Higher-level rules for sequent calculus

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The original axioms-as-rules problem

How to incorporate inference rules encoding axioms into existing proof systems for classical and intuitionistic logics?

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Gentzen: Add mathematical theories to first-order logic.

Consistency of the arithmetic without complete induction.

Equality:	
$\forall x\,(x = x)$	(reflexivity)
$\forall x\,\forall y\,(x = y \supset y = x)$	(symmetry)
$\forall x\,\forall y\,\forall z\,((x = y \,\&\, y = z) \supset x = z)$	(transitivity)
One:	
$\exists x\,(\mathrm{One}\;x)$	(existence of 1)
$\forall x\,\forall y\,((\mathrm{One}\;x \,\&\, \mathrm{One}\;y) \supset x = y)$	(uniqueness of 1)
Predecessor:	
$\forall x\,\exists y\,(x\mathrm{Pr}y)$	(existence of successor)
$\forall x\,\forall y\,(x\mathrm{Pr}y \supset \neg\,\mathrm{One}\;y)$	(1 has no predecessor)
$\forall x\,\forall y\,\forall z\,\forall u\,((x\mathrm{Pr}y \,\&\, z\mathrm{Pr}u \,\&\, x = z) \supset y = u)$	(uniqueness of successor)
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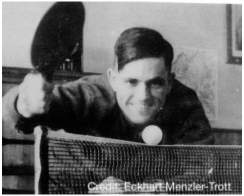

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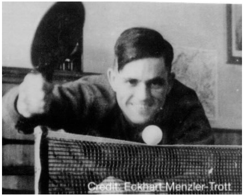

"If our arithmetic is inconsistent, there exists a [cut-free] LK *derivation with endsequent*

$$\mathfrak{U}_1, \ldots \mathfrak{U}_n \vdash$$

where $\mathfrak{U}_1, \ldots \mathfrak{U}_n$ *are arithmetic axiom formulae."*

The original axioms-as-rules problem

How to incorporate inference rules encoding axioms into existing proof systems for classical and intuitionistic logics?

A naive attempt: Add non-logical axioms.

Assume $\vdash P \supset Q$ and $\vdash P$. Then

$$\dfrac{\dfrac{}{\vdash P} \qquad \dfrac{\dfrac{}{\vdash P \supset Q} \quad \dfrac{\dfrac{}{P \vdash P} \quad \dfrac{}{Q \vdash Q}}{P, P \supset Q \vdash Q} \supset_l}{P \vdash Q}\ cut}{\vdash Q}\ cut$$

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The *Hauptsatz* fails for systems with proper axioms.

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A better approach: Add non-logical rules of inference

$$\frac{\Gamma, Q \vdash C}{\Gamma, P \vdash C}\ P \supset Q \qquad \frac{\Gamma, P \vdash C}{\Gamma \vdash C}\ P$$

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The sequent $\vdash Q$ now has the (cut-free) proof

$$\dfrac{\dfrac{\overline{Q \vdash Q}}{P \vdash Q}\ P \supset Q}{\vdash Q}\ P$$

Polarities of connectives

Polarization is a feature of linear logic: $\otimes$**,** $\&$**,** $\oplus$**,** $⅋$

- If the right-introduction rule is invertible, the connective is negative.
- If the left-introduction rule is invertible, the connective is positive.
- De Morgan duality flips polarity. Polarity for atoms is assigned arbitrarily.

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First-order classical and intuitionistic language:

$$A ::= P(x) \mid A \wedge A \mid t \mid A \vee A \mid f \mid A \supset A \mid \exists x\, A \mid \forall x\, A$$

Polarized connectives:

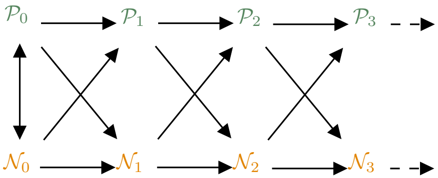
- In classical logic
  - positive and negative versions of the logical connectives and constants:

$$\wedge^-, \wedge^+, t^-, t^+, \vee^-, \vee^+, f^-, f^+$$

- In intuitionistic logic
  - polarized classical connectives and constants where $f^-, \vee^-$ do not occur;
  - negative implication: $\supset$.
- First-order quantifiers: $\forall$ negative and $\exists$ positive.
- A formula is positive if it is a positive atom or has a top-level positive connective.
- A formula is negative if it is a negative atom or has a top-level negative connective.

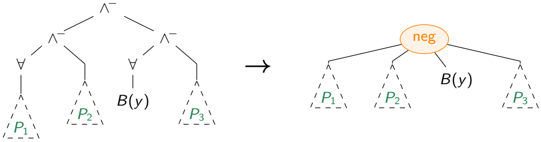
A fresh view to an old problem

Combining the polarities' hierarchy [Ciabattoni et al., 2008] with a



e.g., if B is in $\mathcal{P}_0$ then $\forall x \exists y \forall z.B$ is in $\mathcal{N}_3$.

$(\forall x P_1 \wedge^- P_2) \wedge^- (\forall y B(y) \wedge^- P_3)$



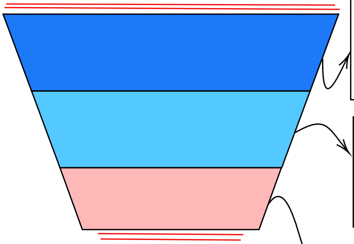
Bipolar $= \mathcal{N}_2$
(polarities flip at most twice)

A fresh view to an old problem

Combining the polarities' hierarchy [Ciabattoni et al., 2008] with a

systematic construction of synthetic rules from axioms using focusing [Andreoli, 1992], justifies the introduction of the class of bipolar axioms. Here, $B \in \mathcal{N}_2$.

$$\Gamma, B, \Gamma_1 \Uparrow \cdot \vdash \cdot \Uparrow \Delta_1 \quad \cdots \quad \Gamma, B, \Gamma_n \Uparrow \cdot \vdash \cdot \Uparrow \Delta_n$$



$$\frac{\Gamma, B \Downarrow B \vdash \Delta}{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow \Delta}\ D_l$$

Store: $C \in \mathcal{N}_{n-2}$. Thus, $\forall i$, $\Gamma_i \subseteq \mathcal{N}_{n-2}$

$$\frac{C, \Gamma \Uparrow \Theta \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow C, \Theta \vdash \cdot \Uparrow \Delta}\ s_l$$

Invertible phase (formulas in $\mathcal{P}_{n-1}$)
Invertible rules are applied eagerly

$$\frac{\Gamma \Uparrow B_1, B_2 \vdash \cdot \Uparrow \Delta}{\Gamma \Uparrow B_1 \wedge^+ B_2 \vdash \Uparrow \Delta}\ \wedge_l^+$$

Non-invertible phase (B is $\mathcal{N}_n$)
Focusing persists

$$\frac{\Gamma \vdash B_i \Downarrow \Delta}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash \Delta}\ \wedge_l^+$$

If $\Gamma \subseteq \mathcal{N}_n$ then $\Gamma_i \subseteq \mathcal{N}_{n-2}$, for all $i = 1, \ldots, n$.

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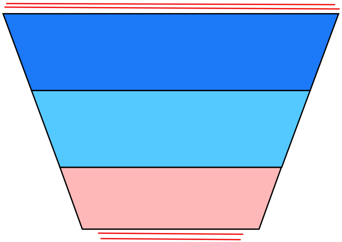

$$\frac{\Gamma, B \Downarrow B \vdash \Delta}{\Gamma, B \Uparrow \cdot \vdash \cdot \Uparrow \Delta} D_l$$

Corresponding synthetic rule

(in LK or LJ)

$$\frac{\Gamma, \Gamma_1 \vdash \Delta_1 \quad \ldots \quad \Gamma, \Gamma_n \vdash \Delta_n}{\Gamma \vdash \Delta} B$$

If $\Gamma \subseteq \mathcal{N}_n$ then $\Gamma_i \subseteq \mathcal{N}_{n-2}$, for all $i = 1, \ldots, n$.

The main results [Marin, Miller, Pimentel & Volpe, 2022]

Theorem 1. Synthetic rules built from bipolar ($\mathcal{N}_2$) axioms involve only atomic formulas.

Theorem 2. The cut rule is admissible in the extension of LK/LJ with synthetic rules corresponding to bipolar axioms.

Example: Various clauses as bipoles

$$\forall \bar{z}(P_1^+ \wedge^+ \ldots \wedge^+ P_m^+ \supset Q^+) \qquad \frac{\overline{P}, Q, \Gamma \vdash \Delta}{\overline{P}, \Gamma' \vdash \Delta}\ FC$$

Forward-chaining [Simpson, Negri, Ciabattoni]

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$$\forall \bar{z}(P_1^- \wedge^- \ldots \wedge^- P_m^- \supset Q^-) \qquad \frac{\Gamma \vdash P_1, \Delta \quad \ldots \quad \Gamma \vdash P_m, \Delta}{\Gamma \vdash Q, \Delta'}\ BC$$

Back-chaining [Viganò]

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Back-chaining [Viganò]

Other examples: Geometric, co-geometric, universal axioms, ...

$$\forall\overline{z}(P_1^\pm \wedge^\pm \ldots \wedge^\pm P_m^\pm \supset \exists\overline{x}_1 \hat{M}_1 \vee^\pm \ldots \vee^\pm \exists\overline{x}_n \hat{M}_n)$$
$$\forall\overline{z}(\forall\overline{x}_1 \hat{M}_1 \wedge^\pm \ldots \wedge^\pm \forall\overline{x}_n \hat{M}_n \supset P_1^- \vee^- \ldots \vee^- P_m^-)$$
$$\forall\overline{z}(P_1^\pm \wedge^\pm \ldots \wedge^\pm P_m^\pm \supset Q_1^\pm \vee^\pm \ldots \vee^\pm Q_n^\pm)$$

Recapitulation

- **Polarity of connectives:** invertibility vs non-invertibility of introduction rules
- **Focusing:** uses polarity to organize proofs into a two-phase structure.

  These features of proofs arose within linear logic. The LKF and LJF proof systems apply these features to classical and intuitionistic logics. [Liang & Miller, 2009]
- **Synthetic inference rules:**
  - **Bipoles:** A flexible means exists to translate bipoles ($\mathcal{N}_2$) to inference rules involving only atomic formulas: see [Marin, Miller, Pimentel, & Volpe, 2022].

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- **Synthetic inference rules:**
  - **Bipoles:** A flexible means exists to translate bipoles ($\mathcal{N}_2$) to inference rules involving only atomic formulas: see [Marin, Miller, Pimentel, & Volpe, 2022].
  - **Non-bipoles:** The topic of the rest of this talk. Two approaches:
    - Transform non-bipoles into bipoles.
    - Use a higher-level system of rules.

One approach to treating non-bipoles: Remove them

Transform non-bipolar formulas into bipolar formulas by introducing new predicate symbols.

- Tseitin [1960's], Mints et al. [1982].
- Andreoli: skolemization [1992], bipolarization [2001].
- Dyckhoff & Negri: geometrisation [2015]
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With higher-order quantification, provability can be maintained.

$$u \supset ((p \supset q) \supset r) \supset s \dashv\vdash \exists x. \begin{pmatrix} (u \supset (x \supset r) \supset s) \wedge \\ (x \supset (p \supset q)) \wedge \\ ((p \supset q) \supset x) \end{pmatrix}$$

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$$u \supset ((p \supset q) \supset r) \supset s \dashv\vdash \exists x. \left(\begin{array}{c} (u \supset (x \supset r) \supset s) \wedge \\ (x \supset (p \supset q)) \wedge \\ ((p \supset q) \supset x) \end{array} \right)$$

If you drop $\exists x$ for a new predicate symbol, the expressions are equisatisfiable.

$$u \supset ((p \supset q) \supset r) \supset s \;=\!\mid\!\models \left(\begin{array}{c} (u \supset (x \supset r) \supset s) \wedge \\ (x \supset (p \supset q)) \wedge \\ ((p \supset q) \supset x) \end{array} \right)$$

N.B.: With only implications, B is of order n if and only if $B \in \mathcal{N}_n$.

Another approach to treating non-bipoles: Higher-level of rules

Let C denote $u \supset ((p \supset q) \supset r) \supset s$. (Assume that s has negative polarity.)

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma, C \Uparrow \cdot \vdash \cdot \Uparrow u}{\Gamma, C \Uparrow \cdot \vdash u \Uparrow \cdot}\ s_l}{\Gamma, C \vdash u \Downarrow}\ R_r \qquad \dfrac{\dfrac{\dfrac{\Gamma, C, p \supset q \Uparrow \cdot \vdash \cdot \Uparrow r}{\Gamma, C \Uparrow p \supset q \vdash r \Uparrow \cdot}}{\Gamma, C \Uparrow \cdot \vdash (p \supset q) \supset r \Uparrow \cdot}}{\Gamma, C \vdash (p \supset q) \supset r \Downarrow}\ R_l \qquad \dfrac{}{\Gamma, C \Downarrow s \vdash s}\ I_l}{\Gamma, C \Downarrow u \supset ((p \supset q) \supset r) \supset s \vdash s}}{\Gamma, C \Uparrow \cdot \vdash \cdot \Uparrow s}$$

This justifies the synthetic inference rule

$$\dfrac{\Gamma \vdash u \quad \Gamma, p \supset q \vdash r}{\Gamma \vdash s}\ C$$

Unfortunately, this contains an occurrence of a logical connective.

The $\mathcal{N}_1$ formula $p \supset q$ formula can be replaced by an inference rule: which rule depends on polarity.

Higher-level of rules: an example

The synthetic rule for $C = u \supset ((p \supset q) \supset r) \supset s$ (where s has negative polarity).

$$\left[\quad \dfrac{\Gamma \vdash u \qquad \dfrac{\begin{array}{c}\text{(Rule based on } p \supset q)\\ \vdots\end{array}}{\Gamma \vdash r}}{\Gamma \vdash s}\; C \quad\right]$$

The second premise has an inference rule that is available to prove that premise.

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The second premise has an inference rule that is available to prove that premise. The shape of that rule depends on the polarity of p and q. There are four possibilities.

$$\dfrac{\Psi \vdash p}{\Psi \vdash q}\ (p-,\ q-) \qquad \dfrac{\Psi \vdash p \quad \Psi, q \vdash E}{\Psi \vdash E}\ (p-,\ q+)$$

$$\dfrac{}{\Psi, p \vdash q}\ (p+,\ q-) \qquad \dfrac{\Psi, p, q \vdash E}{\Psi, p \vdash E}\ (p+,\ q+)$$

For example,

$$\left[\quad \dfrac{\Gamma \vdash u \qquad \dfrac{\begin{array}{c}\left(\dfrac{\Psi, p, q \vdash E}{\Psi, p \vdash E}\right)\\ \vdots\end{array}}{\Gamma \vdash r}}{\Gamma \vdash s}\ C \quad\right]$$

Higher-level of rules: An example with quantifiers

The (polarized) formula stating the existence of least upper bounds.

$$\forall x \forall y \exists z (x \leq z \wedge^{+} y \leq z \wedge^{+} \forall w (x \leq w \wedge^{+} y \leq w \supset z \leq w)),$$

Focusing on this formula yields the derivation.

$$\frac{\Sigma, z : \forall w (x \leq w \wedge^{+} y \leq w \supset z \leq w),\ x \leq z,\ y \leq z,\ \Gamma \vdash \Delta}{\Sigma : \Gamma \vdash \Delta}$$

Sequents are prefixed with a list of eigenvariables Σ which are bound over the sequent.

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The assumption $\forall w (x \leq w \wedge^+ y \leq w \supset z \leq w)$ can be converted to an inference rule (depending on the polarity of the $\leq$ predicate). For example,

$$\frac{\left(\dfrac{\Sigma, z : \Gamma \vdash x \leq w \qquad \Sigma, z : \Gamma \vdash y \leq w}{\Sigma, z : \Gamma \vdash z \leq w}\right) \\ \vdots}{\dfrac{\Sigma, z : \Gamma, x \leq z,\ y \leq z \vdash \Delta}{\Sigma : \Gamma \vdash \Delta}}$$

There are no logical constants. The scope of variables is getting complicated.

Higher-level of rules: Continued

The $\mathcal{N}_3$ formula

$$\forall x \forall y \exists z (x \leq z \wedge^+ y \leq z \wedge^+ \forall w (x \leq w \wedge^+ y \leq w \supset z \leq w)),$$

can be bipolarized by introducing a new predicate lub so that the atomic formula (lub x y z) denotes the fact that z is the least upper bound of x and y.

$$\forall x \forall y \exists z . [(x \leq z \wedge^+ y \leq z \wedge^+ \text{lub } x \ y \ z)] \wedge^-$$

$$\forall x \forall y \forall z . [\text{lub } x \ y \ z \equiv \forall w . (x \leq w \wedge^+ y \leq w \supset z \leq w)]$$

Focusing on this formula yields the derivation.

$$\frac{\Sigma, z : \text{lub } x \ y \ z, \ x \leq z, \ y \leq z, \ \Gamma \vdash \Delta}{\Sigma : \Gamma \vdash \Delta}$$

It seems more natural to use this formulation with a new predicate than the rule with a new scoped inference rule.

Note: If the order relation is also known to be antisymmetric, then the lub predicate actually defines a function. Moving from relations to functions is another topic.

Higher-level of rules in natural deduction

If we limit ourselves to

- negative connectives ($\supset$, $\wedge^-$, $\forall$) and
- negative polarized atoms,

then the sequent calculus is essentially natural deduction: [Herbelin 1994], [Espírito Santo 2007].

Systems of higher-level rules in natural deduction have been considered long ago.

- Schroeder-Heister, A Natural Extension of Natural Deduction, 1984.
- Avron, Gentzenizing Schroeder-Heister's Natural Extension of Natural Deduction, 1990.
- Harper, Honsell, Gordon Plotkin, "A Framework for Defining Logics", 1993.

Higher-level of rules in sequent calculus

Using the papers [Marin, Miller, Pimentel, & Volpe, 2022] and [Miller & Pimentel, 2013], we should be able to describe systems of higher-level rules for the sequent calculus that accounts for

- intuitionistic, classical, linear proof systems,
- additive and multiplicative connectives,
- forward-chaining and back-chaining polarities, and
- first-order quantification and eigenvariables.

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However: Capturing these features without using logical connectives seems a questionable pursuit since logical formulas have evolved to capture all these features (except for the polarity of atoms).

- The implications $\multimap$, $\Rightarrow$ do not have natural counterparts in inference rules.
- The nesting of scopes for quantifiers ($\forall$, $\exists$) is natural in logical formulas, while writing explicit binders in inference rules is eschewed.

Future plans

- We plan to consider how higher-level rules can be organized to capture the richness of inference in the sequent calculus for (at least) classical, intuitionistic, and linear logics.

- We will need to understand the trade-offs between bipolarizing formulas (with the introduction of new predicates) or not.
  - Developing these approaches in an interactive theorem prover (such as Abella) might provide an interesting setting to explore these trade-offs in various simple mathematical theories.

- We also hope to understand the relations between higher-level rules and more "exotic" proof systems: hypersequents [Ciabattoni & Genco, 2018].