∞-categorical models of linear logic

Elies Harington   Samuel Mimram

École Polytechnique

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Goal

∞-categories are rich in interesting phenomena, especially some with a *linear flavor*.
Goal : axiomatize categorical models of linear logic in ∞-categories.

- ∞-categories : objects, morphisms, higher morphisms between morphisms, etc.
- categorical models of linear logic : lots of heavy categorical structures

The **property** of a diagram commuting is replaced by the **data** of a higher isomorphism.
Such data must itself be subject to further conditions, that become more data, etc.

Arguments based on explicit computations don't generalize well to this setting.
The ideas and concepts that easily generalize are the more **unbiased, abstract** ones.

Remark

Here, ∞-category means $(\infty, 1)$-category: all morphisms of dimension > 1 will be invertible.

Outline

How to do denotational semantics in a category $\mathcal{C}$:

Syntax	Categorical semantics
Formulae A	Object $[\![A]\!]$ of $\mathcal{C}$
Proof π of $A \vdash B$	Morphism $[\![\pi]\!] : [\![A]\!] \to [\![B]\!]$ in $\mathcal{C}$
Cut elimination $\pi \rightsquigarrow \pi'$	Equality of morphisms $[\![\pi]\!] = [\![\pi]\!]'$
Additional syntactic constructions	Additional categorical structure

Intuitionistic linear logic

Formulas

$$\begin{aligned} F ::= & A \mid B \mid \dots \\ & \mid A \mathbin{\&} B \\ & \mid A \otimes B \\ & \mid A \multimap B \\ & \mid 1 \mid \top \\ & \mid !A \\ & \mid \dots \end{aligned}$$

Contexts $\Gamma ::= A_1, \dots, A_n$
Judgements $\Gamma \vdash B$

Examples of rules

$$\frac{}{A \vdash A}\,(\mathrm{ax}) \qquad \frac{\Gamma \vdash A \qquad \Delta, A \vdash C}{\Gamma, \Delta \vdash C}\,(\mathrm{cut})$$

$$\frac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes\mathrm{L}) \qquad \frac{\Gamma \vdash A \qquad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\,(\otimes\mathrm{R})$$

$$\frac{\Gamma, A_i \vdash B}{\Gamma, A_1 \mathbin{\&} A_2 \vdash B}\,(\&\mathrm{L}_i) \qquad \frac{\Gamma \vdash A \qquad \Gamma \vdash B}{\Gamma \vdash A \mathbin{\&} B}\,(\&\mathrm{R})$$

$$\frac{\Gamma \vdash A \qquad \Delta, B \vdash C}{\Gamma, \Delta, A \multimap B \vdash C}\,(\multimap\mathrm{L}) \qquad \frac{\Gamma, A \vdash B}{\Gamma \vdash A \multimap B}\,(\multimap\mathrm{R})$$

Linear and non-linear implications

Linear implication:

$$A \multimap B$$

Cannot duplicate or erase hypothesis A in proof

Non-linear (intuitionistic) implication:

$$!A \multimap B$$

Can duplicate or erase hypothesis A in proof.

Categorical semantics of linear logic: $\otimes$

Formulas A interpreted as objects $[\![A]\!] \in \mathcal{C}$.
$[\![A \otimes B]\!] = ?$
Need a *(symmetric) monoidal structure* on $\mathcal{C}$:
A functor $- \otimes - : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and an object $1 \in \mathcal{C}$ with natural isomorphisms

$$X \otimes Y \simeq Y \otimes X,$$
$$(X \otimes Y) \otimes Z \simeq X \otimes (Y \otimes Z),$$
$$X \otimes 1 \simeq X \simeq 1 \otimes X$$

satisfying some axioms.
Due to $\dfrac{\Gamma, A, B \vdash C}{\Gamma, A \otimes B \vdash C}\,(\otimes\text{L})$, can define $[\![A_1, \dots, A_n]\!] := [\![A_1 \otimes \cdots \otimes A_n]\!] = [\![A_1]\!] \otimes \cdots \otimes [\![A_n]\!]$

Categorical semantics of linear logic: $\multimap$

We can use the rules

$$\frac{A, B \vdash C}{A \otimes B \vdash C}\,(\otimes \mathrm{L}) \qquad\qquad \frac{A, B \vdash C}{A \vdash B \multimap C}\,(\multimap \mathrm{R})$$

to show we need bijections

$$\mathrm{Hom}_{\mathcal{C}}(\llbracket A \rrbracket \otimes \llbracket B \rrbracket, \llbracket C \rrbracket) \simeq \mathrm{Hom}_{\mathcal{C}}(\llbracket A \rrbracket, \llbracket B \multimap C \rrbracket)$$

Ask for $\mathcal{C}$ to be monoidal closed : $(X \otimes -) \dashv (X \multimap -)$.

$$\mathrm{Hom}_{\mathcal{C}}(X \otimes Y, Z) \simeq \mathrm{Hom}_{\mathcal{C}}(X, Y \multimap Z)$$

Categorical semantics of linear logic: &

The proofs

$$\dfrac{\dfrac{}{A_i \vdash A_i}\,(\mathrm{ax})}{A_1 \mathbin{\&} A_2 \vdash A_i}\,(\&\mathrm{L}_i)$$

will be interpreted as "projection" morphisms $\pi_i : [\![A_1 \mathbin{\&} A_2]\!] \to A_i$.
Thus we interpret & as the cartesian product in $\mathcal{C}$.

$$[\![A \mathbin{\&} B]\!] := [\![A]\!] \times [\![B]\!]$$

Categorical semantics of linear logic: !

Rules for the exponential

$$\frac{\Gamma, A \vdash B}{\Gamma, !A \vdash B}\ (\text{der})$$

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}\ (\text{prom})$$

$$\frac{\Gamma, !A, !A \vdash B}{\Gamma, !A \vdash B}\ (\text{contr})$$

$$\frac{\Gamma, !A \vdash B}{\Gamma \vdash B}\ (\text{weak})$$

The exponential ! $\rightsquigarrow$ a functor $! : \mathcal{C} \to \mathcal{C}$.
Promotion and dereliction rules $\rightsquigarrow$! is a *comonad*.

$$\dfrac{\dfrac{}{!A \vdash !A}\ (\text{ax}) \qquad \dfrac{}{!A \vdash !A}\ (\text{ax})}{\dfrac{!A, !A \vdash !A \otimes !A}{!A \vdash !A \otimes !A}\ (\text{contr})}\ (\otimes\text{R})$$

Similarly, $!A \vdash 1$.
Cut elimination shows that this gives a *comonoid* structure on $[\![!A]\!]$.

The goal

Many ways to package all the previous structures in simpler axiomatizations.

Goal: find an axiomatization **that can easily be transposed to the ∞-categorical setting.**

Seely categories

Definition ([See97])

A *Seely category* is a

1. symmetric monoidal closed category $(\mathcal{C}, \otimes, 1, \multimap)$
2. with finite products ($\&$ and $\top$),
3. a comonad $(!, \delta, \varepsilon) : \mathcal{C} \to \mathcal{C}$,
4. isomorphisms $m^2_{A,B} : !(A \,\&\, B) \simeq !A \otimes !B$ and $m^0 : !\top \simeq 1$ so that $! : (\mathcal{C}, \&) \to (\mathcal{C}, \otimes)$ is a *symmetric monoidal functor*
5. such that the following diagram commutes

$$\begin{array}{ccccc} !A \otimes !B & & \xrightarrow{\delta_A \otimes \delta_B} & & !!A \otimes !!B \\ \downarrow{\scriptstyle m^2_{A,B}} & & & & \downarrow{\scriptstyle m^2_{!A,!B}} \\ !(A \,\&\, B) & \xrightarrow[\delta_{A\&B}]{} & !!(A \,\&\, B) & \xrightarrow[!\langle !\pi_1, !\pi_2 \rangle]{} & !(!A \& !B) \end{array}$$

Point 5 is too ad hoc to have a natural ∞-categorical generalization.

Linear categories

Definition ([BBDPH97])

A *linear category* is :

- a symmetric monoidal closed category $(\mathcal{L}, \otimes, 1)$,
- together with a *lax symmetric monoidal comonad* $((!, m), \delta, \varepsilon)$,
- and a natural commutative comonoid structure $d_A : !A \to !A \otimes !A$, $e_A : !A \to 1$,

such that d_A and e_A are coalgebra morphisms for ! and δ is a comonoid morphism.

Less ad hoc, but still a lot of structure.

Linear-non-linear adjunctions

Every linear category $(\mathcal{L}, \otimes, 1, !, \dots)$ induces $(\mathcal{L}^!, \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathcal{L}, \otimes)$.
$\mathcal{L}^!$ category of coalgebras for the comonad !.
The morphisms in $\mathcal{L}^!$ represent the non-linear morphisms of linear logic ($!A \multimap B$).

Definition ([Ben95])

A *linear-non-linear adjunction* is an adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\perp}}}} (\mathcal{L}, \otimes)$$

between a cartesian category $\mathcal{M}$ and a symmetric monoidal closed category $\mathcal{L}$, where the left adjoint $L : \mathcal{M} \to \mathcal{L}$ is strongly monoidal $L(X \times Y) \simeq LX \otimes LY$.

$\mathcal{L}$ "linear" category, $\mathcal{M}$ "multiplicative" (non-linear) category.

Linear-non-linear adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

Induced comonad $LM : \mathcal{L} \to \mathcal{L}$ makes $\mathcal{L}$ into linear category.
Multiple choices of $\mathcal{M}$ may yield the same comonad : there is **more** structure than strictly needed.
But it is packaged in a more **minimalistic** way.
Only notions needed: monoidal functor, cartesian products, adjunctions.

A special case : Lafont categories

$!A$ must be a (commutative) comonoid.

Definition

$(\mathcal{L}, \otimes, !)$ is a *Lafont category* if $!A$ is the *cofree commutative comonoid on* A for every A.

Definition

Write $\mathrm{Comon}(\mathcal{L})$ for the category of commutative comonoids in $\mathcal{L}$.

Proposition

The category $\mathrm{Comon}(\mathcal{L})$ is cartesian. If $\mathcal{L}$ is Lafont, there is a linear-non-linear adjunction

$$(\mathrm{Comon}(\mathcal{L}), \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\bot}} (\mathcal{L}, \otimes).$$

Example: the relational model

The category Rel :

- Objects : sets $X, Y, \ldots$
- Morphisms : relations $R \subseteq X \times Y$
- Tensor product : cartesian product of underlying sets $X \times Y$
- Linear implication : also cartesian product of underlying sets, since

$$\mathrm{Rel}(X \times Y, Z) \simeq \mathrm{Rel}(X, Y \times Z)$$

- Cartesian product : disjoint union of underlying sets $X \sqcup Y$
- Exponential comonad : multisets $\mathrm{Mul}(X)$ on X (finite lists up to reordering, finite subsets with repetitions)

Proposition

$(\mathrm{Rel}, \times, \mathrm{Mul})$ is Lafont.
i.e. $\mathrm{Mul}(X)$ is the cofree commutative comonoid X in Rel.

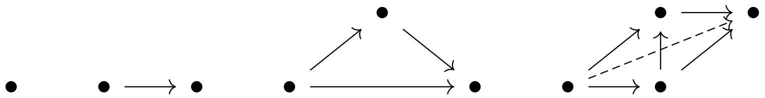
Outline

Almost all results in this section are from Lurie's work [Lurb, Lura, Lur18] or straightforward corollaries.

Shapes for higher morphisms

In categories, there is a unique way to compose morphisms.
In an ∞-category, various compositions may exist, and they are only related by higher isomorphisms between them.
To define ∞-categories, we need "shapes" for morphisms (cells) of arbitrary dimensions, and how they relate to one another.

Many possible choices, but the most developed one is that of **simplices**.



Simplicial sets

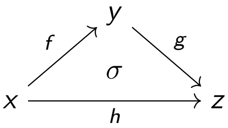
Definition (Simplex category)

$\mathbf{\Delta}$ denotes the category with objects the linear orders $[n] = \{0 < \cdots < n\}$ with $n \in \mathbb{N}$, and monotonous maps between them.

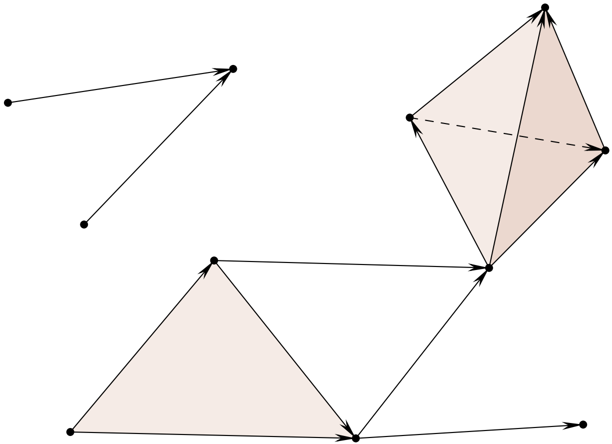
Definition

A *simplicial set* is a functor $X : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Set}$. Their category is written sSet.

- elements of X_0 are thoughts of as vertices of X
- $f \in X_1$ is thought of as an edge. The inclusions $\{0\} \hookrightarrow \{0,1\}$ and $\{1\} \hookrightarrow \{0,1\}$ give F a source and target vertices $d_0 f$ and $d_1 f$.
- $\sigma \in X_2$ is thought of as a filled triangle witnessing that "h is a composition of f and g"



Simplicial sets: a drawing



Examples of simplicial sets

Definition

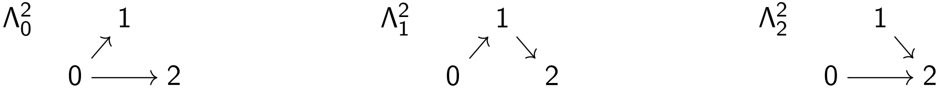
For every $n \in \mathbb{N}$ there is a simplicial set Δ^n such that

$$\forall X, \mathrm{Hom}(\Delta^n, X) \simeq X_n$$

. Δ^n is called the standard n-simplex.

Definition

Let $n > 1$, $0 \leq k \leq n$. The *horn* Λ^n_k is the subsimplicial set of Δ^n obtained by removing the unique cell of dimension n and the cell of dimension $(n-1)$ opposite to the vertex k.
The horn is an *inner horn* if $0 < k < n$, and an *outer horn* if $k = 0$ or $k = n$.



Categories as simplicial sets

Definition

Every category $\mathcal{C}$ determines a simplicial set $N\mathcal{C}$ called its **nerve**.
The n-simplices in $N\mathcal{C}$ are given by sequences of composable morphisms in $\mathcal{C}$.

$$d_0\sigma \longrightarrow d_1\sigma \cdots\cdots\rightarrow d_{n-1}\sigma \longrightarrow d_n\sigma$$

The action of morphisms in Δ is given by composition and discarding in $\mathcal{C}$.

Example

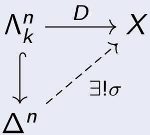
The inclusion $\{0,1,2\} \simeq \{1,2,4\} \hookrightarrow \{0,1,2,3,4\}$ gives the action

$$x_0 \xrightarrow{f} x_1 \xrightarrow{g} x_2 \xrightarrow{h} x_3 \xrightarrow{k} x_4 \qquad \mapsto \qquad x_1 \xrightarrow{g} x_2 \xrightarrow{h \circ k} x_4$$

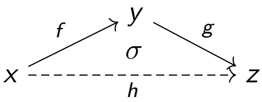
Categories as simplicial sets

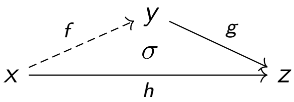
Proposition

A simplicial set X is isomorphic to the nerve of a category if and only if for every $0 < k < n$, $n > 1$, and morphism $D : \Lambda^n_k \to X$, there **exists a unique** cell $\sigma \in X_n$ making the following diagram commute.



It is the nerve of a groupoid if and only if this condition also applies when $0 \geq k \geq n$, $n > 0$.





$\Lambda^2_1 \qquad h = g \circ f$ $\qquad\qquad$ $\Lambda^1_0 \qquad f = g^{-1} \circ h$

∞-categories, ∞-groupoids

Definition

An ∞-category is a simplicial set X such that there **exists a** **(non-necessarily unique)** lift with respect to every inclusion of inner horn $\Lambda^n_k \hookrightarrow \Delta^n$:

$$\begin{array}{ccc} \Lambda^n_k & \xrightarrow{D} & X \\ \big\downarrow & \overset{\exists \sigma}{\nearrow} & \\ \Delta^n & & \end{array}$$

It is an ∞-groupoid if it admits lifts also for outer horn inclusions.
The vertices of an ∞-category are called *objects*, its edges are called *morphisms*.

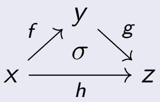
Example

The nerve of a category is an ∞-category, the nerve of a groupoid is an ∞-groupoid.
The nerve functor N : Cat $\to$ ∞Cat is fully faithful.

Composition of morphisms

Definition

In an ∞-category $\mathcal{C}$, given a triangle $\sigma \in \mathcal{C}_2$



we say that σ witnesses that **h is a composition of g and f.**

Proposition

In an ∞-category $\mathcal{C}$, composition of morphisms always exists, and is generally not unique.

Homotopy between morphisms

Let $\mathcal{C}$ be an ∞-category.

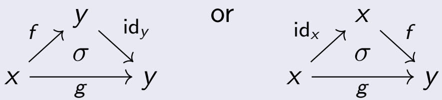
Definition

Let $x \in \mathcal{C}_0$. There is an identity morphism $\mathrm{id}_x : x \to x$ x given by the action of $\mathcal{C}$ on the only map $\{0, 1\} \to \{0\}$.

Let $f, g : x \to y$ be morphisms in $\mathcal{C}$.

Definition

A homotopy between f and g is a 2-cell $\sigma \in \mathcal{C}_2$ of the following shape

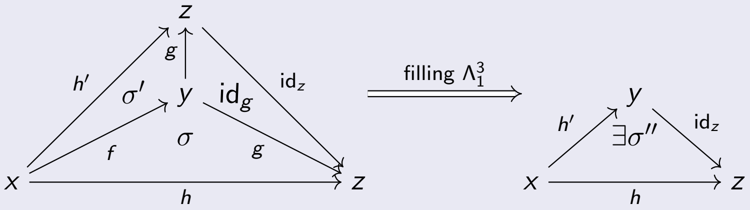


f and g are *homotopic* (written $f \sim g$) if there exists a homotopy between f and g.

Proposition

Let $f : x \to y$, $g : y \to z$, and $h, h' : x \to z$ two compositions of g and f. Then $h \sim h'$.

Proof.



□

Homotopies and composition

Proposition

The relation $\sim$ is an equivalence relation.

Proposition

Composition is unique up to homotopy.

Proposition

Composition is associative and unital up to homotopy.

All proofs: playing with horn filling conditions

In particular, can define the **homotopy category** $h\mathcal{C}$ with same objects as $\mathcal{C}$, and morphisms are morphisms in $\mathcal{C}$ up to homotopy.

Functor ∞-categories

A *functor* between ∞-categories is just a morphism of simplicial sets.

Proposition

The category sSet is cartesian closed (as a presheaf category), with internal hom given by

$$Fun(X, Y)_n := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times X, Y)$$

Proposition

If Y is an ∞-category (resp. ∞-groupoid), then $\mathrm{Fun}(X, Y)$ is an ∞-category (resp. ∞-groupoid).

The objects of $\mathrm{Fun}(X, Y)$ are exactly the morphisms of simplicial sets $X \to Y$.

Natural transformations, equivalences

Definition

Let $F, G : X \to Y$ be morphisms of simplicial sets, with Y an ∞-category.
A natural transformation is morphism $\alpha : F \to G$ in $\mathrm{Fun}(X, Y)$.
Equivalently, $\alpha : \Delta^1 \times X \to Y$ such that $\alpha_{|\{0\}\times X} = F$ and $\alpha_{|\{1\}\times X} = G$.

Definition

A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of ∞-categories if there exists $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $G \circ F \to \mathrm{id}_{\mathcal{C}}$, $F \circ G \to \mathrm{id}_{\mathcal{D}}$.

Hom ∞-groupoid

Let x, y be objects of an ∞-category $\mathcal{C}$. Write $\mathrm{Hom}_{\mathcal{C}}(x, y)$ or simply $\mathrm{Hom}(x, y)$ for the following pullback in sSet.

$$\begin{array}{ccc} \mathrm{Hom}_{\mathcal{C}}(x,y) & \longrightarrow & \mathrm{Fun}(\Delta^1, X) \\ \downarrow & \lrcorner & \downarrow \text{restriction to endpoints} \\ \Delta^0 & \xrightarrow[(x,y)]{} & \mathrm{Fun}(\Delta^0 \sqcup \Delta^0, X) \end{array}$$

Proposition

$\mathrm{Hom}_{\mathcal{C}}(x, y)$ is an ∞-groupoid whose objects are given by morphisms $f : x \to y$ in $\mathcal{C}$ and whose morphisms are given by homotopies.

Proposition

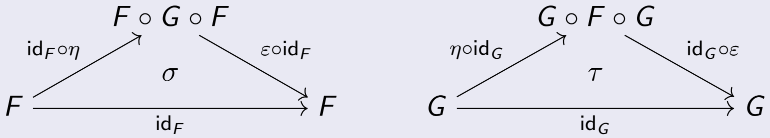
The existence of composite of morphisms in $\mathcal{C}$ can be enhanced to the choice of a functor $\mathrm{Hom}(x, y) \times \mathrm{Hom}(y, z) \to \mathrm{Hom}(x, z)$.

Adjunctions

Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$ be functors between ∞-categories, and $\eta : \mathrm{id}_{\mathcal{C}} \to G \circ F$, $\varepsilon : F \circ G \to \mathrm{id}_{\mathcal{D}}$ be natural transformations.

Definition

(η, ε) is a *unit-counit pair* for F and G if there exist compositions



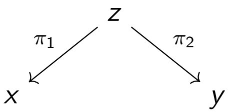
in $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ and $\mathrm{Fun}(\mathcal{D}, \mathcal{C})$.

Definition

F is left adjoint to G (and G right adjoint to F) if there exists a unit-counit pair for F and G.

(Co)limits

Let $x, y \in \mathcal{C}$ and ∞-category. A product of x and y is a diagram



such that for all $z' \in \mathcal{C}$, the induced map

$$\mathrm{Hom}(z', z) \to \mathrm{Hom}(z', x) \times \mathrm{Hom}(z', y)$$

is an equivalence of ∞-groupoids.

Remark

The universal property is up to equivalence, while in 1-categories it's up to isomorphism.

General limits and colimits can be defined along those lines.

Summary of ∞-category theory so far

- ∞-categories have objects, morphisms, homotopies
- existence of compositions
- uniqueness up to homotopy
- Hom-∞-groupoids instead of Hom-sets
- universal properties are up to equivalence
- adjunctions can be defined as usual

Monoids in categories

In a category $\mathcal{C}$ with finite products, a commutative monoid is an object M together with maps $\mu : M \times M \to M$, $\eta : 1 \to M$, such that the following commute.

associativity

$$\begin{array}{ccc} M\times M\times M & \xrightarrow{\mathrm{id}_M\times\mu} & M\times M \\ {\scriptstyle \mu\times\mathrm{id}_M}\downarrow & & \downarrow{\scriptstyle\mu} \\ M\times M & \xrightarrow[\mu]{} & M \end{array}$$

unitality

$$\begin{array}{ccccc} M & \xrightarrow{\mathrm{id}_M\times\eta} & M\times M & \xleftarrow{\eta\times\mathrm{id}_M} & M \\ & {\scriptstyle\mathrm{id}_M}\searrow & \downarrow & \swarrow{\scriptstyle\mathrm{id}_M} & \\ & & M & & \end{array}$$

commutativity

$$\begin{array}{ccc} M\times M & \xrightarrow{\langle\pi_2,\pi_1\rangle} & M\times M \\ {\scriptstyle\mu}\downarrow & \swarrow{\scriptstyle\mu} & \\ M & & \end{array}$$

In an ∞-category, need further coherence conditions on the data of homotopies, in every dimension.

How to specify everything in a homogeneous way ?

Monoids in categories

The previous definition of commutative monoid is *biased* : many other operations than μ and η exist in monoids.

$$M^5 \times M^2$$

$$(x_1, x_2, x_3, x_4, x_5) \mapsto (\mu(x_3, x_1), \mu(x_2, x_5))$$

Every partial map $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$ induces a map

$$M^m \to M^n$$

$$(x_i)_{1 \leq i \leq m} \mapsto \Big(\prod_{f(i)=j} x_i \Big)_{1 \leq j \leq n}$$

Write FinSet_* for the category of finite sets $\{1, \ldots, n\}$ and partial maps.

Proposition

Commutative monoids in $\mathcal{C}$ correspond to functors $F : \mathrm{FinSet}_* \to \mathcal{C}$ such that $F(\{1, \ldots, n\}) \simeq F(\{1\})^n$.

Monoids in ∞-categories

Definition

A commutative monoid in an ∞-category $\mathcal{C}$ is a functor $F : N\mathrm{FinSet}_* \to \mathcal{C}$ such that $F(\{1, \ldots, n\}) \simeq F(\{1\})^n$.

This is for monoids with respect to **cartesian products.**

Symmetric monoidal ∞-categories

Fun fact

A symmetric monoidal category is exactly a commutative (pseudo)monoid in the bicategory of categories.

Proposition

There is an ∞-category $\infty\mathrm{Cat}$ whose objects are ∞-categories, morphisms are functors, homotopies are natural isomorphisms, etc.
This ∞-category admits cartesian products, given by the cartesian product of the underlying simplicial sets.

Definition

A symmetric monoidal ∞-category is a monoid M in $\infty\mathrm{Cat}$, i.e. $M : N\mathrm{FinSet}_* \to \infty\mathrm{Cat}$. Its underlying ∞-category is $M(1)$.

Monoidal functors and monoids

With more effort, possible to define :

- commutative monoids in symmetric monoidal ∞-categories
- strong monoidal functors ($F(x \otimes y) \simeq F(x) \otimes F(y)$ + higher structure)
- lax monoidal functors (with maps $F(x) \otimes F(y) \to F(x \otimes y)$ + higher structure)

and show (lax) monoidal functors preserve monoids.

Outline

Content of our article [HM25]

∞-linear-non-linear adjunction

Definition

An LNL adjunction in ∞-categories is an adjunction

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

between a cartesian ∞-category $\mathcal{M}$ and a symmetric monoidal closed ∞-category $\mathcal{L}^{\otimes}$, such that the left adjoint L is strong monoidal.

Proposition

The right adjoint is lax monoidal.

Sanity check : comonoid structure on !A

Proposition

In a cartesian ∞-category, every object admits a unique commutative comonoid structure. (comultiplication is given by the diagonal map $X \to X \times X$)

Since strongly monoidal functors preserve commutative comonoids, we get

Corollary

In an LNL adjunction between ∞-categories,

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

For every object $x \in \mathcal{L}$, $!x := LMx$ inherits a canonical commutative comonoid structure.

Sanity check : Seely isomorphisms

Let

$$(\mathcal{M}, \times) \underset{M}{\overset{L}{\underset{\longleftarrow}{\overset{\longrightarrow}{\bot}}}} (\mathcal{L}, \otimes)$$

be an LNL adjunction between ∞-categories, where $\mathcal{C}$ has cartesian products.
Since right adjoints preserve limits, M is strongly monoidal from $(\mathcal{L}, \times)$ to $(\mathcal{M}, \times)$. Hence the composite $! = LM : \mathcal{L} \to \mathcal{L}$ is strongly monoidal $(\mathcal{L}, \times) \to (\mathcal{L}, \otimes)$.

In particular this gives Seely isomorphisms $!(A \times B) \simeq !A \otimes !B$, $!\top = 1$.

Lafont ∞-categories

A monoidal structure on an ∞-category $\mathcal{C}$ determines a monoidal structure on $\mathcal{C}^{\mathrm{op}}$ via the self-equivalence $\mathrm{op} : \infty\mathrm{Cat} \to \infty\mathrm{Cat}$.

Definition (Commutative comonoids)

Given a SM∞C $\mathcal{C}$, the ∞-category $\mathrm{Comon}(\mathcal{C})$ is defined as $\mathrm{Mon}(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}}$.

Theorem

The ∞-category $\mathrm{Comon}(\mathcal{C})$ *is cartesian and the forgetful functor* $\mathrm{Comon}(\mathcal{C}) \to \mathcal{C}$ *is strongly monoidal from the cartesian structure to the monoidal one.*

Corollary

If $\mathrm{Comon}(\mathcal{C}) \to \mathcal{C}$ *has a right adjoint, it induces an LNL adjunction* $(\mathrm{Comon}(\mathcal{C}), \times) \underset{\longleftarrow}{\overset{\longrightarrow}{\perp}} (\mathcal{C}, \otimes)$

Definition

In that case, we say that $\mathcal{C}$ is a *Lafont ∞-category*.

An explicit formula for cofree comonoids

The following has been shown in 1-category theory by [MTT].

Theorem

Let $(\mathcal{L}, \otimes)$ be a symmetric monoidal ∞-category, and $X \in \mathcal{C}$.
If for all $A \in \mathcal{C}$,

$$A \otimes \prod_{n \in \mathbb{N}} (X^{\otimes n})^{\mathfrak{S}_n} \to \prod_{n \in \mathbb{N}} (A \otimes X^{\otimes n})^{\mathfrak{S}_n}$$

is an equivalence, then

$$\prod_{n \in \mathbb{N}} (X^{\otimes n})^{\mathfrak{S}_n}$$

is the cofree commutative comonoid on X.

It follows easily from more general results of Lurie [Lura].

Another criterion for existence of cofree comonoids

Definition

An ∞-category $\mathcal{C}$ is *presentable* if

- it is closed under small colimits
- there is a small set of objects $S \subset \mathcal{C}_0$ such that every object is a *filtered colimit* of objects of S

Theorem

Let $\mathcal{C}$ be a symmetric monoidal presentable ∞-category such that $\forall x \in \mathcal{C}$, the functor $x \otimes - : \mathcal{C} \to \mathcal{C}$ preserves small colimits. Then $\mathcal{C}$ is Lafont (it admits cofree comonoids).

But in general there is no nice formula in this context.

Example : ∞-categorical generalized species

(∞)-category	Rel	Prof
Objects	Sets X, Y	∞-categories $\mathcal{C}, \mathcal{D}$
Linear morphisms	Relations $R : X \times Y \to \mathsf{Bool}$	∞-profunctors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \infty\mathsf{Grpd}$
Lafont exponential	$\mathsf{Mul}(X)$ multisets on underlying set	$\mathsf{Sym}(\mathcal{C})$ free symmetric monoidal ∞-category
Non-linear morphisms	"multi-relations" $\mathsf{Mul}(X) \times Y \to \mathsf{Bool}$	"∞-generalized species" $\mathsf{Sym}(\mathcal{C}) \times \mathcal{D}^{\mathrm{op}} \to \infty\mathsf{Grpd}$
Extensional objects	Complete lattices $P(X) = \mathsf{Bool}^X$	Presheaf ∞-categories $\mathcal{P}(\mathcal{C}) := \mathsf{Fun}(\mathcal{C}^{\mathrm{op}}, \infty\mathsf{Grpd})$
Extensional morphisms	Maps $P(X) \to P(Y)$ that preserve arbitrary joins	Functors $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$ that preserve small colimits
Extensional non-linear morphisms	?	Analytic functors ?

Conclusion

Conclusion :

- Building upon the heavy machinery of ∞-categories developed, we generalized two notions of models of linear logic to the ∞-categorical setting (Lafont categories and LNL adjunctions).
- We constructed new such models analogous to variants of the relational model and bicategorical models of species and polynomials.

Future work :

- Give direct definitions of linear ∞-categories and Seely ∞-categories, and show they induce LNL adjunctions.
- Explicit comparison of our generalized ∞-species and analytic functors.
- Generalize to $(\infty, 2)$-categorical setting to model differential linear logic.
- Try to fit advanced homotopical constructions with linear flavour (Goodwillie calculus ?) into this new setting.

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A reminder on the 1-categorical story

Intensional	Extensional
category Rel	full subcat of SupLat on the $P(X)$, $X \in$ Set
category Porel	full subcat of SupLat on the $P(X)$, $X \in$ Poset
Mul(X) free commutative monoid on underlying (po)set	free commutative comonoid in SupLat
non-linear maps Mul(X) $\to Y$	?
FC(X) free poset with finite joins on X	$!_S P(X)$ exponential induced by LNL adjunction $\text{Scott} \underset{\hookleftarrow}{\overset{\longrightarrow}{\perp}} \text{SupLat}$
non-linear maps FC(X) $\to Y$	Scott-continuous maps $P(X) \to P(Y)$

∞-categories with colimits

Let $\mathbb{K}$ be a class of simplicial sets. Write $\infty\mathrm{Cat}_{\mathbb{K}}$ for the sub-∞-category of $\infty\mathrm{Cat}$ on ∞-categories that admit colimits indexed by simplicial sets in $\mathbb{K}$, and functors that preserve such colimits.

Special cases : $\infty\mathrm{Cat}_{\mathrm{cc}}$ for $\mathbb{K} =$ all simplicial sets ("cc" for cocontinuous), $\infty\mathrm{Cat}_{\mathrm{filtr}}$ for filtered simplicial sets, $\infty\mathrm{Cat}_{\mathrm{sift}}$ for sifted simplicial sets.

Proposition

The ∞-category $\infty\mathrm{Cat}_{\mathbb{K}}$ admits a symmetric monoidal closed structure whose tensor products classifies functors $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ that preserve $\mathbb{K}$-colimits *independently in both variables*. Moreover, if $\mathbb{K} \subseteq \mathbb{K}'$, the forgetful functor $\infty\mathrm{Cat}_{\mathbb{K}'} \to \infty\mathrm{Cat}_{\mathbb{K}}$ admits a strongly monoidal left adjoint.

Proposition

If $\mathbb{K}$ consists only of sifted simplicial sets, then the previous monoidal structure is cartesian. That is the case for $\infty\mathrm{Cat} = \infty\mathrm{Cat}_{\emptyset}$, $\infty\mathrm{Cat}_{\mathrm{filtr}}$ and $\infty\mathrm{Cat}_{\mathrm{sift}}$.

Cocompletion-based LNLs

There is a chain of strongly monoidal left adjoints

$$\mathcal{S} \underset{\perp}{\rightleftarrows} \infty\mathsf{Cat} \underset{\perp}{\rightleftarrows} \infty\mathsf{Cat}_{\mathsf{filtr}} \underset{\perp}{\rightleftarrows} \infty\mathsf{Cat}_{\mathsf{sift}} \underset{\perp}{\rightleftarrows} \infty\mathsf{Cat}_{\mathsf{cc}}$$

where the monoidal structures on all but $\infty\mathsf{Cat}_{\mathsf{cc}}$ are cartesian.
Moreover they are all monoidal closed, in particular we get 4 LNL adjunctions, and hence 4 exponential comonads on $\infty\mathsf{Cat}_{\mathsf{cc}}$.
Write $!_{\mathsf{f}}$ for the one induced by the adjunction with $\infty\mathsf{Cat}_{\mathsf{filtr}}$ and similarly for $!_{\mathsf{s}}$ and $\infty\mathsf{Cat}_{\mathsf{sift}}$.

Theorem

For a small ∞-category $\mathcal{C}$, $!_s\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathcal{C}^{\sqcup})$, where $\mathcal{C}^{\sqcup}$ is the free cocompletion of $\mathcal{C}$ under finite coproducts.

Theorem

For a small ∞-category $\mathcal{C}$, $!_f\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathcal{C}^{fin})$, where $\mathcal{C}^{fin}$ is the free cocompletion of $\mathcal{C}$ under finite colimits.

Intensional point of view

We defined $!_s$ and $!_f$ at the extensional level (cocomplete ∞-categories).

Intensional	Extensional
Profunctors $\mathcal{C} \times \mathcal{D}^{\mathsf{op}} \to \mathcal{S}$	Cocontinuous functor $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$
Completion under finite coproducts comonad on Prof	$!_s$ comonad on $\infty\mathrm{Cat}_{cc}$
Completion under finite colimits comonad on Prof	$!_f$ comonad on $\infty\mathrm{Cat}_{cc}$

At the level of posets, finite coproducts and finite colimits coincide. Hence we have two generalizations of the comonad *FC* on Porel.

Free exponential on $\infty\mathrm{Cat}_{cc}$

Theorem

The full sub-∞-category of $\infty\mathrm{Cat}_{cc}$ on presheaf ∞-categories admits cofree commutative comonoids.
Moreover, the presheaf construction $\infty\mathrm{Cat} \to \infty\mathrm{Cat}_{cc}$ maps free commutative monoids to cofree commutative comonoids : $!\mathcal{P}(\mathcal{C}) = \mathcal{P}(\mathrm{Sym}(\mathcal{C}))$, where $\mathrm{Sym}(\mathcal{C}) := \coprod_{n\in\mathbb{N}} \mathcal{C}^n /\!/ \mathfrak{S}_n$ is the free symmetric monoidal ∞-category on $\mathcal{C}$.

Intensional	Extensional
Profunctors $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$	Cocontinuous functors $\mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D})$
Free symmetric monoidal category comonad on Prof	Cofree commutative comonoid on $\infty\mathrm{Cat}_{cc}$
Non-linear morphisms	
Genralized ∞-species $\mathrm{Sym}(\mathcal{C}) \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$	Analytic ∞-functors ?

0-∞ analogy

0-categories	∞-categories
set $X \in \mathrm{Set}$	∞-groupoid $X \in \mathcal{S}$
poset E	∞-category $\mathcal{C}$
Fibred relation $R \subseteq X \times Y$	Span $Z \to X \times Y$
Indexed relation $X \times Y \to \mathrm{Bool}$	Functor $X \times Y \to \mathcal{S}$
Monotonous relations $E \times F^{\mathrm{op}} \to \mathrm{Bool}$	Profunctor $\mathcal{C} \times \mathcal{D}^{\mathrm{op}} \to \mathcal{S}$
Free suplattice $P(E) := (E^{\mathrm{op}} \to \mathrm{Bool})$	Presheaf ∞-category $\mathcal{P}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{S})$
Suplattice morphism	Small colimit-preserving functor
Scott-continuous map	Filtered-colimit preserving functor (or sifted-colimit preserving)