▸ DALE MILLER AND ELAINE PIMENTEL, *Higher-level rules for sequent calculus.*
Inria-Saclay & LIX, École Polytechnique, Palaiseau, France.
*E-mail*: `dale.miller@inria.fr`.
Department of Computer Science, University College London, 6-72 Gower St, UK.
*E-mail*: `e.pimentel@ucl.ac.uk`.

Theories over a base logic are typically represented in two ways: as collections of formulas accepted as axioms and assumptions or as collections of inference rules that do not involve logical connectives.

Schroeder-Heister [8] and Avron [1] showed the equivalence in propositional intuitionistic logic between conventional natural deduction (where formulas can be assumed and discharged) and natural deduction with higher-level rules (where atomic formulas and rules can be assumed and discharged). The LF [4] specification framework generalized this work on natural deduction to quantificational intuitionistic logic while simultaneously encoding the extended natural deduction proofs using a term representation within a dependently typed $\lambda$-calculus.

Moving from natural deduction to sequent calculus permits developing approaches to higher-level rules for both classical and intuitionistic logic proofs [3, 6]. In particular, it is natural to ask whether or not an $LJ$ or $LK$ proof of the sequent $\Gamma \vdash \Delta$ can be built where the formulas in $\Gamma$ (the assumptions) are replaced with inference rules. It was shown in [9, 10, 6] that this replacement is possible when assumptions are *geometric formulas*. For example, if the multiset of assumptions is $\Gamma, \forall x \forall y \forall z.[Rxz \supset Rzy \supset Rxy]$ then the assumption stating the transitivity of the predicate $R$ can be replaced by one of the following inference rules.

$$\frac{\Gamma \vdash Rxz \quad \Gamma \vdash Rzy}{\Gamma \vdash Rxy}\ \text{backchain} \qquad \frac{\Gamma, Rxz, Rzy, Rxy \vdash B}{\Gamma, Rxz, Rzy \vdash B}\ \text{forwardchain}$$

In this way, we have replaced a formula containing five occurrences of logical connectives with one of these two rules, neither containing logical connectives. If the only assumptions are, say, Horn clauses, it is possible to have a complete proof system for atomic consequences of such clauses using a proof system involving only atomic formulas.

While it is possible to generalize the restriction to geometric formulas a bit to *bipoles* by considering polarity [5], it is interesting to ask to what extent can non-bipole formulas be replaced with higher-level rules in the sequent setting. Such a question was addressed in [7], where *generalized geometric formulas* are treated using a *system of rules*, a setting in which an inference rule can allow some of its premises to have additional inference rules available. These additional inference rules are scoped over particular proofs of premises. We will show that such a scoping of inference rules is a direct reading of inference rules in a polarized proof system. For example, focusing on the (polarized) formula that states the existence of least upper bounds

$$\forall x \forall y \exists z (x \leq z \wedge^+ y \leq z \wedge^+ \forall w (x \leq w \wedge^+ y \leq w \supset z \leq w)),$$

yields the synthetic inference rule (terminology from [5])

$$\frac{\Sigma, z : \forall w (x \leq w \wedge^+ y \leq w \supset z \leq w), x \leq z, y \leq z, \Gamma \vdash \Delta}{\Sigma : \Gamma \vdash \Delta}\ (*).$$

Here, sequents are prefixed with a list of variables, e.g., $\Sigma$, that are the eigenvariables that may appear in the formulas of that sequent: this prefix is intended to *bind* variables over the sequent it precedes. Note that the prefix of the conclusion is different from the prefix of the premise. The availability of the additional assumption $\forall w (x \leq w \wedge^+ y \leq$

$w \supset z \leq w$) in the premise corresponds exactly to having either one of the following two inference rules scoped over that premise

$$\frac{\Sigma, z : \Gamma \vdash x \leq w \quad \Sigma, z : \Gamma \vdash y \leq w}{\Sigma, z : \Gamma \vdash z \leq w} \qquad \frac{\Sigma, z : \Gamma, x \leq w, y \leq w, z \leq w \vdash \Delta}{\Sigma, z : \Gamma, x \leq w, y \leq w \vdash \Delta} ,$$

depending on whether or not the polarity of the $\leq$ predicate is negative or positive, respectively. Thus, we can rewrite the inference figure $(*)$ so that it does not mention any logical connectives by stipulating that one of these two inference rules is available to prove that premise.

We will also point out how Tseitin predicate symbols, often introduced into formulas to reduce their logical complexity [2], can be used to make scoped rules available globally.


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