Sequent Calculus: Focused proof systems (Lecture 4)

Dale Miller

INRIA Saclay & Ecole Polytechnique, France

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Presenting and applying a focused proof system for classical logic.

Invertible rules and the negative phase

Some inference rules are *invertible*, *e.g.*,

$$\frac{A, \Gamma \longrightarrow B}{\Gamma \longrightarrow A \supset B} \qquad \frac{\Gamma \longrightarrow A \qquad \Gamma \longrightarrow B}{\Gamma \longrightarrow A \wedge B} \qquad \frac{\Gamma \longrightarrow B[y/x]}{\Gamma \longrightarrow \forall x.B}$$

First focusing principle: when proving a sequent, apply invertible rules exhaustively and in any order.

This is the *negative phase* of proof search: if formulas are "processes" in an "environment," then these formulas "evolve" without communications ("asynchronously") with the environment.

Non-invertible rules and the positive phase

Some inference rules are not generally invertible, *e.g.*,

$$\frac{\Gamma_1 \longrightarrow A \quad \Gamma_2 \longrightarrow B}{\Gamma_1, \Gamma_2 \longrightarrow A \wedge B} \qquad \frac{\Gamma \longrightarrow B[t/x]}{\Gamma \longrightarrow \exists x.B}$$

Some *backtracking* is generally necessary within proof search using these inference rules.

Second focusing principle: non-invertible rules are applied in a "chain-like" fashion.

This is the *positive phase* of proof search.

Extending the neg/pos distinction to atoms

Focusing proof systems generally extend the neg/pos distinction to atoms.

We shall assume that somehow all atoms are given a *bias*, that is, they are either positive or negative.

A *positive formula* is either a positive atom or has a top-level connective whose right-introduction rule is not invertible.

A *negative formula* is either a negative atom or has a top-level connective whose right-introduction rules is invertible.

Various focusing-like proof system

Uniform proofs [M, Nadathur, Scedrov, 1987] describes goal-directed search and backchaining.

LLF: [Andreoli, 1992]: a focused proof system for linear logic.

LKT/LKQ/LK^{η}*:* focusing systems for classical logic [Danos, Joinet, Schellinx,1993]

LJQ [Herbelin, 1995] permits forward-chaining proof. *LJQ'* [Dyckhoff & Lengrand, 2007] extends it.

λ*RCC* [Jagadeesan, Nadathur, Saraswat, 2005] mixes forward chaining and backward chaining (in a subset of intuitionistic logic).

LJF [Liang & M, 2009] allows forward and backward proof in all of intuitionistic logic. LJT, LJQ, λRCC, and LJ are subsystems.

LKF (following) provides focusing for all of classical logic.

The full picture behind focusing

Andreoli (1992) was the first to give a focused proof system for a full logic (linear logic).

The proof system for MALL (multiplicative-additive linear logic) is remarkably elegant and unambiguous.

Some complexity arises from using the exponentials (!, ?): in particular, exponentials terminate focusing phases.

We now present two comprehensive focused proof systems for classical logic.

- LKF for *classical logic*
- LKF for *classical logic* with fixed points and equality

Classical logic and one-sided sequents

Two conventions for dealing with classical logic.

- Formulas are in *negation normal form*.
  - $B \supset C$ is replaced with $\neg B \vee C$,
  - negations are pushed to the atoms
- Sequents will be one-sided. In particular, the two sided sequent

$$\Sigma : B_1, \ldots, B_n \vdash C_1, \ldots, C_m$$

will be converted to

$$\Sigma : \quad \vdash \neg B_1, \ldots, \neg B_n, C_1, \ldots, C_m.$$

We also drop the "$\Sigma :$" prefix on sequents.

LKF: Focusing for Classical Logic

Formulas are *polarized* as follows.

- atoms are assigned bias (either $+$ or $-$), and
- $\wedge$ $\vee$, t, and f are annotated with either $+$ or $-$.
  Thus: $\wedge^-$, $\wedge^+$, $\vee^-$, $\vee^+$, t^-, t^+, f^-, f^+.

LKF is a focused, one-sided sequent calculus with the sequents

$$\vdash \Theta \Uparrow \Gamma \qquad \text{and} \qquad \vdash \Theta \Downarrow B$$

Here, Θ is a multiset of positive formulas and negative literals, Γ is a multiset of formulas, and B is a formula.

LKF : focused proof systems for classical logic

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A \qquad \vdash \Theta \Uparrow \Gamma, B}{\vdash \Theta \Uparrow \Gamma, A \wedge^- B}$$

$$\frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A, B}{\vdash \Theta \Uparrow \Gamma, A \vee^- B} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A[y/x]}{\vdash \Theta \Uparrow \Gamma, \forall x A}$$

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A \qquad \vdash \Theta \Uparrow \Gamma, B}{\vdash \Theta \Uparrow \Gamma, A \wedge^- B}$$

$$\frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A, B}{\vdash \Theta \Uparrow \Gamma, A \vee^- B} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A[y/x]}{\vdash \Theta \Uparrow \Gamma, \forall x A}$$

$$\frac{}{\vdash \Theta \Downarrow t^+} \qquad \frac{\vdash \Theta \Downarrow A \qquad \vdash \Theta \Downarrow B}{\vdash \Theta \Downarrow A \wedge^+ B} \qquad \frac{\vdash \Theta \Downarrow A_i}{\vdash \Theta \Downarrow A_1 \vee^+ A_2} \qquad \frac{\vdash \Theta \Downarrow A[t/x]}{\vdash \Theta \Downarrow \exists x A}$$

LKF : focused proof systems for classical logic

$$\frac{}{\vdash \Theta \Uparrow \Gamma, t^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A \qquad \vdash \Theta \Uparrow \Gamma, B}{\vdash \Theta \Uparrow \Gamma, A \wedge^- B}$$

$$\frac{\vdash \Theta \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, f^-} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A, B}{\vdash \Theta \Uparrow \Gamma, A \vee^- B} \qquad \frac{\vdash \Theta \Uparrow \Gamma, A[y/x]}{\vdash \Theta \Uparrow \Gamma, \forall x A}$$

$$\frac{}{\vdash \Theta \Downarrow t^+} \qquad \frac{\vdash \Theta \Downarrow A \qquad \vdash \Theta \Downarrow B}{\vdash \Theta \Downarrow A \wedge^+ B} \qquad \frac{\vdash \Theta \Downarrow A_i}{\vdash \Theta \Downarrow A_1 \vee^+ A_2} \qquad \frac{\vdash \Theta \Downarrow A[t/x]}{\vdash \Theta \Downarrow \exists x A}$$

Init $\quad$ Store $\quad$ Release $\quad$ Decide

$$\frac{}{\vdash \neg P_a, \Theta \Downarrow P_a} \qquad \frac{\vdash \Theta, C \Uparrow \Gamma}{\vdash \Theta \Uparrow \Gamma, C} \qquad \frac{\vdash \Theta \Uparrow N}{\vdash \Theta \Downarrow N} \qquad \frac{\vdash P, \Theta \Downarrow P}{\vdash P, \Theta \Uparrow \cdot}$$

P positive; P_a positive literal; N negative;
C positive formula or negative literal.

About the structural rules in LKF

The only form of *contraction* is in the Decide rule

$$\frac{\vdash P, \Theta \Downarrow P}{\vdash P, \Theta \Uparrow \cdot}$$

The only occurrence of *weakening* is in the Init rule.

$$\frac{}{\vdash \neg P_a, \Theta \Downarrow P_a}$$

Thus negative non-atomic formulas are treated *linearly* (in the sense of linear logic).

Only positive formulas are contracted.

The abstraction behind focused proofs

We can ignore the internal structure of phases and consider only their boundaries.

We can now move from *micro-rules* (introduction rules) to *macro-rules* (pos or neg phases).

The *decide depth* of an LKF proofs is the maximum number of *Decide* rules along any path starting from the end-sequent.

This measures counts "bi-poles": one positive phase followed by a negative phase.

Results about LKF

Let B be a first-order logic formula and let $\hat{B}$ result from B by placing $+$ or $-$ on t, f, $\wedge$, and $\vee$ (there are exponentially many such placements).

Theorem. B is a first-order theorem if and only if $\hat{B}$ has an LKF proof. [Liang & M, TCS 2009]

Thus the different polarizations do not change *provability* but can radically change the *proofs*.

Recall the Fibonacci series example: one specification yielded an exponential time algorithm or a linear time algorithm depending only on bias assignment.

An example

Let a, b, c be positive atoms and let Θ contain the formula $a \wedge^+ b \wedge^+ \neg c$.

$$
\cfrac{
  \cfrac{}{\vdash \Theta \Downarrow a}\ \mathit{Init}
  \qquad
  \cfrac{}{\vdash \Theta \Downarrow b}\ \mathit{Init}
  \qquad
  \cfrac{\cfrac{\vdash \Theta, \neg c \Uparrow \cdot}{\vdash \Theta \Uparrow \neg c}}{\vdash \Theta \Downarrow \neg c}\ \mathit{Release}
}{
  \cfrac{\vdash \Theta \Downarrow a \wedge^+ b \wedge^+ \neg c}{\vdash \Theta \Uparrow \cdot}\ \mathit{Decide}
}\ \mathit{and}
$$

This derivation is possible iff Θ is of the form $\neg a, \neg b, \Theta'$. Thus, the "macro-rule" is

$$
\frac{\vdash \neg a, \neg b, \neg c, \Theta' \Uparrow \cdot}{\vdash \neg a, \neg b, \Theta' \Uparrow \cdot}
$$

Two certificates for propositional logic: negative

Use $\wedge^-$ and $\vee^-$. Their introduction rules are invertible. The initial "macro-rule" is huge, having all the clauses in the conjunctive normal form of B as premises.

$$\dfrac{\cdots \quad \dfrac{\dfrac{}{\vdash L_1, \ldots, L_n \Downarrow L_i}\ \mathit{Init}}{\vdash L_1, \ldots, L_n \Uparrow \cdot}\ \mathit{Decide} \quad \cdots}{\dfrac{\vdots}{\vdash \cdot \Uparrow B}}$$

The proof "certificate" can specify the complementary literals for each premise or it can ask the checker to *search* for such pairs.

Proof certificates can be tiny but require exponential time for checking.

Two certificates for propositional logic: positive

Use $\wedge^+$ and $\vee^+$. Sequents are of the form $\vdash B, \mathcal{L} \Uparrow \cdot$ and $\vdash B, \mathcal{L} \Downarrow P$, where B is the original formula to prove, P is positive, and $\mathcal{L}$ is a set of negative literals.

Macro rules are in one-to-one correspondence with $\phi \in DNF(B)$. Divide ϕ into ϕ^- (negative literals) and ϕ^+ (positive literals).

$$\cfrac{\cfrac{\{\vdash B, \mathcal{L}, N \Uparrow \cdot \quad | \quad N \in \phi^-\}}{\vdash B, \mathcal{L} \Downarrow B}}{\vdash B, \mathcal{L} \Uparrow \cdot}\ \textit{Decide} \quad \text{provided } \neg\phi^+ \in \mathcal{L}$$

Proof certificates are sequences of members of $DNF(B)$. Size and processing time can be reduced (in response to "cleverness").

Positives allow "clever" choices

To illustrate the trade-off between proof-size and proof-checking time consider the following simple example.

Let B be a propositional formula with a large conjunctive normal form. Let B^- (respectively, B^+) be the result of annotating all the connectives in B negative (respectively, positively).

Consider the tautology $C = (p \vee B) \vee \neg p$.

A *negative focused proof* results from computing the conjunctive normal form of C and then observing that each disjunct is trivial.

There are many *positive focused proof* but one has decide depth 2: first move through C to pick $\neg p$ and then move again through C to pick p.

Herbrand's Theorem proved

Herbrand's Theorem.
Let B be a quantifier-free first-order formula. $\exists\bar{x}.B$ is a theorem if and only if there is an $n \geq 1$ and substitutions $\theta_1, \ldots, \theta_n$ such that $B\theta_1 \vee \cdots \vee B\theta_n$ is tautologous.

This theorem is easily proved by the completeness of LKF.