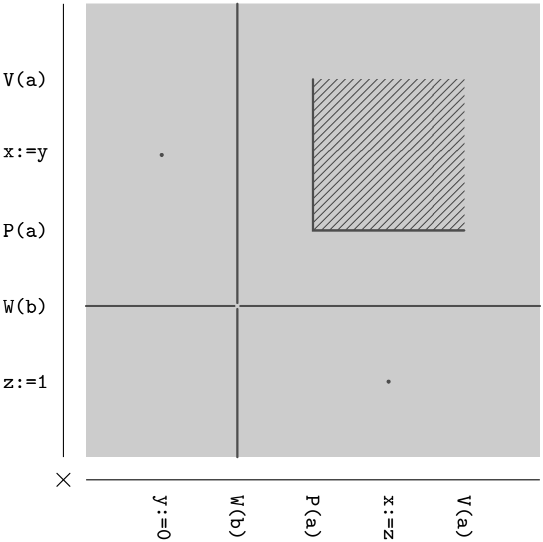

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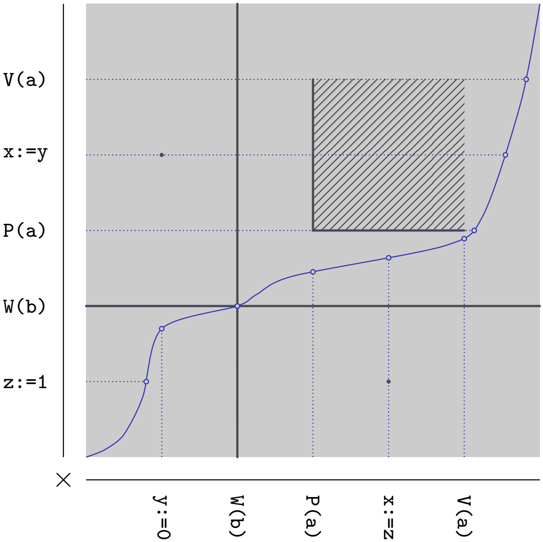

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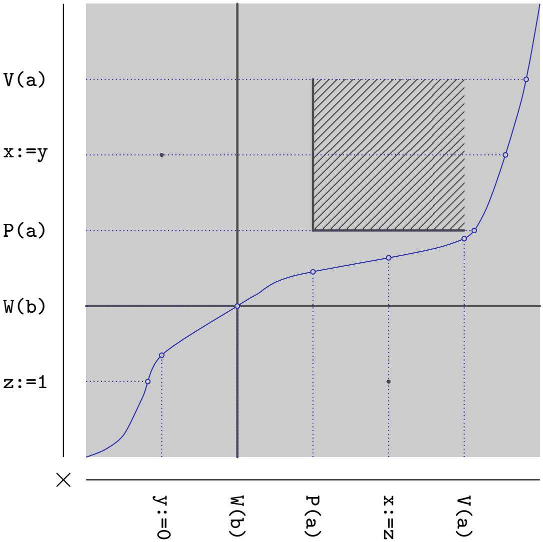

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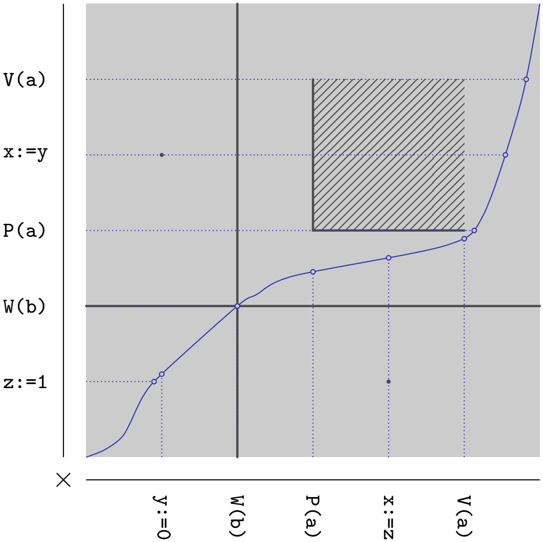

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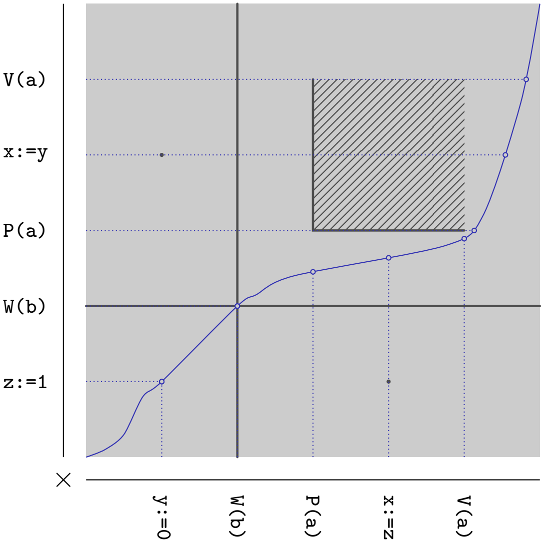

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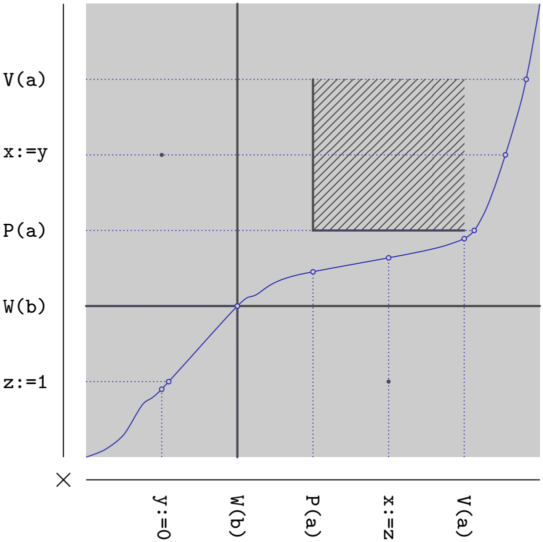

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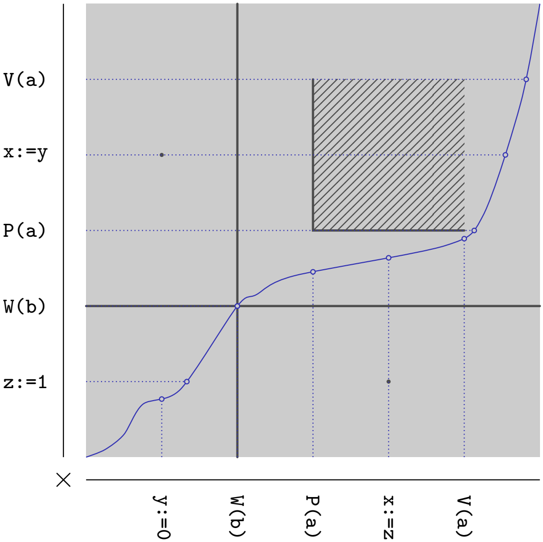

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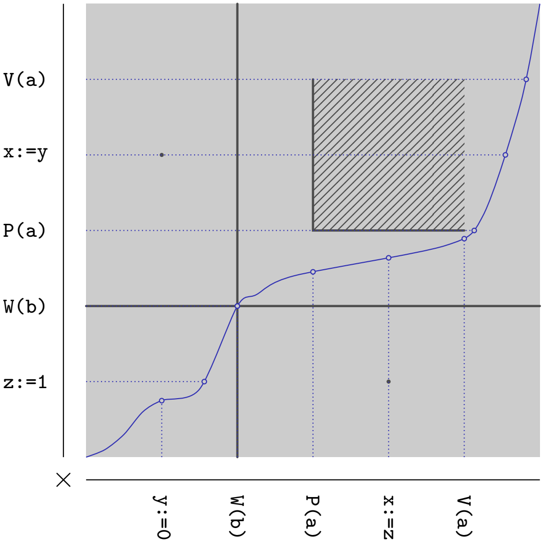

Weakly directed homotopy of directed paths

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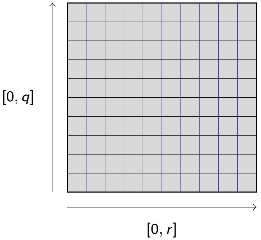

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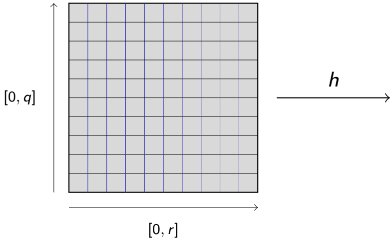

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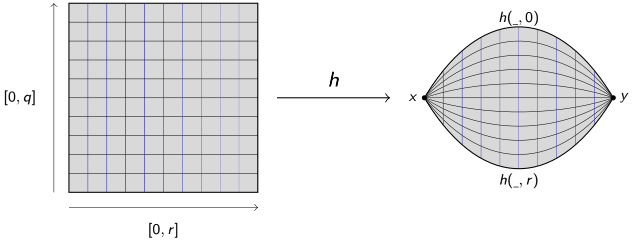

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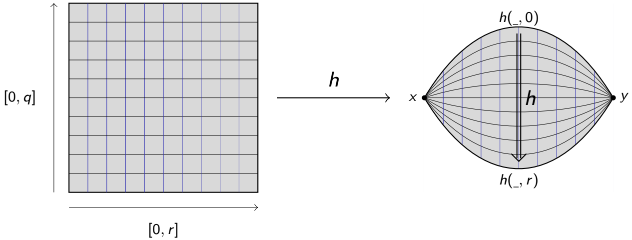


Substantiating the continuous models

Main theorem

Adequacy

The “actions” of weakly dihomotopic directed paths are the same. A directed path is an execution trace iff it is weakly dihomotopic with an execution trace.

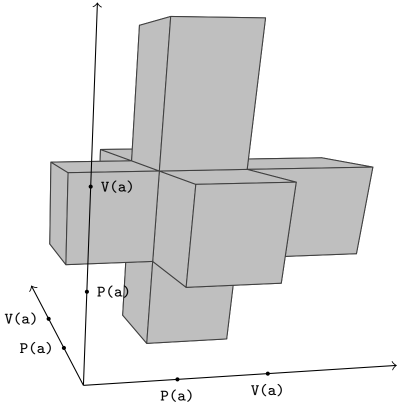
Tetrahemihexacron

a.k.a. 3D Swiss Cross

```
sem:  1 a
```

```
proc:
  p = P(a);V(a)
```

```
init:  3p
```



Floating cube

influence of arity

```
sem:  2 a
```

```
proc:
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```
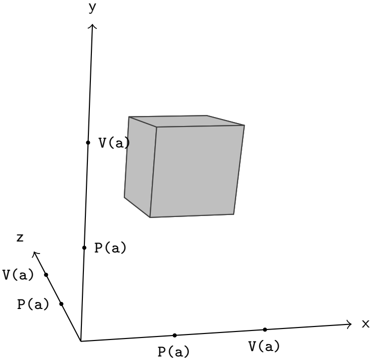

```
init:  3p
```



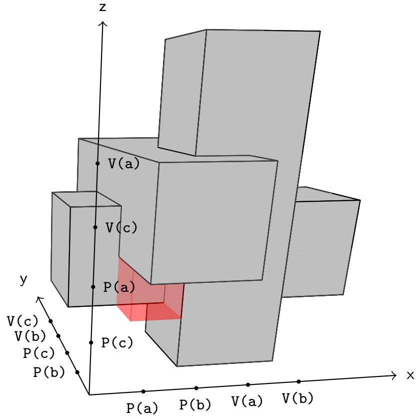
The dining philosophers

with its deadlock attractor

```
sem:  1 a b c
```

```
proc:
  x = P(a);P(b);V(a);V(b)
  y = P(b);P(c);V(b);V(c)
  z = P(c);P(a);V(c);V(a)
```

```
init:  x y z
```



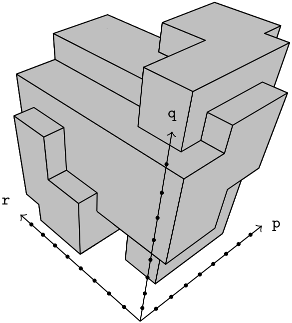
The Lipski algorithm

has no deadlock

```
sem:  1 x y z u v w
```

```
proc:
  p = P(x);P(y);P(z);V(x);P(w);V(z);V(y);V(w)
  q = P(u);P(v);P(x);V(u);P(z);V(v);V(x);V(z)
  r = P(y);P(w);V(y);P(u);V(w);P(v);V(u);V(v)
```

```
init:  p q r
```



Regions

over $G_1, \ldots, G_d$

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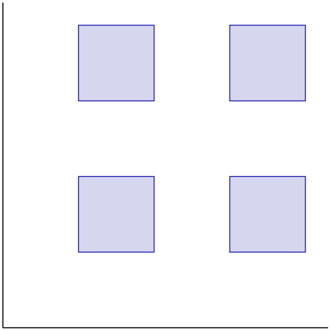
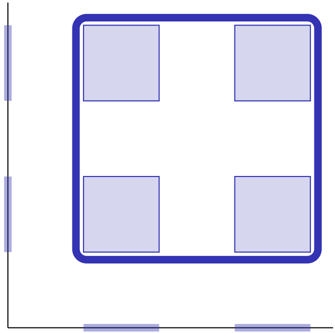
If X and Y are regions over $G_1, \ldots, G_d$ and $G'_1, \ldots, G'_{d'}$ then $X \times Y$ is a region over $G_1, \ldots, G_d, G'_1, \ldots, G'_{d'}$.

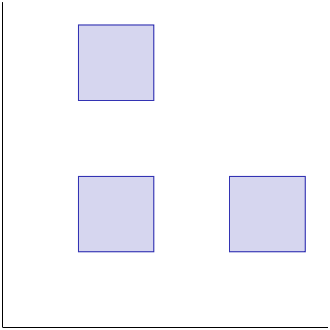
Maximal blocks



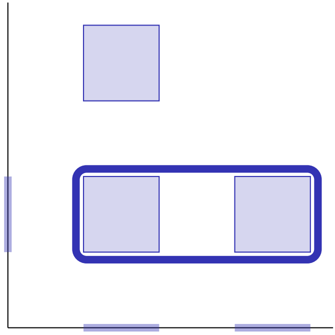
Maximal blocks



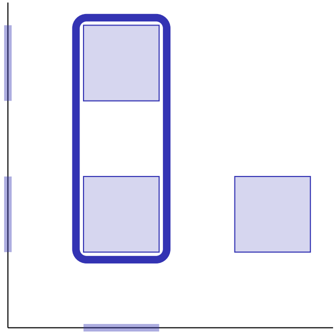
Maximal blocks



Maximal blocks



Maximal blocks



Main results

Maximal subblocks and Boolean structure

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Maximal subblocks

$X \subseteq \upharpoonleft G_1 \downharpoonleft \times \cdots \times \upharpoonleft G_d \downharpoonleft$ is a region iff it has finitely many maximal subblocks.

Main results

Maximal subblocks and Boolean structure

Maximal subblocks

$X \subseteq \upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_d \downharpoonright$ is a region iff it has finitely many maximal subblocks.

Boolean structure

The collection of regions over $G_1, \ldots, G_d$ form a Boolean subalgebra of the powerset of $\upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_d \downharpoonright$.

Main results

Unique decomposition

Prime decomposition

Up to coordinates reordering, any region can be written as a Cartesian product of irreducible regions in a unique way. This is the prime decomposition of it.

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Parallelization of code

The prime decomposition of the continuous model of some program provides a decomposition of the program as a parallel compound of "observationally independent" programs.

Main result

Effectiveness

An algorithm (Nicolas Ninin)

Let $M_1, \ldots, M_b$ be the maximal subblocks of $X^c = \upharpoonleft G_1 \downharpoonright \times \cdots \times \upharpoonleft G_d \downharpoonright \setminus X$. Let $\sim$ be the equivalence relation on $\{1, \ldots, d\}$ generated by $i \sim j$ when there exist $k \in \{1, \ldots, b\}$ such that

$$\mathrm{proj}_i(M_k) \neq \upharpoonleft G_i \downharpoonright \quad \text{and} \quad \mathrm{proj}_j(M_k) \neq \upharpoonleft G_j \downharpoonright$$

The prime decomposition of X is given by the $\sim$-equivalence classes.

Parallelizing a program

```
sem:  1 a b
sem:  2 c
```

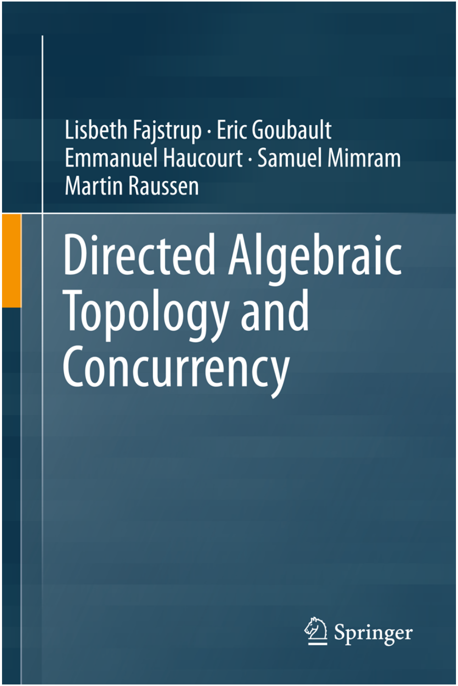
```
proc:
  p = P(a);P(c);V(c);V(a)

proc:
  q = P(b);P(c);V(c);V(b)
```

```
init:  2p 2q
```

Parallelizing a program

sem: 1 a b sem: 2 c	sem: 1 a b sem: 2 c
proc: p = P(a);P(c);V(c);V(a)	proc: q = P(b);P(c);V(c);V(b)
init: 2p	init: 2q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