Ad Hoc Networks: Study of Protocol Behaviour

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Chapter 1

Introduction and Thesis Structure

1.1 Introduction

The main subjects of this thesis are wireless ad hoc networks and their mathematical models. A wireless ad hoc network is a self organizing network, where mobile users are connected in an ad hoc manner and message routes form multihop paths. The possible applications for mobile ad hoc networks include the solutions for military, emergency and rescue operations, since they are often conducted in areas without any existing telecommunication infrastructure and they demand decentralized network management to increase reliability.

At the present time the number of wireless networks in the world is increasing rapidly. More and more new products for wireless local area networks (WLANs) are presented every day. Most of current laptops are now sold with an integrated wireless network card. A large number of personal digital assistants and mobile phones also have access to wireless technology. The introduction of inexpensive WLAN routers allowed many households to install Internet access points. In fact, WLANs start competing with cell phones in the domain of Internet connections. If several years ago the only way to access the Internet from a mobile device was to use an internal modem inside a cell phone, nowadays, in many town "spots", like cafés and libraries, a connection via LEAN is also possible. Even in the sphere of voice connections mobile cell phones start being pressured by WLANs, as the number of voice-over-IP (VoIP) solutions grow.

To illustrate this we present an excerpt from ComputerWeekly.com wireless special report "Remote revolution" (14 March, 2006): *Analyst firm Gartner warns that by 2009 there will be a fundamental shift in the economics and value of the voice connection. It predicts that voice over IP and wireless will dominate and 99% of new voice connections and 70% overall will be wireless. IDC research shows that 66% of the European working population is equipped with mobile devices and predicts that by 2007 there will be 99.3 million mobile-enabled workers in Europe.* Also it is worth noting that the general tendency of converging between WLAN and cell phone services and technologies is accelerating.

The mobility of wireless devices opens the way to completely new applications. A new term *m-business* has appeared to describe the commercial transactions done via mobile devices. *According to the study, there were 94.9 million mobile commerce users in 2003 who will grow to 1.67 billion users by 2008* says strategies research group Telecom Trends International in 2004. There is a vast literature on the mobile services and commerce, see, for example, [16].

One common feature that unites practically all current wireless solutions is the existence of a base stations infrastructure. Mobile devices need a fixed base station or an access point in order

to be connected and form a network. The fixed station serves as a network management device, it synchronizes the mobile devices, authorises them to participate in the network, and routes the messages. Such an organisation has several drawbacks, including a poor *scaling capacity*. A simple illustration of the latter is when a mobile cell phone stops working because of the large number of people trying to access the same base station, for example, during big concerts, festivals and demonstrations. It might seem as an issue of small importance to the audience of a concert, but the law enforcement or medical assistance forces may need a more reliable network cover.

A different problem is the need itself in the base stations. The emergency response teams during large scale natural disasters or the army during military operations may lack completely of the infrastructure and support for the information networks. Therefore, there is a demand for an *ad hoc* network organisation that would connect participants without additional installations. At the same time such network should, as it mentioned above, allow a massive scaling in number of users.

A possible answer to these problems are *multihop mobile ad hoc networks* (MANET). It is useful sometimes to think of the wireless network organizations on the examples of another shared communication medium: audio waves. Let us consider, a railway station full with people and borrow the names from cryptological literature. If Alice, who is at this moment at one corner of the station wants to talk to Bob, whose exact position is not known, she has several possibilities:

1. Try to shout her message to Bob. The first problem of such an approach is that other people inside will have difficulty to speak to each other as the environment will become quickly too noisy.

2. Another simple solution is to move closer to Bob. The problems arising are that Alice, maybe, cannot move at this particular moment because she is busy talking to her neighbor Carol. Or she simply does not know where Bob is.

3. There is another solution: to address the message to an information desk, which would listen to the message of Alice and relay it to Bob via loudspeakers installed in the station. This example is analogue to the current widespread solutions based on Wi-Fi. Such an approach allows to extend the network to the outer world and keep the communication organized.

4. Multihop solution: Alice knows a chain of people inside the station, who are able to relay her message to Bob. Such a chain would normally start with a neighbor of Alice, therefore, she starts by sending her message to that neighbor. After several relays the message would reach Bob.

The last approach is analogue to the way the multihop ad hoc radio networks behave. This reduces the interference within the network and allows to economize the energy consumption. The latter is highly important for small devices, who have limited battery resources and this opens the way to another type of application for the wireless connectivity, *the sensor networks*.

Different electronic devices, like fire or intrusion detection sensors, can be made of a size of a couple of millimeters and remain inexpensive in production cost. Distributed over a large area, a network of such devices can monitor different events. The problem then arises: how to collect the data? Miniature devices can be equipped with wireless access cards, but their energy resources will be limited because of their form factor. Therefore, the routing should be optimized. Again, the base station managing the whole network cannot always be adequate.

If the number of bases is small, then the reliability of the detection system will suffer and the sensors will need to cover long distance transmissions. If this number is increased, then the solution might become obtrusive and costly. This implies the need for multihop paths, in order to deliver the monitored information to a control center. The sensor network are distinguished in a different class from the MANET, as they are not mobile. The monitoring activity makes the traffic flow somewhat predictable, opening, thus the way to a greater degree of optimization.

A number of protocols were created to manage the communications within MANETs. Some of them inherit the ad hoc nature of the network and use simple route discovery schemes, routing the messages as soon as paths are discovered. Others create a logical structure over the physical network optimizing the communications or searching for the optimal routes.

There is no absolute winner between the different types of the routing methods. Different practical scenarios may favour different styles of routing. The number of participants, mobility, traffic pattern, location of the nodes, environment obstacles, all these parameters are important to consider for an optimal routing. To unify these values together into a single model is a difficult, but important task. In this thesis we will see three layers of ad hoc networks: link availability (physical layer), media access (MAC) layer and network layer, which characterised by ad hoc network routing protocols. Therefore, TCP and IP, despite all their interest, are not included into the current study. Let us start with reviewing some historical examples, how analytical models were useful to the real world applications.

1.2 Analytical Modeling of Communication Networks

A mathematical modeling of communication networks started with the works of Agner Krarup Erlang on the telephone traffic at the beginning of 20th century [40, 41]. At that time each telephone call had to be re-directed manually from the telephone station to its proper destination. Erlang showed how to predict the load on a telephone station and estimate the necessary equipment depending on the number of subscribers. His works founded the whole domain of applied probability, now called queuing theory. The results from the queuing theory are applied routinely in the design of new network protocols. Erlang studied the calls from the following point of view. The calls waiting to be re-directed or *served*, form *the queue* and depend on the way and the frequency the calls appear and on the speed and ability of an operator. It also depends on the number of operators the station employ. The queuing theory describes call arrivals and the duration of each service by corresponding stochastic processes. As it was concluded from experimental data a call is best described by a Poisson process and its duration by an exponential law distribution. The number of telephone operators necessary to serve a given number of subscribers can be calculated as a function of the parameters of these two laws.

Another cornerstone work was done by Claude Shannon, who presented the mathematical theory of communication in 1948 [85]. He demonstrated the theoretical limits on the speed of information transmission over a noisy channel. He writes in his article: "The fundamental problem of communication is that of reproducing at one point either exactly or approximately a message selected at another point". Although he was not the first to measure information numerically ([74, 75, 47]), his mathematical theory [85, 86] gave a formal definition of the communication system and showed how message coding can be accounted for and introduced the *entropy* for information. As we will see it later in Sec. 2.2 this theory is particularly important for highly competitive wireless medium, where the noisy channel can be used to model the interfering signals from all neighboring devices.

In most cases, the network analytical models consider asymptotic behaviour for a great

number of nodes. As the figures in the introduction suggest, it is not completely unrealistic scenarios for future applications. In the field of wireless communications, analytical modeling takes a particular importance as thorough simulations are difficult to accomplish. Firstly, their computational complexity is high. Secondly, as a complex system it has a large number of parameters and their inter-relations or impact on the network behaviour are not necessary known. Whereas, analytical models might often shed light on those relations and simplify the choice of correct parameters. In general, their computational complexity, even in difficult situations, when no closed form solution can be found, stays lower than those of simulations. Therefore, often such models serve as a first round approximation to the real life situations. However, they allow to demonstrate many properties of the corresponding networks and expose their potential problems. The scaling properties of network design is a natural example, where analytical models may help a lot. In fact, it corresponds to finding asymptotic behaviour of the corresponding system and as it often happens in mathematics, asymptotic expressions are easier to find than the exact solutions. The simulations in their turn, normally, suffer from an increase in number of nodes or in transmitted packets, their running time is increased correspondingly to their algorithms.

An interesting issue connected to the wireless medium is the impact of flooding on the performance. Flooding is a way to make a message reach all nodes in a network by forcing every node to repeat the message. It is used mostly to exchange necessary routing information, for example, to find a new path. In this case special *control* messages flood the whole network. If in a wired network such a method can be quite productive, in a wireless environment this can cause massive congestion. Since the effect of a single transmission cannot be limited only to the sender-receiver pair, but its signal can disturb the transmissions of other neighbouring nodes. It seems that the faster, flooding is going, the more nodes transmit at the same time, therefore, the noisier the environment gets. Imagine, a railway station, where all people start quickly repeating the same message straight after they learn the message from somebody else. The noise will be such that the normal dialogs will be interrupted from time to time, by such flooding waves. The noisy environment limits a possible transmission distance and it can completely block a link between two nodes. Therefore, the first problem is that this diffusion mechanism is error prone, because the message will not necessary arrive to all nodes in the network. The second problem is far more important – flooding may invalidate existing routes. If flooding is not progressing too rapidly, interference noise is less, but it introduces additional communication delays. The impact of the flooding is difficult to predict without some generalised view on the network and its traffic. This opens the field for analytical models. A large part of the work in this thesis concerns the issues of flooding in a wireless multihop network.

Mobility of ad hoc networks constitutes an additional challenge for a study of their performance. The exact pattern of movements is rarely known and it depends on network employment. There are several models used mostly by simulation software to predict the mobility of nodes. They include random waypoints model, random walk, random walk on a fixed geometric graph and others. Sometimes, it is possible to find some characteristics of these models, for example, the link duration, which can be calculated knowing the velocity vectors of the nodes. However, the probability, that one node will eventually meet another or the time, when it might happen, are the properties difficult to estimate. For the purposes of analytical modeling, the mobility is usually taken into account implicitly. It is often assumed, that nodes move relatively slowly and during the studied period of time their positions practically do not change. Hence, the network is represented by a random snapshot of the positions of the nodes. Another assumption may be to include mobility as the probability of link breakage, the higher mobility is, the higher is this probability. The explicit inclusion of the mobility properties is difficult and in this thesis

we take the implicit approach.

In the following sections, let us present the structure of the thesis and the contributions it brings.

1.3 Thesis Organization

One of the goals of this thesis is to present the analytical models of different parts of communication systems and to show how they can be made working together. The structure of the thesis reflects this idea. We start with the link availability modeling or what is called *a physical layer*, followed by a description of protocols for the multiple access to shared medium, then we study the propagation of a flooding wave in a multihop network. Then, we consider two classical protocols of ad hoc networks and compare them in analytical terms. We finish with a work on the scale-free growing network model and PageRank. Let us present the content of the chapters in greater details.

Starting with the physical layer, we study in Chapter 2 link availability models in the context of wireless ad hoc networks. The existence of a direct link between two nodes in networks normally means that a direct transmission of a signal between these two nodes is possible. But in the wireless environment the success of a transmission depends on many events. The distance between the nodes, obstacles in the way of wave propagation, walls surrounding the sources and the power of the signal, all these parameters can vary and the link availability will be changing with them. It is especially true for a mobile environment. We distinguish two types of general models of the wireless environment: outdoor and indoor models. In the former we find that the signal power at reception depending on the distance between the sender and the target. It corresponds to the physical properties of the propagation of a radio signal. In the latter this power does not depend on the distance, since the indoor environment is characterized by a high number of obstacles, that make the distance rather irrelevant to the signal propagation.

Chapter 3 contains a review on *medium access* (MAC) protocols and some performance analysis examples. Medium access protocols are responsible for providing mechanisms of shared utilisation of the common medium by multiple participants. Often they include some error correction technique to fight the collisions between multiple transmissions. In particular, we present the most popular MAC protocols used in current wireless networks. We study some properties of these protocols and their performance models. As an illustration we build a model for the CSMA/CA protocol using an existing Markovian representation of its exponential backoff mechanism on top of a link availability model presented in the previous chapter. Then, we address the issue of the MAC layer performance for the case of a classical flooding. Flooding is used to broadcast a message through the whole network. In its classical, non-optimized form every node is forced to repeat the message once. It was remarked, that it is a source of great instability of the network performance. However, its simplicity attracted many protocols to use it, at least in a limited form.

Flooding, as a part of route discovery mechanism, belongs to higher level protocols than MAC protocols. In Chapter 4 we study a problem of path lengths in multihop networks, where these paths are discovered using a classical flooding mechanism. The goal of this work was to extend the one-dimensional study [32] to the plane motion. It was partially achieved and we present a Markovian model of path discovery via flooding. The probabilistic process represents a "winning" node that is to be included into a new path between a sender and a destination. We show how to find the distribution of its stationary state, obtaining thus the value of an average hop length for a path, under these restrictions.

In Chapter 5 we apply the physical model of geometrical random graph based on the SIR inequality to the comparison of two different protocols for ad hoc networks, AODV [80] and OLSR [31]. The chapter starts with general description of the protocols for wireless ad hoc networks. Then, we provide details on AODV and OLSR mechanisms. We study their performance using a fixed point approach. We assume, that the network is balanced and working under the same load for both protocols. As they create different amounts of control traffic, the total traffic will be also different, affecting the transmission range in the SIR based link availability model. But the amount of traffic in turn depends on the transmission range. To cover the same distance with a smaller transmission range the nodes will be forced to relay messages more often. Thus we obtain equations describing total traffic of these protocols and, then, we study their solutions.

Chapter 6 studies PageRank [25] reputation measure on a scale free growing network model of Barabasi and Albert [14]. The model was proposed with the goal to better understand the evolution of World Wide Web and to explain the power law for degree distribution of Web pages. It is based on *preferential attachment* mechanism, where newly appearing nodes connect themselves to the existing ones with probability proportional to the number of links those nodes have. PageRank can be interpreted as a frequency of Web page visits by a random surfer and thus it reflects the popularity of a Web page. In the chapter six we find an analytical expression for the expected PageRank value in a scale free growing network model as a function of the age of the growing network and the age of a particular node. PageRank can also be seen as *a reputation system*, where a popularity of arbitrary node serves him as a reputation measure for all other nodes [30]. Reputation system are seen as a new paradigm for authorisation and security in different kinds of networks. For example, the famous auction site ebay.com has a reputation system that helps sellers and buyers to reduce risks during transactions. Of course, it is difficult to expect from an ad hoc network that it would evolve as Web graph. However, ad hoc network assumes quite a high degree of collaboration among the users. We can imagine that, in systems, where users would like to secure their transmission and make them more reliable, they will be forced to choose some *friendly* nodes to do the relaying of their messages. Therefore, kind of upper layer will be needed and it will be based on the relations between the users. Therefore, Web graph and ad hoc networks are the examples of *social networks*. As a general result it was remarked that many social networks also have power law distribution of the node degrees. This connects our study on PageRank and scale free growing network models to the world of ad hoc networks.

1.4 Contributions

In 2.5.2 we present a new model of short range communication network that is based on a random graph, but where the link availability depends on the traffic load in the network. The model allows to replace a difficult SIR based calculation of short range model by a random graph representation. But contrary to classical random graph model, where the edge existence is controlled by a random variable that depends only on the number of nodes, this new model makes the edges dependent on the network load. It is analogous to the model unifying the geometric random graph and SIR based models, described in [49]. To illustrate its application, we present an example of this model for a particular random distribution in 2.5.3.

The study of the path lengths for flooding based protocols is presented in Sec. 4.3.2. In [32] the authors show how to find a hop length of a path discovered via flooding in one dimensional case. We have studied this problem in two dimensions. Under the assumptions that we impose,

the stationary distribution is found for the Markov chain, that describes a sequence of points, which lay on a newly discovered route. The hop length follows from this result. The work was published in [63]. The difference of this work from other path length studies is that, they were mostly interested in a shortest route obtainable in a given model of link availability. Whereas, the flooding imposes an additional constraint as the nodes on the path cannot be chosen globally. Rather, they are established hop by hop via a process, that can be seen as a local competition.

A new framework for comparison of ad hoc protocols in a realistic physical model is presented in Section 5.5. We consider the overhead of control messages of each protocol. Assuming that both of them work at a given level of a useful load, the total load of the network at arbitrary time slot is then calculated. The link availability depends on the load at the same settings, thus making the comparison possible. The work was published in [62]. In its overhead estimation part, our solution is similar to the work [54], where the overhead of OLSR was found, but without relation to any link availability model. In general our approach is similar to the one used in [3] for local evaluation of link state routing in ad hoc networks, but contrary to this work we are looking to calculate the situation globally.

In Chapter 6 the exact expression is found for the average PageRank as a function of the age of nodes and the age of the model in a scale-free growing network model. The mean field method is then applied to estimate the probability distribution of PageRank in this model. Compared to the work [77] where the similar solution was estimated, the exact expression we find allows us to have a deeper insight into the evolution of PageRank with the age and capture some important details, including the dependence of PageRank on the damping factor constant. Also, the probability distribution obtained via mean field method gives a good approximation, checked with simulations, when it is based on the exact solution for the average PageRank. This work has been done together with Konstantin Avrachenkov [6].

Chapter 2

Link Availability Models of Wireless Multihop Networks

In this chapter we collect together different analytical models used to predict the behaviour of the multihop radio networks. The reader will find the following topics in this chapter:

- *Physical aspects: the wireless environment constraints, signal fading with the distance, interference issues related to highly competitive scenarios.*
- *Transmission modeling: link availability with the respect to limitations of a physical layer. It includes the Bernoulli random graph model, the geometric random graph model, STIRG and the SIR based geometric random graphs.*

2.1 Introduction

In general an analytical model of a wired or wireless network contains two important parts

- *Traffic pattern.* Network behaviour depends on the traffic amount and on such characteristics as message duration, frequency, the network total load or the number of active connections. In general, traffic pattern is unknown, therefore, these characteristics are described using some kind of stochastic modeling and the tools of queue analysis.
- *The link availability mechanism.* To analyse a network we need to know how to estimate that a message can be successfully sent between two nodes. For wired networks the solution is to consider the graph that corresponds to the wire connections. In wireless networks the link availability is more difficult to manage. In addition to the natural constraints on the link performance that are distance between the nodes, mobility of the nodes and energy consumption there is another limitation put on the wireless links. The link availability depends not only on the two nodes participating in a given link but also on other nodes whose transmissions can interfere with the messages sent over the link. Such are the characteristics of the shared wireless medium.

These uncertain links make a fundamental difference in the representations of wired and wireless networks. While the former graphs can be modeled by normal graphs, the latter, in general, can not be defined as graphs with fixed edges, some uncertainty is needed.

2.2 Physical Layer. Link Availability Modeling

Successful reception of a message depends on many conditions. Antenna design, signal coding technique, available radio frequencies, multiplexing determine whether a signal can be successfully received and decoded. But such parameters constraints include the geographical positions of the nodes and the obstacles on the way of the wave propagation.

Signal propagation and decoding constraints are often modeled by the following conditions:

- *Condition on Distance.* It is based on the simple condition that we can successfully decode a signal if it is sufficiently powerful. Therefore, we fix a threshold P_{th} and we compare the signal with this bound. This approach is good enough for the low interference (competitive) environment. It ignores most of the details of the signal reception especially the interference and, therefore, it is contested by many researchers as a realistic multihop network model.

- *SIR Inequality* Signal treatment more adapted for the competitive ad hoc network nature is to use signal to interference noise ratio (SINR[1]) condition of the successful reception of a signal from a node i to arbitrary node j is defined by the following formula:

$$\frac{P_i \kappa_{ij}}{\sum_{k \neq i} \theta_{ki} P_k \kappa_{kj} + N} \geq K \tag{2.1}$$

  where

  - N is the receiver thermal noise power at the access port
  - P_k is the transmitter power used by k
  - κ_{ij} is the power gain[2] between i and j. It models the signal fading and possible obstacles on the way between i and j. As it is difficult to include into the consideration all real obstacles, therefore, κ_{ij} is often taken as a function of the distance $d(i,j)$ between i and j and, then, it takes form of

$$\kappa_{ij} = l\left(d(i,j)\right) \tag{2.2}$$

    where $l(\cdot)$ is called attenuation function and it is often taken in form of $l(x) = x^{-\alpha}$. Although it is often noted that the power gain should be having more compact support rather than simply a power function [12].
  - θ_{ki} is the normalised crosscorrelation between the signals from the nodes k and i at the access port receiver. It is the effective fraction of the received signal power from transmitter k that contributes to the interference. This depends mostly on the multiple access scheme that was chosen to control the network:
    - TDMA(time division multiple access) and FDMA (frequency division multiple access) are orthogonal schemes. Therefore, θ_{ki} can take only two values 0 and 1.
    - For CDMA θ_{ki} can take values in between of 0 and 1.

    We describe the multiple access schemes in Sec. 3.1 of Chapter 3.

[1] Sometimes we meet the following abbreviation: SNR for *signal to noise ratio* and SIR for *signal to interference ratio*. The former then assumes that the interference is included in N. The latter corresponds to the case of $N = 0$.

[2] κ stands for $\kappa\varepsilon\rho\delta o\varsigma$, "gain" in Greek

In fact, the condition 2.1 asymmetric: if a node i can hear some other node j, it does not necessary mean that the node j can also hear i under the same conditions. Being rather specific for the wireless medium, this asymmetry actually turn into a challenge for many routing protocols, as their algorithms need to explicitly state that the links are assumed symmetric. For example, in [82] author argues that unidirectional links can jeopardise the performance of distance vector (see Section 5.1) type of routing protocols.

The second condition follows from the fundamental result of C. Shannon [85] in information theory, that the capacity of a noisy channel is bounded by C

$$C = W \log \left(1 + \frac{P}{WN_0}\right) \tag{2.3}$$

where W is bandwidth of the channel in Hz and P is the received signal power in watts, N_0 is noise spectral density in W/Hz, the term $N = WN_0$ is the total noise power in watts. Note, that in $P + N$ should be a total signal received by a station and it is true for (2.1). Another assumption hidden in (2.3) is that the SNR does not depend on the frequency and in this case it is effectively proportional to W. In many models, the condition (2.3) is implicit, mostly because the spectrum width is assumed constant and the SIR inequality $K < \frac{P}{N}$ (in this expression N expresses interference as noise), with K constant, provides a lower bound for the capacity of a single link.

2.3 Unit disk model

It was probably one of the first models proposed for ad hoc networks. The model is based on the simple approach that a link between two nodes exists if they are within a distance r. Therefore, what determines characteristics of such a model is the position of the nodes. In many cases the model is considered with the nodes distributed within a unit disk, which gives the model its name. The distribution of the points is described by a uniform distribution. It is a *binomial point process* Ψ of n points distributed in a compact region $\mathcal{X}$, such way that the number of the nodes in some subregion $A \subset \mathcal{X}$ follows the binomial distribution:

$$\mathbb{P}[k \text{ points inside of } A] = \left(\frac{|A|}{|\mathcal{X}|}\right)^k \left(1 - \frac{|A|}{|\mathcal{X}|}\right)^{n-k}$$

where $|\cdot|$ is the area.

Definition 2.1. A binomial point process Ψ *of n points is formed by n independent points $\xi_1, \ldots, \xi_n$ uniformly distributed in the same compact set $\mathcal{X}$:*

$$\mathbb{P}\left(\xi_1 \in A_1, \ldots, \xi_n \in A_n\right) = \frac{|A_1| \ldots |A_n|}{|\mathcal{X}|} \tag{2.4}$$

for $A_1, \ldots, A_n$ Borel subsets of $\mathcal{X}$.

A binomial point process is close to an intuitive notion of a random distribution of nodes in some region. However, it is sometimes difficult to handle due to the fact that the points are not completely independent. For example, the distribution of the number of points of Ψ in some region $A \subset \mathcal{X}$ is not independent from the number of points in $\mathcal{X} \setminus A$. Thus, a standard and often implicit approach is to approximate the binomial point process by *a Poisson point process* (P.p.p.) of corresponding intensity $n/|\mathcal{X}|$.

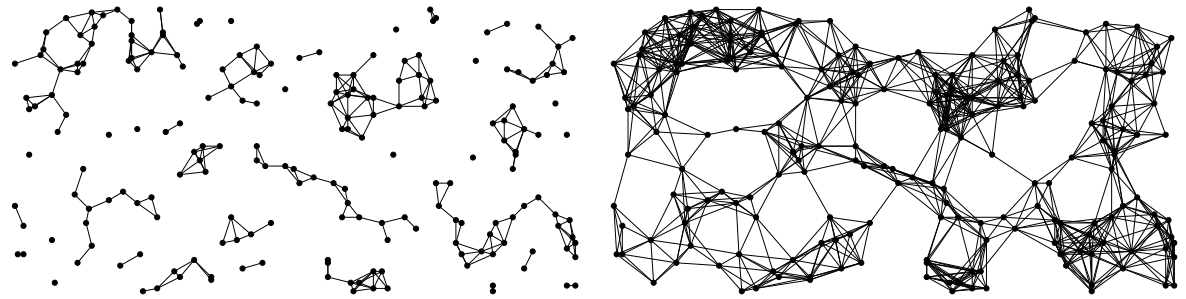


Figure 2.1: Illustration to the unit disk (geometric random graph) model.

Definition 2.2. A stationary Poisson point process Φ *can be defined as a point process with Poisson distribution of point-counts: the number of points of* Φ *in a bounded Borel set* B *has a Poisson distribution of mean* $\lambda|B|_d$ *for some constant* λ*. And* λ *is called intensity or density of* Φ*.*

We denote the vertices of the P.p.p. Φ as $\{V_i\}$. Two nodes V_i and V_j are considered to have a direct link if

$$d(V_i, V_j) \leq r \tag{2.5}$$

where $d(\cdot, \cdot)$ is the Euclidean distance in $\mathbb{R}^2$.

Thus we obtain a graph $G(V, E)$ where the edges E are between directly connected nodes. Sometimes this graph is called a *random geometric graph* $G_r(V, E)$. On Fig. 2.1 we present an illustration to this model. For the right image the transmission range r was increased, leaving the node positions intact. The number of links has also increased in the graph $G(V, E)$. Remark, that this model has links symmetrical by its definition.

The unit disk model is a particular case of *the stationary Boolean model* in $\mathbb{R}^2$:

Definition 2.3. *(see [89]) Suppose* $\Phi = \{x_1, x_2, x_3, \ldots\}$ *is a stationary Poisson point process of intensity* ν*. For a given dimension* d *let* $\Xi_1, \Xi_2, \Xi_3, \ldots$ *be a sequence of independent identically distributed random compact sets in* $\mathbb{R}^d$*, independent from* Φ*. The* Boolean model Ξ *is then constructed as follows:*

$$\Xi = \bigcup_{n=1}^{\infty} (\Xi_n + x_n) \tag{2.6}$$

given that the following condition is satisfied

$$\mathbb{E}(|\Xi_0 \oplus K|_d) < \infty \text{ for all compact } K \tag{2.7}$$

where Ξ_0 *denotes a random compact set of the same distribution as* Ξ_n*, but independent of them and of the process* Φ*;* $|\cdot|_d$ *is Lebesgue measure in* $\mathbb{R}^d$ *and* $\oplus$ *is Minkowski addition:* $A \oplus B = \{y \mid y = a + b, a \in A, b \in B\}$

The Boolean model is also known as the *Poisson germ-grain model*. The unit disk model can be represented as a particular case of the Boolean model, where Ξ_i are the disks of a fixed radius r with probability 1 (or $r/2$, in which case we consider the disks touching each other to correspond to two connected nodes). We list some properties of this model(see [89]):

- The Boolean model is stationary (translation-invariant) and isotropic (rotation-invariant).

- The probability that a point $o \in \mathbb{R}^2$ is within the transmission range of one of the nodes of the model is equal to
$$P[o \in \Xi] = 1 - \exp(-\nu\pi r^2) \tag{2.8}$$

- The spherical contact distribution function, which can be defined as the distribution of the distance from the origin of coordinates to Ξ, conditioned by the fact that the origin of coordinates is not in Ξ writes:
$$H_s(x) = 1 - \exp(-\nu x\pi(2r + x)) \tag{2.9}$$
In network terms it is the distance between any point to a closest region covered by transmissions from arbitrary node.

These properties can be directly generalised to the cases when the transmission regions are represented by a disk of random radius or by a random convex figure. A random disk corresponds to differences in energy supply of the nodes, i.e. the nodes with stronger antenna or more battery capacity can transmit a signal farther away. Whereas the random convex figure can also model natural obstacles or antenna performance [10].

The unit disk model has two principle parameters: the intensity of the underlying Poisson process ν and the transmission radius r. In the following sections we will concentrate on the following properties of the ad hoc network represented as a unit disk model:

- *The connectivity*: whether the network has several components, or isolated nodes or it is fully connected. This question can also be reformulated as "find conditions on r and ν such that the network is connected with high probability."

- *Throughput capacity* of the network. It can be defined as the theoretically possible throughput of a connection between two random nodes. The upper level protocols are not taken into consideration and the capacity is calculated as a volume of the information that can be possibly transfered between two nodes over a multihop route.

One of the earliest works on the connectivity in multihop networks is [28]. The question was studied in one dimension: consider the Poisson point process of intensity λ in $\mathbb{R}$; all the stations can transmit a message to the right and at some moment one node starts a transmission; then the farthest to right point within a transmission range relays the message further. The problem consist of determining the minimal radius with which the message will be spread to $+\infty$. As the authors called it is a "broadcast percolation problem". They obtain a recursive expression for the critical transmission radius.

Almost ten years later independently P. Gupta, P.R. Kumar [44] and M. Penrose [78] published their studies of the problem in two dimensions. They base their solutions also on the percolation theory of P.p.p. in two dimensions [67]. They follow the common scheme of the approximation of the binomial point process of n points inside of some compact region of unit area in $\mathbb{R}^2$ (unit disk [44] and unit square [78]) by a Poisson point process of a corresponding density n. We will show here the theorem from P. Gupta and P.R. Kumar [44]. Let $D_{1/\sqrt{\pi}}$ be a disk of unit area in $\mathbb{R}^2$ (its radius is $1/\sqrt{\pi}$). Then by $\mathcal{G}(n, r(n))$ we will understand a geometric random graph formed by n points uniformly in $D_{1/\sqrt{\pi}}$ with transmission threshold $r(n)$.

Theorem 2.1. *(from [44]) Graph $\mathcal{G}(n, r(n))$ with $\pi r^2(n) = \frac{\log n + c(n)}{n}$ is connected with probability one as $n \to \infty$ if and only if $c(n) \to \infty$)*

The percolation theory used in two dimensions concerns a geometric random graph $\mathcal{G}^P(\lambda, r(\lambda))$ in $\mathbb{R}^2$, where the nodes distributed according to the Poisson law of intensity λ and the radius is $r(\lambda)$. Also there is a condition on $\mathcal{G}^P(\lambda, r(\lambda))$ that the origin of coordinates is a node of this graph. The result states that the origin is either isolated or it is connected to an infinite graph component with probability 1 when the intensity λ increases $\lambda \to \infty$. It reduces the problem of the connectivity to the existence of isolated nodes. However, the case of a geometric random graph inside a compact region, for example $D_{1/\sqrt{\pi}}$, carries an additional difficulty of the border effects.

From the condition: a point is isolated if there is no others points within the distance r, we can do the following estimate. As all the points are independent, the probability for *a typical*[3] point to be connected to any other point of P.p.p of intensity n is

$$\mathbb{P}[\text{no neighbors}] = \exp(-n\pi r^2) \tag{2.10}$$

In a region of a unit area the average number of points is n and we want the probability that n points are not isolated to be equal to 0

$$1 - n\exp(-n\pi r^2) = 0 \tag{2.11}$$

therefore we obtain

$$\pi r^2 = \frac{\log n}{n} \tag{2.12}$$

The theorem 2.1 is much more precise as it gives the necessary and sufficient conditions. Now if we write the direct estimation for $r(n)$ we obtain

$$r(n) = O\left(\sqrt{\frac{\log n}{n}}\right) \tag{2.13}$$

In addition to a simple connectivity, in [78] M. Penrose study also the k-connectivity of the random geometric graph. This work can be useful for routing problems, such as *multi-path* routing.

The problem of the network connectivity can also be set differently: how many neighbors do the nodes need to keep the graph connected? The answer to this question was presented in [93], where the authors showed that the nodes need $O(\log n)$ neighbors. In simplified terms it can be seen from (2.12) as $\pi r^2 n$ is the average number of neighbors. In [93] we find a proof that the logarithmic order on number of neighbors is the sufficient and necessary condition for the connectivity.

At this point we can consider the throughput capacity of a geometric random graph, these results were presented by Gupta and Kumar in [45]. The *throughput* of $\lambda(n)$ bit/s for each node is *feasible* if there is a spatial and temporal scheme for scheduling transmissions, such that by operating the network in multi-hop fashion and buffering at intermediate nodes when awaiting transmission, every node can send $\lambda(n)$ bits/s on average to its chosen (random) destination node. That is, there is a $T < \infty$ such that in every time interval $[(i-1)T, iT]$ every node can send $T\lambda(n)$ bits to its corresponding destination node.

The *throughput capacity* is defined as the order of almost surely feasible throughput. The interference was included into the consideration by restricting the nodes allowed to transmit at the same time to be at the distance cr where c is some constant. Often such approach to

[3]This term can be made precise by the means of the Palm distribution theory [89, 11]. But for the case of stationary Poisson point process it stays close to what intuition suggests

modeling gets the name *protocol* model, as it is also the case in [45]. They impose the following two conditions: the distance between communicating nodes is less or equal to r and the distance from other nodes to the receiver is greater or equal to $(1+\Delta)r$, where $\Delta > 0$. The destination nodes are chosen at random.

Theorem 2.2. *(see [45]) In the case of a planar disk the order throughput capacity is*

$$\lambda(n) = O\left(\frac{1}{\sqrt{n \log n}}\right) \tag{2.14}$$

The authors showed fine estimations for the constants in $\frac{1}{\sqrt{n \log n}}$. The routing scheme is based on the use of Voronoi tessellation, where size of Voronoi cells is chosen to fit some constraints. The source-destination pairs are chosen at random and the path between them is approximated by the Voronoi cells. In Section 4.2, we will shortly review a result on the length of the paths, based on the Voronoi tessellation, where corresponding definitions will be given. But here, we just note that the exact proof of Theorem 2.2 is quite laborious. However, to see where this asymptotic arises from we may follow the logic used to prove the lower bound in the Theorem 2.2: the possible number of the nodes transmitting at the same time is of order $\frac{1}{\pi c^2 r^2(n)}$. Suppose that the average euclidean length of the path is $\bar{L}$, then the length of the path in hops is $\bar{L}/r(n)$. Therefore we can express the total charge of n routes on the network as

$$\frac{\bar{L}}{r(n)}\lambda(n)n = \frac{C}{r^2(n)} \tag{2.15}$$

as the transmission radius of the connected network is of order $\sqrt{\frac{\log n}{n}}$ then we obtain

$$\lambda(n) = \frac{C}{\bar{L}nr(n)} = O\left(\frac{1}{\sqrt{n \log n}}\right)$$

As $n \to \infty$ the throughput capacity of every path is reducing to 0, although the overall capacity of the network is of order $\sqrt{n/\log n}$ and it is growing to infinity ([50]).

2.4 SIR based Modeling for Geometrical Models

The flaw of the unit disk model is its insensitiveness to the traffic load changes. The situation, when the network is highly loaded and all nodes have many messages to transmit, can be compared to a room with many people speaking altogether. We know that in this case we can possibly be heard by only our closest neighbors. Whereas, if nobody in the room pronounce a word, our whisper would reach everybody. The similar behaviour we would expect from the network in the shared wireless medium. But these scenarios are impossible in the unit disk model, as our messages can be received only within a disk of fixed radius r and they do not spread or disturb any other communications farther away.

More realistic signal reception (link availability) model is based on the SIR inequality. As in the unit disk model, the positions of the nodes are represented by a Poisson point process. There is no really a widely accepted title for this model, therefore, we will call it mostly SIR based model.

2.4.1 Links availability in SIR model

We describe this model, balancing between the definitions and results that can be found in the works of Ph. Jacquet [49] and F. Baccelli, B. Błaszczyszyn and P. Muhlethaler [12].

Let us consider the Poisson point process $\mathcal{X} = \{X_i\}$ of intensity λ in $\mathbb{R}^2$. The emitting power of the nodes is denoted by P_i, in general case these are the random variables [12]. In other terms $\{X_i, P_i\}$ is a marked Poisson point process, where X_i are the points and P_i are their marks. We also denote by C_i the cell around the node i, where the transmission from i can be successfully received:

$$C_i = \left\{ x : \frac{P_i l\left(d(x, X_i)\right)}{N + \gamma \sum_{j \neq i} \theta_{i,j} P_j l\left(d(x, X_j)\right)} \geq K \right\} \tag{2.16}$$

Then, the coverage probability is the probability that a point x, located at distance R from the point 0, is inside of C_0:

$$p_R = \mathbb{P}[x \in C_0] = \mathbb{P}\left[P_0 \geq K \frac{N + \sum_{i>0} \theta_{0,i} P_i l(d(i,0))}{l(R)} \right] \tag{2.17}$$

Note, that the sum $\sum_{i>0} P_i l(d(i,0))$ is the Poisson shot-noise [12] (we denote it by $I_{\mathcal{X}}$) and the Laplace transformation $\phi_{I_{\mathcal{X}}}$ is known.

$$\phi_{I_{\mathcal{X}}}(\xi) = \exp\left(-\lambda \int_{\mathbb{R}^2} 1 - \mathbb{E}[e^{-\xi P l(|x|)}] \right) \tag{2.18}$$

For general distribution of $\{P_i\}$, p_R is difficult to find [12].

- But in the case, when P_i has an exponential distribution with the mean $1/\mu$, it can be calculated directly (with some simplification: $\theta_{ij} \equiv 1$ and $N \equiv 0$):

$$p_R = \int_0^\infty e^{-\mu s K / l(R)} dP[I_{\mathcal{X}} < s] = \phi_{I_{\mathcal{X}}}(\mu K / l(R)) \tag{2.19}$$

  For the case, when $l(x) = x^{-\alpha}$

$$p_R = \exp\left(-\lambda R^2 K^{2/\alpha} (2\pi \Gamma(2/\alpha) \Gamma(1 - 2/\alpha))/2 \right) \tag{2.20}$$

- In the case when the $P_i \equiv \text{const} = 1$ [49], it takes more difficult form.

$$p_R = \int_0^\infty \frac{K}{l(R)} dP[I_{\mathcal{X}} < s] \tag{2.21}$$

  As it is not the exponential distribution, one need to find the inverse Laplace transformation of $\phi_{I_{\mathcal{X}}}$ to compare the interference with signal received. It was calculated in asymptotic ($r \to \infty$) in [49]:

$$p_R = 1 - \lambda \pi K^{2/\alpha} R^2 + O(R^3) \tag{2.22}$$

Note, that the expressions above are relevant to the cases of $\alpha > 2$. In fact, it is the condition that an average shot noise is finite. As the number of points is unbounded, it is not always the case. On average, there are $\lambda\pi(2R+1)$ points of P.p.p. of intensity λ in a stripe between two disks D_{R+1} and D_R. Therefore, their collective power is approximately $\frac{2\pi}{R^{\alpha-1}}$ and the total interference from all nodes of the network is $2\pi \sum_R \frac{1}{R^{\alpha-1}}$. This sum converges only if $\alpha > 2$.

2.4.2 Connectivity

First of all we need to define the connectivity of the SIR based model. Normally, it is understood as a possibility for a message issued by some node in the network to reach any other node. This property depends on how close positions of the nodes are, but it also depends on the level of communication traffic generated across the network. In a busy network we expect the links between the neighbours to be often interrupted by surrounding nodes. The approach taken in [37] to consider the question of connectivity is the following. The authors define Poisson Signal To Interference Ratio Graph (STIRG), which is based on the nodes distributed according Poisson point process and the links are defined using SIR inequality. An edge between two nodes X_i and X_j exists in STIRG iff the node $X_i \in C_i$ and $X_j \in C_i$, where C_i and C_j are the cells where the signals from X_i and X_j can successfully received, as it is defined in (2.16). Thus the links are symmetric.

A superposition of multiple STIRGs over some finite period of time gives a straightforward approach in defining the link availability and the network connectivity. With this definition the authors in [37] find the numerical solution for the connectivity threshold function which depends on the constant γ of decoding abilities of the protocol in use and the threshold K. In their work the authors of [37] showed that each node of STIRG can have at most $1 + 1/\gamma K$ neighbors. This result holds in general case when the signal attenuation function is not fixed and the transmission powers of the nodes can be different. Thus degree of the STIRG has an upper bound equal to $1 + \frac{1}{K\gamma}$.

With the help of the percolation theory they prove that in the STIRG based model of an ad hoc network a feasible throughput of a random multihop path is $O(\frac{1}{\sqrt{n \log n}})$, where n is a number of nodes in a unit disk. This corresponds to the result of P. Gupta and P.R.Kumar [45] presented in the next section.

2.4.3 Throughput Capacity in the SIR based model

The fundamental result of P. Gupta and P.R. Kumar [45] says that

Theorem 2.3. *Two constants c and c' exist, such that*

$$\mathbb{P}\left[\lambda(n) = \frac{cR}{\sqrt{n \log n}} \text{ is feasible}\right] = 1 \tag{2.23}$$

and

$$\mathbb{P}\left[\lambda(n) = \frac{c'R}{\sqrt{n}} \text{ is feasible}\right] = 0 \tag{2.24}$$

The proof is based on a presentation of the scheduling scheme, that makes corresponding to (2.23) throughput feasible.

The limitation on the bandwidth of ad hoc network is coming from the fact that to reduce the interference noise the nodes are to reducing the transmission range, but this leads to the longer routes and thus more transmissions. The idea of Grossglauser and Tse[43] is to distribute the parts of a message between many relay nodes. These nodes will come close at some point of time to the corresponding receiver and then they will transmit its part of the message. Interestingly, it was shown that the simplest schedule, when every sender waits until it is close to the receiver node and just then it transmits the message, does not give a satisfying result. Without relaying the achievable throughput is still decreasing to 0 at least as fast as $n^{\frac{1}{1-\alpha/2}}$. But for a scheduling policy that allows two hop relaying the authors showed that the feasible throughput converges

to a positive constant $O(1)$ as $n \to \infty$. The nodes move accordingly to the process $\{X_i(t)\}$ which are independent, stationary and ergodic.

Now we will give some details how this result was achieved. The chosen scheduling policy is the following: fix a parameter θ. At any time slot t there are θn sender nodes chosen randomly. The rest $(1-\theta)n$ nodes are potential receivers. A sender transmits its message using a unit transmit power (e.i. for any i, $P_i = 1$). The receivers are chosen as a random set of size θn from the "non-senders" and then only those feasible sender-receiver pairs retained, for which the transmission successful (2.1 condition is satisfied). Let the number of such pairs be n_t. In the article it was shown that $\frac{\mathbf{E}[n_t]}{n}$ converges to a positive limit. It is based on the fact that the received power at a nearest neighbor is of the same order as the interference noises. Which is in its turn is based on the property of the sequence of n i.i.d. random variables, that if the cumulative density function $F(w)$ of these variables decays slower than w^{-1} as $w \to \infty$, then the largest of them is of the same order as the sum.

Such scheduling assures one hop communications. But in order to provide two hop relaying there is a further two step scheme proposed. At even time slots the senders send parts of their original messages to other nodes to relay (and the destinations if they happen to be close enough). At odd time slots some chosen senders relay the previously stored messages to the destination nodes, if they come close.

The biggest flaw of this approach is the long delays of the communication. It can take a long time before a relay node will meet the destination node, even if they should meet as the movement process is stationary and ergodic.

2.4.4 SIR based random geometric graph

Consider the network represented by P.p.p of intensity ν. Then, we will call *the traffic density* the average number of packets transmitted per time slot per unit square. If the packet duration is equal to 1 slot, then it means the average number of the nodes transmitting at any point. If these nodes are random, then we can represent the transmitting nodes by P.p.p. with intensity λ equal to the traffic density.

We assume that two nodes are connected, if they lay within the distance R and the corresponding coverage probability p_R (cf. (2.17)) is greater than some fixed threshold value p_0, for example $p_0 = 1/3$. This link model was first presented in [3]. From (2.22) we can find r: $p_r = 1/3$ at arbitrary level of traffic density

$$r(\lambda) = \frac{r_0}{\sqrt{\lambda}} \tag{2.25}$$

where r_0 is the connectivity radius for $\lambda = 1$. For the values $\alpha = 2.5$ and $K = 10$, it was calculated $r_0 \approx 0.12$ [3] (see appendix C in [8] for a short and elegant proof that $r = O(1/\sqrt{\lambda})$.)

Therefore, we can take as an assumption for this neighborhood model that two nodes of the P.p.p. of intensity ν are connected if they are within the distance $r(\lambda)$ from each other. This replaces the STIRG graph with the geometrical graph where contrary to the unit disk model, the transmission distance depends on the traffic density in the network.

The number of neighbors necessary for the connectivity of this graph can be estimated by the order of $O(\log n)$, because such situation corresponds to the unit disk model, as $r(\lambda)$ is fixed. It means that we can estimate the maximum traffic density that leaves the network connected: $\pi r^2(\lambda_{max})\nu = \pi\nu\frac{r_0^2}{\lambda_{max}} = C\log\nu$, or

$$\lambda_{max} = O\left(\frac{\nu}{\log\nu}\right) \tag{2.26}$$

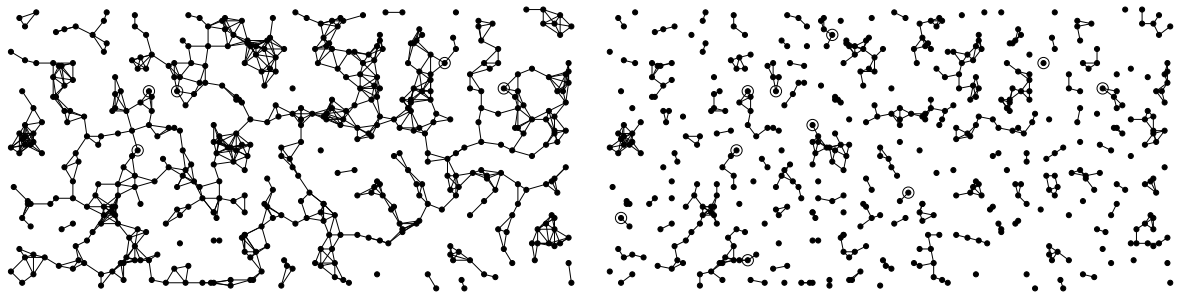


Figure 2.2: Illustration to SIR based geometric random graph model.

Let us return to the graph G_λ described in 2.4.2 and the expression 2.25:

$$r(\lambda) = \frac{r_0}{\sqrt{\lambda}} \tag{2.27}$$

This model inherits all the positive points of the classical unit disk model and in addition it answers the problems described at the beginning of this section. The radius of the transmission range depends on the load of the network. Therefore, when the network is not loaded the transmission range increases to infinity. Otherwise, if the load is high the range is decreasing.

The equation (2.26) with the estimation of the maximum traffic density can lead to the capacity of a multihop network [50]. Note, that the average length of multihop routes at some compact area of surface S can be estimated via $O(\frac{\sqrt{S}}{r(\lambda)}) = O(\sqrt{\lambda})$. The number of sender-receiver pair is $O(\nu)$. The throughput capacity C can be found as the total number of the packets sent over the whole network divided by the average path length and the number of sender-receiver pairs. It leads us to the result of P. Gupta and P.R.Kumar [45]:

$$C_{max} = O\left(\sqrt{\lambda_{max}}/\nu\right) = O\left(\sqrt{\frac{1}{\nu \log \nu}}\right) \tag{2.28}$$

Let us consider this model in greater details. It has two main parameters: the density of all nodes ν and the density of transmitters λ. All nodes are modeled by the P.p.p. of intensity ν. It is a geometric graph model, where the transmission range is defined by $\frac{r_0}{\sqrt{\lambda}}$. The transmitters can be described as via $\frac{\lambda}{\nu}$-thinning of this P.p.p. It means that transmitters are chosen according to Bernoulli probability distribution with the success probability λ/ν. This makes the transmitters also distributed accordingly to P.p.p. of intensity λ [89]. On Fig. 2.2 we present an illustration to this model. Every step there are λ transmitting nodes per unit area on average. 500 points were randomly distributed in a 200×100 rectangle. The left image contains 5 transmitting nodes and the right image has 10 nodes. We see on the right image, that the graph becomes highly disconnected, because of the double increase of traffic density λ, though the node density is not changed.

The model presented in [3] unites the random geometric model with the SIR inequality. This approach allows to estimate the behaviour of the model with the respect of the network traffic load. At the same time, it stays feasible with the respect of calculation complexity. We illustrate the application of this model in Chapter 5.

2.4.5 Transmission, interference, carrier sense ranges

The volatility of STIRG makes it difficult to handle and, also, calculating only the transmission radius does not give all the information needed, for example, to find the area where a signal is

effectively interfering with others. Limiting them to the neighbours is not too realistic. Although it is not true for a general signal attenuation function, but for those that decrease rapidly to 0, one can think of avoiding taking into account the emitting nodes starting with some threshold distance.

Thus, sometimes we can find the description of the multihop network links with the help of two or three different ranges.

1. *the transmission range* R: the range inside which the nodes are able to receive or overhear the packet transmissions.

2. *the carrier sensing range* R_{cs}: the range within which the nodes are able to sense the signal, but not necessary decode it correctly.

3. *the interference range* R_i: the range where any other transmission will interfere with correct reception.

The following inequalities are always true: $R < R_{cs}$ and $R < R_i$. However [55] the relations between R_{cs} and R_i are not clear in the case of R_i not being fixed and depend on the traffic in the network. For example, in the network simulator NS2 [66] the default setting is $R_i = 2.2R_{cs}$.

The range of R_i is closely related to the spatial reuse property of the network. In [4] the authors study the triangular mesh networks, where the distance between the nodes is equal to R_i. They estimate the ratio between the ranges:

$$\frac{R_i}{R} \geq \frac{2}{\sqrt{3}} \left(6K\zeta(\alpha - 1)\right)^{1/\alpha} \tag{2.29}$$

where $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is zeta function of Riemann. This estimation can be used as the lower bound on R_i for any node pattern, for which the nodes are at least R_i away from each other [12].

2.5 Geometry-free Models

Under *geometry-free* we understand the networks links, which are not dependent on the distances between the nodes. Such assumption is relevant to the short range networks, where many nodes are located, for example, in a single building. The nodes are close from the geometrical point of view. But random obstacles and walls partially block signals inside the building, as a consequence the links between the nodes are rather random. A model analogue to the unit disk model, in the geometry-free case, is *random graph*.

Let us consider a graph G_p, with n vertices. Its edges are defined in the following way: there is an edge between vertices i and j, with probability p. This graph is called *the Erdös-Rényi random graph*[4]. We present an illustration to the model on Fig. 2.3. We fix $n = 50$ and consider the probability p equal to 0.05 on the left image and $p = 0.1$ on the right. Unsurprisingly, the right image has a higher connectivity.

There is some ambiguity in modeling a network with some random structure. It is reflected in the following example: let us consider n network users located together at some building. The links between them are modeled via i.i.d. Bernoulli r.v. with probability of a link equal to p. Then we can consider just one realisation of such distribution, and with the n growing just add the users and their links to the model. In this case the probability p should not be

[4]Sometimes it is also called *Bernoulli random graph*

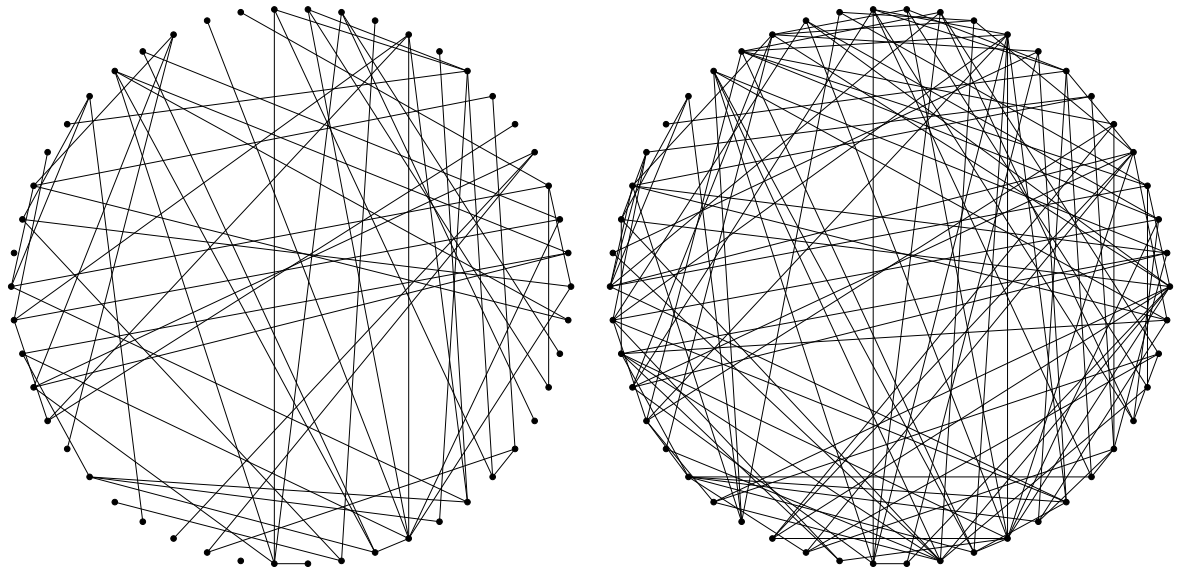


Figure 2.3: Random graph model illustration.

changing as all users are equal in their capacity and luck in front of the environment obstacles. The other way is to consider the graph with edges possibly changing every time and with the probability p that can decrease with the growth of n. It is the difference at following a random process (graph G_p) and its realisation.

The random graph model is rich with results as it was thoroughly studied since the middle of 20th century. One of the most important questions for the link availability models are whether the network is connected. The famous result of Erdös and Rényi (1959) [39] states that the $p = log(n)/n$ is the threshold for the graph connectedness

Theorem 2.4.

Let $c \in \mathbb{R}$ be fixed and let $p = \frac{\log n + c + o(1)}{n}$. Then

$$\mathbb{P}(G_p \text{ is connected }) \to \exp(-e^{-c}) \tag{2.30}$$

clearly, if $c \to \infty$, then this probability converges to 1.

If the probability p is fixed, then the average path length on a random graph is bounded. But if the probability satisfies some additional conditions the average path length can be found using the following theorem

Theorem 2.5. *(F.Chung and L. Lu[29]) If $np \geq c > 1$ for some constant c, then almost surely the average distance of G_p is $(1 + o(1))\frac{\log n}{lognp}$, provided that $\frac{\log n}{lognp} \to \infty$ as $n \to \infty$.*

Note, that the same expression was found for the diameter of a random graph by Erdös and Rényi.

The configuration, which corresponds to a minimal number of links in a wireless network whilst keeping the network connected, is often considered as a best configuration for maximising the throughput capacity. Since it minimises the interference between the nodes, what allows to a higher number of nodes to transmit simultaneously. For the geometric random graph such function is the connectedness threshold $p(n) = \sqrt{log(n)/n}$. Therefore, we can suppose that the average path length for this optimal case is $\log n / \log \log n$.

The problem of finding a maximal number of possible simultaneous transmissions in a random graph appears to be a complicated task. The problem can be defined in the following way: find a maximal number of pairs (transmitters and receivers), such that any receiver is connected to a corresponding unique transmitter and it is not connected to any other transmitter. Although, it is possible to estimate the probability that a random graph G_p contains a given subgraph (cf. [5]), but the second part of the condition, which means that receivers should not be connected to other transmitters to avoid interference, seems to be difficult to handle. To the best of our knowledge this problem stays open.

2.5.1 Network with Random Connections

An interesting model was proposed in [42]. It is based on the random graph model, but the probability of a successful transmission over a link is made dependent on the interference from other nodes in the neighborhood.

Consider a complete graph $\mathcal{G}_n{}'$ with $|V| = n$ nodes. Every edge (i, j) has a weight $\gamma_{i,j} = \gamma_{j,i}$, where $\{\gamma_{i,j}\}$ are positive i.i.d. random variables. Fix a parameter β_n and denote by $\mathcal{G}_n(V, E)$ the subgraph of $\mathcal{G}_n{}'$, that has the same vertex set, but the edges are chosen by the following condition

$$E = \{(i, j) : \gamma_{i,j} > \beta_n\} \tag{2.31}$$

.

These edges are called *good*. If $\gamma_{i,j}$ represent the power of the signal coming from the node i and received at the node j, the condition means that the link i, j is sufficiently good and does not contain many obstacles. Therefore, the nodes i and j will attempt to exchange their messages directly. The links are assumed symmetrical, as $\gamma_{i,j} = \gamma_{j,i}$. In fact, this construction $\mathcal{G}_n(V, E)$ is the classical random graph G_p with n vertices and $p = \mathbb{P}[\gamma > \beta_n]$. In order to keep such graph connected, β_n should satisfy the following equation

$$1 - F_n(\beta_n) = \frac{\log n + w_n}{n} \tag{2.32}$$

where $w_n \to \infty$ and $F_n(\gamma)$ is the cumulative distribution function of $\gamma_{i,j}$.

Next step is to define the routes or the scheduling of the communications. Consider k vertex-disjoint paths for k random source-destination pairs s_i, d_i. With high probability [7], there exist $k \geq \alpha_1 n \frac{\log np}{\log n}$ vertex-disjoint paths, where α_1 is a constant. Note, that the diameter of the network is $O\left(\frac{\log n}{\log np}\right)$ therefore, this number of the paths is close to the maximum.

A message will be successfully transmitted along a path $i < k$: $i_0 = s_i, i_1, \dots i_h = d_i$ from node s_i to node d_i, if SIR inequality (2.1) is independently verified at every relay node on this path. We denote the event that SIR is verified for a link i_j, i_{j+1} by E_j

$$E_j = \left[\frac{P\gamma_{i_j,i_{j+1}}}{\sigma^2 + P\sum_{l\neq i_j}\gamma_{l,i_{j+1}}} > K\right] \tag{2.33}$$

where P is the transmission power equal for all stations, and K is the SIR threshold constant. Actually, K depends on n, therefore, we will write K_n. The throughput is defined in [42] as

$$T = (1 - \epsilon)\frac{k}{h}\log(1 + K_n) \tag{2.34}$$

where k/h the number of simultaneous transmissions during a single time slot (number of paths divided by the path length in hops), and ϵ is fraction of dropped messages. In the context of (2.34) ϵ is the probability that SIR condition is verified for all hops ($\epsilon = \mathbb{P}[\wedge_{i=1}^{h} E_i]$). The value $\log(1 + K_n)$ is a sustainable throughput via a successful link, it is required since K_n is considered variable and, therefore, we need to take into account the limit on the channel capacity established by the fundamental theorem of Shannon on the capacity C_m of a noisy channel, stated in (2.3). In many models, this condition is implicit, mostly because the spectrum width is a constant and the SIR inequality $K < \frac{P}{N}$, with K constant, provides a lower bound for the capacity of a single link. But in this model, if $K_n \to 0$, then the possible throughput of a single transmission is decreasing with the same speed K_n and, therefore, it needs to be taken into account for the total throughput calculation.

The authors in [42] find the throughput for the general case of the distribution F_n, but for our review we will just cite two resulting expressions.

- One is for the case when the probability distribution is given by

$$f(\gamma) = (1-p)\mathbb{1}(\gamma = 1) + p\mathbb{1}(\gamma = 0) \tag{2.35}$$

  This equation suggests the natural choice of the threshold $\beta_n = 1$ for the good edges: those who has non-zero gain. For the optimal network utilisation p is chosen equal to the connectivity threshold: $p = \frac{\log n + w_n}{n}$. It provides the order of $k = O(n\frac{\log\log n}{\log n})$ simultaneous transmissions. The optimal SIR threshold is chosen to be of order $O(\frac{1}{\log\log n})$

  Hence the throughput is given by [42]

$$T = a\alpha\frac{n}{\log n}\frac{\log(\log n + w_n)}{\log n + w_n} + O\left(\frac{1}{\log(\log n + w_n)}\right) \tag{2.36}$$

  where a and α are constants and $w_n \to \infty$.

- Another example is $f_n(\gamma) = e^{-\gamma}$. It was shown that the throughput

$$T = \frac{a\alpha\log n}{4} + O\left(\frac{\log n}{n}\right) \tag{2.37}$$

  is achievable for $n \to \infty$, where $\alpha < 1$ and $a < 1$ are constants.

It is important to note that the threshold K should depend on n in order to achieve the maximal throughput.

2.5.2 SIR based Random Graph

In the previous section we have seen a model where the routing can be done on edges of a random graph, but the condition of the successful transmission was imposed in addition to the existence of the edges and it depended on the transmissions of other nodes in the network. As its authors remark [42] the routing is not necessarily limited to the *good* edges. Let us explore this possibility slightly further, first re-writing the conditions determining the neighbors in such model and then presenting the way of reducing the model to a straight random graph model by integrating the SIR condition into the edge existence probability. This extension repeats the logic of SIR based model for a geometric random graph proposed in the work [3], which was studied in Section 2.4.2.

Let us fix the SIR threshold K, as it is a parameter that depends mostly on the hardware and multiplexing, and also coding schemes. We consider the network of n nodes and assume that the signal gain $\kappa_{i,j}$ between the nodes i and j is i.i.d. random variable defined by its cumulative probability function F and the density function $f = F'$. We suppose that the probability density function exists, although for a general settings this condition can be removed. Also for generality we make the power of the nodes P_i depend on the nodes. A transmission between two nodes i and j is successful if the interference from other *transmitting* nodes is low:

$$\frac{P_i \kappa_{i,j}}{\sum_{l \neq i} \mathbb{1}(E_l) P_l \kappa_{l,j}} > K \tag{2.38}$$

where E_l is an event, that the node l is transmitting at this time interval. In order to estimate this event we will consider the following traffic pattern: we assume that all nodes at any time have a message to transmit and the probability of the transmission at any step is equal to q, where q is an external parameter that can depend on the load of the nodes or the physical and MAC layer constraints. Therefore, we can write $\mathbb{P}[E_l] = q$ and on average at arbitrary step there are qn messages transmitted across the network. This probability q (or rather the value qn) corresponds to the *traffic density* in [49, 3].

In short our goal is to replace the graph with random weights $\kappa_{i,j}$ by a non-weighted graph model, where an edge exists iff the corresponding weighted edge has a weight κ, such that κ compared to the accumulated interference from other transmitting nodes satisfies the SIR inequality with probability higher than some fixed parameter p_0. If weight κ satisfies this condition, then a bigger weight $\kappa' > \kappa$ also satisfies it. We denote by κ_0 the minimal weight κ for which this condition holds. As the weights are random, we calculate the probability that a given edge would have a weight greater than κ_0. We denote this probability by p. Therefore, instead of a graph with random weights $\{\kappa_i\}$, the model is described by a random graph with parameter p. Similar way in Section 2.4.4 we replaced a weighted geometric graph with random weights (power gains), by a geometric random graph. Let us describe this model in more formal definitions.

We fix a threshold probability $0 < p_0 \leq 1$ and assume for the simplicity $P_i \equiv 1$. We will include into our graph only the edges i, j, that have such gain $\kappa_{i,j}$, that the probability of a successful transmission between i and j with traffic load q is greater than p_0. If κ_0 is the minimal gain, for which this probability is greater than the threshold p_0, then for all $\kappa_{i,j} > \kappa_0$ the corresponding edge i, j will exist in the new model. Thus its existence depends on the distribution F and the parameter q. Let us denote by p the probability $\mathbb{P}[\kappa_i, j > \kappa_0]$. As $g_{i,j}$ are distributed according to F, then

$$p = 1 - F(\kappa_0) \tag{2.39}$$

and κ_0 is determined by

$$\kappa_0 = \min \left(\kappa \,:\, \mathbb{P}[\kappa > K \sum_{l \neq j} \mathbb{1}(E_l) \kappa_{i,l}] = p_0 \right) \tag{2.40}$$

Note, that the graph defined in this way is a classical random graph G_p. For the given threshold value p_0 and the distribution F, the probability p depends on the traffic parameter q. In fact, it is this dependence $p = p(q)$ we are most interested in finding. We would expect that a growing $p(q)$ is a monotonically decreasing function.

The explicit calculations of $p(q)$ depend on F. As the term $\sum_{l\neq j}\mathbb{1}(E_l)\kappa_{i,l}$ is a sum of i.i.d. random variables it can found in an explicit form for some distributions F using the Laplace transform. Because of the independence of E_i and $\kappa_{i,j}$, the Laplace transform of the terms $\mathbb{1}(E_l)\kappa_{i,l}$ is simply

$$\mathbb{E}e^{-\mathbb{1}(E_l)\kappa_{i,l}s} = q\mathbb{E}e^{-\kappa_{i,l}s} + 1 - q \tag{2.41}$$

Therefore,

$$\hat{\phi}_n(s) = \mathbb{E}e^{-\sum_{l\neq j}\mathbb{1}(E_l)\kappa_{i,l}s} = (q\mathbb{E}e^{\kappa_{i,l}s} + 1 - q)^{n-2} = (q\hat{f}_n(s) + 1 - q)^{n-2} \tag{2.42}$$

where by $\hat{f}$ we denote Laplace transform of f. It worths noting here, that we suppose that the receiver node j is not transmitting.

To find probability of the success of a transmission over arbitrary link i, j we need to find the inverse of the Laplace transformation and, then, estimate the probability that the gain $\kappa_{i,j}$ will be K times greater than the noise:

$$P_{\mathcal{S}}(\kappa) = \mathbb{P}[\text{successful transmission over a link with the gain } \geq \kappa] = \int_0^{\kappa/K}\phi(t)dt \tag{2.43}$$

Then, we need to solve

$$P_{\mathcal{S}}(\kappa) = p_0$$

to find the κ_0.

The probability $p = 1 - F(\kappa_0)$ will give the parameter of the random graph G_p. This resulting random graph G_p has the edges with the probability of successful transmission higher than p_0.

The probability p we obtain this way is a function of q and n: $p(q, n)$. The classical theory provides several threshold values of p for different graph properties: connectivity, coloring. For example, the connectivity limit $\log n/n$ can be taken as the value $p(q, n)$. Therefore, q is constrained to be a particular function of n, that corresponds to the graph property.

Let us illustrate this approach with an example.

2.5.3 Illustration: Exponential Power Attenuation

Let us assume that gains $\kappa_{i,j}$ are distributed exponentially: $F(x) = 1-\exp(-x)$. For a complete SIR based model we would need to estimate whether a link gain $\kappa_{i,j}$ will outweigh the sum of noise at a node j coming from other nodes in the network at the same moment. In this case, the existence of the edge (i, j) changes all the time and depends on the communications among other nodes. Therefore, network state is not easily described.

One approach to restrict such network to a graph is to leave only those edges that provide the probability of a successful transmission high enough. Therefore, let us fix p_0.

The Laplace transform of the noise perceived in j coming from all other nodes $\sum_i \kappa'_{i,j} = \sum_i \mathbb{1}(E_i)\kappa_{i,j}$ can be found as

$$\mathbb{E}e^{-\sum_{l\neq j}\mathbb{1}(E_l)\kappa_{i,l}s} = \left(\frac{q}{s+1} + 1 - q\right)^{n-2} = \left(1 - q\frac{s}{s+1}\right)^{n-2} \tag{2.44}$$

where $s > -1$. It is possible to consider instead of (2.44) its asymptotic estimate:

$$\int_{\gamma-i\infty}^{\gamma+i\infty}\left(1 - q\frac{s}{s+1}\right)^n ds \approx \int_{\gamma-i\infty}^{\gamma+i\infty} e^{-qn\frac{s}{s+1}}ds \tag{2.45}$$

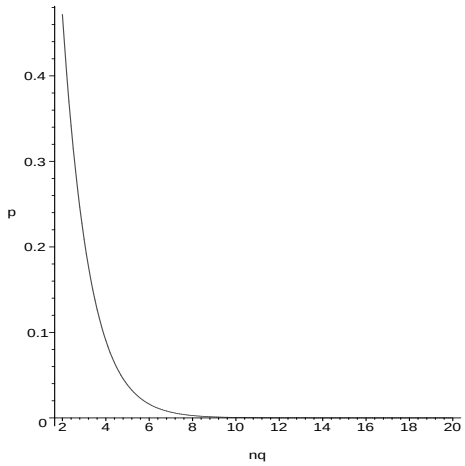


Figure 2.4: Numerical solution to (2.48) power gain equation

where $\gamma > -1$, $n \to \infty$ and $q \to 0$. This estimation holds as function $(1 - z/n)^n$ converges uniformly to $exp(-nz)$ for z defined on a compact region. In our case the function $\frac{z}{1+z}$ maps the vertical line $(\gamma - \infty i, \gamma + \infty i)$ to a circle, which is a compact space. As the functions inside the integral converge uniformly, the integrals also converge.

Its inverse transform $\phi_n(t)$, which is an estimation for the probability density function of a sum noises can be found as

$$\phi_n(t) = \sqrt{\frac{nq}{t}} e^{-nq} e^{-t} I_1(2\sqrt{nqt}) + \delta(t) e^{-t} e^{-nq} \tag{2.46}$$

where I is a modified Bessel function of the first kind and δ is Dirac function. Then we need to find κ_0, which satisfies the following equation

$$\int_0^{\kappa_0/K} \phi_n(t)dt = \int_0^{\kappa_0/K} \sqrt{\frac{nq}{t}} e^{-nq} e^{-t} I_1(2\sqrt{nqt})dt + e^{-nq} = p_0 \tag{2.47}$$

Although (2.47) can be simplified:

$$e^{-nq} \left(e^{-\kappa_0/K} I_0\left(2\sqrt{nq\kappa_0/K}\right) + \int_0^{\kappa_0/K} e^{-t} I_0\left(2\sqrt{nqt}\right) dt \right) = p_0 \tag{2.48}$$

the closed form seems to be difficult to find and we will stop the calculations for this example at this point. As an illustration, we provide a graph $p(q) = exp(-\kappa_0)$ based on the numerical solution to (2.48) on Fig. 2.4 ($K = 1$ and $p_0 = 0.5$). On x-axe we located nq, as it is clear that κ_0 depends rather on this value than simply on q. Note, that nq is an average number of the nodes transmitting at arbitrary time slot. Therefore, the probability of a successful transmission depends on the average number of competitors and not on the total number of nodes.

In this section we described a model of a short range multihop wireless network based on the random graph, where the neighborhood depends on the traffic level. Though it seems to be a promising approach the practical calculations are difficult to handle. For example, the capacity of the network could be estimated via $p(q) = \frac{\log n}{n}$ condition, but $p = p(q, n)$ should be obtained in the closed form first. Often, the hard part of the calculations is to find the inverse Laplace transform of (2.42), even in the asymptotic form $\exp(nq\hat{f}_n(s) - nq)$.

This asymptotic approximation suggests that the traffic scheduling probability p should depend on nq. In other words, $p(q,n)$ is a function of the average number of neighbors $p(q,n) \approx p(nq)$.

A possible generalization of the model comprise modeling the transmission power. It can be done by introducing a random distribution of the power. Of course, it may create an additional difficulty for the analysis.

2.6 Short Classification of the Link Availability Models

The list of the models we present in this chapter does not pretend to be complete. There are exist more specific models of physical link availability and protocol behaviour, for example, for the fixed graph network structure. The goal of this presentation is to find the common and also the distinctive characteristics of the models. In general, the link availability is based on the distribution of power gains among the nodes and on the conditions of successful transmissions, whether the former is distance based or not, and the latter is decided via a fixed bound comparison or SIR inequality.

Therefore, the dependence of the power gains between nodes on *distance* is one of the criteria of model classifications. In the models based on point processes in the plane, the geometrical properties are respected and power gains are modeled by a signal attenuation function, which depends on the distance distribution. The models of small networks in an environments with obstacles are the opposite examples, there the gains are distributed randomly without any geometrical constraints.

Another property of the power gain distribution is existence for a given node of a bound on the number of nodes, whose transmission power may affect message reception of the given node. In other words, whether the model should consider an arbitrary single transmission as potentially capable to influence the communications between any other nodes or it is limited only to neighboring nodes. In the case of the geometrical models this constraint can correspond to considering the attenuation functions with an unbounded or a compact support. Usually the neighbors of A are those nodes that can receive a message from A. Therefore, there is a gap between the limitation of the signal influence to the neighboring nodes and all other nodes. We consider a model where a signal disturbs not only the neighbors, but *some* other nodes as not a part of the models with a *limited signal influence*.

In some cases link availability is considered dependent on the actual level of *traffic load*. In this cases additional information characterising the traffic is needed. In many cases it is described by a probability p of a node to transmit at arbitrary time slot. It gives a possibility to describe the overall noise coming from nodes in the network at the same time, thus the transmission range is also influenced. If the traffic load is not taken into consideration, then transmission range is considered to be constant.

To summarise the models from this chapter let us present them and their properties in a single table:

	Geometrical	Traffic Load	Limited Signal Influence
Unit Disk	+	−	+
STIRG	+	+	−
Unit Disk + SIR	+	+	+
Random Graph	−	−	+
Random Gains	−	+	−
Random Graph + SIR	−	+	+

As an interesting development for the analytical modeling of wireless network links we can think of a hybrid model, which could integrate together short and long range network properties. At least two ways can be suggested. The first one is to use a hierarchical approach, where geometry dependent model connects multiple instances of geometry-free models. The second approach for building a hybrid model can be a more complex representation of power gains, where they would include two components: distance dependent random value and an additional distance-independent random value. The further investigation can be an interesting subject for a future work.

Chapter 3

Models of Media Access (MAC) Layer

The goal of this chapter is firstly to present several current MAC protocols and, secondly, to study the properties of flooding in a wireless network with respect to MAC protocols.

We will review mostly Aloha and 802.11 DCF (CSMA/CA) protocols, to recall some well known results on their performance evaluation and also to see how these protocols can be represented via probabilistic models and stochastic geometry. We also include some new development for the Markov model of CSMA/CA, considered in the framework of the SIR based geometric graph link availability model.

At the second half we will consider modeling flooding together with MAC layer. In many MAC models the standard assumption is heavy traffic*, i.e. nodes are busy all the time and they constantly compete for the medium. Therefore, we try to find out how this assumption can be close to the events happening in a flooding wave. We also consider in details how a flooding wave can behave locally, supporting our ideas with simulations results. The latter will be used in Chapter 5 for a comparison of two ad hoc network protocols.*

3.1 Common MAC protocols in wireless networks

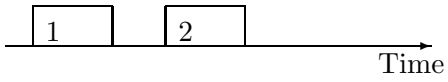
Medium access protocols are the algorithms, that allow to share a common medium among many users [73]. In general the medium access protocols can be divided in three categories fixed assignment (TDMA, FDMA), random access (ALOHA, CSMA/CA) and demand assignment protocols. The last category is normally out of the scope of the ad hoc networks, as they assume a central authority to schedule the transmission around. The IEEE 802.11 standard for wireless local area networks (WLAN) encompasses the physical layer (PHY) and the lower portion of the data link layer. Medium Access Controller (MAC) sublayer is the lower portion of the data link layer.

The importance of the MAC protocols for ad hoc networks can be seen from the following observation: the need in multihop routes is born out of the problems of sharing a medium. Therefore, a good understanding of ad hoc network can not be obtained without taking into account the corresponding MAC protocol. In general, MAC protocols apart of the access control can include some features of the error detection or avoidance.

A number of MAC mechanisms is based on multiplexing schemes for the wireless and cellular networks. There are several ways to separate user transmissions sharing the same medium : in time, in frequency or using advanced signal coding. It corresponds to four well known multiple

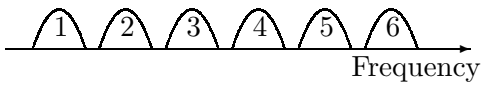
access propositions:

- TDMA - Time Division Multiple Access, Time division multiple access (TDMA) is a transmission technology that allows a number of users to access a single radio-frequency (RF) channel without interference by allocating unique time slots to each user within each channel. GSM protocol for cell phones is one of the most known applications for TDMA in wireless.



- FDMA - Frequency Division Multiple Access. a transmission technology where the assigned frequency band for a network is divided into sub-bands which are allocated to different users. This way every user has its own frequency band, which is not intersecting with others.

  One of the biggest problem of FDMA is frequency band utilisation. The channels cannot be too close one to another, because transmitters output some energy outside the band allocated to the channel. Therefore, some space between the channel is needed.



- OFDMA - Orthogonal frequency division multiple access is a communications technique that divides a communications channel into a number of equally spaced frequency bands. A subcarrier carrying a portion of the user information is transmitted in each band. Each subcarrier is orthogonal (independent of each other) with every other subcarrier. Contrary to FDMA, the bands can intersect. This approach allows a better utilization of the frequency band allocated to the network. At the present moment the number of OFDM solutions is increasing rapidly. OFDMA multiple access is based on OFDM multicarrier mechanism, which allows to send a message by splitting it in parts and then slowly sending these parts in parallel over many low-rate (orthogonal) carriers. This approach leads to a greater degree of robustness and a better frequency band re-utilization compared to FDMA.

  In [69] we find the list of the following advantages of the OFDM scheme :

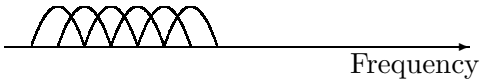
  – It is an efficient way to deal with *multipath.* In a radio channel a signal, because of reflections and refractions, is received from many different paths. Therefore, it can interfere with itself, thus confusing a receiver at its decoding stage. The received copies of the signal might have different amplitudes and phases.
  – it is possible to enhance the capacity of a channel by adapting the data rate per subcarrier according to the SNR ratio.

- it is robust against a narrowband interference, because only a small part of subcarriers is affected by such event.
- it makes a single-frequency networks possible, which is attractive for the broadcasts.

On the contrary, the drawbacks include

- OFDM is more sensitive to frequency offset and phase noise.
- OFDM has a relatively large peak-to average power rate.

OFDM was included into design of the 802.11a,802.11g and also HiperLAN/2 protocols



for WLANs.

- CDMA - in Code Division Multiple Access scheme every user has its one code sequence, which is used to encode its message. The code allows to spread a signal along a large spectrum of frequencies (spread-spectrum modulation). Therefore, the transmission bandwidth should be larger than the information bandwidth (we might think of it in terms of Shannon theorem 2.3). The receiver, knowing the code, is able to decode the signal and recover the message. The cross-correlation between different codes are made minimal and this allows to establish multiple access mechanism. Signals coming from different sources can be distinguished, provided that competition is not too high. There is a number of ways to perform such spread modulation: Direct-Sequence (DS), Frequency-Hopping (FH), Time-hopping (TH) and several hybrid types. A dense description of CDMA can be be found in [69].

All these multiple access mechanisms are based on their multiplexing analogues, which have the names TDM, FDM, OFDM and CDM ("M" for multiplexing). The goal of multiplexing techniques is to encode a single transmission signal, optimizing its speed and reliability, allowing an efficient use of network resources. In order to do that, they divide the transmission into pieces and multiplex them for an optimized transmission. This division of a single message can be, then, replaced by multiple users leading to the corresponding multiple access scheme.

Of course, these schemes can receive a further optimization. For example, to improve the reliability of the frequency division based multiple access mechanisms can include a frequency hopping. The users change the allocated frequency band following some earlier defined schedule. Then, in the environments where some parts of the frequency band meet more obstacles, such scheduling will introduce an additional reliability.

In addition to medium sharing techniques described earlier, the MAC protocols can include collision resolution mechanisms. One of the first MAC protocols for the packet radio networks is Aloha. The protocol was developed by N. Abramson for the packet radio network that existed in 1970 ([1]) at the University of Hawaii. The nodes separate their transmissions in time. If a collision between several nodes occurs, then any node suffered from this collision resends the packet later. The delay it takes is random and it is chosen uniformly from the interval $[1, T_{aloha}]$, where T_{aloha} is the maximum duration of the delay and it is a parameter of the protocol.

Carrier Sense Multiple Access/Collision Avoidance (CSMA/CA) is one of the MAC protocols in the Wireless LAN standard 802.11. As its name states the protocol attempts to avoid the

collisions. It introduces a two or four way handshake scheme, where the sender reserves a bandwidth for some period of time and the receiver needs to confirm this reservation. Other nodes in the neighborhood wait with their transmission until the reservation time will be finished. If a reservation has not been successful, the sender node waits for a random time period.

Two classical difficulties, related to the collisions in a highly competitive wireless network, the MAC protocols need to confront are the following:

- Hidden Terminal Problem. It is caused by asymmetric information, when nodes not knowing about each other are trying to transmit their messages to the same node.
- Exposed Terminal Problem: a node assumes that it cannot transmit because it senses a transmission, but, actually, the receiver is not in the range of this transmission and, thus, it can receive it.

Aloha Protocol

Aloha protocol has two versions : the classical and the slotted Aloha.

- Let us consider a simplified model of the Aloha protocol. The overall load on the channel is represented by a Poisson process of intensity Λ. The packet length is fixed. A transmission of a packet of duration T starting at time t will be successful, if there are no other nodes transmitting at the period $[t-T, t+T]$. Therefore, the probability of the successful transmission can be found as
$$\mathbb{P}[success] = e^{-2T\Lambda}$$
Thus, the rate of the successful packets is $\Lambda\mathbb{P}[success]$. As the throughput λ_{th} is the fraction of time when a useful information is carried, we obtain
$$\lambda_{th} = \Lambda T e^{-2\Lambda T}$$
. This function is maximized at the point $\Lambda T = 0.5$, which corresponds to $\max \lambda_{th} \approx 0.18$.
- Slotted Aloha: time is divided in slots and the duration of packet is equal to this time slot. Before transmitting every node waits for the beginning of the slot, thus the packet either overlaps within the same slot, or does not overlap at all. This allows to improve the throughput up to $\lambda_{th} = \Lambda T \exp(-\Lambda T)$ and $\max \lambda_{th} \approx 0.36$.

A good introduction to the multiple access techniques with some performance analysis can be found at [87]

To continue analysis of the Aloha protocol we consider a different constraint on the traffic pattern of the network instead of Poisson arrivals: let us we assume *a heavy traffic* pattern. In other words, a node has an infinite queue and as soon as it transmits a packet from its top, it has another packet to transmit. In this case the Aloha protocol described earlier can be considered as a simpler algorithm: the node at every slot decides whether to transmit the packet at random, with probability p it does it and with probability $1-p$ it does not.

Aloha studied in SIR based model

Let us consider Aloha protocol in terms of the SIR based geometric graph model of Section 2.4.4. We assume, that the network modeled by P.p.p. for the node positions and the traffic pattern is the heavy traffic. Therefore, the emitting nodes (P.p.p. of intensity λ_{em}) always

have a packet to transmit. In this case Aloha transmission schedule can be defined by Medium Access Probability (MAP) p, i.e. the probability that at arbitrary slot any node will transmit a packet. Therefore, the traffic density is equal to $\lambda = p\lambda_{em}$.

Then we take into the play the expressions found for the SIR model and determine the probability function $p_R(\lambda)$, that is the probability that a signal will be successfully decoded for the nodes at distance R in the network with the traffic density λ. The information we can obtain is, for example, the maximal MAP p, such that the network has the greatest possible number of transmissions per time slot or, in other words, best spatial reuse. The average number of the emitters per unit area capable of transmitting within the distance R is equal to (see [12])

$$\mathbb{E}N(\lambda) = \lambda p_R(\lambda). \tag{3.1}$$

Let us consider the simplified attenuation function $l(R) = R^{-\alpha}$ and, in the case of the constant transmitting power $P \equiv 1$ from (2.25) we obtain:

$$p\lambda_{em} = \frac{r_0^2}{R^2} \tag{3.2}$$

The expression (3.2) gives the relation between MAP p and the transmission radius R in a network controlled by Aloha MAC scheme, under the assumption of the heavy traffic. A small step further is to suppose that MAP p is related to the number of neighbors M competing for the channel: $p = \frac{C_2}{M}$ (C_2 is some constant, which can be defined in an implementation of Aloha)

In SIR based model the average number of those neighbors can be estimated by $M = \lambda_{em}\pi(CR)^2$, where $C = R_{cs}/R$ is the ratio of carrier sense (or interference) range and the transmission range (cf Section 2.4.5). Combining these expressions together with (3.2) gives the following dependence between C and C_2 in case of $P \equiv 1$:

$$C_2 = r_0^2 C^2 \tag{3.3}$$

It gives us directions on how to choose an optimal C_2 for Aloha implementation, because C is more a hardware characteristic. Recall, that r_0 in (2.25) depends on the probability of a successful reception, therefore, on the quality of the link. Therefore, Choosing C_2 under this limits will mean to under-utilise the bandwidth available or, in its turn, to increase reliability. Otherwise, taking C_2 larger than in (3.3) will reduce the quality of the link. Knowing the dependence of r_0 on p_0 (see [3]), we can find the dependence of the reception probability on C_0. But as the closed form of this dependence is unknown, we leave p_0 hidden inside of r_0 in (3.3).

CSMA/CA

Carrier Sense Multiple Access with Collision Avoidance (CSMA/CA) protocol is in the base of Distributed Coordination Function (DCF), which is a part of 802.11 protocol for Wireless Local Area Networks. The default scheme of CSMA/CA is two-way handshake scheme - the destination node immediately confirms successful reception with an acknowledgment message (ACK). In addition to this rule there is an optional four way handshake scheme, with request-to-send/clear-to-send (RTS/CTS) packets mechanism. The source node attempts to reserve the channel with RTS packet and the destination, if it can, acknowledges this reservation by sending CTS packet. Then the message is transmitted and acknowledged as in the previous scheme. This approach not only allows to reduce the duration of collisions, as they mostly occur during the RTS packets, which are short, but also to fight the hidden terminal problem. Retransmission of collided packets is managed according to binary exponential backoff rules. If

a collision occurs, the timer is chosen uniformly at random from the interval $[1, W]$, if a second attempt is unsuccessful, the delay is chosen again uniformly at random but from the interval $[1, 2W]$, and so on.

In reality the protocol is more complex, especially in its exponential backoff algorithm. For example, consider a sender node, that started its random timer. The timer will only run down when the channel is idle. There is also a number of additional delays, for example, the one after transmissions, when the sender node is waiting with its timer stopped. When the delay runs out, the node waits again before starting its reservation activity.

CSMA/CA studied under heavy traffic assumption

In work [17] G. Bianchi studies the backoff mechanism 802.11 DCF in what can be considered as unit disk model. The traffic pattern is supposed to be heavy load. He presents a Markov chain model that approximates the evolution of the delay timer with an important assumption that at each transmission attempt, regardless of the number of retransmissions suffered, each packet collides with constant and independent probability p. There are two other parameters: the number of neighbors n (that makes from this model the "unit disk model") and the minimal timeout duration W.

Then, the parameter that can be calculated in this model, is the probability τ that the station transmits a packet in a generic randomly chosen slot. The author shows that

$$p = 1 - (1 - \tau)^{n-1} \tag{3.4}$$

and

$$\tau = \frac{2(1-2p)}{(1-2p)(W+1) + pW(1-(2p)^m)} \tag{3.5}$$

where m is the maximal number of backoff steps ($m = \log_2 \max \text{backoff}/W$). This system of two equations has a unique solution $\tau^*$for $p \in (0, 1)$ and $\tau \in (0, 1)$. These expressions have the interest on their own, but also they allow to find the throughput and the maximal saturation throughput. However, these exact expressions are quite long and the interested reader will find them in [17]. Another interesting result in this work is that the maximal throughput does not depends on number of competing nodes. The following approximation of the probability $\tau = \frac{1}{nK}$ is given, where K is square root of the average time in slots the channel is sensed busy during a collision, which is the function of the packets' lengths. Then, the probability P_{tr} that there is at least one transmission at the considered time slot can be expressed as

$$P_{tr} = 1 - (1-\tau)^n = 1 - \left(1 - \frac{1}{nK}\right)^n \approx 1 - \exp\left(-\frac{1}{K}\right) \tag{3.6}$$

and the probability that a transmission occurring at the channel is successful is equal to

$$P_s = \frac{n\tau(1-\tau)^{n-1}}{P_{tr}} \approx \frac{1}{K(e^{1/K} - 1)} \tag{3.7}$$

This leads to the approximation of maximal achievable throughput S_{max}:

$$S_{max} = \frac{E[P]}{T_s + \sqrt{\sigma T_c} + T_c(\sqrt{\frac{T_c}{2\sigma}}(e^{\frac{2\sigma}{T_c}} - 1) - 1)} \tag{3.8}$$

where T_s is average time the channel is sensed busy because of successful transmission, T_c is the average time the channel is sensed busy by each station during a collision $K = \sqrt{T_c/\sigma}$, σ is the duration of a slot, $E[P]$ average length of packets.

Another model analysing the DCF mechanism was proposed in [88]. It is simplified with the respect the one in [17]. The Markovian model is used to describe the behavior of a node surrounded by competitive neighbors. The time is slotted. The nodes assumed to be heavily loaded, i.e. they always have some information to transmit. One of the main assumptions of the model is the same as we find for most of the models, that the probability of a collision p_c for a node that is trying to transmit is constant.

Then, the backoff periods are modeled by a Markov chain where a state is the number of the attempts in the backoff mechanism $1, 2, 3, \ldots$ (The maximal number of attempts m is assumed unlimited). This chain is described by the following probabilities: $P_{i,i+1} = p_c$ (principle assumption) and $P_{i,1} = 1 - p_c$ (the heavy load is assumed, therefore, the node has a new message to transmit as soon as it sends its previous message); for all other i and j: $P_{i,j} = 0$. Also a node will stay for an average time $\frac{W_0 2^i + 1}{2}$ at state i.

A priori p_c is unknown. But there is another parameter closely related to it: p which is the probability that a given node will transmit at an arbitrary time slot. The relation between p and p_c is as follows:

$$p_c = 1 - (1 - p)^N \tag{3.9}$$

p,p_c are the functions of the initial size of the backoff maximal timeout W_0, number of the nodes N competing with the given node and the exponent of the backoff mechanism, which is often equal to 2. The model is shown to be stable and the numerical values for p and p_c are obtained. We will cite one of the representations for p:

$$p = \frac{2(1 - 2p_c)}{W_0(1 - p_c) + 1 - 2p_c} \tag{3.10}$$

after using (3.9) we obtain

$$p = \frac{2(-1 + 2(1 - p)^N)}{W_0(1 - p)^N - 1 + 2(1 - p)^N} \tag{3.11}$$

whose solution in closed form is unknown. But, of course, any standard numerical method for solving non-linear equations can be used for its solution.

Let us make a step further in this analysis and use the carrier sense range to evaluate the number of nodes participating in the competition. We assume that the emitting nodes are distributed according to P.p.p. with intensity λ. On average their number can be obtained

$$N = \lambda \pi R_{cs}^2$$

Analogous to our previous study of Aloha, we can make an assumption that $R_{cs} = CR_t$ for some C (cf. Section 2.4.5). Therefore, using (2.25) to estimate the transmission range R_t, we obtain

$$N = \pi C^2 r_0^2 \frac{\lambda}{\lambda_{\text{exp\_backoff}}} \tag{3.12}$$

At this point let us go back to our model, that states that the effect of the backoff mechanism on the network can be reduced to a scaling of the number of emitting nodes to $p\lambda$ at an arbitrary time slot. It has, in fact, the equivalent in the field of the stochastic geometry: $p-$thinning

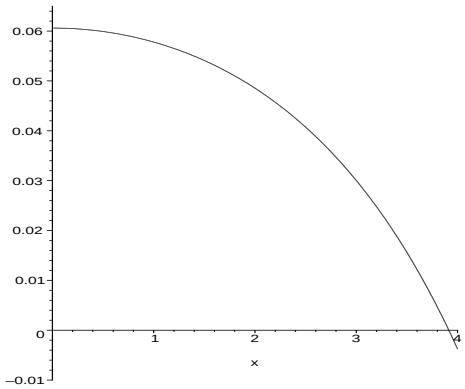


Figure 3.1: MAP probability p as function of C in (3.13)

of the P.p.p. The p-thinning Φ of a point process Φ_b can be obtained by deleting the points with probability $1-p$, this deletion is independent of locations and possible deletions of any other points of Φ_b [89]. A p-thinning of a Poisson point process of intensity λ is a Poisson point process with intensity $p\lambda$. Therefore, we can consider

$$\lambda_{\text{exp\_backoff}} = p\lambda$$

thus obtaining

$$p = \frac{2(-1+2(1-p)^{\pi C^2 r_0^2/p})}{W_0(1-p)^{\pi C^2 r_0^2/p} - 1 + 2(1-p)^{\pi C^2 r_0^2/p}} \tag{3.13}$$

The numerical solution for p as function of C in the equation (3.13) is presented on the Fig. 3.1. We can observe, that even when the carrier sense range is small and, therefore, there is no competition with other nodes, the backoff mechanism is

SIR modeling of CSMA/CA

In article [12] the authors make a comparison of Aloha protocol with CSMA/CA. The CSMA/CA modeling is based on the spatial reuse property of this mechanism that allows to transmit to the nodes that are R_{cs} (carrier sense range) away from each other.

Then, at arbitrary slot, the transmitting nodes under the control of CSMA/CA, can be approximated via Mattern hard-core process [89]. It is a dependent thinning of Poisson stationary process Φ_b of intensity λ_b. The points of Φ_b are marked independently by random numbers uniformly distributed over $(0,1)$. The dependent thinning retains the point $x \in \Phi_b$ with mark $m(x)$ if the sphere $b(x,h)$ contains no points of Φ_b with marks smaller than $m(x)$. Formally we can write that the Mattern hard-core process Φ is given by

$$\Phi = \{x \in \Phi_b \,:\, m(x) < m(y) \text{ for all } y \in \Phi_b \cap b(x,h)\{x\}\} \tag{3.14}$$

The intensity of Φ is equal to $\lambda = p\lambda_b$, where

$$p = \frac{1-\exp(-\lambda_b \pi h^2)}{\lambda_b \pi h^2} \tag{3.15}$$

(in general case of $\mathbb{R}^d$, πh^2 should be replaced by the volume of the corresponding disk)

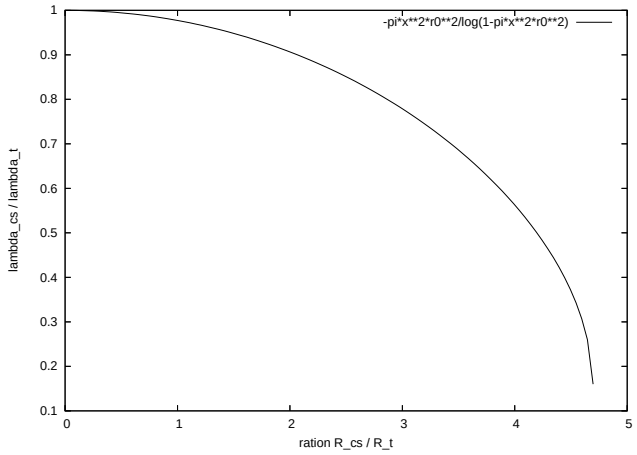


Figure 3.2: MAP probability p as function of C in (3.17)

In its application to CSMA/CA, this property approximates the RTS/CTS reservation scheme, where some nodes reserve the channel and within the distance $h = R_{cs}$ there is no other nodes authorised to transmit. Let us fix the density of emitting nodes λ and the transmission radius R. R_{cs} can be estimated by (2.29), therefore, the spatial intensity of CSMA/CA is $p\lambda$, with $h = R_{cs}$ in (3.15). The authors of [12] compare the two protocols for different values of the degree α of the attenuation function. In particular, for α just greater than 2, the optimised Aloha outperforms the CSMA/CA. Then, approximatively at the point $\alpha = 2.6$, two protocols are equal. After this point CSMA/CA takes a lead.

Up to this point we were considering R fixed, but thinking of $R_{cs} = CR_t = C\frac{r_0}{\sqrt{\lambda_{\text{csma}}}}$, we obtain

$$\lambda_{\text{csma}} = \frac{1 - \exp\left(-\pi\lambda C^2 \frac{r_0^2}{\lambda_{\text{csma}}}\right)}{\pi} \frac{\lambda_{\text{csma}}}{C^2 r_0^2} \tag{3.16}$$

$$\lambda_{\text{csma}} = -\frac{\pi C^2 r_0^2}{\log(1 - \pi C^2 r_0^2)}\lambda \tag{3.17}$$

Let us underscore the differences in the models in [17] and [88] from the Mattern hard-core process modeling. One can think of first type models as the representation of the exponential backoff mechanism, without taking into account the RTS/CTS hand-shake. Whereas, the second model ignores the backoff scheme completely and it describes the ways the nodes can be separated in space by the hand-shake scheme. In fact the spatial reuse (3.14) corresponds to the optimal CSMA/CA implementation, as it only describes the possibility to fit the nodes separated by the distance R_{cs} in CSMA/CA-style. Therefore, it can be seen as the maximum efficiency CSMA/CA can achieve using different scheduling. Fig. 3.3 shows the difference in the probability of a successful transmission for the CSMA/CA hand-shake and the exponential backoff timer approaches. We can see that *the exponential backoff is under-utilising the carrier sense zones, that are available under the hand-shake scheme.*

3.2 Some insights on the flooding process

Consider the network, where the nodes participating in the flooding are represented by P.p.p. of intensity ν in $\mathbb{R}^2$. The classical flooding means that all of these nodes should broadcast the

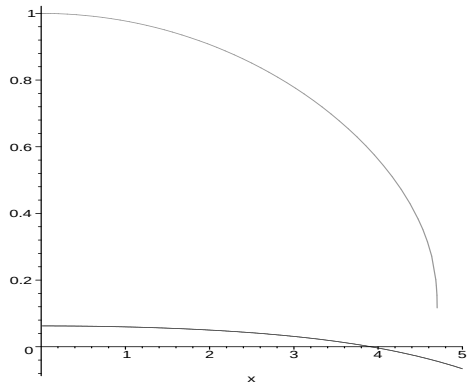


Figure 3.3: Comparison of MAP p as function of C in (3.13) and (3.17)

message upon receiving it. As we have seen at the previous chapter the MAC protocol controls the ability of the nodes to transmit the message and, therefore, it influences the flooding process. Let us demonstrate how the flooding can be seen from the point of view of the models with fixed and traffic-dependable transmission radius.

Suppose that in some bounded region S all the nodes have already received the message, but the number of nodes retransmitted the message is small enough so that we can assume that all the nodes within S have yet to transmit the message. At every step there is a part of the nodes that frees itself of the message after transmitting it, therefore, the intensity ν is reducing with every step. Of course, this statement only makes sense in the case that the process stays P.p.p.

Consider the model where we consider the transmission radius dependable on the traffic density λ: $r = \frac{A}{\sqrt{\lambda}}$, where A is a constant , this is the case, for example, when the we consider the SIR model for the constant and exponentially distributed transmission power (cf (2.19) and (2.21)). Therefore, r depends on density of the emitting nodes at the particular slot. Let us assume that an arbitrary node transmits its message at time slot k with probability denoted by p_k. Then, the transmitting nodes and the silent nodes form Poisson point processes of the intensities, which we will denote by λ_i and λ'_i accordingly. Clearly, $\lambda_i = p_i\lambda'_i$, $\lambda'_{i+1} = (1-p_i)\lambda'_i$ and $\lambda'_0 = \nu$. The traffic density coincide with λ_i and its evolution can be written as

$$\begin{aligned} \lambda_1 &= p_0\nu \\ \lambda_2 &= p_1(1-p_0)\nu \\ \lambda_3 &= p_2(1-p_1)(1-p_0)\nu \\ &\dots \end{aligned} \tag{3.18}$$

The average number of the nodes N_i competing with a given node for the medium at step i can be found as $N_i = \lambda'_i\pi R_{cs}(i)$, where $R_{cs}(i)$ is the carrier sense range at the step i. Let us find how N_i is changing with the time.

We assume that the carrier sense radius is proportional to the transmission radius $R_{cs} = CR_t = Cr$. Then N_i can be written as

$$N_i = \lambda'_i\pi R_{cs}^2(i) = \lambda'_i\pi C^2\frac{A^2}{\lambda_i} = \frac{\pi C^2A^2}{p_i} \tag{3.19}$$

Therefore, if the corresponding MAC protocol would have a property of keeping MAP constant at any time slot: $p_i \equiv p$, then the flooding could be represented in terms of heavy load

assumption. In fact, the number of competing nodes in this case would be kept constant. For example, a MAC protocol, maximising the spatial reuse of a hand-shake scheme, has this property, as we can see it from (3.17). For such protocol simplified flooding assumption corresponds to the heavy load assumption.

3.3 Flooding in Simplified Probabilistic MAC protocol

This section can be seen as a preliminary work for a comparison of two ad hoc network protocols in Chapter 5. As a path discovery phase of one of these protocols (AODV) is based on flooding, we would like to learn some details of the flooding process. The framework we build in Chapter 5 for higher level network modeling is missing several crucial parameters. As their entirely analytical estimation, to the best of our knowledge, is unknown, we will try to estimate them from a sample of simulations based on a highly simplified MAC protocol. One such parameter is path length and another is the probability of an error in a transmission over a single link caused by a flooding wave.

As flooding represents one of the highest loads a network can experience, its influence on the transmission error rate has a particular interest. Let us consider the following situation: when nodes in some bounded area start relaying a flooding message, other messages should suffer of a such increase in the traffic level. As we have seen it in Chapter 2, an increase in traffic decreases transmission range between the nodes of the network. For a given link, we are interested in investigating how long the transmission range will stay less than the length of the link, because of the flooding message transmissions. We first present a simplified MAC protocol, then we present the data from a simulation, where a message is disseminated via flooding across a bounded area with a fixed number of nodes. We follow the evolution of the traffic density in the chosen area and we estimate the results with analytical or semi-analytical expressions, where it seems possible. The reason we present this work in the current chapter, despite its close relation to Chapter 5, is that flooding performance depends heavily on MAC layer protocol and the thesis structure reflects the different layers of an ad hoc network.

3.3.1 The flooding model

One of the important parameters for a flooding evolution is the node density ν. The more dense the network is, the higher is the load per unit area. Consider a situation, when all nodes in some neighbourhood almost simultaneously start re-transmitting a message. If there is no controlling MAC protocol, the traffic density raise up to highest level - ν, which means that all nodes transmit at the same time. The flooding in this case would stop after a first step or two. Due to their own interference the retransmitting nodes would not have any neighbours capable of receiving them.

There are multiple solutions for sharing the medium by the waiting nodes. In case of the 802.11 MAC layer, it is the Distributed Coordination Function (DCF [76]): prior to initiating a transmission, a node chooses a random delay from a fixed interval. Then, at the end of its waiting period, if the node finds the network still busy it chooses another random waiting time from the double of the initial interval. This is called an *exponential backoff algorithm*. Some protocols can use their own random delay (*jitter*) mechanisms.

What characteristics of the flooding are we interested in? From the point of view of the network load it is the evolution of the traffic density, when a wave of flooding goes over some bounded region S'. In particular, we consider the period of time necessary for the traffic to decrease to pre-flooding level. One of the major assumptions in this section is that on average

the flooding wave, after several initial steps, moves over the network uniformly. More precisely a gain of the flooding wave in number of new nodes acquiring the message is constant on average. We denote this gain by X. Intuitively, a flooding spreads, in average, in all directions at the same pace. We have performed simulations to see what happens to a network when a flooding wave goes across a bounded area.

3.3.2 Simulation Conditions

To illustrate principal ideas and to simplify analysis, we make use of a simple randomised medium access protocol of Aloha type, where the nodes transmit a message

- when they have something to transmit
- with probability p at any time slot

this probability does not depend on the time slot and fixed for the whole network.

Let us describe the parameters we have chosen for the simulation sample. In order to simplify our simulations we consider a model where all nodes try to transmit the message with a constant probability $p = 0.025$. This algorithm plays the role of the coordination function between the nodes.When a node succeeds in transmitting a message it does not repeat it any more. The transmit power of all nodes is 1 and the signal attenuation is $r^{-\alpha}$, where $\alpha = 2.5$. The network is contained in a square of size $200m$x$200m$. The total number of nodes is 1000 ($\nu = 0.025$). We are interested in an average propagation of the flooding wave. Starting in a single point such wave, on average, will be spreading as a growing disk. In some small region, an arc of a large circle can be approximated by a straight line. Therefore, we start our simulations with 20 nodes on the left edge, ready to transmit a flooding message.

We also impose on the network a homogeneous background noise of power $0.0001dB$. As we can see from (2.1), this level of noise allows the direct communication at distance $d \approx 15.8m$ or about 20 neighbours for the SIR threshold constant $K = 10$. In other terms we can find the traffic density that would allow such radius of communication from (2.25): $\lambda_{noise} = r_0^2/d^2 \approx 0.0000576$. This corresponds to 2.3 messages in average in time slot over all network of 1000 nodes. This provides a reasonable value of the threshold λ_{noise} that we can set as the parameter for the flooding. We will be interested in the time until the traffic density of the flooding falls under this level. The results of the simulations are presented in Fig. 3.4 and 3.5.

Let us discuss Fig. 3.4 in more details. By our assumption of linearity of the flooding progression this curve should be composed of two straight lines: one corresponds to the event when the flooding is gaining new nodes and another one to the event when there are no new nodes left. This is not the case in this simulation due to the border effects. Nevertheless, approximating the straight part of this curve in the interval [20,80] we can calculate that the flooding is moving with the average gain $X \approx 8.08$ nodes per time slot. Let us now analyse the diagram on Fig. 3.5

On Fig. 3.5, we present the traffic density during the simulation. The horizontal line on the graph corresponds to the background noise level λ_{noise}. On the graph we can observe how flooding crosses a bounded region. The second point, where the curve intersects the background noise level, is the moment when the flooding traffic density has decreased sufficiently. For this simulation it takes on average 164 steps. This value, of course, depends on the probability p. When this probability is higher, the peak of the traffic density is also higher, but it takes less time for the flooding to pass. Nevertheless, there is a bound after which some nodes of the network can never receive the message. As we have seen in the example earlier, when $p = 1$, the

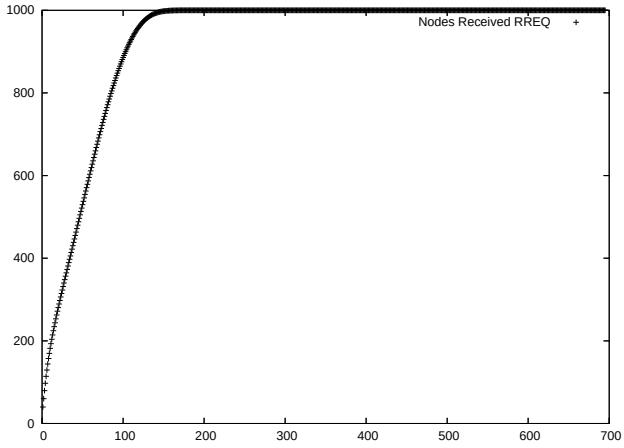


Figure 3.4: The number of nodes having received the message as a function of time

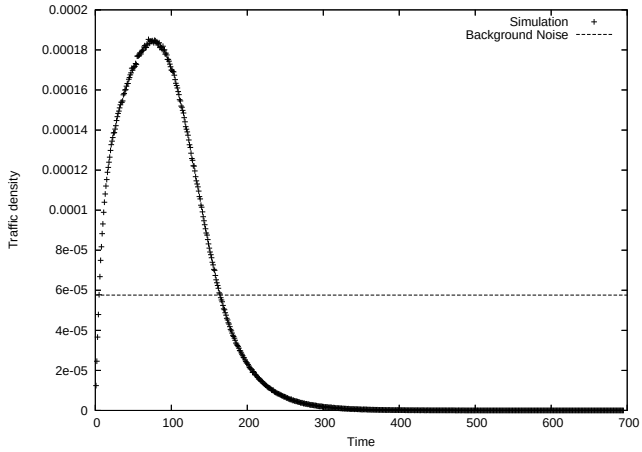


Figure 3.5: Traffic density evolution (number of packets per unit square) with time

flooding will stop almost instantly as there would be too much noise created by the neighbouring nodes for anyone to hear them. Therefore, the process of increasing p up to the bound at which flooding is yet possible can be seen as the way to optimise the network.

3.3.3 Analytical Explanation of the simulation

The curve on Fig. 3.5 can be explained as follows. First of all, it can be decomposed into two parts: ascending and descending. The ascending part corresponds to the progress of the wave through the network, whereas the descending one reflects the event that there is no new nodes receiving the message and the number of nodes wishing to retransmit decreases at rate p. Also it is worth noting here that under the conditions of this model the average number of nodes awaiting to transmit the message is proportional to the average number of transmitting nodes (with coefficient $1/p$).

During the ascending phase, if we denote by x_n the average number of nodes awaiting to transmit the message, there is following recursive expression for x_{n+1}: $x_{n+1} = x_n(1-p) + X$, where X is the average number of new nodes gained by the flooding at an arbitrary step (here

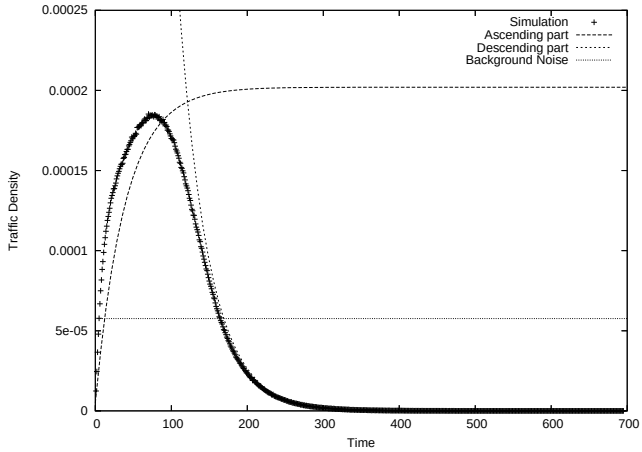


Figure 3.6: Traffic density estimated by the semi-analytical curves from (3.21)

$X = 8.08$). We take $x_1 = X$ because of the linearity assumption. Therefore

$$x_{n+1} = \frac{1-(1-p)^{n+1}}{p} X \tag{3.20}$$

Let us denote by n_0 the moment when all N nodes received the message. As some of them have already re-transmitted it, we have $x_{n_0} < N$. At the moment, when the sequence x_n starts to decrease: $x_{n+1} = x_n(1-p)$ (for $n > n_0$). Therefore the sequence $\{x_n\}$ can be written as

$$x_0 = X \tag{3.21}$$

$$x_{n+1} = \frac{1-(1-p)^{n+1}}{p} X\,, \quad 0 < n \le n_0 \tag{3.22}$$

$$x_{n+1} = x_{n_0}(1-p)^{n-n_0}\,, \quad n > n_0 \tag{3.23}$$

The value of n_0 can be derived from the average gain of the flooding wave that we have denoted by X: $n_0 = \frac{\nu S}{X}$. We compare these two curves of x_n with the simulation results on Fig. 3.6.

To find the point when the traffic level decreases to λ_{noise} we need to solve the following equation:

$$\lambda_{noise} = x_{n_0}(1-p)^{n-n_0}\frac{p}{S}$$

We put the coefficient p/S here to scale the number of nodes, wishing to transmit, to the traffic density (number of transmitting nodes per unit area). We obtain

$$n = n_0 + \frac{\log(\lambda_{noise} S) - \log(x_{n_0} p)}{\log(1-p)} \tag{3.24}$$

Therefore, using the average gain of the flooding wave X we can calculate the value of x_{n_0}:

$$x_{n_0} = \frac{1-(1-p)^{\frac{\nu S}{X}+1}}{p} X \tag{3.25}$$

Substituting (3.25) into (3.24), we obtain

$$n = n_0 + \frac{\log(\lambda_{noise} S) - \log\left(\left(1-(1-p)^{\frac{\nu S}{X}+1}\right) X\right)}{\log(1-p)} \tag{3.26}$$

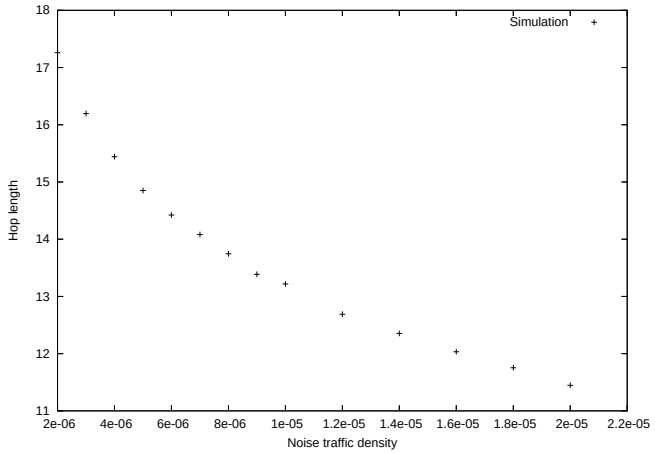


Figure 3.7: Hop length dependence on the background noise

Using (3.26) we can determine the average interval of time necessary for flooding to pass. We suppose that during this interval it is difficult for all other messages to be successfully transmitted as the interference coming from flooding is too high. A general approach for transport layer protocols to decide when a message should be dropped is to compare its waiting time with a maximal threshold parameter (like, for example, NEXT_HOP_WAIT in RFC 3561 of AODV). When the time is expired, the protocols stop re-transmission attempts and the message is considered failed. Therefore, a longer period of flooding wave increases the probability that messages will wait too long and they will eventually be dropped. Although, we do not have the exact distribution for x_{n_0}, only the average value, but in Chapter 5 we will try to estimate message failure probability using the general ideas outlined in this section.

3.3.4 Hop Length Estimation

This section is more related to Chapter 5. For purposes of the comparison of ad hoc network protocols we trace a simple curve approximating the hop length in the settings of this section. The goal is to find the dependence of average hop length discovered via flooding process on the background noise inside the system. This noise exclude the interference coming from the flooding messages and, therefore, it is related to other communication messages transmitted across the network.

The simulations, run with the same parameters as in Section 3.3.2, show that the average value of the hop length can be found between the transmission range corresponding to the background noise and the transmission range, which corresponds the traffic density of flooding. h_{AODV} depends on the node density ν, background noise λ_{noise} and probability p, which is one of the characteristics of the model. As in this model ν and p are fixed, the dependence we need to evaluate is h_{AODV} a a function of λ_{noise}. We approximate the simulation results on Fig. 3.7 with the following expression:

$$h_{AODV}(\lambda_{noise}) = \frac{8.2}{\lambda_{noise}^{0.1}} - 11.7 \tag{3.27}$$

In fact, this curve fails to fit into the expression $h = O(\lambda_{noise}^{-1/2})$, as we hoped it would. This can be explained by a border effect. In this zone, the traffic density of different subregions is

not the same, while λ_{noise} is an averaging measurement constant for all subregions. Hence the logic $r = O(1/\sqrt{\lambda})$ does not work and we are forced to use the results of these simulations.

The study of flooding in this section can be seen as a preparation to the Chapter 5, where we compare two routing ad hoc network protocols, AODV and OLSR. One can find flooding mechanism in both protocols, but the former use it in a classical non-optimised way and all nodes in the network participate. The latter, on the contrary, defines a subset of nodes, which will provide relaying of diffused message. Therefore, in order to obtain such characteristics of AODV as hop length and the flooding wave duration, we will use the results from this section. In fact, in Chapter 4 we study the question of the path length, but the link availability model used is the unit disk model. As we will see it, even in the case of this *simpler* model, the problem stays difficult.

3.4 Conclusions

This chapter consists of three main subjects. The first topic is a presentation of medium access MAC layer protocols and their features. We show different existing multiple access schemes based on time and wavelength division, and also on advanced coding and multiplexing techniques. Then, we consider collision resolution part of MAC protocols, presenting some known analytical results concerning their performance. We also extend these models to include more global view of a wireless network, where the number of neighbors or competitors for the shared medium depends on the traffic level.

The last subject is to present some simulation results of a simplified MAC protocol. Its goal is to illustrate how flooding can influence the network performance. As flooding is a common mechanism for many ad hoc network protocols, the data we retrieve from these simulation will be later used in a framework for comparison of ad hoc network protocols presented in Chapter 5.

Chapter 4

Path Length Studied in a Unit Disk Model

In this chapter we study the lengths of paths obtained via classical flooding in the unit disk model. The following illustration explains the approach taken in this chapter: consider the moment when a destination node receives a request for a route from a source node. For many routing protocols, the destination node is obliged to reply immediately back along the newly found path. Therefore, the chosen route is the first found path. While this path was being discovered, it was the first node re-transmitting the request, that will enter into the resulting path. It is not necessary an optimal solution. This node can be seen as a random winner of the competition over the shared media. In this section we study how its position is distributed within a transmission region. This study can be seen as a generalisation of [32]

4.1 Introduction

A Unit Disk Model is one of the simplest ways to describe a wireless multihop network. It represents the nodes as points and the transmission region of every node is a disk of a fixed radius. It is based on the assumption that two nodes are directly connected if the distance between them is less than some threshold bound r. Working with this model we can scale all distances so that $r = 1$ and therefore every node of the network has unit disk region where its transmissions will be successfully received.

Such assumption, of course, is a simplification of the concept of the link availability between two nodes in the real world. Without additional restrictions it does not take into account the interference of the nodes. But on other hand the multihop nature of ad hoc networks is well taken in the consideration. As the mobile devices tend to be limited in power consumption, their radio range stays bounded and the threshold r is the parameter corresponding to this limitations. In general the power restrictions of the nodes are not all the same. To take into account this case there is a non-homogeneous version of the unit disk model where nodes $1, 2, 3, \ldots$ have different thresholds $r_1, r_2, r_3, \ldots$. The further generalisation can include the liberation of the covering region shapes, to reflect different landscapes where the signal of a transmission is spreading not equally in all directions.

As it is a model of link availability, many other parameters of the real life network can be added to it in order to better reflect different scenarios that are modeled. The nodes' positions can be either set fixed to form some graph, or be described by a random process. The node mobility can also be included in this framework taking the coordinates of the nodes

as functions of time. But at any moment of time the ad hoc network will be described by a graph where the edges are determined by the distances between the nodes. Therefore if a node transmits a message at some moment, all its neighbors on this graph are supposed to receive this message correctly. The interference can be partially imposed here by restriction on two neighbors transmitting at the same time. Such restrictions are more of a protocol level of the model, as it is a rule of the behaviour for the nodes. In the further sections we describe the analytical results known at this moment about the unit disk model in the stationary case (without mobility directly included) and then we show the results on the length of the paths in ad hoc networks which are using the flooding as the path discovery mechanisms.

4.2 Path Length Studies in Wireless Networks

The average path length is a property of the network which depends on the corresponding routing protocol, which determines the routes using available information about the network topology. Those protocols, which reduce routing overhead by limiting the topology learning mechanisms, often choose a simple algorithm for path calculation, like, for example, first found path approach. The protocols, which possess a necessary information about node links, try to optimise the routes, finding a shortest (in the number of hops).

Analytical expressions for path length are difficult to find. In a unit disk model a lower bound estimation for a shortest path length can be estimated as a fraction of euclidean distance and the transmission range. But this approach is not taking into account the positions of the nodes.

As another approach in a shortest path estimation we can consider the paths on Delaunay graph. In a greatly simplified manner we can say that Delaunay graph of a set of points $\{x_i\}$ is composed in plane, by connecting every point to its closest neighbors. In this case the path between two arbitrary points x_i and x_j is chosen to approximate the corresponding segment $[x_i, x_j]$. Let us provide more formal definitions and cite the corresponding result from [90], where these paths received a thorough examination.

Voronoi diagrams paths

The model which is described here is different from the unit disk model. Although the vertex set is the same and it is represented by P.p.p., the links between the nodes are defined differently.

Definition 4.1. The Voronoi diagram *with the nuclei set* $\{x_i\}$, $x_i \in \mathbb{R}^d$, *for a given dimension* d *is a family of subsets of* $\mathbb{R}^d$ *defined by*

$$C(x_i) = \{y \in \mathbb{R}^d \, : \, \|y - x_i\| \leq \|y - x_j\|, i \neq j\} \tag{4.1}$$

The Delaunay graph is a dual object to the Voronoi diagram.

Definition 4.2. The Delaunay graph *constructed on the vertex set* $\{x_i\}$ *is the graph connecting all the pairs of points* (x_i, x_j) *such that the Voronoi cells* $C(x_i)$ *and* $C(x_j)$ *share a* $(d-1)$*-facet.*

In the model which is considered here $d = 2$ and the nuclei set $\{x_i\}$ is distributed accordingly to P.p.p. of a given intensity ν.

Let s and t be two points in $\mathbb{R}^2$. Let $C(x_{i_1}), C(x_{i_2}), \ldots, C(x_{i_j})$ be the sequence of the cells successively crossed by the segment $[s, t]$. The sequence of the nuclei $x_{i_1}, x_{i_2}, \ldots, x_{i_j}$ defines a *Markov path* l on the Delaunay graph from x_{i_1} to x_{i_j}. The length of this path is the sum of the distances between the points : $|l| = \sum_{k=2}^{j} \|x_{k-1}, x_k\|$.

In [90] it was shown that the length of average Markov path l between two given points s and t in $\mathbb{R}^2$ is expressed as

$$\mathbb{E}|l| = \frac{4}{\pi}\|s - t\| \tag{4.2}$$

4.3 Path length in the flooding based protocols

4.3.1 Introduction

In this section we will study one of the principle characteristics of flooding path discovery mechanism - the path length in number of hops. We will propose a simplified theoretical model to estimate the average path length for the routing protocols which are using flooding during their path discovery phase. The starting point for this study is the article of Clausen, Jacquet and Viennot [32] where a one-dimensional flooding model is evaluated. Here we study extensions to a more complex and realistic model in two dimensions.

Flooding is an intuitive way to make a message reach all the users of a network by making every station repeat the message once it receives it from a neighbor. The broadcast capabilities of ad hoc networks inherited from a wireless layer help to implement this scheme. Many protocols are using flooding as the main tool in their initial path discovery phase. In its general form it can be found in AODV [80] , DSR [56] and other protocols. (Protocols, such as OLSR [31] and TBRPF [83], with apparent mechanisms of route optimisation, are not in the scope of this analysis.) But there are also a number of relatively new protocols that include randomness in their flooding schemes, for example Gossip based protocols [46]. These protocols are much closer to the model described here, especially if we consider its non-uniform case. The study of the general flooding scheme in terms of redundancy, contention and collisions can be found in [72] and its comparison to the variety of optimisation techniques in [92].

The one-dimensional case

The one-dimensional model analysed in [32] is the following. All mobile stations or users are displayed on a straight line. A source node, located at coordinate 0, is looking for a route to some destination node N. Thus it starts transmitting a route request to its neighbors, which are trying to relay this request further until the destination node N is reached.

The set of points corresponding to the possible station locations is dense in $[0, \infty)$. The covering region is the geometrical area where all "neighbors" of the station are located. In this model the covering region $\mathcal{C}(x)$ for an emitting point x is $[x, x + 1]$. Thus the optimal routing distance for the point x is equal to $\lfloor x \rfloor + 1$: it is the case if we choose the sequence of intermediate nodes at the points $1, 2, \ldots, \lfloor x \rfloor$.

Flooding is modeled by a random process $X_0, X_1, \ldots, X_n, \ldots$ where X_n is chosen uniformly at random from $\mathcal{C}(X_{n-1}) = [X_{n-1}, X_{n-1} + 1]$. This random choice is applied in order to model the interference between the stations within the same zone.

In order to find the path from 0 to N we actually select the subsequence of $\{X_n\}$, such that there is no i with $X_{i+2} - X_i \leq 1$. Starting with $i = 2$ we do not take into account a point X_{i+1} if the next point X_{i+2} is also covered by the transmission of X_i. Every X_n has also an attributed *hop counter* mark M_n. Many routing protocols include such a counter field in their route request messages. When a station receives the request for the first time it increases this counter by 1 before retransmitting the message further.

In this model the sequence M_n based on X_n is defined through the following rule: $M_n = M_{\alpha^1(n)} + 1$, for $n \geq 1$, where $\alpha^1(n) = \min\{k < n : X_n - X_k \leq 1\}$ (the sequence $\{X_n\}$

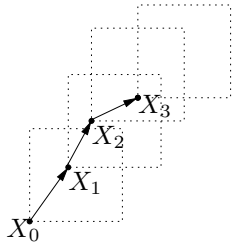


Figure 4.1: Route search model

is non-decreasing). So the index $\alpha^1(n)$ corresponds to the station from which the request message has been received by X_n for the first time. Equivalently we could define $\alpha^1(n)$ as $\alpha^1(n) = argmin\{M_k : X_n - X_k \leq 1\}$.

The analysis performed in [32] shows that on the average the route length for the destination at distance N is equal to $4/3N$ as $N \to \infty$. That is 1/3 of the relay nodes can be excluded from an average $2N$-hop path from 0 to N.

4.3.2 Model Settings and Results

We consider the ad hoc network in the unit disk model framework. The transmission range is fixed and we will go even further by simplifying the transmission region shape.

Let's first describe what we take as flooding in the case of the planar motion. The following scheme explains the search for a new route in an ad hoc network :

- The source s broadcasts to its neighbors within a covering area $\mathcal{C}(s)$ a request for the route to some destination node d. The request message contains a hop counter, set to 1 at the beginning.
- When a non-destination node receives a request it checks whether it sees it for the first time: if not it ignores the message. Otherwise it increases the message hop counter by 1 and broadcasts this updated request further.
- When the destination d receives the request message, it sends a confirmation packet back along the discovery path. We assume that the stations possess all necessary information to implement this backtracking: the actual path is established during this phase. Thus d chooses the first route discovered and the length of the route is equal to the value of the hop counter of the message received by d.

At any time $k = 0, 1, 2, \ldots$ there is only one node X_k in this model which transmits the message to all its neighbors inside the covering area $\mathcal{C}(X_k)$. The source node $X_0 = (0, 0)$ is located at the origin of coordinates in $\mathbb{R}^2$. We also need to define what the form of the transmission region $\mathcal{C}(\cdot)$ is and to define the rule how to choose a new broadcasting point in this region. We make the assumption that nodes in the network form a dense set in the plane as it was in the one-dimensional model described in the previous section.

We define $\mathcal{C}(X)$ as a square $[x(X), x(X)+1] \times [y(X), y(X)+1]$, where $x(X)$ and $y(X)$ are the corresponding x- and y-coordinates. The square as generic shape simplifies the mathematical evaluation; considering only one quadrant models oriented diffusion with a linear growth, since only this quadrant will be useful for the route. At time n we choose X_n uniformly at random from the area $\mathcal{C}(X_{n-1})$ covered by the previous point X_{n-1} (see Fig.4.1).

Taking such transmission region shape we guarantee that the path obtained via such algorithm will be loop free. It allows us to work with Markov chains as a loop free property limits the hop number dependence of a node n to two precedent nodes $n-1, n-2$ and not the whole history before $n-1, n-2, n-3, \ldots$. As we discuss it later, we could have taken any shape that fits one quadrant, but the square simplifies a lot the calculations.

Let us now introduce the variable M_n which contains the value of the hop counter at point X_n and hence it is the distance between 0 and n (in hops). According to the rule stated previously $M_0 = 0$ and M_n is defined by:

$$M_n = min\{M_k | X_n \in \mathcal{C}(X_k)\} + 1 \tag{4.3}$$

Our restrictions on the model in the choice of the covering area imply that M_n is a non-decreasing sequence (loop-free property of the path). Our objective now is to find the probability P_n,

$$P_n = \mathbb{P}[M_n = M_{n-1} + 1]$$

for any given n. It will allow us to estimate the average value of the hop counter after n steps and therefore the average distance to the point X_n.

The induced Markov chain

The form of transmission region $\mathcal{C}$ provides us with the nice property that for all $i < j \leq n$, $\mathcal{C}(X_i) \cap \mathcal{C}(X_n) \subset \mathcal{C}(X_j) \cap \mathcal{C}(X_n)$. Thus the probability P_{n+1} depends only on the two points X_n and $X_{\alpha(n)}$, where $\alpha(n) = max\{k | M_k = M_n - 1\}$. If X_{n+1} falls inside $\mathcal{C}(X_{\alpha(n)})$ then $M_{n+1} = M_n$ and otherwise it increases. Thus, we have $P_{n+1} = 1 - |\mathcal{C}(X_{\alpha(n)}) \cap \mathcal{C}(X_n)|$.

Furthermore, we can free the sequence $\{M_n\}$ from the dependence on the coordinates of $\{X_n\}$. Let us denote by Y_n the random variable that corresponds to the position of the right-upper corner of the square $\mathcal{C}(X_{\alpha(n)})$ expressed in the coordinates centered at X_n (see Fig.4.2). Let us denote by X_n^Δ a point chosen uniformly at random in the unit square $\mathcal{C}(0)$: X_{n+1} can be represented as $X_{n+1} = X_n + X_n^\Delta$.

Thus we can now define the sequence $\{Y_n\}$ in terms of the X's and X^Δ's in the following way

$$Y_{n+1} = \psi(Y_n, X_n^\Delta) \tag{4.4}$$

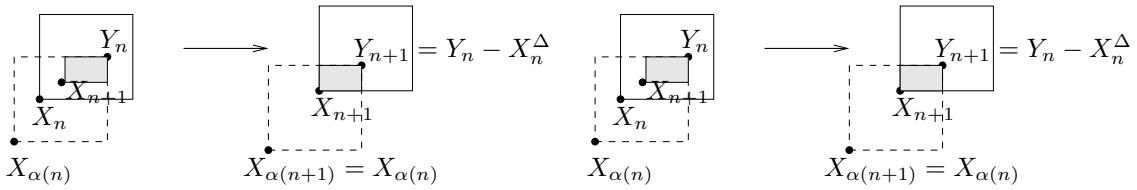
where $\psi : \mathcal{C}(0) \times \mathcal{C}(0) \to \mathcal{C}(0)$ is the function defined by

$$\psi(Y, X) = \begin{cases} (x(Y) - x(X), y(Y) - y(X)), \\ \qquad \text{if } x(X) \leq x(Y) \text{ and } y(X) \leq y(Y) \\ (1 - x(X), 1 - y(X)), \qquad \text{otherwise} \end{cases} \tag{4.5}$$

where $x(\cdot)$ and $y(\cdot)$ denote the corresponding $x-$ and $y-$ coordinates.

In the first case, X_{n+1} takes the place of X_n as a relay in $\mathcal{C}(X_{\alpha(n)})$ and $\alpha(n+1) = \alpha(n)$; in the second case, a new square is created with origin X_{n+1} and the new point Y_{n+1} is such that $\alpha(n+1) = n$. The two situations are shown in Fig. 4.2.

From (4.4) we can consider Y_n as a general Markov chain, satisfying a stochastic difference equation, with a continuous state space and discrete time.



(a) Hop counter does not increase

(b) Hop counter increases by 1

Figure 4.2: Evolution of sequence $\{Y_n\}$

The analytical solution

To prove that there is a stationary distribution of Y_n and find it we use the stochastic kernel of this Markov process. We suppose that Y_n has a density function which we denote by f_n. Because of the uniformity of X_n^Δ inside $\mathcal{C}(0)$ we have the following recurrence equation for f_n:

$$\begin{aligned} f_n(x,y) &= \iint_{G(x,y)} f_{n-1}(u,v)dudv \\ &\quad + \iint_{F(x,y)} f_{n-1}(u,v)dudv \end{aligned} \tag{4.6}$$

where $F(x,y)$ is $\mathcal{C}(0) \setminus [1-x,1] \times [1-y,1]$ and $G(x,y)$ is the rectangle $[x,1] \times [y,1]$ (Fig. 4.3). The region G corresponds to a non-increasing hop number and F to an increasing one. The region G is the set of all possible places for the point Y_n to be at the North-East of a random point X_{n+1} drawn randomly in $\mathcal{C}(X_n)$. Similarly, F is defined as the complement of this set, as illustrated in Fig. 4.3. Note that on the left image $Y_n = Y_{\alpha(n)}$ and (x,y) are the coordinates of Y_n relatively to X_n.

Let us observe the following facts for f_n.

1. If f_n is a density function we can rewrite (4.6) as

$$\begin{aligned} f_{n+1}(x,y) &= \int_x^1 \int_y^1 f_n(u,v)dudv + 1 \\ &\quad - \int_{1-x}^1 \int_{1-y}^1 f_n(u,v)dudv. \end{aligned} \tag{4.7}$$

2. If f_0 is a density function then f_n also is a density function :

$$\int_0^1 \int_0^1 f_n(x,y)dxdy = 1. \tag{4.8}$$

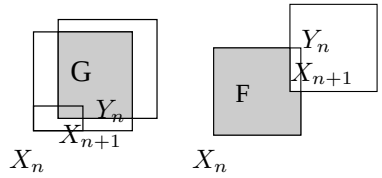


Figure 4.3: Integration areas for the recurrence equation

3. The substitutions $x \mapsto 1-x$ and $y \mapsto 1-y$ lead to

$$f_n(x,y) + f_n(1-x,1-y) = 2 \tag{4.9}$$

The recurrence takes the form

$$\begin{aligned} f_{n+1}(x,y) &= 1 - 2xy + \int_x^1 \int_y^1 f_n(u,v)dvdu \\ &+ \int_0^x \int_0^y f_n(u,v)dvdu \end{aligned} \tag{4.10}$$

Theorem 4.1. *The density function of the stationary distribution f^* exists and $f_n \to f^*$, $n \to \infty$. Also f^* is differentiable.*

Proof. As we prove in Appendix A functions $F_n(x,y) = \int_0^x \int_0^y f_n$ converge to F^*. As we can see it from (4.10), these functions are differentiable and bounded (it can be proved by induction), therefore, the limit is also differentiable. We denote $f^* = \frac{\partial^2 F}{\partial x \partial y}$.

The distribution functions converge in variation by Proposition A.1 , therefore, their densities converge to the density of the limit ([23], Th. 7, appendix 3):

$$\int |f_n - f^*| = \int |d(P_n - P)| \to 0$$

Therefore f^* is the limit of f. f^* is determined *almost surely.* □

As $f^*(x,y)$ exists it should satisfy

$$\begin{aligned} f^*(x,y) &= \int_x^1 \int_y^1 f^*(u,v)dudv + 1 \\ &- \int_{1-x}^1 \int_{1-y}^1 f^*(u,v)dudv \end{aligned} \tag{4.11}$$

After differentiation we obtain the partial differential equation

$$f^*_{xy} = -2 + 2f^* \tag{4.12}$$

This equation can be solved analytically using the Riemann method for hyperbolic partial differential equations (see [94]). The general form of the solution can be expressed as

$$\begin{aligned} f^*(x,y) &= \int_0^y I_0(2\sqrt{2x(y-\eta)})\phi(\eta)d\eta \\ &+ \int_0^x I_0(2\sqrt{2y(x-\zeta)})\theta(\zeta)d\zeta + 1 \end{aligned} \tag{4.13}$$

where I_0 is the modified Bessel function of order 0, ϕ and θ being the boundary conditions :

$$\begin{aligned} f^*_x &= \phi(x), \quad \text{for } y = 0 \\ f^*_y &= \theta(x), \quad \text{for } x = 0 \end{aligned}$$

In our case $f^*(x,y)$ is a symmetric function, therefore, $\phi(x) = \theta(x)$ and, from (4.9), we obtain $f(x,1-x) = 1$. Function ϕ is, then, defined implicitly as a solution of the following integral equation:

$$\int_0^{1-x} I_0\left(2\sqrt{2x(1-x-\eta)}\right)\phi(\eta)d\eta + \int_0^x I_0\left(2\sqrt{2(1-x)(1-\eta)}\right)\phi(\eta)d\eta = 1 \tag{4.14}$$

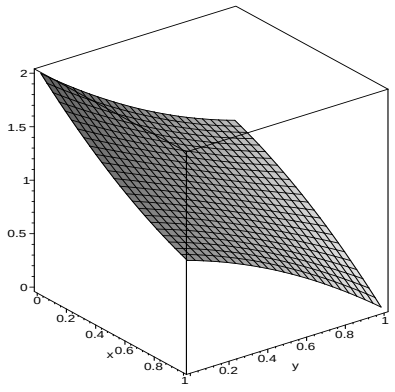


Figure 4.4: The density function $f^*(x, y)$

Here, we find a numerical estimation of the solution using equation (4.7) iteratively; after a several steps the estimation of the density function f_n is

$$f^*(x, y) \approx 2 - 1.428x - 1.428y + 0.417x^2 + 2xy + 0.417y^2 \tag{4.15}$$

Then the steady state probability of increasing the hop number P_* can be estimated numerically as

$$P_* = 1 - \int_0^1 \int_0^1 xyf^*(x, y)dxdy \approx 0.82176 \tag{4.16}$$

Thus knowing a number N of the points participating in the flooding defined by $\{X_n\}$, we can estimate the hop distance as P_*N:

Theorem 4.2. *The steady state probability of hop counter increase can be found as*

$$P_* = 1 - \int_0^1 \int_0^1 xyf^*(x, y)dxdy$$

Generalisation to non-uniform densities

In previous sections we have been considering the uniform distribution for the choice of the next broadcasting node. It was a way to model the interference competition for the neighboring nodes. But in a number of flooding algorithms the choice of the nodes that are going to relay the message request is probabilistic and not necessary uniform. For instance, the nodes closer to the borders of the covering area can have higher probability to broadcast next ([26, 84, 46]). We can adapt our model to reflect these situations, by replacing the uniform distribution for $\{X_n^\Delta\}$ with some other having a density function φ.

We recall the definition of Y_n given by equation (4.4)

$$Y_n = \psi(Y_{n-1}, X_n^\Delta) \tag{4.17}$$

where X_n^Δ has a distribution function with a density φ on $\mathcal{C}(0)$. The function φ should satisfy the condition that the chain Y_n converges to a stationary distribution represented by a density function $f_\mathcal{N}^*$. If we want to keep the proof of Prop. A.1 we need to impose the following restriction on φ: there should exist $\xi : \varphi(D_\xi(0) \cap \mathcal{C}(0)) > 0$, where $D_\xi(0)$ is a disk of radius ξ

and with a centre at 0. In this case we can take the probability measure μ as $\mu(D_\xi(0) \cap \mathcal{C}(0)) = 1$ and the proof will follow.

The recurrence relation (eq. 4.18) takes into account the change of the measure for X_n^Δ on $\mathcal{C}$

$$\begin{aligned} f^*_{\mathcal{N}}(x,y) &= \int_x^1 \int_y^1 f^*_{\mathcal{N}}(u,v)\varphi(u,v)dudv + 1 \\ &\quad - \int_{1-x}^1 \int_{1-y}^1 f^*_{\mathcal{N}}(u,v)\varphi(u,v)dudv \end{aligned} \tag{4.18}$$

In this case P_* can also be represented as

$$P_* = 1 - \int_0^1 \int_0^1 \Phi(x,y) f^*_{\mathcal{N}}(x,y)dxdy \tag{4.19}$$

where $\Phi(x,y) = \mathbb{P}\left[(X_n^\Delta)_x < x, (X_n^\Delta)_y < y\right]$.

For example, in the case of the distribution defined by the density function $\varphi = \frac{x+y}{2}$ we obtain, by numerical computation, that $P_* \approx 0.9342$.

4.3.3 Conclusions and Discussion

The model we have introduced and analysed has the following interpretation : consider a highly dense community of users of an ad hoc network. Some node starts searching for an interlocutor and it broadcasts around a request for a such connection. Other stations try to repeat this request as soon as they receive it. At some time the destination node is reached. Then the backtracking process is initiated which corresponds to a selection from the stations participating in the planar flooding of one branch $X_N, X_{N-1}, \ldots, X_0$ such that X_N is the destination, X_0 is the source and X_k is located in the covering area $\mathcal{C}(X_{k-1})$ of X_{k-1} for all $0 < k \leq N$. Then this path is shortened by eliminating points X_k such that $X_{k-1} \in \mathcal{C}(X_{k+1})$, since even without these points the connection between X_0 to X_N still holds.

Provided that one can estimate N and the probability P_* of the elimination of the "useless" nodes from the established connection then the path length can be estimated as P_*N. This estimations are closely related to the possible covering areas.

The oriented covering area, chosen in this model, reflects its biggest frontal gain during a planar flooding expansion toward the destination.

We compared the path length predicted by this model with the results of the simulation of the generalised flooding $\{Z_n\}$ with the oriented square covering area. The flooding was implemented choosing points Z_n uniformly at random from $\cup_{0 \leq i \leq n-1} \mathcal{C}(Z_i)$ - entire region previously covered. We measured the number of hosp of the connections obtained via $\{Z_n\}$ to the points located on the diagonal $x = y$. The results of this comparison are displayed in Fig.4.5.

The limitation of the covering area shape to the unit square might seem to be too restrictive, but it gives an easy way for computing and estimating the optimisation factor.

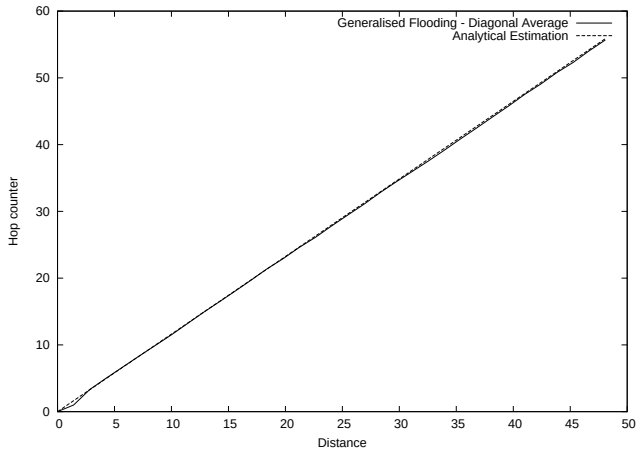


Figure 4.5: Average path length comparison

Chapter 5

Comparing Ad Hoc Network Protocols

This chapter contains the description of the framework for comparison of ad hoc network protocols. The result is based on two models:

1. *modeling the corresponding protocol by calculating the overall traffic composed of all control and information-related messages. This approach was taken in [54] to estimate the overhead of the OLSR protocol.*

2. *link availability modeling, where the transmission range depends on the traffic density, as it was discussed in the first chapter.*

This method is similar to the one in [9]. In short, it can be illustrated in the following way: the overall traffic generated by the protocol depends on the control and information messages issued by the nodes of the network. The number of control messages can depend on the interference level and, therefore, on the transmission range. For each protocol, the number of times these messages are relayed is also a function of the transmission range. The transmission range in its turn depends on the traffic density (in SIR based link mode), which, on the average, can be estimated by the number of the control and information messages together transmitted at an arbitrary time slot. Therefore, the traffic density can be expressed as a function of itself with the network parameters fixed.

The goal of this chapter is to find the overall traffic density as a function of the density of the nodes and of the network useful load.

5.1 Ad Hoc Network Protocols. General Overview.

Since the end of 1990s, when a multihop approach in the wireless networks has been growing in popularity, a number of protocols for mobile ad hoc networks were proposed. The protocols are mainly divided into two groups *reactive* and *pro-active*. The former try to find routes only when a corresponding demand appears, thus limiting the routing information the nodes need to store. In general, when a need for a new path arises, they flood the network with route request messages. Whereas, the latter tend to fill the routing tables with the necessary information before any potential demand comes up. In general pro-active routing is organised via some mechanism of a regular exchange of the routing information.

The common opinion on the two types of protocols is that the reactive protocols provide a better performance for highly mobile scenarios, where the paths are changing really rapidly

and where the traffic load is not too high. They suffer from the increase of the traffic load and of the number of nodes. The pro-active protocols, on the contrary, reduce delays for new routes and they react better to the increasing number of nodes, but they suffer from too rapid changes of topology. There exists also a variety of other ad hoc network protocols, that includes hierarchical, geographical and hybrid protocols. Some of these solutions are based on a way of learning the geographical positions of the participating nodes, for example, using the global positioning system (GPS). Some other mix different routing strategies to optimise the performance.

In this chapter we will study two protocols representing the reactive and the pro-active routing solutions, ad-hoc on-demand distance vector routing (AODV) [80] and optimised links state routing (OLSR) [31]. They are chosen because they include most of the features common to their respective groups. Also these two protocols are chosen by IETF MANET group as candidates for the future standards in mobile ad hoc networks.

Before presenting the protocols let us mention the general mechanisms they are based upon: *the distance vector routing* and *the link state routing*. These algorithms are largely used in wired networks (see, for example, [59]).

Distance vector routing is based on Bellman-Ford shortest path search algorithm. Nodes keep a routing table of records of the form <*destination, cost, next hop*>. The neighboring nodes will have cost 1. Assuming that the identity of other nodes is known, the routing cost for them is set to ∞. Every node periodically broadcasts its routing table to its neighbors. When arbitrary node receives such information from a neighbor, it updates its routing table checking, whether the current cost can be decreased by routing via this neighbor. If it is possible, the node updates its routing table to reflect this change.

Link state routing proposes to the nodes to keep a copy of the network map and update it regularly. The routing tables then contain the information on the status of the links between all nodes. The nodes diffuse a set of their link-state information throughout the whole network. Such packets would contain the identity of the issuing node, its neighbors and the cost of routing to them. After the end of the initial network discovery routine, all nodes should have the same network image in their link-state tables. Then, if a need arises, every node can find the entire shortest path in the table using Dijkstra algorithm.

Both routing methods have their strong and weak sides. But the wireless multihop environment changes the goals and, thus, the evaluation of a routing paradigm. Therefore, it is better to look at particular protocols and try to compare them. In what concerns AODV vs. OLSR, there is a number of works that have done the performance comparison, for example, see [34, 48].

5.2 AODV in Details

The protocol AODV (Ad Hoc On Demand Distance Vector) was presented by C.E. Perkins and E. Belding-Royer in their work [80]. It was developed on the base of the protocol named DSDV (Destination Sequenced Distance Vector) [79], which was an implementation of distance vector routing method applied to ad hoc networks and which was a pro-active protocol. At the present moment, the development of AODV is continued with protocol named DYMO [27], which includes some further optimisations compared to AODV. Let us describe the different phases of AODV functioning.

Path Discovery

AODV is reactive (on demand) protocol based on distance vector routing. Therefore, when the need arises and if a route to a destination node is unknown, the protocol starts the route discovery process. It broadcasts the Route Request (RREQ) message throughout the network. Every node, which does not know the route to the destination, should repeat the RREQ. When a node, that has the route in its routing table or the destination itself receives the RREQ, the reply message (RREP) is sent back backtracking the path to the source node. Thus RREP is delivered via unicast, contrary to the broadcast of the RREQ. The RREQ contains route sequence number in order to avoid getting back the information about the stale paths.

Processing RREPs, the source and all intermediate nodes cache the information about the route to the destination in their respective routing tables. These records contain the destination, the next hop node IP addresses and a route (destination) sequence number. This number is used to prevent loops and to avoid obsolete entries. Additional information includes the list of the precursor nodes, these nodes use the table owner node to relay the messages to the corresponding destination. Each row has also lifetime entry associated with it.

To prevent unlimited broadcast throughout the whole network, the RREQ has a maximal hop parameter. If the destination is not found within this limit, the RREQ is repeated with the parameter increased.

Path maintenance

The protocol has route error detection and repair mechanism. If a node discovers that some active link is broken it generates Route Error (RRER) message, which is broadcast to the precursor nodes. The RRER is then relayed to the concerned source nodes. If the path is needed, then the source node re-initiates route discovery.

Local Connectivity

To be independent from the MAC layer the AODV includes local connectivity management mechanism. The nodes issue “Hello” messages to remind their neighbours that they are still available for routing. Then the neighbours update their routing tables to refresh the lifetime marks of the paths routed via these nodes. If the network layer under AODV (MAC layer) has its own mechanism of the local connectivity control (it is the case, for example, for 802.11 protocol), then “Hello” messages need not to be used.

5.3 OLSR in Details

OLSR [31] is a proactive protocol, it means that the nodes update regularly the routing information and, when a demand arises, they have the routing paths ready for the use.

OLSR adopts the link state routing strategy, optimising it for the wireless ad hoc networks. An OLSR node broadcasts periodically the Topology Control (TC) messages containing the list of its neighbours and these messages are diffused across the network. This is the classical link state approach. OLSR improves the performance by making use of the broadcast nature of the wireless networks, as it tries to minimise the number of nodes necessary to diffuse the TC messages across the network. Every node selects a subset of its neighbours, called Multipoint Relay (MPR) nodes. The MPR nodes are chosen to cover with their radio transmission range all two-hop neighbours (neighbours of the neighbours). Therefore, the flooding of the network with TC messages is done only via the MPR structure.

OLSR does not demand the MPR set to be minimal and it was, actually, proved that to find the minimal MPR set is NP problem ([91]). The technique of the MPR nodes selection proposed in OSLR is based on the greedy style algorithm, which, in one interface case, can be resumed as

1. First add to the MPR set the nodes that provide the unique links to some two-hop neighbours.

2. Consequently add the nodes with the greatest number of the links with the two-hop neighbours that are not covered by the previously selected MPR nodes.

When a node receives a TC message, it can add or update its information on the links accordingly. The paths are then calculated using a shortest path algorithm (for example, Dijkstra algorithm).

Local Connectivity Control

The local connectivity management is based on the periodical distribution of "Hello" messages by all participating nodes. These messages contain the information on neighbours and on the MPR nodes selection. Also there is an additional functionality included to manage the asymmetrical links, but as they are out of scope of our further analysis, therefore, we avoid their description.

When a node receives the "Hello" message from a neighbour it updates its neighbour list, two-hop neighbour and *MPR selector* lists. MPR selectors are those nodes, that have chosen the given node as an MPR.

5.4 Studies of the Protocol Performance

OLSR

There is a number of works analysing the OLSR performance. We start our review with [51], which is one of the first works where OLSR is compared to DSR. The network connections are modeled by a classical random graph. It means that a direct link between two edges exists with a probability p, which is the determining parameter of the model. This makes the model mostly relevant to the indoor network where all nodes located within the transmission radius of each other and where the random perturbations does not allow some nodes to correctly receive transmissions. There is no geographical information about the positions of the nodes included into the consideration.

The MPR selection is one of central parts of the protocol and it received a close attention. Its algorithm was thoroughly studied in [91, 61] for its complexity. In [53] the analysis showed that the average size of the MPR set tends to be smaller than $O(M^{1/3})$, where M is the number of neighbours. Therefore, the overall structure of the MPR nodes that covers the network allows to obtain a significant improvement in the diffusion of the information over the network, compared to the simple flooding. The flooding techniques were analysed in [52, 9] in terms of the overhead. In the second article the broadcast stage of OLSR was compared to its analogue in Distributed Dynamic Routing (DDR) protocol.

The network performance under OLSR control was investigated in [33, 3]. In these works the protocol traffic was estimated separating it into different types of messages ("Hello", Topology Control). Then there is an estimation on the number of the messages and their frequency.

5.5 Direct Comparison at the Packet Level

AODV

AODV is a complex protocol with several types of control packets circulating between the nodes. In our analysis, we will concentrate on the following subset of them:

- "Hello" messages. They were introduced into AODV to maintain the local connectivity. These messages are transmitted periodically. We denote this frequency rate by χ.
- Route Request (RREQ) messages. These messages are participating in the discovery of new routes and in repairing the old ones. Therefore they are "on demand" packets which compose the flooding. We do not consider the local repair mechanism of AODV and new path creation. The latter can be easily included into the additive traffic density expression. We suppose that over some long period of time the frequency of flooding waves is ϕ, which depends on the rate of the errors which occur due to the wireless traffic congestion or interference problems.
- Normal traffic packets. They depend on the communication pattern of each network. We assume that in our case we have Λ paths regularly communicating, with the frequency ω.

These messages compose the main part of the traffic. The equation of the traffic density over some long period of time in the AODV network can be expressed in following way.

$$\lambda_{AODV} = \frac{N\chi}{S} + \frac{\phi N}{S} + \frac{L\Lambda\omega}{hS}, \text{ or} \tag{5.1}$$

$$\lambda_{AODV} = \nu\chi + \phi\nu + \frac{C_1\Lambda\omega}{h\sqrt{S}} \tag{5.2}$$

where N is the number of nodes, S is the area ($N/S = \nu$), L is the average euclidean distance between sender-destination pair of Λ paths (it is of order $O(\sqrt{S})$, i.e. $L = C_1\sqrt{S}$, where the constant C_1 characterises the region), h is an average hop length. In this expression there are two parameters that depends on other ones: flooding rate ϕ and hop length h.

The flooding rate ϕ should be proportional to the number of useful packets $L\Lambda\omega/h$. These packets or links can be interrupted by two events: by a random perturbation (thermal noise) or by a flooding that interrupted the link. The latter can be considered as self-inflicted damage. Let us denote the thermal component of ϕ by θ and the self-inflicted one by ϵ.

$$\phi = \theta + \epsilon \tag{5.3}$$

θ is a parameter of our model, whereas ϵ has to be determined using a flooding model we present in the next section.

The hop length h also depends on the flooding characteristics, node density and the current traffic density λ. This is because flooding is exactly the way AODV finds the paths.

OLSR

OLSR is a proactive protocol and as a consequence there is a lot of periodic activity to establish the paths across the network. The "Hello" messages mechanism informs the nodes about their 2-hop neighbourhood. Then every node fixes a subset (Multi Point Relay nodes, MPR) of its neighbours allowed to re-transmit its messages. This subset should be able to reach with its

transmissions all the nodes at a two hop distance and it is also optimised in size. The node choosing its MPRs is called *MPR selector*.

After this step, every node advertises itself across the network. This is performed by a flooding the MPR neighbours of the node, then MPRs of MPRs, etc. Therefore, compared to the classical AODV flooding the number of nodes and messages involved is much less. A good optimisation is achieved using the following protocol rule: a node forwards a message only received *for the first time* from its selector. We include into our simplified model the following packets of OLSR (see [54], [53] for similar models)

- "Hello" messages. Their frequency rate is denoted by χ. In the case, when it is necessary to avoid confusion with the AODV notations we will use subscripts $_{AODV}$ and $_{OLSR}$.
- Topology Control (TC) messages. They compose the OLSR flooding and contain the information about the issuing node and its MPR nodes. We denote the rate of TC messages by τ.
- Normal information packets. We denote the number of communicating pairs by Λ, the average distance between sender and receiver by $L = C_1\sqrt{S}$, where S is the area of the network and the average hop length we denote by h. The rate of the communication is denoted by ω.

We can express the traffic density in the whole network over some long period of time by

$$\lambda_{OLSR} = \frac{\chi N}{S} + \frac{N_{MPR} N \tau}{S} + \frac{L\Lambda\omega}{hS}, \tag{5.4}$$

where N_{MPR} is the number of nodes participating in MPR flooding. The study [53] of the MPR nodes in the network showed that for M neighbours there are about $M^{1/3}$ of MPR nodes. Therefore we can estimate N_{MPR} in the following way: it is the probability of every node to become an MPR multiplied by the number of nodes, i.e. $\frac{M^{1/3}N}{M}$ ([2]). As we can see it from (2.25), the number of neighbours in its turn is determined by the node density and the traffic density:

$$M = \frac{\pi r_0^2}{\lambda} \tag{5.5}$$

We suppose that the equation (5.4) describes the behaviour of the network in equilibrium. When a perturbation occurs the network readjusts itself and if necessary, it recreates new routes. Under such assumptions we can conclude that the hop length of the routes in OLSR corresponds to the current level of traffic density and therefore can be taken as $r_0/\sqrt{\lambda_{OLSR}}$.

Re-writing (5.4) (dropping the subscript of λ_{OLSR}) we obtain

$$\lambda = \chi\nu + 5\left(\nu\frac{\pi r_0^2}{\lambda}\right)^{-2/3} N\nu\tau + \frac{\sqrt{\lambda}C_1\Lambda\omega}{r_0\sqrt{S}} \tag{5.6}$$

$$\lambda = \chi\nu + 5\left(\pi r_0^2\right)^{-2/3}\lambda^{2/3}N\nu^{5/3}\tau + \sqrt{\lambda}\frac{C_1\Lambda\omega}{r_0\sqrt{S}} \tag{5.7}$$

Although this equation is not solvable in radicals for λ, we can obtain its numerical solution, which will be used in the comparison of two protocols further in this section.

Protocol comparison

In this section we compare the traffic density of two protocols with the same useful load Λ. For OLSR we will use directly the equation (5.7). However, the AODV equation (5.2) needs some further calculations.

We assume that the network is in equilibrium. Therefore, there are Λ working connections with the average traffic density λ_{AODV}. We suppose that the network uses for the flooding packets the same *probability* coordination and we can apply the model of the section 3.3 to AODV. We fix probability $p = 0.025$ and λ_{noise} is the average traffic density across the network, i.e. $\lambda_{noise} = \lambda_{AODV}$. In this case we can use the estimation for the hop length from (3.27), as the values of the traffic density traced in the approximated simulations are the same as λ_{AODV}.

The error rate ϵ is the second parameter of AODV and we have not found its explicit form. This is the rate of the failures caused by flooding. It can be estimated as it corresponds to the probability (rate) of the active link meeting a wave of the flooding.

$$\epsilon = \frac{C_1 \Lambda \omega \sqrt{S} \phi}{h} P_e \tag{5.8}$$

where P_e is the probability of the event that flooding will interrupt the transmission at arbitrary node. The interpretation of (5.8) is the following: the number of errors caused by flooding is proportional to the number of links (the number of useful connections multiplied by the average path length) $\Lambda \omega C_1 \sqrt{S}/h$ and to the flooding rate ϕ. Substituting in the expression (5.3) the value of ϵ by (5.8) and solving the resulting equation we obtain

$$\phi = \frac{\theta}{1 - \frac{C_1 \Lambda \omega \sqrt{S}}{h} P_e} \tag{5.9}$$

At this point we need another assumption to continue further our investigation. We assume that P_e is a probability function which depends on the duration of the peak of the flooding wave in some bounded region (taken from (3.26)) and it depends on T^*, the period which AODV waits before declaring a packet lost. P_e is ascending in flooding peak duration and descending in T^*. We choose P_e to correspond these constraints as following:

$$P_e = \exp\left(C_2 \left(n_0 + \frac{\log(\lambda_{noise} S) - \log\left(\left(1 - (1-p)^{\frac{\nu S}{X} + 1} \right) X \right)}{\log(1-p)} \right) / T^* + C_3 \right) \tag{5.10}$$

where C_2 and C_3 are normalising constants. This function can influence the form of the final curve. But as long as the thermal noise θ is small enough, this influence is limited.

On Fig. 5.2 we see the final results that follow from OLSR (5.7) and AODV traffic density equations. We restrain from writing the AODV function in whole as it is composed of multiple parts: equation (5.2), (3.26),(3.27), (5.10), (5.9) and the parts corresponding to P_e (5.10) and to h_{noise} (3.27) lack argumentation. Nevertheless, these curves allows to estimate the comparative behaviour of two protocols.

The x-axe is the percentage of the nodes involved in some active connection (number of the connections is then $xN/100$). Most of the parameters are shown in the table in Fig. 5.1. The curves of AODV are calculated through a numerical computation with the values of $\theta = 1/100000$ and $\theta = 1/10000$. Whereas, OLSR has two curves with $\tau = 1/20000$ and $1/10000$. These two parameters also make the comparison difficult. The values of θ and τ are related in the sense that when there is θ error rate in the network, OLSR should raise the rate τ sufficiently high, to repair the broken links with a minimal delay.

Value	AODV	OLSR	Description
	0.00001	0.00001	Time slot duration
N	1000	1000	Number of nodes
S	200	200	Surface
C_1	0.5	0.5	$C_1 = S/\sqrt{L}$
r_0	0.12	0.12	
ω	1/10000	1/10000	Useful load rate
χ	1/2000	1/2000	Hello rate
τ	NA	1/20000	OLSR TC refresh rate
T^*	0.005		AODV Link failure detection time
v	1.616		Speed of the flooding wave
p	0.025		Probability of the node transmission

Figure 5.1: The different parameters of two protocols used in comparison

On Fig. 5.2 the slope of AODV is greater than those of OLSR. The reason for this is that the length of the paths of AODV are longer than those of OLSR. This is the L/h part in the initial equations for both protocols (5.4) and (5.1). When θ increases AODV curve makes almost a parallel shift up. OLSR has a similar reaction to the change of τ.

We can conclude from the graphs that at some values of θ and τ the protocols have a load at which they perform equally. Before this point AODV performs better in terms of the overhead. After this point it is OLSR that has less overhead. It is not too surprising as we can see it on two extreme examples. In one case there is a zero useful traffic, therefore AODV does nothing, which is much better than OLSR establishing the structure of the network. In the other case the useful load is high, that makes the paths in AODV longer and the error rate higher, therefore OLSR optimisation of the resources allows a better overhead. The approach of estimating the traffic density of OLSR and AODV that we present here allows to consider the case in greater details.

The framework presented in this chapter is not a complete analytical model, because there are two important parameters that are approximated: AODV hop length and probability of flooding to break a link. However, we present a deeper insight on the mechanism of the interaction between flooding and the existing links and this approach allows us to compare the curves of the traffic level of two different protocols.

Let us also note, that this framework is built around several models, which were described in the previous chapters: SIR based random geometric graph and MAC layer modeling. Note, that the use of a fixed random geometric graph (unit disk model) would not be a good choice for such analysis, as our approach is based on the adaptation of an ad hoc network to changing traffic conditions and unit disk model is insensitive to these changes. The Erdös-Rényi random graph model suffers from the same shortage. However, the SIR based random graph described in Section 2.5.2 potentially could be used in such framework. But its problem is difficulty of finding the closed form equation, equivalent to (2.25) : $r = O(1/\sqrt{\lambda})$. The numerical computations can be done, but they would be lacking the clarity of analytical expressions. In fact, such comparison was possible because of the additive property of the traffic density. It allowed to combine the different types of messages described in the protocols.

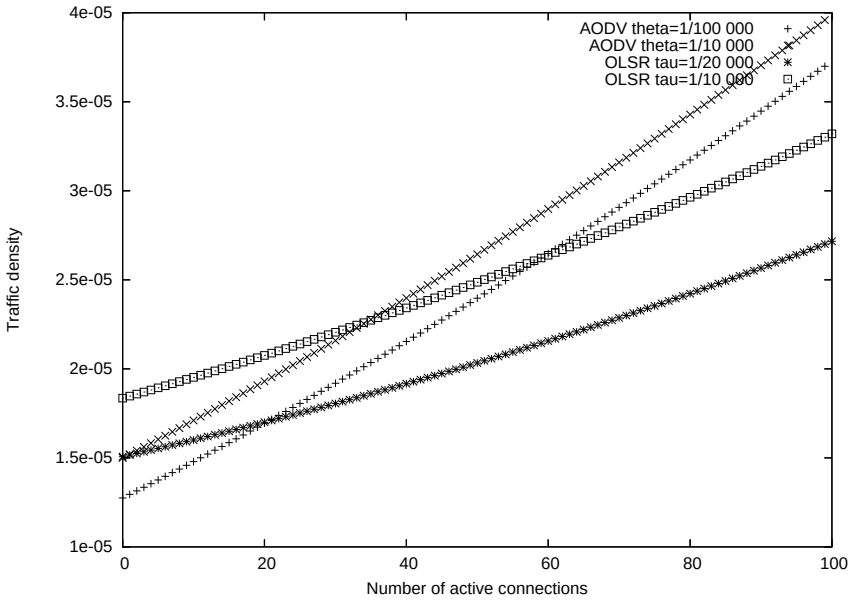


Figure 5.2: Traffic density of AODV ($\theta = 1/10000$ and $\theta = 1/100000$) and OLSR ($\tau = 1/20000$ and $\tau = 1/10000$)

Chapter 6

PageRank of Scale Free Growing Networks

At first sight this last chapter is not relevant to ad hoc networks. The object of its study is the Web graph and PageRank. However, the link between them exists. Starting with the general ideas: ad hoc networks can be seen as cooperative societies. In the real world it is unlikely that the nodes will link themselves to any unknown neighbors and rely their message forwarding on them. Therefore, some kind of authorisation system will be needed, which will create inter-dependency among the users and, therefore, a social network*. One of the mechanisms actively discussed at the present moment is* reputation systems *(see [58] for a up-to-date review). These systems are based on the feedbacks of the members of a community on each other. PageRank, as a measure of popularity of a particular node, is closely related to reputation evaluation.*

This work was co-authored with K. Avrachenkov

6.1 Reputation Systems in Ad Hoc Networks

Large scale ad hoc networks need a mechanism, that would insure some level of network security and also of collaboration among its users. *Reputation systems* may be a possible answer to this demand.

As wikipedia.org defines it: *A* **reputation system** *is a type of collaborative filtering algorithm which attempts to determine ratings for a collection of entities, given a collection of opinions that those entities hold about each other.* An important working example of an explicit reputation system, put in place by a commercial site, is the famous auction site *ebay.com*. After each transaction users are invited to leave their opinions on their counterpart. Those opinions are open for review to any other user. Therefore, potential customers (and sellers) can verify whether other clients were happy with a given seller (customer) and, therefore, base their decision about what price to pay or about even participation in the bidding on this information.

Another example of reputation systems is PageRank. As the google.com site says it: *PageRank relies on the uniquely democratic nature of the web by using its vast link structure as an indicator of an individual page's value. In essence, Google interprets a link from page A to page B as a vote, by page A, for page B.* Also, every page has its own weight, which it puts into the vote. Therefore, PageRank proposes a numerical evaluation of reputation [30].

The problems of authorisation and security are notorious for wireless networks. The absence of any central point or any other stable infrastructure make things even more difficult for the mobile ad hoc networks. There is a number of solutions for these problems proposed in the

literature and one of possible approaches consist of establishing a reputation system in the network, where the nodes would share the information of each other (see, for example, [64]).

In this chapter we study evolution of PageRank on a scale free growing network model. The growing network model cannot be directly applied to the link availability modeling in ad hoc networks, because it describes a social side of the Web graph evolution. Therefore, the results we obtain are rather specific for some models of social networks. Nevertheless, this study might have its interest for the future applications in ad hoc networks, for their reputation systems at the protocol level.

6.2 Introduction to PageRank

Surfers on the Internet frequently use search engines to find pages satisfying their query. However, there are typically hundreds or thousands of relevant pages available on the Web. Thus, listing them in a proper order is a crucial and non-trivial task. The original idea of Google presented in [25] is to list pages according to their PageRank which reflects popularity of a page. The PageRank is defined in the following way. Denote by n the total number of pages on the Web and define the $n \times n$ hyperlink matrix P as follows. Suppose that page i has $k > 0$ outgoing links. Then $p_{ij} = 1/k$ if j is one of the outgoing links and $p_{ij} = 0$ otherwise. If a page does not have outgoing links, the probability is spread among all pages of the Web, namely, $p_{ij} = 1/n$. In order to make the hyperlink graph connected, it is assumed that a random surfer goes with some probability to an arbitrary Web page with the uniform distribution. Thus, the PageRank is defined as a stationary distribution of a Markov chain whose state space is the set of all Web pages, and the transition matrix is

$$\tilde{P} = cP + (1 - c)(1/n)E, \tag{6.1}$$

where E is a matrix whose all entries are equal to one and $c \in (0, 1)$ is the probability of following a link on the page and not jumping to a random page (it is chosen by Google to be 0.85). The constant c is often referred to as a damping factor. The Google matrix $\tilde{P}$ is stochastic, aperiodic, and irreducible, so there exists a unique row vector π such that

$$\pi\tilde{P} = \pi, \quad \pi\underline{1} = 1, \tag{6.2}$$

where $\underline{1}$ is a column vector of ones. The row vector π satisfying (6.2) is called a PageRank vector, or simply PageRank. If a surfer follows a hyperlink with probability c and jumps to a random page with probability $1 - c$, then π_i can be interpreted as a stationary probability that the surfer is at page i.

Barabási and Albert [13] have proposed a scale free growing network model to understand the evolution of the World Wide Web and in particular to explain the power law for in and out degree distributions. Then, Bollobás et al. [20] have refined their model and proved rigorously that in and out degree distributions satisfy power laws. Pandurangan et al. [77] applied the "mean-field" heuristics from [13, 14, 15] to show that the PageRank distribution in the scale free growing network model satisfies the power law with exponent 2. They have also proposed a model where new nodes attach with weighted probability that takes into account the in degree as well as PageRank. By studying two large samples of the Web, the authors of [77] found that PageRank closely follows a power law with exponent 2.1.

In the present work we find an analytical expression for the expected PageRank value in a scale free growing network model as a function of the age of the growing network and the age of

a particular node. We prove that the average PageRank value does not depend on the number of outgoing links. This fact helps us significantly, since we can deal with tree graphs instead of directed acyclic graphs. Then, we derive asymptotics that shows that PageRank follows closely a power law with exponent 2.08 in the middle range of its values. Finally, our expressions give a mathematical insight for the choice of the damping factor c.

The structure of the chapter is as follows: in Section 6.3 we describe the scale free growing network model, which is used in the present work, and its relation to the other scale free growing network models. In Section 6.4 we derive an explicit formulae for PageRank for directed acyclic graphs and tree graphs. In Section 6.5 we prove that in our model the average PageRank does not depend on the number of outgoing links. Sections 6.6 and 6.7 provide auxiliary results on the moment generating function of the nodes' heights in subtrees and on the subtree size distribution, which lead to the final results and asymptotics given in Section 6.8. The study is concluded by Section 6.9, where we discuss the results and compare them with the related results from the literature. Some techniques that we use in the present work are explained in more detail in Appendices.

6.3 Scale Free Network Models

Inspired by the power law in and out degree distributions of the World Wide Web, Barabási and Albert [13] have proposed growing network model with preferential attachment. In their model a new node is attached to some old nodes with probability proportional to the in degree of the old nodes. The authors of [13, 14, 15] have developed the "mean-field" heuristics, which allowed them to derive approximations to the power law degree distributions. Then, Bollobás et al. [20] have added some missing parts to the Barabási-Albert model and have shown rigorously that the degree distributions of the scale free growing network model indeed satisfy power laws. The model of [20] allows self loops and multiple links.

It turns out that there is an explicit analytic expression (it is given in the next section) for the PageRank of directed acyclic graphs. Furthermore, Google when computes the PageRank disregards the hyperlinks within the same Web page. Taking into account the above two reasons, we have decided to work with the following scale-free growing network model: The time is discrete. The network grows at the speed of one node per time step. We fix a parameter m, the number of outgoing links from each node. At each time step a new node creates m links to the existing nodes. Let us denote the growing network at arbitrary time step n by G_n^m. At this point we need to define the way the links of a new node connect to the existing nodes. We denote by $d_v(n)$ the *in* degree of node v at time step n.

- At step 0 the initial node 0 is created and it has no links. The initial node has weight m by definition.
- Then, at the next time step 1 a new node has no other choice but to connect its m links to the initial node. Node 1 receives the weight m and the weight of node 0 becomes $2m$.
- A new node that appears after time step 1 connects each of its m edges independently with probability proportional to the existing nodes' weights equal to *in* degrees plus m. Namely, the probability that node n connects to node v, $v < n$, is given by

$$\mathbb{P}[n \to v | G_{n-1}^m] = \frac{d_v(n-1)+m}{\sum_{k=0}^{n-1}(d_k(n-1)+m)} = \frac{d_v(n-1)+m}{2m(n-1)+m}. \tag{6.3}$$

For instance, node 2 connects with probability 2/3 to the initial node 0 and with probability 1/3 to node 1.

It is easy to see, that in the case of $m = 1$ the growing network G_n^1 is a tree. This fact will be used extensively later in the chapter. We would like to note that the scale-free growing network model of [36] is the closest model to ours. An interested reader can find a detail overview of growing network models in the surveys [22, 35, 71].

6.4 PageRank of Growing Networks

Let us study the PageRank for growing networks with fixed outdegree m. We would like to emphasize that in this section we do not assume any preferential attachment of new nodes. It is only assumed that at each time step a new node is added to the network and makes m links to previously created nodes. Thus, if the initial node does not have any outgoing links, a growing network realization is a Directed Acyclic Graph (DAG) at each time step. To calculate the PageRank one needs to attribute some outgoing links to the initial node. There are two natural options: either to make a self loop in the initial node or to connect the initial node to all nodes in the network. Essentially, the difference between these two cases can be reduced to the total number of in- and out-going degrees (for nodes $v \geq 1$) [60]. Since it turns out that the expression we obtain for the values of PageRank is much simpler in the case of the initial node with self loop, we choose the first option in the present work.

We denote by $\pi_v(n)$ the PageRank of node v after the n-th step of the growing network evolution. Of course, $n \geq v$. We note that at time step n the PageRank value of a newly created node n is minimal and is given by $\pi_n(n) = \frac{1-c}{n+1}$.

Let us denote by $L_v(n)$ the set of all paths from nodes $v+1, ..., n$ to v and by $|l|$ the length of a path l. Then, the PageRank vector of a growing network realization can be calculated by an explicit formula given in the next theorem.

Theorem 6.1. *The PageRank of a growing network realization of node v, $v > 0$, at time step n is given by equation*

$$\pi_v(n) = \frac{1-c}{n+1}\left(1 + \sum_{l \in L_v(n)} \left(\frac{c}{m}\right)^{|l|}\right), \tag{6.4}$$

and the PageRank of the initial node $v = 0$ is given by

$$\pi_0(n) = \frac{1}{n+1}\left(1 + \sum_{l \in L_0(n)} \left(\frac{c}{m}\right)^{|l|}\right). \tag{6.5}$$

Proof. The PageRank vector of any network can be expressed by the formula [68, 18, 60]

$$\pi = \frac{1-c}{n+1}\underline{1}^T[I - cP]^{-1}, \tag{6.6}$$

where $\underline{1}^T$ is the row vector of ones and P is the hyperlink matrix as in (6.1). We can rewrite the inverse matrix as a power series

$$[I - cP]^{-1} = I + cP + c^2P^2 + ... ,$$

Next we note that the (i, j) element of matrix $[I - cP]^{-1}$ corresponds to the sum of $(c/m)^{|l|}$ over all possible paths from node i to node j. The premultiplication of $[I - cP]^{-1}$ by vector $\underline{1}^T$

gives the sum of all paths to node j. In the case $v > 0$, there are no loops and hence we obtain formula (6.4). In the case $v = 0$, each path to the initial node ends with a self loop. Because of this self loop each term $(c/m)^{|l|}$ is multiplied by the series $1 + c + c^2 + ...$ The sum of the later series is equal to $1/(1 - c)$, which cancels the factor $1 - c$ in (6.6) and results in the particular expression (6.5) for the PageRank of the initial node. □

Next we note that if $m = 1$, every realization of the growing network becomes a tree. This simplifies further the formulae (6.4) and (6.5). In the case $m = 1$, let us denote by $T_v(n)$ the subtree of the growing network with the root in node v at time step n, $n > v$. Also we denote by $X_n(v, w)$ the hop-distance between v and w at step n (of course, we should have $w \in T_v(n)$). We shall also call $X_n(v, w)$ *height* of w in $T_v(n)$. Let us denote the number of nodes in $T_v(n)$ by $Y_v(n)$. Then, we have the following corollary from Theorem 6.1.

Corollary 6.1. *If all the distances between the root node v and all nodes in $T_v(n)$ are known, then $\pi_v(n)$, the PageRank of node v, $V > 0$, can be expressed explicitly as follows:*

$$\pi_v(n) = \frac{1-c}{n+1}\left(1 + \sum_{\alpha \in T_v(n)} c^{X_n(v,\alpha)}\right), \tag{6.7}$$

or in its alternative local time form with respect to the subtree $T_v(n)$,

$$\pi_v(n) = \frac{1-c}{n+1}(1 + \sum_{k=1}^{Y_v(n)} c^{X_n(v,k)}), \tag{6.8}$$

and the PageRank of the initial node 0 is given by

$$\pi_0(n) = \frac{1}{n+1}\left(1 + \sum_{k=1}^{n} c^{X_n(0,k)}\right). \tag{6.9}$$

6.5 The case $m > 1$ can be reduced to the case $m = 1$

It follows from Corollary 6.1 that the calculation of PageRank is much simpler in the case of tree graphs than in the case of directed acyclic graphs. In particular, in the case of tree graphs there is a one-to-one correspondence between the paths and the nodes. Fortunately, as the following Theorem 6.2 demonstrates, the expected values of PageRank in the cases $m > 1$ and $m = 1$ are equal for the corresponding nodes of the same age. Denote by $\mathbb{E}\pi_v^m(n)$ the expected value of PageRank of node v at time step n for our growing network model G_n^m.

Theorem 6.2. *In the present scale free growing network model G_n^m, the average PageRank of node v does not depend on m. Namely, we have*

$$\mathbb{E}\pi_v^m(n) = \mathbb{E}\pi_v^1(n), \quad v < n. \tag{6.10}$$

Proof. The proof is done by induction on the node age. Thus, we fix v and consider time steps $n = v + 1, v + 2, ...$.

As the induction base, consider node v at time step $v + 1$. There is a new node $v + 1$ that is being added to the network and this new node has m links with j links to node v, $0 \le j \le m$,

and $m-j$ links to the rest of the nodes. Let us find the expected value of PageRank for node v:

$$\mathbb{E}\pi_v^m(v+1) = \sum_{j=0}^{m} \frac{jc}{m}\frac{1-c}{v+2}\mathbb{P}\left[v+1 \text{ has } j \text{ links to } v\right] + \frac{1-c}{v+2}, \tag{6.11}$$

it is equal to

$$\mathbb{E}\pi_v^m(v+1) = \frac{c}{m}\frac{1-c}{v+2}\sum_{j=0}^{m} j\mathbb{P}\left[v+1 \text{ has } j \text{ links to } v\right] + \frac{1-c}{v+2}. \tag{6.12}$$

The probability that a link will be created from $v+1$ to v is equal to $\frac{m}{2mv+m} = \frac{1}{2v+1}$. Therefore, the sum in (6.12) is the average number of the links from $v+1$ to v or, in other words, the average number of the successes in m Bernoulli trials with the probability of the success equal to $\frac{1}{2v+1}$. Therefore, we can write

$$\mathbb{E}\pi_v^m(v+1) = \frac{c}{m}\frac{1-c}{v+2}\frac{m}{2v+1} + \frac{1-c}{v+2} = \frac{c(1-c)}{(v+2)(2v+1)} + \frac{1-c}{v+2}. \tag{6.13}$$

Thus, $\mathbb{E}\pi_v^m(v+1)$ does not depend on m and the induction base is proved.

Next we consider node v at its age of t, or equivalently, at time step $n = v+t$, and we suppose, that all the average PageRanks $\mathbb{E}\pi_k^m(v+t)$ of the nodes k, $v < k \leq v+t$ do not depend on m. The nodes k, $v < k \leq v+t$, are the nodes that are "younger" than node v. We shall prove that $\mathbb{E}\pi_v^m(v+t)$ the expected value of PageRank of node v at time step $v+t$ also does not depend on m.

Let us denote a realization of the network G_n^m at time step $v+t-1$ as λ. At time step $v+t$ a new node $v+t$ is born which connects itself with m links to the older nodes according to the preferential attachment rule. The PageRank of node v at time step $v+t$, knowing that the configuration at time step $v+t-1$ was λ, is given by

$${}^{\lambda}\pi_v^m(v+t) = \sum_{k=v+1}^{v+t} \frac{c}{m}{}^{\lambda}\pi_k^m(v+t)\mathcal{M}\{k \to v,\ \lambda\} + \frac{1-c}{v+t+1}, \tag{6.14}$$

where $\mathcal{M}\{k \to v,\ \lambda\}$ is the number of edges from node k to node v. In particular, we note that the PageRank of an arbitrary node depends only on those nodes that appear later in time. Now we consider the expectation of (6.14) over all possible realizations λ:

$$\mathbb{E}\pi_v^m(v+t) = \sum_{k=v+1}^{v+t} \frac{c}{m}\mathbb{E}\left({}^{\lambda}\pi_k^m(v+t)\mathcal{M}\{k \to v,\ \lambda\}\right) + \frac{1-c}{v+t+1}. \tag{6.15}$$

We claim that ${}^{\lambda}\pi_k^m$ and $\mathcal{M}\{k \to v,\ \lambda\}$ are independent.

In fact, as mentioned above, the PageRank ${}^{\lambda}\pi_k^m$ of node k depends on the nodes that appear in time later than node k, whereas the number of the links between k and v depends only on the nodes that appeared before node k due to the preferential attachment rule.

Therefore, $\mathbb{E}\left({}^{\lambda}\pi_k^m(v+t)\mathcal{M}\{k \to v,\ \lambda\}\right) = \mathbb{E}\left({}^{\lambda}\pi_k^m(v+t)\right)\mathbb{E}\left(\mathcal{M}\{k \to v,\ \lambda\}\right)$, and hence, we can write

$$\begin{aligned}
\mathbb{E}\pi_v^m(v+t) &= \sum_{k=v+1}^{v+t} \frac{c}{m}\mathbb{E}\pi_k^m\mathbb{E}\mathcal{M}\{k \to v\} + \frac{1-c}{v+t+1} \qquad (6.16)\\
&= \sum_{k=v+1}^{v+t-1} \frac{c}{m}\mathbb{E}\pi_k^m\mathbb{E}\mathcal{M}\{k \to v\} + \frac{c}{m}\mathbb{E}\mathcal{M}\{v+t \to v\}\frac{1-c}{v+t+1} + \frac{1-c}{v+t+1}
\end{aligned}$$

Since each outgoing link from node k is created independently, we have

$$\mathbb{E}\mathcal{M}\{k \to v\} = m\mathbb{P}[\text{one link from } k \text{ to } v].$$

Due to the preferential attachment rule (see (6.3)), the probability $\mathbb{P}[\text{one link from } k \text{ to } v]$ does not depend on m, if the expected weight of node v is proportional to m. Let us show this:

$$\mathbb{E}(d_k(n) + m | d_k(n-1)) = d_k(n-1) + m + m\frac{m + d_k(n-1)}{2m(n-1) + m}, \tag{6.17}$$

taking the average over all possible network realizations, we get

$$\mathbb{E}(d_k(n) + m) = \mathbb{E}(d_k(n-1) + m) + \frac{\mathbb{E}(d_k(n-1) + m)}{2n - 1}. \tag{6.18}$$

Knowing that $\mathbb{E}(d_k(k) + m) = m$ we conclude, even without calculating the final expression for $\mathbb{E}d_k(n)$, that it is proportional to m.

Since $\mathbb{P}[\text{one link from } k \text{ to } v]$ does not depend on m and $\mathbb{E}\pi_k^m(v+t)$ for $k = v+1, ..., v+t-1$ also does not depend on m by the induction hypothesis, the induction step is proved. This marks the end of the proof.

□

The theorem allows us to concentrate the study of PageRank of the growing network model G_n^m on the case $m = 1$, when each realization of the growing network is a tree.

Let us provide clarifications to the claim that the case $m = 1$ is much simpler than the case $m > 1$, thus outlining the steps of the ensuing analysis presented in the next sections. It follows from Corollary 6.1 that the PageRank of a given node v depends on the *number of nodes* in the subtree $T_v(n)$ and on the *distances* from these nodes to node v. Both these values can be described using the Markov type random processes:

- The size of the tree $T_v(n)$ is a random variable and it is easy to see that in a growing network model with preferential attachment mechanism the evolution of the size of $T_v(n)$ is a Markov chain: at every step n the size $|T_v(n)|$ of the tree $T_v(n)$ depends only on the size of the tree at the previous step $n-1$. Inside the tree $T_v(n)$ all nodes are connected to each other, therefore, the probability of a new node to link itself to the tree $T_v(n)$ can be calculated directly and it is equal to $2|T_v(n)| - 1$ divided by the corresponding normalising factor. The term -1 is explained by the fact, that we consider the node v to be inside the tree $T_v(n)$, but it participates only with its *out* degree 1. Further details on the evolution of the tree $T_v(n)$ are given in Section 6.6.

- Let us consider the tree $T_v(\cdot)$ at the moment, when it has n' nodes. By the above arguments about the tree formation, we can limit our consideration only to the nodes that belong to the tree and ignore the rest of the network. In particular, node v becomes the initial node and the moments of attachment of new nodes to the tree can be considered as the local time of $T_v(\cdot)$. When a new node is connected to some already existing node in the tree $T_v(\cdot)$, its distance to the root (or its height) depends only on the height of that node. Therefore, in the model with the preferential attachment mechanism, the probability of a new node to be at some height h depends on the number of the existing nodes with height $h-1$ and their "popularity" (the number of the nodes at height h). Actually, we can express this probability as a number of the nodes with heights $h-1$ and h divided by the number of the nodes n'. It does not depend on other details, for example, how the nodes are exactly connected inside the tree. Using this fact, we calculate the moment generating function of the nodes' heights in Section 6.7.

In (6.8) we have a sum of a random number of random variables. To find the expectation of this sum we need some generalization of the Wald's equation. Such a generalization is provided by Kolmogorov-Prokhorov equation (Appendix B), which allows to obtain the final result in Section 6.8 by combining the expressions for the tree size distribution and the moment generating function of the nodes' heights.

6.6 Distribution of the subtree size $Y_v(n)$

We start with the lemma that gives an explicit expression for the distribution of the subtree size.

Lemma 6.1. *The probability that at time step n the subtree rooted in node v has k nodes is given by*

$$\mathbb{P}[Y_v(n) = k] = \frac{\Gamma(n-v+1)\Gamma(k+1/2)\Gamma(n-i)\Gamma(v+1/2)}{\Gamma(n-v-k+1)\Gamma(k+1)\Gamma(1/2)\Gamma(v)\Gamma(n+1/2)} \tag{6.19}$$

Proof. We show that the evolution of the subtree size $Y_v(n) = |T_v(n)|$ can be described by the Pólya-Eggenberger urn model (see Appendix C).

There are balls of two colors, black and white, in one urn. Initially the urn contains $b = 1$ black balls and $w = 2v$ white ones. At every step one ball is drawn at random from the urn. Then it is returned back together with $s = 2$ balls of the same color.

The balls correspond to the in and out degrees of the nodes. The number of the balls is the sum of the degrees. The black balls correspond to the nodes from the subtree $T_v(n)$. The white balls, therefore, correspond to the nodes outside the subtree. Every existing edge (k, l) in G_n^1 corresponds to two balls in the urn model. Namely, one ball corresponds to the out degree of node k and the other ball corresponds to the in degree of node l. Therefore, the Pólya-Eggenberger distribution can be used to estimate the number of the black (or white) balls in the urn at time step n.

The choice of a black ball from the urn corresponds to the event, that a new $(n+1)$-th node connects itself to the subtree of v. Otherwise, the new node connects itself to some node outside of the subtree $T_v(\cdot)$, and, therefore, neither this node nor its subtree nodes will ever connect themselves to v with a path lying in the subtree of v.

We specify the expression for the Pólya-Eggenberger distribution (see Appendix C) for our problem

$$\begin{aligned}\mathbb{P}[Y_v(n) = k] &= \binom{n-v}{k} 1(1+2)\dots(1+2n) \times \\ &\quad \times \frac{2v(2v+2)\dots(2(n-k))}{(1+2v)(1+2v+2)\dots(1+2n)},\end{aligned} \tag{6.20}$$

or, equivalently, in its Gamma function form it gives the expression (6.19). □

Let us illustrate the application of the urn model to the growing network formation by a simple example (see Figure 6.1). The upper row of the balls corresponds to the outgoing degree of the nodes marked with their own numbers and the second row corresponds to the incoming degrees of the nodes. At time step 3 we have an urn with 7 balls: 6 white and 1 black. Node 0 has the in degree $d_0 = 2$, therefore, there are 3 balls bearing the mark 0. If we draw from the urn a white ball, like on Fig. 6.1 (b), no matter which number it has (here it is 2) we fall out of T_3. Therefore, two white balls are added. On contrary, if we choose a black ball, then

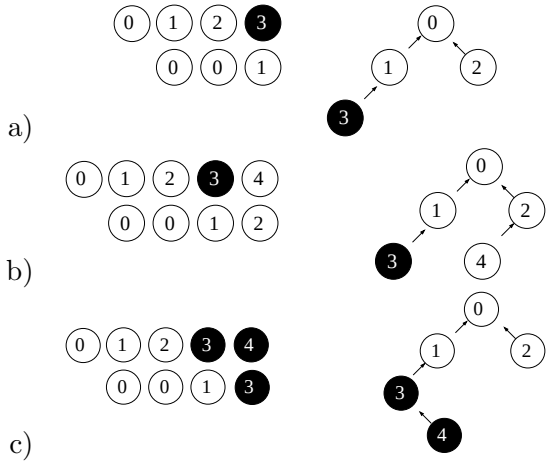


Figure 6.1: Illustration to the urn model: (a)Growing network after 3 steps. We choose to follow the subtree T_3. (b) Node 4 is not linked to T_3. (c) Node 4 is linked to T_3.

the new node falls inside the tree T_3, and, therefore, we add two black balls. Now it is easy to see that if we erase the number of marks from the balls but leave the ball colors, then we will not change anything in the formation of the number of balls of each color. Thus, the evolution of $Y_3(n)$ does not depend on the topology of T_3, but it depends only on the number of nodes inside and outside the subtree T_3.

6.7 Moment Generating function of the nodes' heights in subtrees of G_n^1

In this section we obtain the moment generating function of the nodes' heights inside the subtree $G_{n'}^1 = T_v(n)$, where $n' = |T_v(n)|$. As we already remarked in the previous section, n' can be seen as a local time inside $T_v(n)$. Since $G_{n'}^1$ and G_n^1 have the same distribution, we can consider without loss of generality only the distribution theof the heights in G_n^1. The later explains why we have taken a particular care for the choice of the initial weight for the initial node.

Lemma 6.2. *If X_n is the distance between the initial node and node n, then*

$$\mathbb{E}c^{X_n} = \frac{\Gamma(n+\frac{c}{2})\sqrt{\pi}}{\Gamma(n+\frac{1}{2})\Gamma(\frac{c}{2})}. \tag{6.21}$$

Proof. The evolution of X_n can be described without reference to any particular network realization, sub-lying graph or tree structure. If node n has height k in G_n^1, then it means that it is connected to a node with the height $k-1$. The conditional probability of such event is the the number of nodes with height $k-1$ plus the number of nodes with height k, normalised by $2n-1$, that is,

$$\begin{aligned}\mathbb{P}[X_n = k | X_{n-1}, \ldots, X_0] = \\ \frac{\sum_{i=0}^{n-1} \mathbb{1}(X_i = k) + \sum_{i=0}^{n-1} \mathbb{1}(X_i = k-1)}{2n-1},\end{aligned} \tag{6.22}$$

where $\mathbb{1}(\cdot)$ is an indicator function.

Using (6.22), we can calculate the conditional moment generating function of the nodes' heights as follows:

$$\begin{aligned}
\mathbb{E}(c^{X_n}|X_{n-1}, X_{n-2}\ldots, X_0) &= \sum_{k=0}^{n} c^k \mathbb{P}[X_n = k|X_{n-1}, \ldots, X_0] && (6.23)\\
&= \frac{\sum_{k=0}^{n} c^k \mathbb{1}(X_{n-1} = k) + \sum_{k=1}^{n} c^k \mathbb{1}(X_{n-1} = k-1)}{2n-1} + \\
&\quad + \frac{2n-3}{2n-1}\mathbb{E}(c^{X_{n-1}}|X_{n-2}\ldots, X_0) && (6.24)\\
&= \frac{\sum_{k=0}^{n} c^k \mathbb{1}(X_{n-1} = k) + c\sum_{k=0}^{n-1} c^k \mathbb{1}(X_{n-1} = k)}{2n-1} + \\
&\quad + \frac{2n-3}{2n-1}\mathbb{E}(c^{X_{n-1}}|X_{n-2}\ldots, X_0), && (6.25)
\end{aligned}$$

where $\mathbb{1}(X_i = n) = 0$ for all $i < n$. Next, applying the double expectation value rule, we obtain the following recurrent equation

$$\mathbb{E}c^{X_n} = (1 - \frac{1-c}{2n-1})\mathbb{E}c^{X_{n-1}}. \tag{6.26}$$

The above recurrent equation gives

$$\mathbb{E}c^{X_n} = \prod_{k=1}^{n}\left[1 - \frac{1-c}{2k-1}\right] = \frac{\Gamma(n+\frac{c}{2})\sqrt{\pi}}{\Gamma(n+\frac{1}{2})\Gamma(\frac{c}{2})}, \tag{6.27}$$

which completes the proof. □

Using the derivations in the proof of Lemma 6.2, we can also estimate the average tree height. Namely, we have

$$\mathbb{E}X_n = \mathbb{E}X_{n-1} + \frac{1}{2n-1}$$

and, consequently,

$$\mathbb{E}X_n = \sum_{k=1}^{n}\frac{1}{2k-1}. \tag{6.28}$$

The equation (6.28) can be interpreted as follows:

Lemma 6.3. *The average height of the scale free growing network model G_n^1 after n time steps is of order* $\log(n)$.

This result is in line with the results of [21].

Now we can already calculate the expected PageRank value of the initial node. After taking the expectation in (6.9), we substitute in (6.9) the expression for $\mathbb{E}c^{X_n(0,k)}$ given in (6.21). Then, simplifying the sum, we obtain

$$\mathbb{E}\pi_0(n) = \frac{1}{1+n}\left(\frac{1}{c+1} + \frac{2\sqrt{\pi}\Gamma(n+\frac{c}{2}+1)}{(c+1)\Gamma(\frac{c}{2})\Gamma(n+1/2)}\right). \tag{6.29}$$

6.8 Final Result Statement and Asymptotics

The expected value of PageRank is provided by the following theorem.

Theorem 6.3. *The expected value of PageRank $\pi_v(n)$ of node v at time step n in the present growing network model G_n^m is given by*

$$\mathbb{E}\pi_v(n) = \frac{1-c}{1+n}\left(\frac{1}{1+c} + \frac{c\Gamma(v+\frac{1}{2})\Gamma(n+\frac{c}{2}+1)}{(1+c)\Gamma(v+\frac{c}{2}+1)\Gamma(n+\frac{1}{2})}\right), \tag{6.30}$$

for $v > 0$, and

$$\mathbb{E}\pi_0(n) = \frac{1}{1+n}\left(\frac{1}{c+1} + \frac{2\sqrt{\pi}\Gamma(n+\frac{c}{2}+1)}{(c+1)\Gamma(\frac{c}{2})\Gamma(n+1/2)}\right), \tag{6.31}$$

for the particular case of $v = 0$.

Proof. First, we reduce the case $m > 1$ to the case $m = 1$ by Theorem 6.2.

Then, we apply Kolmogorov-Prokhorov Theorem (see Appendix B) to equation (6.8). Namely, we have

$$\mathbb{E}\pi_v(n) = \frac{1-c}{n+1}\left(1 + \sum_{i=1}^{\infty}\mathbb{P}[Y_v(n) \geq i]\mathbb{E}c^{X_i}\right). \tag{6.32}$$

Using the property that the height of a node will not be greater than the size of the tree, i.e., $\mathbb{P}[Y_v(n) > n - v] = 0$, we transform equation (6.32) to

$$\begin{aligned}\mathbb{E}\pi_v(n) &= \frac{1-c}{n+1}\left(1 + \sum_{i=1}^{n-v}\left(\sum_{k=i}^{n-v}\mathbb{P}[Y_v(n) = k]\right)\mathbb{E}c^{X_i}\right) \\ &= \frac{1-c}{n+1}\left(1 + \sum_{i=1}^{n-v}\mathbb{P}[Y_v(n) = i]\left(\sum_{k=1}^{i}\mathbb{E}c^{X_k}\right)\right).\end{aligned} \tag{6.33}$$

Next, we substitute the expressions obtained for X_n and $Y_v(n)$, equations (6.21) and (6.19), respectively, into (6.33) to obtain

$$\begin{aligned}\mathbb{E}\pi_v(n) = \frac{1-c}{n+1}\Bigg(1 + \sum_{i=1}^{n-v}\frac{\Gamma(n-v+1)}{\Gamma(n-v-i+1)\Gamma(i+1)} \times \\ \times \frac{\Gamma(i+1/2)\Gamma(n-i)\Gamma(v+1/2)}{\Gamma(v)\Gamma(n+1/2)}\sum_{k=1}^{i}\frac{\Gamma(k+c/2)}{\Gamma(k+1/2)\Gamma(c/2)}\Bigg).\end{aligned} \tag{6.34}$$

Simplifying the internal sum in the above equation, we obtain the following expression

$$\begin{aligned}\mathbb{E}\pi_v(n) &= \frac{1-c}{n+1}\Bigg(1 + \frac{\Gamma(n-v+1)\Gamma(v+1/2)}{\Gamma(v)\Gamma(n+1/2)}\times \\ &\quad \sum_{i=1}^{n-v}\frac{\Gamma(n-i)\Gamma(i+1/2)}{\Gamma(n-v-i+1)\Gamma(i+1)}\left(\frac{2\sqrt{\pi}\Gamma(i+1+c/2)}{(1+c)\Gamma(i+1/2)\Gamma(c/2)} - \frac{c}{c+1}\right)\Bigg) = \\ &= \frac{1-c}{n+1}\Bigg(1 + \frac{2\sqrt{\pi}\Gamma(n-v+1)\Gamma(v+1/2)}{(1+c)\Gamma(c/2)\Gamma(v)\Gamma(n+1/2)}\sum_{i=1}^{n-v}\frac{\Gamma(n-i)\Gamma(i+1+c/2)}{\Gamma(n-v-i+1)\Gamma(i+1)} \\ &\quad - \frac{c\Gamma(n-v+1)\Gamma(v+1/2)}{(1+c)\Gamma(v)\Gamma(n+1/2)}\sum_{i=1}^{n-v}\frac{\Gamma(n-i)\Gamma(i+1/2)}{\Gamma(n-v-i+1)\Gamma(i+1)}\Bigg).\end{aligned} \tag{6.35}$$

By using the Zeilberger's algorithm and his package EKHAD for Maple, we prove (see Lemma D.1 in Appendix D) the following hypergeometric identity

$$\sum_{i=1}^{n-v} \frac{\Gamma(n-i)\Gamma(i+1+c/2)}{\Gamma(n-v-i+1)\Gamma(i+1)} = \frac{\Gamma(v)\Gamma(n+c/2+1)\Gamma(1+c/2)}{\Gamma(v+c/2+1)\Gamma(n-v+1)} - \frac{\Gamma(n)\Gamma(1+c/2)}{\Gamma(n-v+1)}. \tag{6.36}$$

We can apply this identity to the both sums in (6.35), since we can think of the second sum as a particular case of the first one with $c = -1/2$. After some simplifications we obtain the final result (6.30). □

The expression (6.30) is already simple enough. However, it can be made even more transparent by using the following asymptotics:

$$\Gamma(x+a)/\Gamma(x) \approx x^a,$$

when $0 < a < 1$ and $x \to +\infty$. Thus, we have

$$\mathbb{E}\pi_v(n) \approx \frac{1-c}{1+n}\left(\frac{1}{1+c} + \frac{c}{1+c}(v+\frac{1}{2})^{-\frac{1+c}{2}}(n+\frac{1}{2})^{\frac{1+c}{2}}\right). \tag{6.37}$$

or, neglecting the first term,

$$\mathbb{E}\pi_v(n) \approx \frac{1-c}{1+c} c v^{-\frac{1+c}{2}} n^{-\frac{1-c}{2}}. \tag{6.38}$$

In particular, for the zero node, we have

$$\mathbb{E}\pi_0(n) \approx \frac{2\sqrt{\pi}}{(1+c)\Gamma(\frac{c}{2})} n^{-\frac{1-c}{2}}. \tag{6.39}$$

As one can see from Figure 6.2, the asymptotics (6.37) indeed closely follows the exact expression (6.30).

6.9 Discussion and comparison with related work

First, let us compare our results with the results of Pandurangan et al. [77]. In the present work we have obtained exact analytical expression and asymptotics for the expected value of the PageRank as a function of the age of the growing network and the age of a particular node. In Pandurangan et al. [77] the authors have used "mean-field" approach [13, 14, 15] to obtain an approximation for the PageRank distribution. Let us use our results on the expected value of PageRank for the "mean-field" calculations of [77]. Specifically, suppose that n is fixed, PageRank depends continuously on v and the node age is uniformly distributed. Then, using our asymptotic expression (6.37), we obtain

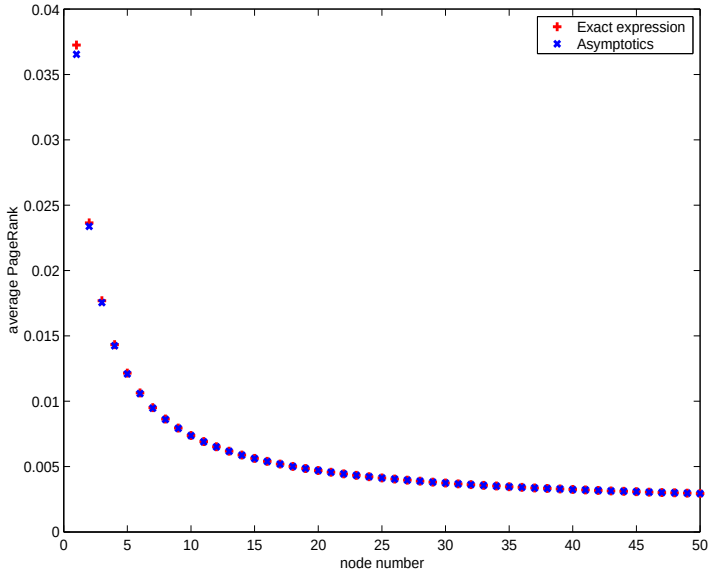


Figure 6.2: Comparison between the asymptotics (6.37) and the exact expression (6.30).

$$\begin{aligned} P(x) &= \mathbb{P}[\pi_v < x] \\ &\approx \mathbb{P}\left[v > \left(\left(\frac{1+n}{1-c}x - \frac{1}{1+c}\right)\frac{1+c}{c}(n+\frac{1}{2})^{-\frac{1+c}{2}}\right)^{-\frac{2}{1+c}} - 1/2\right] \\ &= 1 - \left(\left(\left(\frac{1+n}{1-c}x - \frac{1}{1+c}\right)\frac{1+c}{c}(n+\frac{1}{2})^{-\frac{1+c}{2}}\right)^{-\frac{2}{1+c}} - 1/2\right)/n \\ &= 1 + \frac{1}{2n} - \left(1+\frac{1}{2n}\right)c^{\frac{2}{1+c}}\left(\frac{1+c}{1-c}(n+1)x - 1\right)^{-\frac{2}{1+c}}. \end{aligned} \tag{6.40}$$

In particular, we note that $P(\frac{1-c}{n+1}) = 0$, as $x = \frac{1-c}{n+1}$ is the minimal value of PageRank. Taking the derivative of (6.40), we obtain the density distribution function of the PageRank value

$$p(x) = \frac{2}{1-c}(n+1)\left(1+\frac{1}{2n}\right)c^{\frac{2}{1+c}}\left(\frac{1+c}{1-c}(n+1)x - 1\right)^{-\frac{3+c}{1+c}}. \tag{6.41}$$

For large values of n and for values of x, which are not too small and not too close to one, the expression (6.41) is close to the power law

$$p(x) \asymp \frac{1}{x^{\frac{3+c}{1+c}}}.$$

For instance, for the dumping factor $c = 0.85$, we can conclude that the density distribution of PageRank for nodes, whose numbers are not too small and not too close to n, can be approximated by a Power law with the exponent 2.08. Note that the "mean-field" approximation of Pandurangan et al. [77] gives the exponent 2 and the experiments with the real Web data in Pandurangan et al. [77] give the exponent 2.1.

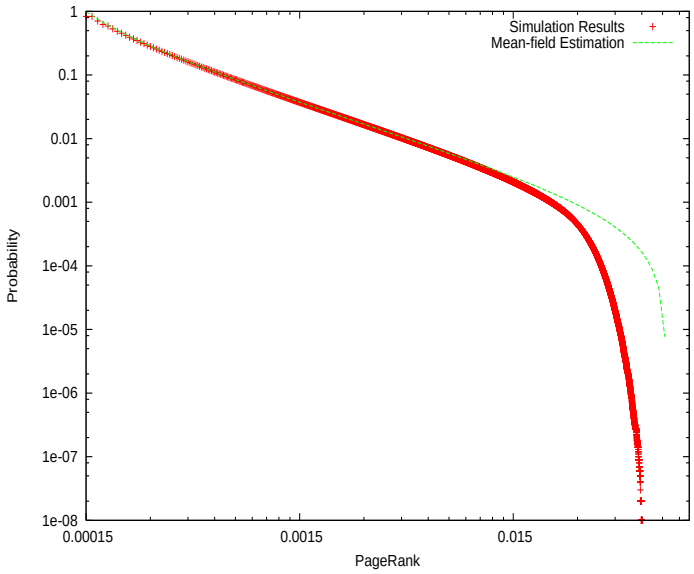


Figure 6.3: Cumulative complimentary distribution function: Simulation results compared to the mean-field estimation.

To test the mean-field estimation (6.40), we have run simulations of our growing network model. The network was grown up to $n = 1000$ for 100000 simulation runs with $m = 10$. In Figure 6.3 the mean-field estimation (6.40) is compared with the cumulative complimentary distribution function $\mathbb{P}[\pi_v > x] = 1 - \mathbb{P}[\pi_v < x]$ obtained from the simulations. As pointed out in [70], when dealing with power laws, it is preferable to work with the cumulative complimentary distribution function rather then with the density distribution function or the histogram. The cumulative distribution function of a power law $x^{-\alpha}$ also follows the power law, but with the exponent $x^{-\alpha+1}$. When calculating the PageRank, we have used $c = 0.85$. We note that plot is indeed close to a straight line for the middle segment of the PageRank range. In [77] the authors also noticed that PageRank follows a power law except those pages with very small PageRank. This phenomenon can easily be explained with the help of (6.41). The term $\frac{1+c}{1-c}(n+1)x$ becomes comparable with 1 in (6.41) for values of x too close to the minimal PageRank $\frac{1-c}{n+1}$ and the distribution density function can not be, in this case, approximated by $O(x^{-\alpha})$. The mismatch for large values of PageRank can be explained as follows: the "mean-field" approach cannot be applied to the nodes with large PageRank because there are simply not enough such nodes to use the "averaging" argument.

As it can be observed from (6.31) and (6.30), the zero node is special. As n grows, its PageRank converges to 0, but, nevertheless, its value is bigger than the PageRank of other nodes. We can normalize the expected value of PageRank of all nodes by $\mathbb{E}\pi_0$. In fact, we have

$$\tilde{\pi}_v = \lim_{n\to\infty} \frac{\mathbb{E}\pi_v(n)}{\mathbb{E}\pi_0(n)} = \frac{(1-c)c\Gamma(\frac{c}{2})}{2\sqrt{\pi}} \frac{\Gamma(v+\frac{1}{2})}{\Gamma(v+\frac{c}{2}+1)}. \tag{6.42}$$

Let us call $\tilde{\pi}$ the relative PageRank. We would like to emphasize that the *relative PageRank* does not depend on time. The relative PageRank closely follows a power law except some initial nodes.

Recall that Google divides the whole range of PageRank in 10 intervals using logarithmic scale. Curiously enough, if PageRank exactly followed a power law, then this division would be independent of c and the exponent of the power law, but would depend only on n, the age of

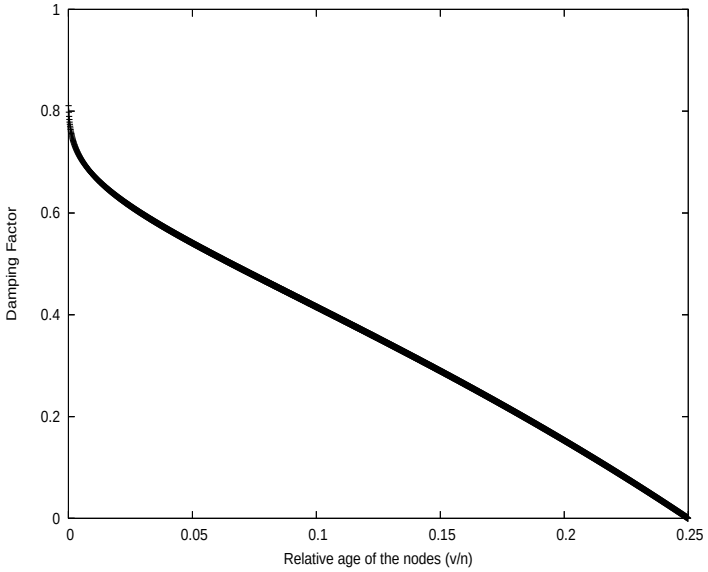


Figure 6.4: The optimal value of c as a function of v/n.

the network. Specifically, in such a case, the following formula holds for the boundaries of the ranking intervals:

$$v_k^* = (n)^{\frac{k}{10}}, k = 1, ..., 10.$$

The above observation justifies further the scale free term for the growing network model.

The authors of [19] investigated the effect of damping factor c on PageRank. In their numerical example they have noticed that PageRank for some nodes attains a maximal value for some value of c. Let us investigate the dependence of PageRank on c in our growing network model. In our case the value of c which maximizes the PageRank expression (6.37) depends on the ratio v/n. In Figure 6.4 we plot the optimal value of c as a function of the ratio v/n. As an example, in Figure 6.5 we plot the expected value of PageRank for node 1 and node 2 when $n = 10000$. If the World Wide Web has 8 billion pages, then the present model suggests that the pages that mostly benefit from the value of $c = 0.85$ are around the node $v = 46212$. Thus, it looks like the damping factor $c = 0.85$ benefits only a small fraction of old pages. Thus, to give a better ranking to less established Web pages and to distribute PageRank more fairly, it is necessary to decrease the value of c. Of course, this will also have a positive effect on the convergence of the numerical methods for PageRank computation. The question by how much the damping factor can be reduced merits a careful special investigation.

Finally, we would like to note that the choice of the initial weight for the zero node was a crucial factor for the derivation of simple explicit expressions. This choice affects only the preferential attachment process. In fact, all the methods in the present work can be applied to the growing network models with different preferential attachment process. The expression (6.19) would change slightly, but there is no guarantee that one could find a simple closed form of the final expression (6.30).

Curiously enough, we have tested several scale free growing network models and in all our experiments the results were very close. Thus, the analysis of PageRank for different growing network models and further generalization of the results is an interesting perspective research direction.

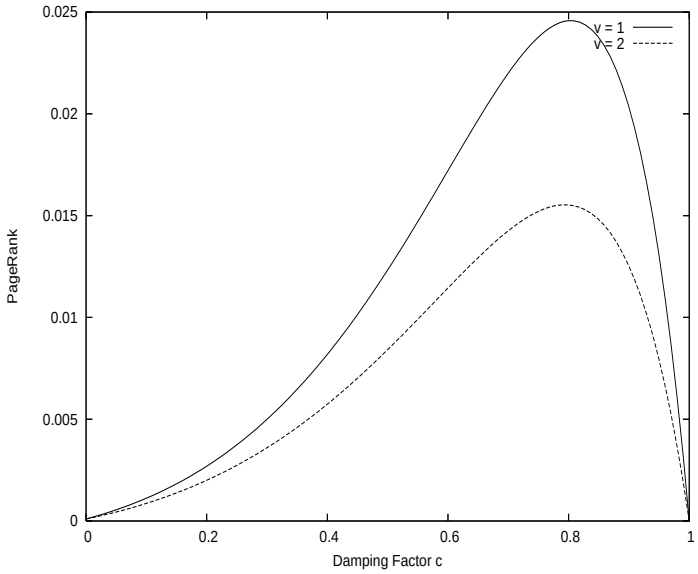


Figure 6.5: The PageRank as a function of c for $v = 1, 2$ and $n = 10000$

Chapter 7

Conclusions

With this thesis we aimed to present the subject of analytical modeling of ad hoc networks. It contains three main topics: general analytical models for multihop networks, flooding impact on these networks and PageRank modeling.

We first reviewed some fundamental results on capacity and connectivity of wireless multihop networks. We started with the structured review of link availability models, which were presented in Chapter 2. Then, we considered MAC protocols and the performance characteristics of some of them in Chapter 3. In Chapter 5 using the SIR based link availability model and some of the results of studies of MAC protocols, we compared two ad hoc network protocols, AODV and OLSR. The equations, which we obtained, estimate the traffic level generated by these protocols. The method can be seen, as an illustration to building from blocks a complex model of an ad hoc network, based on link availability and MAC layers. This approach demonstrates, that SIR based random geometric graph model is a good tool for studying network layer protocols, as it combines the relative simplicity of the unit disk model with the responsive to traffic changes nature of the SIR based models. The additive property of traffic density plays an important role in modeling different types of protocol messages. An interesting continuation of the method seems to be a further investigation of the SIR based random graph model. A combination of these two models, corresponding to outdoor and indoor networks, would also be of extreme interest, as real life wireless networks exhibit properties of both.

Flooding is a mechanism included in many ad hoc network protocols and is responsible for diffusing routing information across networks. Possible synchronisation between the transmissions of the flooding message can cause a heavy load on the network resources. It was a subject of the examination in Chapters 3 and 4. We considered the load of flooding on the network using simulation data. Then we studied the length of paths discovered using this approach. Although the exact solution to this problem is still difficult to obtain, our analysis provides a deeper insight on the flooding process and its influence on the network performance. These results are also used to support the modeling of AODV protocol for its comparison with OLSR in Chapter 5.

In Chapter 6 we take a step into an adjacent domain and we analyse PageRank of scale free growing network model, which is a simplified model of Web graph. The calculation of average PageRank was an interesting problem, as we could represent the evolution of the growing network model with preferential attachments in terms of urn probabilities and we were able to obtain the resulting expression in a closed form. The latter relied on the technique of hypergeometric summation and Zeilberger's algorithm. The analysis of the final expression showed how average PageRank depends on the choice of the damping factor parameter. It

also allows a good estimation of PageRank distribution using a mean field method and, hence, making possible the comparison of the resulting distribution with the power law.

PageRank, apart of its utility for the search engines, is also a reputation system: the pages linked to the nodes with high PageRank also have a high PageRank. In other words, the links from well respected nodes assure a good reputation for their destination node. Reputation systems are likely to play an important role in the world of cooperative and social networks, which includes ad hoc networks. In fact, the ad hoc networks distribute the routing charge among many participating nodes. As we saw earlier, such cooperative behaviour can bring a gain in terms of network performance. But the nodes should have some incentives for providing their services to others and they should be trusted to relay messages. The reputation systems may help in both problems, by attributing a higher reputation to the nodes relaying the messages often and correctly. The reputation systems are intensively studied at the present moment and it is an interesting direction for future works.

Appendix A

Existence of stationary distribution

Before we prove that the chain converges to a stationary distribution we introduce some definitions necessary to work with Markov chains on general spaces [23]. We denote by $\tau_V(x)$ *the hitting time* of V by the Markov chain $\{X_i\}$ started at x:

$$\tau_V(x) = min\{k \geq 1 \ : \ X_k(x) \in V\}$$

Also we denote by P the probability function : $P(x, n, V) = \mathbb{P}[X_n \in V]$

Definition A.1. *Markov chain $\{X_n\}$ defined on $(\mathcal{X}, \mathcal{B})$ is Harris chain, if there is a set $V \in \mathfrak{B}$, probability measure μ on $(\mathcal{X}, \mathcal{B})$ and numbers $n_0 \geq 1$, $0 < p < 1$ such that*
(I_0) $\mathbb{P}(\tau_V(x) < \infty) = 1$ *for all* $x \in \mathcal{X}$
(II) $P(x, n_0, B) \geq p\mu(B)$ *for all* $x \in V$ *and* $B \in \mathfrak{B}$

There are also two more conditions we will need: the condition of *positivity*
(I) $\sup_{x \in V} \mathbb{E}\tau_V(x) < \infty$
and the condition of *aperiodicity*: let $X_k(\mu)$ be our Markov chain with the initial state X_0 drawn according to μ distribution.

$$\tau_V(\mu) = min\{k \geq 0 : X_k(\mu) \in V\}$$

We denote by $n_1, n_2, \ldots$ possible values of $\tau_V(\mu)$,i.e. such values that $\mathbb{P}(\tau_V(\mu) = n_k) > 0$, $k = 1, 2, \ldots$. Then the last condition is
(III) $\exists k$: $G.C.D.(n_0 + n_1, n_0 + n_2, \ldots, n_0 + n_k) = 1$

Definition A.2. *A sequence of distributions $\{P_n\}$ on $(\mathcal{X}, \mathcal{B})$* converges in variation *to P, if*

$$\sup_{B \in \mathfrak{B}} |P_n(B) - P(B)| \to 0, \ n \to \infty \tag{A.1}$$

Proposition A.1. *The chain $\{Y_n\}$ is aperiodic positive Harris chain and it converges (in variation) to a stationary distribution F^*.*

Proof. We need to prove 4 conditions listed above. It is sufficient to present the set V and the numbers n_0 and p such that this conditions will hold:

1. We take for V the whole space $[0, 1] \times [0, 1]$. Therefore the conditions (I_0) and (I) hold.

2. μ is defined the following way: $\mu(A) = 4|A \cap [0, 1/2] \times [0, 1/2]|$. It is the probability measure with its support on the square of area $1/4$. We set $n_0 = 1$ and $p = 1$.

   Take any set $A \in \mathfrak{B}(\mathcal{C})$. The probability $P(x, 1, A) = |A \cap \widehat{F}(x)| + |A \cap \widehat{G}(x)|$, where $|\cdot|$ is Lebesgue measure and $\widehat{F} = [0, x(x)] \times [0, y(x)]$,$\widehat{G} = \mathcal{C} \setminus [1 - x(x), 1] \times [1 - y(x), 1]$ by the definition of the Markov chain $\{Y_k\}$.

   Therefore for all x we have $[0, 1/2] \times [0, 1/2] \subset \widehat{F}(x) \cup \widehat{G}(x)$ (see Fig. to draw). We can conclude that

   $$P(x, 1, A) \geq \mu(A)$$

3. As $\mu(V) = 1 > 0$ and the condition (II) holds with $n_0 = 1$ the condition (III) also holds.

□

Appendix B

Kolmogorov-Prokhorov equation

We present the Kolmogorov-Prokhorov equation following the book of A. Borovkov ([24], chapter 4.4). The Kolmogorov-Prokhorov equation can be seen as a generalization of Wald's equation.

Theorem B.1. *If an integer non-negative random variable ν does not depend on the future with the respect to the sequence of random variables $\{\xi_n\}$ and*

$$\sum_{k=1}^{\infty} \mathbb{P}(\nu \geq k)\mathbb{E}|\xi_k| < \infty,$$

then

$$\mathbb{E}\sum_{k=1}^{\nu} \xi_k = \sum_{k=1}^{\infty} \mathbb{P}(\nu \geq k)\mathbb{E}\xi_k.$$

In our case $\nu = Y_v(n)$ and $\xi_k = c^{X_k}$. The random variable $Y_v(n)$ is independent of $\{X_k\}$, and c^{X_k} are positive, therefore, $\mathbb{E}|c^{X_k}| = \mathbb{E}c^{X_k}$ and the Kolmogorov-Prokhorov equation can be applied to the problem we consider in Chapter 6.

Appendix C

Pólya-Eggenberger urn model

We follow the book [57] in our description of the urn models. The Pólya-Eggenberger urn model starts with one urn where one can find $b + w$ balls of two colors: black and white. Let b be the number of black balls and let w be the number of white balls. At every time step one ball is drawn at random from the urn, then it is returned back together with s balls of the same color. The Pólya-Eggenberger distribution is used to estimate the number of black (or white) balls at time step n. The probability to have k black balls in the urn at time step n can then be expressed as

$$\begin{aligned} \mathcal{P}_{n,k}(w,b,s) &= \binom{n}{k} b(b+s)\dots(b+(k-1)s) \times \\ &\quad \times \frac{w(w+s)\dots(w+(n-k-1)s)}{(b+w)(b+w+s)\dots(b+w+(n-1)s)}, \end{aligned} \tag{C.1}$$

for $k = 0, 1, \dots, n$. Using the gamma function, the above formula can be rewritten as follows:

$$\begin{aligned} \mathcal{P}_{n,k}(w,b,s) &= \binom{n}{k} \frac{\Gamma(\frac{b+w}{s})\Gamma(\frac{b}{s}+k)\Gamma(\frac{w}{s}+n-k)}{\Gamma(b/s)\Gamma(w/s)\Gamma(\frac{b+w}{s}+n)} \\ &= \binom{n}{k} \frac{B(\frac{b}{s}+k, \frac{w}{s}+n-k)}{B(\frac{b}{s}, \frac{w}{s})} \end{aligned} \tag{C.2}$$

It is worthy to note here that the problem of the tree height, which we study in the section 6.7, can also be described in terms of the urn model with a node height value as a mark (or color).

Appendix D

Zeilberger's Algorithm

We follow the book [81] in our description of the Zeilberger's algorithm. Let us consider a sum

$$f(n) = \sum_k F(n,k) \tag{D.1}$$

The goal of the Zeilberger's Algorithm is to find function $G(\cdot,\cdot)$ and coefficients $a_j(n)$ such that

$$\sum_{j=0}^{J} a_j(n) F(n+j,k) = G(n,k+1) - G(n,k) \tag{D.2}$$

This method is also called the method of *creative telescoping*. When such representation is obtained, we can sum the equation (D.2) by k and, if we are lucky with the values F and G, the right part of the sum might collapse to 0 leaving us with an equation of the type:

$$\sum_{j=0}^{J} a_j(n) f(n+j) = 0 \tag{D.3}$$

For example, if $J = 1$, we find the recurrence $a_0(n)f(n) + a_1(n)f(n+1) = 0$ and then $f(n)$ is easy to find.

D. Zeilberger has written the package EKHAD [38] for Maple [65], which implements his algorithm and finds $a_0, \ldots, a_J$ and $R(\cdot,\cdot)$ such that

$$G(n,k) = R(n,k)F(n,k) \tag{D.4}$$

Fortunately, the Zeilberger's Algorithm gives a satisfying result for the sum in (6.34). Let us prove the following lemma.

Lemma D.1.

$$\sum_{i=1}^{n-v} \frac{\Gamma(n-i)\Gamma(i+1+c/2)}{\Gamma(n-v-i+1)\Gamma(i+1)} = \frac{\Gamma(v)\Gamma(n+c/2+1)\Gamma(1+c/2)}{\Gamma(v+c/2+1)\Gamma(n-v+1)} - \frac{\Gamma(n)\Gamma(1+c/2)}{\Gamma(n-v+1)} \tag{D.5}$$

Proof. Consider the internal sum (6.34). We introduce the following notation:

$$F(n,i) = \frac{\Gamma(n+v-i)\Gamma(i+1+c/2)\Gamma(v+c/2+1)\Gamma(n+v+1)}{\Gamma(n-i+1)\Gamma(i+1)\Gamma(v)\Gamma(n+v+c/2+1)\Gamma(1+c/2)} \tag{D.6}$$

It is the summand from (D.5) divided by the result we want to prove (it was guessed from the values of $\mathbb{E}\pi_v$ for $v = 1, 2, 3$), plus we change the variable from $n \to n - v$ and we add the 0th summand. Now, we want to prove that

$$f(n) = \sum_{i=0}^{n} F(n,i) \equiv 1 \tag{D.7}$$

The function *zeil* from EKHAD [38] package in Maple [65] finds the following identities: $a_0 = -1$, $a_1 = 1$ and

$$R(n,i) = -\frac{(n+v-i)i}{(n+v+\frac{c}{2}+1)(n-i+1)} \tag{D.8}$$

Therefore, in our case (D.2) takes the following form

$$F(n,i) - F(n+1,i) = R(n,i+1)F(n,i+1) - R(n,i)F(n,i) \tag{D.9}$$

Then we sum the equation (D.9) for the values $i = 0, \ldots, n-1$ and we find that $R(n,n)F(n,n) - R(n,0)F(n,0) - F(n,n) + F(n+1,n) + F(n+1,n+1) = 0$ for the values F and R in (D.6) and (D.8). Therefore, $f(n) = f(n+1)$. As $f(0) = 1$, it completes the prove.

Note, that it is indeed a proof, because Maple [65] and EKHAD [38] provide the identities (i.e. the values of a_i, $i = 0, 1$ and (D.8)) that can be easily checked. □

List of Figures

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