A proposal for multifocused LJF

Dale Miller, 26 April 2015

Brief background. Multifocused proof systems for classical (one-sided) sequent systems have appeared in print a few places and their design seems to have converged: see [2] for a multifocused version of *LKF* and see [1, 3, 4] for a multifocused versions of MALL. I would like to propose the following design and syntax for a multifocused proof system that generalizes *LJF* [5]. I recently (April 2015) submitted a particular typesetting of *LJF* to Bruno Woltzenlogel Paleo for entry into an Encyclopedia of proof systems: that entry mentions that multifocusing is possible but does not display that aspect of the proof system. I make a proposal for that here in the following figure.

Asynchronous Introduction Rules

$$\frac{\Gamma \Uparrow B_1 \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \supset B_2 \Uparrow} \qquad \frac{\Gamma \Uparrow \cdot \vdash B_1 \Uparrow \quad \Gamma \Uparrow \cdot \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \wedge^- B_2 \Uparrow} \qquad \frac{}{\Gamma \Uparrow \cdot \vdash t^- \Uparrow}$$

$$\frac{\Gamma \Uparrow \cdot \vdash [y/x]B \Uparrow}{\Gamma \Uparrow \cdot \vdash \forall x.B \Uparrow} \qquad \frac{\Gamma \Uparrow [y/x]B, \Theta \vdash \mathcal{R}}{\Gamma \Uparrow \exists x.B, \Theta \vdash \mathcal{R}} \qquad \frac{}{\Gamma \Uparrow f^+, \Theta \vdash \mathcal{R}}$$

$$\frac{\Gamma \Uparrow B_1, B_2, \Theta \vdash \mathcal{R}}{\Gamma \Uparrow B_1 \wedge^+ B_2, \Theta \vdash \mathcal{R}} \qquad \frac{\Gamma \Uparrow \Theta \vdash \mathcal{R}}{\Gamma \Uparrow t^+, \Theta \vdash \mathcal{R}} \qquad \frac{\Gamma \Uparrow B_1, \Theta \vdash \mathcal{R} \quad \Gamma \Uparrow B_2, \Theta \vdash \mathcal{R}}{\Gamma \Uparrow B_1 \vee^+ B_2, \Theta \vdash \mathcal{R}}$$

Synchronous Introduction Rules

$$\frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \Downarrow \mathcal{P}, B_2, \Theta \vdash E}{\Gamma \Downarrow \mathcal{P}, B_1 \supset B_2, \Theta \vdash E} \qquad \frac{\Gamma \Downarrow \mathcal{P}, [t/x]B, \Theta \vdash E}{\Gamma \Downarrow \mathcal{P}, \forall x.B, \Theta \vdash E} \qquad \frac{\Gamma \Downarrow \mathcal{P}, B_i, \Theta \vdash E}{\Gamma \Downarrow \mathcal{P}, B_1 \wedge^- B_2, \Theta \vdash E}\; i \in \{1,2\}$$

$$\frac{\Gamma \vdash B_i \Downarrow}{\Gamma \vdash B_1 \vee^+ B_2 \Downarrow} \qquad \frac{}{\Gamma \vdash t^+ \Downarrow} \qquad \frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \vdash B_2 \Downarrow}{\Gamma \vdash B_1 \wedge^+ B_2 \Downarrow} \qquad \frac{\Gamma \vdash [t/x]B \Downarrow}{\Gamma \vdash \exists x.B \Downarrow}$$

Identity rules

$$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N}\; I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow}\; I_r \qquad \frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \cdot \qquad \Gamma \Uparrow B \vdash \cdot \Uparrow E}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}\; Cut$$

Structural rules (Decide, Release, Store)

$$\frac{\Gamma \Downarrow \mathcal{N} \vdash E}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}\; D_l \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P}\; D_r \qquad \frac{\Gamma \Uparrow \mathcal{P} \vdash \cdot \Uparrow E}{\Gamma \Downarrow \mathcal{P} \vdash E}\; R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \cdot}{\Gamma \vdash N \Downarrow}\; R_r$$

$$\frac{C, \Gamma \Uparrow \Theta \vdash \mathcal{R}}{\Gamma \Uparrow C, \Theta \vdash \mathcal{R}}\; S_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}{\Gamma \Uparrow \cdot \vdash E \Uparrow \cdot}\; S_r$$

Figure 1: A proposal for a multifocused version of *LJF*

Schematic variables. In Figure 1, Θ ranges over multisets of polarized formulas; Γ ranges over lists of polarized formulas; $\mathcal{P}$ ranges over a list of positive formulas; P denotes a positive formula; N denotes a negative formula; C denotes either a negative formula or a positive atom; E denotes either a positive formula or a negative atom; and B denotes an unrestricted polarized formula. The introduction rule for $\forall$ has the usual eigenvariable restriction that y is not free in any formula in the conclusion sequent. Furthermore, in the D_l decide rule, $\mathcal{N}$ denotes a non-empty list in which every element in $\mathcal{N}$ is a negative formula that is an element of Γ.

The sequents. There are two kinds of sequents in this proof system. One kind contains a single $\Downarrow$ on either the right or the left of the turnstile ($\vdash$) and are of the form $\Gamma \Downarrow \Theta \vdash E$ or $\Gamma \vdash B \Downarrow$. In the first case, the formulas in the list Θ are the focus of the sequent whereas in the second case, the formula B is the focus of the sequent. The other kind of sequent has an occurrence of $\Uparrow$ on each side of the turnstile, eg., $\Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2$, and is such that the union of the two multisets Δ_1 and Δ_2 contains exactly one formula: that is, one of these multisets is empty and the other is a singleton. When writing asynchronous rules that introduce a connective on the left-hand side, we write $\mathcal{R}$ to denote $\Delta_1 \Downarrow \Delta_2$.

Various observations

1. I like this syntax since the only additional syntax used in sequents are the up and down arrows. There are no additional symbols, such as the semicolon and brackets. The formulas that are "active" during a phase are those formulas that come between the arrow(s) and the turnstile.

2. During the asynchronous phase, a right introduction rule is applied only when the left asynchronous zone Γ is empty. Similarly, a left-introduction rule in the asynchronous phase introduces the connective at the top-level of the first formula in that context. The scheduling of introduction rules during this phase can be assigned arbitrarily and the zone Γ can be interpreted as a multiset instead of a list.

3. If a synchronous phase contains a left introduction rule, then that phase contains a series of left introduction that stretches from either a R_l rule or I_l at the top to a D_l rule at the bottom. In that case, the size of the zone is the same in both the instances of the R_l (or I_l) and D_l rules. Given the presence of the implication left introduction rule, that phase may also involve right introduction rules.

4. A list is also used during a left synchronous phase: in this case, introduction rules are applied only so that the formula that is introduced is the first (in left-to-right order) non-atomic formula in that zone. (Lists are used in both the synchronous and asynchronous phases since the introduction rules of like-polarized formulas always permute over each other: changing from list to multiset only increases the different ways to construct the same phase.)

5. As with most designs for multifocusing proof system, their soundness and completeness are immediate consequences of these results for the single focus systems.

6. Notice that it is not possible in this proof system to multifocus in such a way that there is a focus on the left and the right: focusing can only happen on one side or the other.

References

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