Proof and refutation in MALL as a game

Dale Miller

INRIA-Saclay & LIX, École Polytechnique
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Joint work with Olivier Delande and Alexis Saurin.
Related papers appeared in MFPS 2006, LICS 2008, TCS 2009.

Outline

Games and proof

Two style of games used in computer science.

Function-argument interaction: function against environment. Models proof normalization. Gets interesting with higher-order computations. *c.f.* Hyland-Ong, Abramsky, full abstraction for PCF.

Dialogue games: Style used in this talk.

"If I have a proof, I can win the argument."

A tradition starting with Lorenzen [1960/61], Hintikka [1968], ...

An small story

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Why use games in logic? So one can relax at PhD defenses!

Prolog and noetherian Horn clauses

Assume that the *noetherian Horn clause program*[1] $\mathcal{P}$ is loaded into Prolog and we ask the query `?- G.`

Prolog will respond by either reporting `yes` or `no`.

If `yes` then Prolog has a proof of G. Such a proof can be represented in the sequent calculus of Gentzen.

If `no` then there is a proof of $\neg G$. Requires the *closed world assumption* or "Clark's completion". Captured in proof theory using *fixed points* (Schroeder-Heister & Hallnäs, Girard, and Baelde & McDowell & Miller & Tiu).

[1] "noetherian" and "Horn" essentially means that G is one big, purely synchronous formula.

Proof and refutation in one computation

Prolog did *one, neutral* computation which yielded a proof of G or a refutation of G (i.e., a proof of $\neg G$).

But the "proof search" explanation of logic programming requires that one

$$\text{start with either} \quad \longrightarrow G \text{ or with} \quad \longrightarrow \neg G.$$

A failed attempt to find a proof does not give, in general, enough information to build a proof of the negation.

Two motivating questions.
(1) Can we *formalize* this in a neutral style? [Use games.]
(2) Can the neutral style be *extended* to richer logics? [Prolog turns out to be a one-move game.]

Contribution: General

We define a game semantics for the multiplicative additive fragment of linear logic (MALL).

- Our approach is distinct from other, well-known game semantics: Hyland & Ong, Blass, Abramsky. We do not model cut-elimination.
- Our game is *positional*.
- Winning strategies correspond to cut-free proofs.
- Games can have three outcomes: winning strategy for player (proof), winning strategy for opponent (proof of the negation), no winning strategy for either player (unprovability of both).

Contribution: Focused proofs

Our games can also help to illuminate focused proofs.

- Why are invertible and non-invertable rules duals of each other?
  - When considering the opponent's move, I have no choice and must consider all of them (the proof step is invertible).
  - When considering my move, I usually have a choice (the proof step is non-invertible).
- In MALL, the treatment of introduction rules is immediate. The treatment of "structural rules" is less clear.
  - Single-focus or multiple-focus? remove all or only some asynchrony?
  - Our games provide a natural choice among such alternatives.

The additive fragment of MALL

Syntax

$$F := F \oplus F \mid 0 \mid F \,\&\, F \mid \top$$

Inference rules

$$\frac{\vdash F}{\vdash F \oplus G}\,\oplus_1 \quad \frac{\vdash G}{\vdash F \oplus G}\,\oplus_2 \quad \frac{\vdash F \quad \vdash G}{\vdash F \,\&\, G}\,\& \quad \frac{}{\vdash \top}\,\top$$

The purely additive fragment of MALL has no room for atoms: since there are no commas, there are no initial rules.

Neutral expressions

Let us define a two-player game for this logic.
A neutral expression E represents a pair of dual formulas.

$$\begin{array}{rcl} G & := & E + E \mid \mathbb{0} \\ E & := & G \mid \updownarrow G \end{array}$$

A neutral expression E has a *positive* and a *negative* translation.

E	$E_1 + E_2$	$\mathbb{0}$	$\updownarrow G$
$[E]^+$	$[E_1]^+ \oplus [E_2]^+$	0	$[G]^-$
$[E]^-$	$[E_1]^- \mathbin{\&} [E_2]^-$	$\top$	$[G]^+$

A simple additive game

Positions are neutral expressions.

Consider the rewrite relation defined by

$$E_1 + E_2 \mapsto E_1 \qquad E_1 + E_2 \mapsto E_2$$

When playing at position E, a player may move to F iff $E \mapsto^* \updownarrow F$.
A player loses at position E if there is no move from E.

Theorem
The player (resp. the opponent) has a winning strategy in E iff $[E]^+$ (resp. $[E]^-$) is provable.

This game is also called a Hintikka game.

Interpretation of moves

A rewrite step (aka "micro-move") $E_1 + E_2 \mapsto E_i$ is seen as

$$\frac{\vdash [E_i]^+}{\vdash [E_1]^+ \oplus [E_2]^+} \oplus_i$$

by the player
(choses which disjunct to prove)

$$\frac{\vdash [E_1]^- \quad \vdash [E_2]^-}{\vdash [E_1]^- \,\&\, [E_2]^-} \,\&$$

by the opponent
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A move (aka "macro-move") $E \mapsto^* \updownarrow F$ is seen as a full layer of introductions of $\oplus$ by the player and as a full layer of introductions of $\&$ by the opponent (in *focused* proof systems, those layers are called *phases*).

Features of the additive game

- The game is *determinate*,
- it is symmetric,
- the players view the game as *dual derivations*,
- a micro-move corresponds to the application of an inference rule, and
- a macro-move corresponds to a full phase.

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Objective

Define a similar game for the more expressive logic MALL without atoms.
Note: MALL with or without atoms is PSPACE-complete.
Note: Delande's PhD & TCS paper presents MALL with atoms.

MALL without atoms

$$F := F \oplus F \mid 0 \mid F \,\&\, F \mid \top \mid F \parr F \mid \bot \mid F \otimes F \mid 1$$

Additives

$$\frac{\vdash F_i, \Delta}{\vdash F_1 \oplus F_2, \Delta}\ \oplus_i \qquad \frac{\vdash F, \Delta \quad \vdash G, \Delta}{\vdash F \,\&\, G, \Delta}\ \& \qquad \frac{}{\vdash \top, \Delta}\ \top$$

Multiplicatives

$$\frac{\vdash F, G, \Delta}{\vdash F \parr G, \Delta}\ \parr \qquad \frac{\vdash \Delta}{\vdash \bot, \Delta}\ \bot \qquad \frac{\vdash F, \Delta_1 \quad \vdash G, \Delta_2}{\vdash F \otimes G, \Delta_1, \Delta_2}\ \otimes \qquad \frac{}{\vdash 1}\ 1$$

Note: initial and cut are admissible.

Neutral expressions

We add a multiplicative neutral connective and its unit.

$$
\begin{array}{rcl}
G & ::= & E + E \mid \mathbb{0} \mid E \times E \mid \mathbb{1} \\
E & ::= & G \mid \updownarrow G
\end{array}
$$

E	$E_1 + E_2$	$\mathbb{0}$	$E_1 \times E_2$	$\mathbb{1}$	$\updownarrow G$
$[E]^+$	$[E_1]^+ \oplus [E_2]^+$	0	$[E_1]^+ \otimes [E_2]^+$	1	$[G]^-$
$[E]^-$	$[E_1]^- \,\&\, [E_2]^-$	$\top$	$[E_1]^- \parr [E_2]^-$	$\bot$	$[G]^+$

Parallelism vs permutability

Consider the two following dual derivations:

$$\dfrac{\dfrac{\vdash F}{\vdash F \oplus G}\oplus_1 \quad \dfrac{}{\vdash 1}1}{\vdash (F \oplus G) \otimes 1}\otimes \qquad\Bigg|\qquad \dfrac{\dfrac{\dfrac{\vdash F^\perp}{\vdash F^\perp, \perp}\perp \quad \dfrac{\vdash G^\perp}{\vdash G^\perp, \perp}\perp}{\vdash F^\perp \mathbin{\&} G^\perp, \perp}\mathbin{\&}}{\vdash (F^\perp \mathbin{\&} G^\perp) \parr \perp}\parr$$

On the left, the player may apply $\oplus_1$ and 1 in any order. On the right, applying $\perp$ before $\&$ would change the derivation.

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$$\dfrac{\vdash F}{\vdash (F \oplus G) \otimes 1} \qquad \Bigg| \qquad \dfrac{\vdash F^\perp \quad \vdash G^\perp}{\vdash (F^\perp \mathbin{\&} G^\perp) \parr \perp}$$

Abstracting away from micro-moves

The internal structure of a macro-move vis-à-vis micro-moves is abstracted from the game.

Focusing allows a similar abstraction in proofs. In particular, permutation of inference rules within a phase are identified in focused proofs.

Focalization

The connectives and units can be classify into the two groups

Synchronous	$\oplus$	0	$\otimes$	1
Asynchronous	$\&$	$\top$	$⅋$	$\bot$

Andreoli's focused proof system [1992] constrained proofs as follows:

1. apply, in any order, asynchronous rules until none are applicable (they all permute over each other);
2. then choose to focus on a (synchronous) formula and apply synchronous rules to it and its descendants until they become asynchronous.

This strategy lacks symmetry: in a synchronous phase we select *one* formula, while in an asynchronous phase we select *all* of them.

Multi-focalization for MALL without atoms

Additives and multiplicatives

$$\frac{\vdash \Gamma \Downarrow F_i, \Delta}{\vdash \Gamma \Downarrow F_1 \oplus F_2, \Delta}\ [\oplus_i] \qquad \frac{\vdash \Gamma \Uparrow F, \Delta \quad \vdash \Gamma \Uparrow G, \Delta}{\vdash \Gamma \Uparrow F \,\&\, G, \Delta}\ [\&]$$

$$\frac{}{\vdash \Gamma \Uparrow \top, \Delta}\ [\top] \qquad \frac{\vdash \Gamma \Uparrow F, G, \Delta}{\vdash \Gamma \Uparrow F \parr G, \Delta}\ [\parr] \qquad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Uparrow \bot, \Delta}\ [\bot]$$

$$\frac{\vdash \Gamma_1 \Downarrow F, \Delta_1 \quad \vdash \Gamma_2 \Downarrow G, \Delta_2}{\vdash \Gamma_1, \Gamma_2 \Downarrow F \otimes G, \Delta_1, \Delta_2}\ [\otimes] \qquad \frac{}{\vdash \Downarrow 1}\ [1]$$

Phase changes

$$\frac{\vdash \Gamma, F \Uparrow \Delta}{\vdash \Gamma \Uparrow F, \Delta}\ [R\Uparrow] \qquad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Downarrow \Delta}\ [R\Downarrow] \qquad \frac{\vdash \Gamma \Downarrow \Delta}{\vdash \Gamma, \Delta \Uparrow}\ [D]$$

$(F \text{ sync.})$ $\qquad (\Delta \text{ async.})$ $\qquad (\Delta \neq \emptyset)$

What's difficult with multiplicatives?

The logic is not complete! The neutral expression $\updownarrow\mathbb{1} \times \updownarrow\mathbb{1}$ translates to two unprovable formulas:

$$\bot \otimes \bot \quad \text{and} \quad 1 \parr 1$$

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$$\bot \otimes \bot \quad \text{and} \quad 1 \parr 1$$

$$\dfrac{\dfrac{\vdash A, \delta_1, \ldots, \delta_k \quad \vdash B, \delta_{k+1}, \ldots, \delta_n}{\vdash A \otimes B, \delta_1, \ldots, \delta_n}}{\vdash (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n} \quad \Bigg| \quad \dfrac{\dfrac{\vdash A^\perp, B^\perp}{\vdash A^\perp \parr B^\perp} \quad \vdash \delta_1^\perp \quad \ldots \quad \vdash \delta_n^\perp}{\vdash (A^\perp \parr B^\perp) \otimes \delta_1^\perp \otimes \ldots \otimes \delta_n^\perp}$$

On the left, the player chooses a partition $\delta_1, \ldots, \delta_n$. This information does not appear on the right.
On the right, you can tell A and B from the δ_i. This information is lost on the left.

Treating the multiplicatives

A game can no longer be determinate. If neither $[E]^+$ nor $[E]^-$ are provable, then no one has a winning strategy starting with E.

Derivations are now explicitly focused. [In the additive-only case, unfocused proofs are actually focused.]

The state of the game cannot be a plain neutral expression any more. We use *neutral graphs* to record multiplicative structure.

An important invariant is maintained:

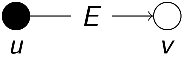
- The $[\cdot]^+$ and $[\cdot]^-$ translations yield the frontiers of slices of two dual derivations, and
- applying only cut rules to these two translations yields the empty sequent.

Vertices as sequents, arcs as formulas

A neutral graph

- is a bipartite graph,
- whose arcs are labeled with neutral expressions,
- with no undirected cycles.

An arc



means that

- the formula $[E]^+$ occurs in the sequent associated with u,
- the formula $[E]^-$ occurs in the sequent associated with v.

A basic example

The frontiers

$$\vdash [E]^{+} \Uparrow \quad \Big| \quad \vdash \Uparrow [E]^{-}$$

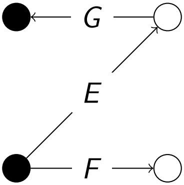
are represented by the neutral graph

A more complex example

The frontiers

$$\vdash [E]^+, [F]^+ \Uparrow \qquad \vdash \Uparrow [G]^- \quad \Big| \quad \vdash [G]^+ \Uparrow [E]^- \qquad \vdash \Uparrow [F]^-$$

are represented by the neutral graph



The full game

The players rewrite a neutral graph.

- A micro-move is seen as the application of a synchronous (resp. asynchronous) rule by the player (resp. the opponent);
- a macro-move is seen as a phase;
- some moves may make one (or both) of the proofs fail;
- a play goes on until both players fail (tie) or the graph becomes empty (win for the player who has not failed).

Theorem

Player (resp. opponent) has a winning strategy from a neutral graph G iff the positive (resp. negative) translation of G is provable.

A dynamic example

$$\dfrac{\dfrac{\dfrac{\dfrac{\delta_1, \ldots, \delta_k \Downarrow A \quad \delta_{k+1}, \ldots, \delta_n \Downarrow B}{\delta_1, \ldots, \delta_n \Downarrow A \otimes B}}{A \otimes B, \delta_1, \ldots, \delta_n \Uparrow}}{\Uparrow A \otimes B, \delta_1, \ldots, \delta_n}}{\color{red}{\Uparrow (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n}}$$

$$\dfrac{\dfrac{\dfrac{\Uparrow A^\perp, B^\perp}{\Uparrow A^\perp \parr B^\perp}}{\Downarrow A^\perp \parr B^\perp} \quad \dfrac{\Uparrow \delta_1^\perp}{\Downarrow \delta_1^\perp} \quad \ldots \quad \dfrac{\Uparrow \delta_n^\perp}{\Downarrow \delta_n^\perp}}{\color{red}{\Downarrow (A^\perp \parr B^\perp) \otimes \delta_1^\perp \otimes \ldots \otimes \delta_n^\perp}}$$

$$\bullet \leftarrow \updownarrow (a \times b) \times \updownarrow d_1 \times \ldots \times \updownarrow d_n - \circ$$

A dynamic example

$$
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$$

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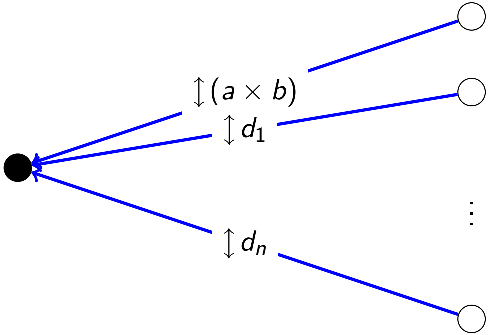

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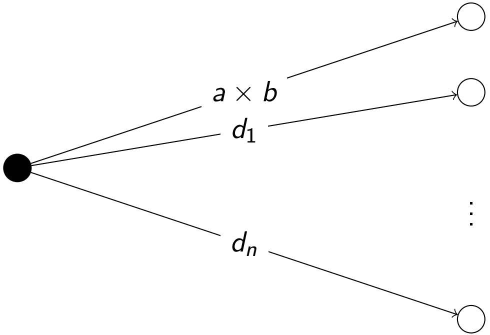

A dynamic example

$$\dfrac{\dfrac{\dfrac{\dfrac{\delta_1, \dots, \delta_k \Downarrow A \quad \delta_{k+1}, \dots, \delta_n \Downarrow B}{\color{red}{\delta_1, \dots, \delta_n \Downarrow A \otimes B}}}{A \otimes B, \delta_1, \dots, \delta_n \Uparrow}}{\Uparrow A \otimes B, \delta_1, \dots, \delta_n}}{\Uparrow (A \otimes B) \parr \delta_1 \parr \dots \parr \delta_n}$$

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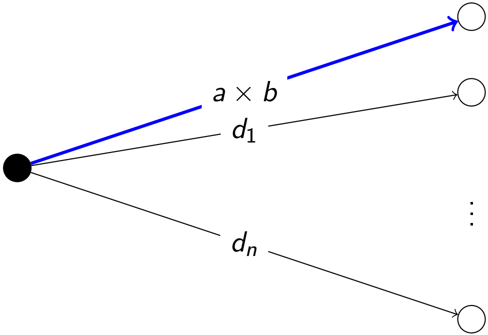

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$$\cfrac{\cfrac{\cfrac{\cfrac{\color{red}{\delta_1, \ldots, \delta_k \Downarrow A \quad \delta_{k+1}, \ldots, \delta_n \Downarrow B}}{\delta_1, \ldots, \delta_n \Downarrow A \otimes B}}{A \otimes B, \delta_1, \ldots, \delta_n \Uparrow}}{\Uparrow A \otimes B, \delta_1, \ldots, \delta_n}}{\Uparrow (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n}$$

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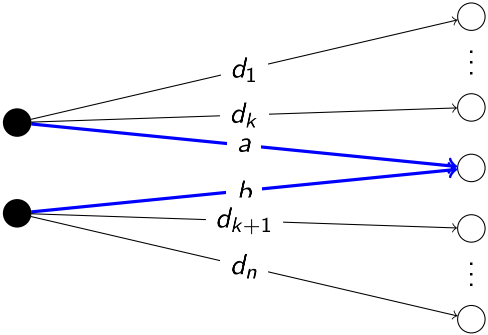


A typical macro-move

In a macro-move

$$G \stackrel{f_0, f_1}{\rightsquigarrow} G'$$

the neutral graph G is rewritten to G', and f_0 (resp. f_1) is a boolean value which is true iff player 0 (resp. 1) fails during the move.

A macro-move can be decomposed in micro-moves: an initial step selecting the neutral expressions to decompose, then small steps corresponding to single rule applications.

$$G \xrightarrow{D} G_0 \xrightarrow{f_0^{(1)}, f_1^{(1)}} G_1 \xrightarrow{f_0^{(2)}, f_1^{(2)}} \cdots \xrightarrow{f_0^{(n)}, f_1^{(n)}} G_n = G'$$

and $f_\sigma = \bigvee_{i=1}^{n} f_\sigma^{(i)}$ ($\sigma \in \{0, 1\}$).

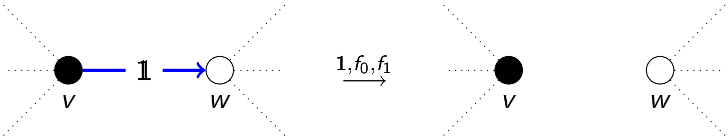
Micro-moves

Micro-move	Sync reading	Async reading
$G \xrightarrow{D} G'$	$[D]$	none
$G \xrightarrow{R} G'$	$[R \Downarrow]$	$[R \Uparrow]$
$G \xrightarrow{+} G'$	$[\oplus]$	$[\&]$
$G \xrightarrow{\times} G'$	$[\otimes]$	$[\parr]$
$G \xrightarrow{0,f_0,f_1} G'$	none	$[\top]$
$G \xrightarrow{1,f_0,f_1} G'$	$[1]$	$[\bot]$

Remark

The micro-moves responsible for failure are those associated with units.

Failure



is seen as the simultaneous application of

Player (black)	Opponent (white)
$$\frac{}{\vdash \Downarrow 1, \Delta_0}\ [1]$$	$$\frac{\vdash \Gamma_1 \Uparrow \Delta_1}{\vdash \Gamma_1 \Uparrow \bot, \Delta_1}\ [\bot]$$
(requires $\Delta_0 = \emptyset$)	(unprovable if $\Gamma_1, \Delta_1 = \emptyset$)

Player fails if v does not become isolated, opponent fails if w becomes isolated.

What would Lakatos say?

Imre Lakatos, "Proofs and Refutations" (1976).

Similarity: Proving and refuting are done as an integrated activity.

Differences: This integrated activity is highly formalistic.

Lakatos would not be happy with this particular project.

Conclusion

- The neutral approach can have three outcomes: winning strategy for a player (proof), winning strategy for her opponent (refutation) or no winning strategy for either (no proof or refutation).
- This game with neutral graphs reveals the complexity of the multiplicatives.
- Every step in the game contributes simultaneously to building a proof and a refutation.
- This positional game yields relative completeness.

Future work

- Extend the games to atoms, fixed points, quantification (See Delande's PhD).
- Capture full completeness (See Delande's PhD).
- Develop connections with ludics.