Classical polarizations yield double-negation translations

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Double-negation translations map formulas to formulas in such a way that if a formula is a classical theorem then its translation is an intuitionistic theorem. We shall go beyond just examining provability by looking at correspondences between inference rules in classical proofs and in intuitionistic proofs of translated formulas. In order to make this comparison interesting and precise, we will examine focused versions of proofs in classical and intuitionistic logics using the LKF and LJF proof systems. We shall show that for a number of known double-negation translations, one can get essentially identical (focused) intuitionistic proofs as (focused) classical proofs. Thus the choice of a common double-negation translation is really the same choice as a polarization of classical logic (of which there are many).

1 Introduction

Applying a double-negation (DN) translation allows to transform any classical proof of a formula A into an intuitionistic proof of the translation of A. The proof transformation itself is, however, usually not well studied – what researchers usually care about in Logic is the existence of this transformation rather than its description.

The approach of the Programming Languages community is more systematic and more focused on the structure of proofs. There, continuation-passing style (CPS) translations of both types (formulas) and terms (proofs) are studied, and these studies show that the transformation of terms (proofs) is sophisticated. Since the correspondence between DN and CPS is immediate—namely, the call-by-name CPS translation is exactly Kolmogorov's DN translation, while the call-by-value CPS translation is almost the same as Kuroda's DN translation—it follows that DN translations as proof transformations are necessarily complicated.

In this paper, we show that the complexity of the DN translation can be tamed if, before the embedding into intuitionistic logic, we first assign a polarization of formulas. More precisely, in Section 4 we show that each of the four well known DN translations of Kolmogorov, Gödel-Gentzen, Kuroda, and Krivine, is factored through a classical polarization together with a "hosting function" that allows a polarized intuitionistic proof system to operate as if it was a polarized classical proof system: in Section 3, this hosting function, independent of a particular polarization, is shown to embed classical sequents into intuitionistic ones yielding structurally almost identical proof trees. In Section 4, we show how the method of generating double negation translations from polarization maps can account for known and new formula transformations.

In order to make this analysis of DN-as-polarization possible and precise, we first need to introduce the focused proofs systems for classical and intuitionistic predicate logics, LKF and LJF [8].

Draft. First appears as https://hal.inria.fr/hal-01354298 on 28 Aug 2016. Minor updates made 5 Jan 2019.

2 Two focused proof systems

2.1 Polarized formulas

We shall assume the usual notion of first-order term. Atomic formulas are structures that result from applying an n-ary ($n \geq 0$) predicate to a list of n first-order terms. First-order classical and intuitionistic logics are based on the logical connectives t, f, $\vee$, $\wedge$, and $\supset$, and the two quantifiers $\forall$ and $\exists$.

When picking a sequent calculus, we have some standard choices to make. For example, sequents can be either one or two-sided (we use one-sided here). There are also some differences in how introduction rules for connectives should be given. For example, disjunction introduction can be written as either

$$\frac{\vdash \Gamma, B_1, B_2}{\vdash \Gamma, B_1 \vee B_2} \qquad \text{or} \qquad \frac{\vdash \Gamma, B_i}{\vdash \Gamma, B_1 \vee B_2}\ i \in \{1,2\}.$$

Given the presence of the contraction and weakening rules, these inference rules are inter-admissible. Since we are interested in having a rich collection of proof structures, we will allow *both* of these inference rules and will distinguish between them by having the first one (which is invertible) introduce the *polarized logical connective* $\vee^-$ and the second one (which is non-invertible) introduce $\vee^+$. Similarly, we introduce $\wedge^-$ and $\wedge^+$ as the de Morgan duals of $\vee^+$ and $\vee^-$, respectively. (The logical constants t and f can also be polarized but we ignore these since they play no role in this paper.) Given this motivation, we introduce *polarized formulas* as expressions built from atomic formulas and negated atomic formulas using the connectives $\vee^-$, $\vee^+$, $\wedge^-$, $\wedge^+$, and the two quantifiers $\forall$ and $\exists$ as constructors also of polarized formulas.

Polarized formulas are either *positive* or *negative* as follows. An atomic formula can be given a polarization in an arbitrary fashion: when we speak about polarized formulas, we will need to specify how atoms are polarized. Generally, we will treat them all the same (*e.g.*,, all atomic formulas are positive) but it is possible to have some be positive and some negative. A formula is *negative* if it is a negative atom, or the negation of a positive atom, or its top-level connective is one of the following: $\vee^-$, $\wedge^-$, $\supset$, $\forall$. Dually, a formula is *positive* if it is a positive atom or the negation of a negative atom or its top-level connective is one of the following: $\vee^+$, $\wedge^+$, $\exists$. We shall say that the unpolarized connectives $\vee$ and $\wedge$ are *ambiguous* since there is more than one version of these connectives as polarized formulas: notice that the implication and the first-order quantifiers are not ambiguous. Finally, we shall need a polarized formula to denote the minimal logic false: we write this as q and it is always treated as a negative atomic formula. Minimal logic negation $\neg_q B$ is defined to be the implication $B \supset q$.

We will also allow positive and negative *delays*, $\partial^-(\cdot)$ and $\partial^+(\cdot)$, into polarized formulas. The idea is that $\partial^-(B)$ is always negative and $\partial^+(B)$ is always positive no matter what the polarity of B is. These delay operators are easily defined using polarized logical connectives: we can take the official definitions to be $\partial^-(B) = \forall x.B$ and $\partial^+(B) = \exists x.B$ (provided that x is not free in B). Alternatively, they can be defined to be the 1-ary version of the binary $\wedge^-$ and $\wedge^+$ connectives, respectively.

Let B be an unpolarized first-order formula. Let $\hat{B}$ be a polarized formula that results from (i) adding either a $+$ or a $-$ to every occurrence of $\wedge$ and $\vee$ in B, (ii) picking some polarization of atomic formulas, and (iii) inserting any number of delays anywhere into the formula. There are, of course, a great many ways that a given formula B can be polarized since following (i) alone leads to an exponential number of such polarizations: if B has n occurrences of $\vee$ and $\wedge$ then there are 2^n ways to annotate them with $+$ or $-$.

ASYNCHRONOUS INTRODUCTION RULES

$$\frac{\vdash \Gamma \Uparrow B_1, \Theta \quad \vdash \Gamma \Uparrow B_2, \Theta}{\vdash \Gamma \Uparrow B_1 \wedge^- B_2, \Theta} \qquad \frac{\vdash \Gamma \Uparrow B_1, B_2, \Theta}{\vdash \Gamma \Uparrow B_1 \vee^- B_2, \Theta} \qquad \frac{\vdash \Gamma \Uparrow [y/x]B, \Theta}{\vdash \Gamma \Uparrow \forall x.B, \Theta}$$

SYNCHRONOUS INTRODUCTION RULES

$$\frac{\vdash \Gamma \Downarrow B_1 \quad \vdash \Gamma \Downarrow B_2}{\vdash \Gamma \Downarrow B_1 \wedge^+ B_2} \qquad \frac{\vdash \Gamma \Downarrow B_i}{\vdash \Gamma \Downarrow B_1 \vee^+ B_2}\ i \in \{1,2\} \qquad \frac{\vdash \Gamma \Downarrow [t/x]B}{\vdash \Gamma \Downarrow \exists x.B}$$

IDENTITY RULES

$$\frac{P \text{ atomic}}{\vdash P^\perp, \Gamma \Downarrow P}\ init \qquad \frac{\vdash \Gamma \Uparrow B \quad \vdash \Gamma \Uparrow B^\perp}{\vdash \Gamma \Uparrow \cdot}\ cut$$

STRUCTURAL RULES

$$\frac{\vdash \Gamma, C \Uparrow \Theta}{\vdash \Gamma \Uparrow C, \Theta}\ store \qquad \frac{\vdash \Gamma \Uparrow N}{\vdash \Gamma \Downarrow N}\ release \qquad \frac{\vdash P, \Gamma \Downarrow P}{\vdash P, \Gamma \Uparrow \cdot}\ decide$$

Here, Γ ranges over multisets of polarized formulas; Θ ranges over lists of polarized formulas; P denotes a positive formula; N denotes a negative formula; C denotes either a positive formula or a negative atom; and B denotes an unrestricted polarized formula. $B^\perp$ denotes the negation normal form of the negation of B. The right introduction rule for $\forall$ has the the usual eigenvariable restriction that y is not free in any formula in the conclusion sequent.

Figure 1: The *LKF* focused classic calculus

2.2 Focused classical logic

The *LKF* proof system (see Figure 1) involves polarized formulas that do not contain $\supset$ and q: these are the classical polarized formulas, also called *LKF*-formulas. There are two kinds of sequents in this proof system, namely, $\vdash \Gamma \Uparrow \Theta$ and $\vdash \Gamma \Downarrow B$, where Γ is a multiset of atomic or positively polarized formulas, B is a polarized formula, and Θ is a list of polarized formulas. The list structure of Θ can be replaced by a multiset but we maintain it as a list. The formula occurrence B in the $\Downarrow$ sequent is called the *focus* of that sequent. Notice that contraction, through the *decide* rule, is only done on positive formulas. The completeness of *LKF* is proved in [8] and can be stated as follows: If B is an (unpolarized) classical logic theorem and $\hat{B}$ is any polarization of B, then $\vdash \cdot \Uparrow \hat{B}$ is provable in *LKF*. Thus, the choice of polarization does not affect provability but it can have a big impact on the structure of proofs.

2.3 Focused intuitionistic logic

The *LJF* proof system (see Figure 2) involves polarized formulas without occurrences of $\vee^-$ and f^-: these are the intuitionistic polarized formulas, also called *LJF*-formulas. There are two kinds of sequents in this proof system. One kind contains a single $\Downarrow$ on either the right or the left of the turnstyle ($\vdash$) and are of the form $\Gamma \Downarrow B \vdash E$ or $\Gamma \vdash B \Downarrow$: in both of these cases, the formula B is the *focus* of the sequent. The other kind of sequent has an occurrence of $\Uparrow$ on each side of the turnstyle, eg., $\Gamma \Uparrow \Theta \vdash \Delta_1 \Uparrow \Delta_2$, and is such that the union of the two multisets Δ_1 and Δ_2 contains exactly one formula: that is, one of these multisets is empty and the other is a singleton. When writing asynchronous rules that introduce a connective on the left-hand side, we use $\mathscr{R}$ to denote $\Delta_1 \Downarrow \Delta_2$.

ASYNCHRONOUS INTRODUCTION RULES

$$\frac{\Gamma \Uparrow B_1 \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \supset B_2 \Uparrow} \qquad \frac{\Gamma \Uparrow \cdot \vdash B_1 \Uparrow \quad \Gamma \Uparrow \cdot \vdash B_2 \Uparrow}{\Gamma \Uparrow \cdot \vdash B_1 \wedge^- B_2 \Uparrow} \qquad \frac{\Gamma \Uparrow \cdot \vdash [y/x]B \Uparrow}{\Gamma \Uparrow \cdot \vdash \forall x.B \Uparrow}$$

$$\frac{\Gamma \Uparrow [y/x]B, \Theta \vdash \mathscr{R}}{\Gamma \Uparrow \exists x.B, \Theta \vdash \mathscr{R}} \qquad \frac{\Gamma \Uparrow B_1, B_2, \Theta \vdash \mathscr{R}}{\Gamma \Uparrow B_1 \wedge^+ B_2, \Theta \vdash \mathscr{R}} \qquad \frac{\Gamma \Uparrow B_1, \Theta \vdash \mathscr{R} \quad \Gamma \Uparrow B_2, \Theta \vdash \mathscr{R}}{\Gamma \Uparrow B_1 \vee^+ B_2, \Theta \vdash \mathscr{R}}$$

SYNCHRONOUS INTRODUCTION RULES

$$\frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \Downarrow B_2 \vdash E}{\Gamma \Downarrow B_1 \supset B_2 \vdash E} \qquad \frac{\Gamma \Downarrow [t/x]B \vdash E}{\Gamma \Downarrow \forall x.B \vdash E} \qquad \frac{\Gamma \Downarrow B_i \vdash E}{\Gamma \Downarrow B_1 \wedge^- B_2 \vdash E}\ i \in \{1,2\}$$

$$\frac{\Gamma \vdash B_i \Downarrow}{\Gamma \vdash B_1 \vee^+ B_2 \Downarrow} \qquad \frac{\Gamma \vdash B_1 \Downarrow \quad \Gamma \vdash B_2 \Downarrow}{\Gamma \vdash B_1 \wedge^+ B_2 \Downarrow} \qquad \frac{\Gamma \vdash [t/x]B \Downarrow}{\Gamma \vdash \exists x.B \Downarrow}$$

IDENTITY RULES

$$\frac{N \text{ atomic}}{\Gamma \Downarrow N \vdash N}\ I_l \qquad \frac{P \text{ atomic}}{\Gamma, P \vdash P \Downarrow}\ I_r \qquad \frac{\Gamma \Uparrow \cdot \vdash B \Uparrow \cdot \quad \Gamma \Uparrow B \vdash \cdot \Uparrow E}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}\ Cut$$

STRUCTURAL RULES

$$\frac{\Gamma, N \Downarrow N \vdash E}{\Gamma, N \Uparrow \cdot \vdash \cdot \Uparrow E}\ D_l \qquad \frac{\Gamma \vdash P \Downarrow}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow P}\ D_r \qquad \frac{\Gamma \Uparrow P \vdash \cdot \Uparrow E}{\Gamma \Downarrow P \vdash E}\ R_l \qquad \frac{\Gamma \Uparrow \cdot \vdash N \Uparrow \cdot}{\Gamma \vdash N \Downarrow}\ R_r$$

$$\frac{C, \Gamma \Uparrow \Theta \vdash \mathscr{R}}{\Gamma \Uparrow C, \Theta \vdash \mathscr{R}}\ S_l \qquad \frac{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow E}{\Gamma \Uparrow \cdot \vdash E \Uparrow \cdot}\ S_r$$

Here, Γ ranges over multisets of polarized formulas; Θ ranges over lists of polarized formulas; P denotes a positive formula; N denotes a negative formula; C denotes either a negative formula or a positive atom; and E denotes either a positive formula or a negative atom; and B denotes an unrestricted polarized formula. The introduction rule for $\forall$ has the usual eigenvariable restriction that y is not free in any formula in the conclusion sequent.

Figure 2: The *LJF* focused intuitionistic sequent calculus

Note that in the asynchronous phase, a right introduction rule is applied only when the left asynchronous zone Θ is empty. Similarly, a left-introduction rule in the asynchronous phase introduces the connective at the top-level of the first formula in that context. The scheduling of introduction rules during this phase can be assigned arbitrarily and the zone Θ can be interpreted as a multiset instead of a list.

Note also that Γ always contains only atomic or negative formulas, and that the contraction, through the *decide*-left rule, is only done on negative formulas.

3 Hosting *LKF* within *LJF*

It is well-known that intuitionistic inference rules are more expressive than classical inference rules. One way to show this explicitly is by using an explicit mechanism for hosting any *LKF* proof within *LJF*. The following function, taken from [2], enables this kind of hosting since it maps between phases in *LKF* and phases in *LJF* (of translated formulas).

Definition 1 (Hosting function ($[\![\cdot]\!]^+, [\![\cdot]\!]^-$)).

$$
\begin{aligned}
[\![a^+]\!]^+ &= a & [\![a^-]\!]^- &= a \\
[\![A \vee^+ B]\!]^+ &= [\![A]\!]^+ \vee [\![B]\!]^+ & [\![A \vee^- B]\!]^- &= [\![A]\!]^- \wedge^+ [\![B]\!]^- \\
[\![A \wedge^+ B]\!]^+ &= [\![A]\!]^+ \wedge^+ [\![B]\!]^+ & [\![A \wedge^- B]\!]^- &= [\![A]\!]^- \vee [\![B]\!]^+ \\
[\![\exists x A]\!]^+ &= \exists x [\![A]\!]^+ & [\![\forall x A]\!]^- &= \exists x [\![A]\!]^- \\
[\![\partial^+(A)]\!]^+ &= [\![A]\!]^+ & [\![\partial^-(A)]\!]^- &= [\![A]\!]^- \\
[\![N]\!]^+ &= \neg_q [\![N]\!]^- & [\![P]\!]^- &= \neg_q [\![P]\!]^+,
\end{aligned}
$$

where N stands for a negative and P for a positive formula.

When we use this hosting function (as we do for the rest of this paper), we assume that atoms in LJF formulas will be polarized positively: the only exception to this is (as mentioned before) the assumption that q is polarized negatively. We also note that since *LJF* contains only positive disjunctions, any occurrence of $\vee$ in a polarized intuitionistic formula should be understood as being $\vee^+$.

Theorem 1. *There is a rule-preserving map of LKF proofs into LJF proofs: that is, for a fixed (negatively polarized) atom q, the following two statements hold:*

$$
\begin{array}{ccc}
\vdash_{LKF} \Gamma \Uparrow \Theta & \xrightarrow{\phi} & [\![\Gamma]\!]^- \Uparrow [\![\Theta]\!]^- \vdash_{LJF} q \Uparrow \\
\vdash_{LKF} \Gamma \Downarrow B & \xrightarrow{\psi} & [\![\Gamma]\!]^- \vdash_{LJF} [\![B]\!]^+ \Downarrow
\end{array}
$$

where the transformations ϕ, ψ are structure-preserving.

Proof. The two statements are proved simultaneously and by induction on the derivation. All but the structural rules of release and decide of LKF are mapped to a single proof rule of LJF: $\wedge^-$ is mapped to $\vee^+$-left, $\vee^-$ is mapped to $\wedge^+$-left, $\forall$ is mapped to $\exists$-left, $\wedge^+$ is mapped to $\wedge^+$-right, $\vee^+$ is mapped to $\vee^+$-right, $\exists$ is mapped to $\exists$-right, *init* is mapped to *init*-right, *store* is mapped to *store*-left. The case for *release*, which is mapped to a *release*-right followed by a $\supset$-right, and the case of *decide*, which is mapped to a *decide*-left, followed by a $\supset$-left, of which one premise ends immediately with an *init*-left on q, is given as follows. Note that because N is negative (reason for the release) $[\![N]\!]^+ = \neg_q [\![N]\!]^-$. Similarly, because P is positive (reason for the decide), $[\![P]\!]^- = \neg_q [\![P]\!]^+$

$$
\dfrac{\vdash \Gamma \Uparrow N}{\vdash \Gamma \Downarrow N}\ release \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma \Uparrow [\![N]\!]^- \vdash q \Uparrow \cdot}{\Gamma \Uparrow \cdot \vdash [\![N]\!]^+ \Uparrow \cdot}\ \supset_r}{\Gamma \vdash [\![N]\!]^+ \Downarrow}\ R_r
$$

$$
\dfrac{\vdash P, \Gamma \Downarrow P}{\vdash P, \Gamma \Uparrow \cdot}\ decide \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma \vdash [\![P]\!]^+ \Downarrow \qquad \dfrac{}{\Gamma \Downarrow q \vdash q}\ init_l}{\Gamma, [\![P]\!]^- \Downarrow [\![P]\!]^- \vdash E}\ \supset_l}{\Gamma, [\![P]\!]^- \Uparrow \cdot \vdash \cdot \Uparrow q}\ D_l
$$

This embedding is not limited to cut-free *LKF* proofs. Indeed, an *LKF* cut is also mapped to an *LJF* cut. The *LKF* cut, seen in figure 1 yields two asynchronous premises, one on a positive formula, one on a negative formula. Take B to be the positive formula, then $B^\perp$ is necessarily a negative formula (the de Morgan dual). The first premise will immediately go into a store, the second premise engages in an asynchronous phase. The LJF cut follows the same structure :

$$\dfrac{\vdash \Gamma \Uparrow B^\perp \qquad \dfrac{\vdash \Gamma, B \Uparrow \cdot}{\vdash \Gamma \Uparrow B}}{\vdash \Gamma \Uparrow \cdot}\ cut \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma \Uparrow [\![B]\!]^+ \vdash q \Uparrow \cdot}{\Gamma \Uparrow \cdot \vdash [\![B]\!]^- \Uparrow \cdot}\supset_r \qquad \dfrac{\Gamma, [\![B]\!]^- \Uparrow \cdot \vdash \cdot \Uparrow q}{\Gamma \Uparrow [\![B]\!]^- \vdash \cdot \Uparrow q}}{\Gamma \Uparrow \cdot \vdash \cdot \Uparrow q}\ Cut$$

The proof follows from this lemma: □

Lemma 1. *For any LKF formula B, $[\![B^\perp]\!]^+ = [\![B]\!]^-$ and $[\![B^\perp]\!]^- = [\![B]\!]^+$*

Proof. By induction on the structure of B. □

Remark 1. Because the output of the hosting embedding is always positive (except for the $\supset$) the rules for the negative connectives ($\forall$-left, $\forall$-right, $\wedge^-$-left, and $\wedge^-$-right) are not used, thus not treated in the proof. Similarly, the only formula stored on the right is q, a negative atom on which decide-right is never used. Release-left does not appear either because the only formulas under focus on the left are of the form $F \supset q$, which yield, in one premise a left-focus on a negative atom (the proof ends immediately with *init*-left on q) and a right-focus on F.

We now proceed to the decomposition of well known double-negation translations in terms of a polarity assignment and the hosting function on formulas.

4 Double-negation translations as polarity assignments

We will treat the double-negation translations of Kolmogorov, Gödel-Gentzen, Kuroda, and Krivine, reusing the notation from the recent paper relating these translation by Oliva and Ferreira [6]; for a classic reference see [11]. The notation $\neg_q B$ stands for the intuitionistic formula $B \supset q$, which is a way to write minimal logic negation.

We start with Krivine's translation $(-)^{\mathrm{Kr}}$ and the corresponding polarization $\underline{(-)}_{\mathrm{Kr}}$:

$$\begin{aligned}
a_{\mathrm{Kr}} &= \neg_q a & \underline{a}_{\mathrm{Kr}} &= a^+ \\
a^\perp{}_{\mathrm{Kr}} &= a & \underline{a^\perp}_{\mathrm{Kr}} &= a^- \\
(A \wedge B)_{\mathrm{Kr}} &= A_{\mathrm{Kr}} \vee B_{\mathrm{Kr}} & \underline{A \wedge B}_{\mathrm{Kr}} &= \underline{A}_{\mathrm{Kr}} \wedge^- \underline{B}_{\mathrm{Kr}} \\
(A \vee B)_{\mathrm{Kr}} &= A_{\mathrm{Kr}} \wedge^+ B_{\mathrm{Kr}} & \underline{A \vee B}_{\mathrm{Kr}} &= \underline{A}_{\mathrm{Kr}} \vee^- \underline{B}_{\mathrm{Kr}} \\
(\forall x A)_{\mathrm{Kr}} &= \exists x A_{\mathrm{Kr}} & \underline{\forall x A}_{\mathrm{Kr}} &= \forall x \underline{A}_{\mathrm{Kr}} \\
(\exists x A)_{\mathrm{Kr}} &= \neg_q \exists x \neg_q A_{\mathrm{Kr}} & \underline{\exists x A}_{\mathrm{Kr}} &= \exists x \partial^-(\underline{A}_{\mathrm{Kr}}),
\end{aligned}$$

where $(-)^{\mathrm{Kr}} = \neg_q (-)_{\mathrm{Kr}}$. By induction on A it is easy to see that

$$A_{\mathrm{Kr}} = [\![\underline{A}_{\mathrm{Kr}}]\!]^- \qquad \text{and, hence,} \qquad A^{\mathrm{Kr}} = \neg_q [\![\underline{A}_{\mathrm{Kr}}]\!]^-. \tag{1}$$

A polarity assignment of classical formulas, thus, generates the double-negation translation.

Next, we express the Kolmogorov translation $(-)^{\mathrm{Ko}}$ as a polarity assignment map for classical formulas $\underline{(-)}_{\mathrm{Ko}}$:

$$
\begin{aligned}
a_{\mathrm{Ko}} &= \neg_q\neg_q a & \underline{a}_{\mathrm{Ko}} &= \partial^-(a^+)\\
a^{\perp}{}_{\mathrm{Ko}} &= \neg_q a & \underline{a^{\perp}}_{\mathrm{Ko}} &= a^-\\
(A\wedge B)_{\mathrm{Ko}} &= (\neg_q\neg_q A_{\mathrm{Ko}}) \wedge^+ (\neg_q\neg_q B_{\mathrm{Ko}}) & \underline{A\wedge B}_{\mathrm{Ko}} &= (\partial^-(\partial^+(\underline{A}_{\mathrm{Ko}}))) \wedge^+ (\partial^-(\partial^+(\underline{B}_{\mathrm{Ko}})))\\
(A\vee B)_{\mathrm{Ko}} &= (\neg_q\neg_q A_{\mathrm{Ko}}) \vee (\neg_q\neg_q B_{\mathrm{Ko}}) & \underline{A\vee B}_{\mathrm{Ko}} &= (\partial^-(\partial^+(\underline{A}_{\mathrm{Ko}}))) \vee^+ (\partial^-(\partial^+(\underline{B}_{\mathrm{Ko}})))\\
\forall x A_{\mathrm{Ko}} &= \neg_q\exists x\neg_q A_{\mathrm{Ko}} & \underline{\forall x A}_{\mathrm{Ko}} &= \forall x\partial^+(\underline{A}_{\mathrm{Ko}})\\
\exists x A_{\mathrm{Ko}} &= \exists x\neg_q\neg_q A_{\mathrm{Ko}} & \underline{\exists x A}_{\mathrm{Ko}} &= \exists x\partial^-(\partial^+(\underline{A}_{\mathrm{Ko}})),
\end{aligned}
$$

where $(-)^{\mathrm{Ko}} = \neg_q\neg_q(-)_{\mathrm{Ko}}$. Again, by simple induction on A we can show that

$$A_{\mathrm{Ko}} = [\![\underline{A}_{\mathrm{Ko}}]\!]^+ \qquad \text{and, hence,} \qquad A^{\mathrm{Ko}} = \neg_q\neg_q[\![\underline{A}_{\mathrm{Ko}}]\!]^+. \tag{2}$$

Note, however, that instead of $\forall x\neg_q\neg_q A_{\mathrm{Ko}}$ we are defining $(\forall x A)_{\mathrm{Ko}}$ by $\neg_q\exists x\neg_q A_{\mathrm{Ko}}$. We use the minimal logic equivalence $\neg_q\exists x A \cong \forall x\neg_q A$, which is also an isomorphism within focused proofs in the sense that using these formulas causes minor differences within phases but causes no differences at phase boundaries. We also needed to use a number of delays, $\partial^+(\cdot)$ and $\partial^-(\cdot)$, to exactly capture Kolmogorov translation, because, as it is well known, this translation is not optimal in the number of negations it uses.

We continue with the polarization $\underline{(-)}_{\mathrm{Ku}}$ of Kuroda's double-negation translation $(-)^{\mathrm{Ku}}$:

$$
\begin{aligned}
a_{\mathrm{Ku}} &= a & \underline{a}_{\mathrm{Ku}} &= a^+\\
a^{\perp}{}_{\mathrm{Ku}} &= \neg_q a & \underline{a^{\perp}}_{\mathrm{Ku}} &= a^-\\
(A\wedge B)_{\mathrm{Ku}} &= A_{\mathrm{Ku}} \wedge^+ B_{\mathrm{Ku}} & \underline{A\wedge B}_{\mathrm{Ku}} &= \underline{A}_{\mathrm{Ku}} \wedge^+ \underline{B}_{\mathrm{Ku}}\\
(A\vee B)_{\mathrm{Ku}} &= A_{\mathrm{Ku}} \vee B_{\mathrm{Ku}} & \underline{A\vee B}_{\mathrm{Ku}} &= \underline{A}_{\mathrm{Ku}} \vee \underline{B}_{\mathrm{Ku}}\\
\forall x A_{\mathrm{Ku}} &= \neg_q\exists x\neg_q A_{\mathrm{Ku}} & \underline{\forall x A}_{\mathrm{Ku}} &= \forall x\partial^+(\underline{A}_{\mathrm{Ku}})\\
\exists x A_{\mathrm{Ku}} &= \exists x A_{\mathrm{Ku}} & \underline{\exists x A}_{\mathrm{Ku}} &= \exists x\underline{A}_{\mathrm{Ku}},
\end{aligned}
$$

where $(-)^{\mathrm{Ku}} = \neg_q\neg_q(-)_{\mathrm{Ku}}$ and we again prefer $\neg_q\exists x\neg_q A_{\mathrm{Ku}}$ over $\forall x\neg_q\neg_q A_{\mathrm{Ku}}$ in the translation of $\forall$. Similar to the previous translations, by induction on A one can easily show that

$$A_{\mathrm{Ku}} = [\![\underline{A}_{\mathrm{Ku}}]\!]^+ \qquad \text{and, hence,} \qquad A^{\mathrm{Ku}} = \neg_q\neg_q[\![\underline{A}_{\mathrm{Ku}}]\!]^+. \tag{3}$$

Finally, we consider the Gödel-Gentzen translation $(-)^{\mathrm{GG}}$,

$$
\begin{aligned}
a^{\mathrm{GG}} &= \neg_q\neg_q a\\
(a^{\perp})^{\mathrm{GG}} &= \neg_q a\\
(A\wedge B)^{\mathrm{GG}} &= A^{\mathrm{GG}} \wedge^- B^{\mathrm{GG}}\\
(A\vee B)^{\mathrm{GG}} &= \neg_q\neg_q(A^{\mathrm{GG}} \vee B^{\mathrm{GG}})\\
(\forall x A)^{\mathrm{GG}} &= \forall x A^{\mathrm{GG}}\\
(\exists x A)^{\mathrm{GG}} &= \neg_q\neg_q\exists x A^{\mathrm{GG}}.
\end{aligned}
$$

Its polarization is given by:

$$\begin{aligned}
\underline{a}_{\mathrm{GG}} &= \partial^-(a^+) & \underline{a}_{\mathrm{GG}_2} &= \partial^+(a^-)\\
\underline{a^\perp}_{\mathrm{GG}} &= a^- & \underline{a^\perp}_{\mathrm{GG}_2} &= a^+\\
\underline{A\wedge B}_{\mathrm{GG}} &= \underline{A}_{\mathrm{GG}} \wedge^- \underline{B}_{\mathrm{GG}} & \underline{A\wedge B}_{\mathrm{GG}_2} &= \underline{A}_{\mathrm{GG}_2} \wedge^- \underline{B}_{\mathrm{GG}_2}\\
\underline{A\vee B}_{\mathrm{GG}} &= \partial^-(\underline{A}_{\mathrm{GG}} \vee^+ \underline{B}_{\mathrm{GG}}) & \underline{A\vee B}_{\mathrm{GG}_2} &= \underline{A}_{\mathrm{GG}} \vee^+ \underline{B}_{\mathrm{GG}}\\
\underline{\forall x A}_{\mathrm{GG}} &= \forall x \underline{A}_{\mathrm{GG}_2} & \underline{\forall x A}_{\mathrm{GG}_2} &= \forall x \underline{A}_{\mathrm{GG}_2}\\
\underline{\exists x A}_{\mathrm{GG}} &= \partial^-(\exists x \underline{A}_{\mathrm{GG}}) & \underline{\exists x A}_{\mathrm{GG}_2} &= \exists x \underline{A}_{\mathrm{GG}}.
\end{aligned}$$

Now, if we write subformulas of the form $\forall x_1 \forall x_2 \cdots \forall x_n \neg_q A$ as the isomorphic $\neg_q \exists x_1 \exists x_2 \cdots \exists x_n A$, and all $\neg_q A \wedge^- \neg_q B$ as $\neg_q (A \vee B)$ we can get the following decomposition of the Gödel-Gentzen translation:

$$A^{\mathrm{GG}} = [\![\underline{A}_{\mathrm{GG}}]\!]^+. \tag{4}$$

We thus have the following theorem.

Theorem 2. *The hosting function for formulas $[\![\cdot]\!]^+, [\![\cdot]\!]^-$ factorizes the four double-negation translations through their polarization maps, that is, for any (non-polarized) first-order formula A we have that:*

$$A^{Ko} = \neg_q\neg_q[\![\underline{A}_{Ko}]\!]^+ \qquad A^{Kr} = \neg_q[\![\underline{A}_{Kr}]\!]^- \qquad A^{Ku} = \neg_q\neg_q[\![\underline{A}_{Ku}]\!]^+ \qquad A^{GG} = [\![\underline{A}_{GG}]\!]^+.$$

Combining Theorem 1 and Theorem 2, we can see that polarized proofs are hosted in a structure-preserving way as follows.

Corollary 1. *There is a structure preserving embedding of an LKF proof of A into an LJF proof of the DN-translation of A, or, more precisely, structure preserving maps of the following kind:*

$$\begin{array}{ccc}
\vdash_{LKF} \Gamma \Uparrow \underline{A}_{Kr} & \longrightarrow & [\![\Gamma]\!]^- \Uparrow \vdash_{LJF} A^{Kr} \Uparrow\\
\vdash_{LKF} \Gamma \Uparrow \underline{A}_{Ko} & \longrightarrow & [\![\Gamma]\!]^- \Uparrow \vdash_{LJF} A^{Ko} \Uparrow\\
\vdash_{LKF} \Gamma \Uparrow \underline{A}_{Ku} & \longrightarrow & [\![\Gamma]\!]^- \Uparrow \vdash_{LJF} A^{Ku} \Uparrow\\
\vdash_{LKF} \Gamma \Uparrow \underline{A}_{GG} & \longrightarrow & [\![\Gamma]\!]^- \Uparrow \vdash_{LJF} A^{GG} \Uparrow.
\end{array}$$

Proof. After a $\supset$-right rule applied, the following instances of Theorem 1 are used:

$$\begin{array}{ccc}
\vdash_{\mathrm{LKF}} \Gamma \Uparrow \underline{A}_{\mathrm{Kr}} & \longrightarrow & [\![\Gamma]\!]^- \Uparrow [\![\underline{A}_{\mathrm{Kr}}]\!]^- \vdash_{\mathrm{LJF}} q \Uparrow\\
\vdash_{\mathrm{LKF}} \Gamma \Uparrow \underline{A}_{\mathrm{Ko}} & \longrightarrow & [\![\Gamma]\!]^- \Uparrow [\![\partial^+(\underline{A}_{\mathrm{Ko}})]\!]^- \vdash_{\mathrm{LJF}} q \Uparrow\\
\vdash_{\mathrm{LKF}} \Gamma \Uparrow \underline{A}_{\mathrm{Ku}} & \longrightarrow & [\![\Gamma]\!]^- \Uparrow [\![\partial^+(\underline{A}_{\mathrm{Ku}})]\!]^- \vdash_{\mathrm{LJF}} q \Uparrow.
\end{array}$$

To see why the last proof-hosting statement is true, observe that $[\![\underline{A}_{\mathrm{GG}}]\!]^+$ is always a formula of the form $\neg_q[\![B]\!]^-$ for some polarized formula B. $\square$

We can thus see that the hosting function $[\![\cdot]\!]^+, [\![\cdot]\!]^-$ actually allows to define a large class of double-negation translations. Namely, given *any* polarization of a formula, the hosting translation of the polarization determines a valid intuitionistic translation of a classical theorem, and moreover such that proofs are hosted in a structure-preserving way. We present two such examples of DN translations obtained in this way.

5 Examples of polarization and corresponding translations

In previous sections, for each of four known double-negation translations, a polarization of classical formulas was given that yielded structurally equivalent focused proofs in the two systems *LKF* and *LJF*. In this section, we take the opposite approach by, first, polarizing a classical formula and, second, giving the corresponding double-negation translation.

Because the starting point was the double-negation translations, and not the classical formula, these polarizations included delays to force the break of focusing corresponding to (sometimes redundant) negations. It is, however, possible to polarize a classical formula without delays, and a corresponding double-negation translation can be given through the hosting embedding given in Definition 1. There is an exponential number of possible polarization of a classical formula bringing up a similar number of possible double-negation translations. This section gives two example. The first example polarizes the ambiguous connectives (conjunction and disjunction) negatively and the second one polarizes those ambiguous connectives positively. The resulting translations resemble Krivine's and Kuroda's translations, respectively, but treats a sequence of quantifiers as a whole, introducing negations only when needed.

5.1 CNF Double-negation translation

Let the polarization $\underline{(\cdot)}_{\text{cnf}}$ be given by

$$\underline{A}_{\text{cnf}} = A \qquad \underline{B \wedge C}_{\text{cnf}} = \underline{B}_{\text{cnf}} \wedge^- \underline{C}_{\text{cnf}} \qquad \underline{B \vee C}_{\text{cnf}} = \underline{B}_{\text{cnf}} \vee^- \underline{C}_{\text{cnf}}$$

$$\underline{A^{\perp}}_{\text{cnf}} = A^{\perp} \qquad \underline{\forall x.C}_{\text{cnf}} = \forall x.\underline{C}_{\text{cnf}} \qquad \underline{\exists x.C}_{\text{cnf}} = \exists x.\underline{C}_{\text{cnf}}$$

(Notice that if B is a propositional formula then the only *LKF* proofs of $\underline{B}_{\text{cnf}}$ resemble the construction of the conjunctive normal form of B.) The double-negation translation induced by this polarization differs from Krivine's in the treatment of sequences of existential. Indeed, the hosting embedding shows that a negation is only needed at phase switches, therefore simply giving a negative polarity when able (i.e. all but the existential) will result in a modified Krivine's translation with fewer negations.

In order to define the double-negation translation that corresponds to this polarization, $(\cdot)^{\text{cnf}}$, we consider the following two functions defined via mutual recursion and let F be a syntactic variable denoting any classical formula that is not a top-level existential quantifier. Then $(B)^{\text{cnf}} = \neg_q (B)^{\text{cnf1}}$ is defined as

$$(\forall x.C)^{\text{cnf1}} = \exists x.C^{\text{cnf1}} \qquad A^{\text{cnf1}} = \neg_q A \qquad A^{\perp\,\text{cnf1}} = A$$

$$(B \wedge C)^{\text{cnf1}} = B^{\text{cnf1}} \vee^+ C^{\text{cnf1}} \qquad (B \vee C)^{\text{cnf1}} = B^{\text{cnf1}} \wedge^+ C^{\text{cnf1}}$$

$$(\exists x.C)^{\text{cnf1}} = \neg_q \exists x.(C)^{\text{cnf2}} \qquad (\exists x.C)^{\text{cnf2}} = \exists x.(C)^{\text{cnf2}} \qquad F^{\text{cnf2}} = \neg_q F^{\text{cnf1}}$$

5.2 Minimal Kuroda's Double-negation translation

Opposite to the previous polarization, connectives are given positive polarities when possible:

$$\underline{A}_{\text{pos}} = A \qquad \underline{B \wedge C}_{\text{pos}} = \underline{B}_{\text{pos}} \wedge^+ \underline{C}_{\text{pos}} \qquad \underline{B \vee C}_{\text{pos}} = \underline{B}_{\text{pos}} \vee^+ \underline{C}_{\text{pos}}$$

$$\underline{A^{\perp}}_{\text{pos}} = A^{\perp} \qquad \underline{\forall x.C}_{\text{pos}} = \forall x.\underline{C}_{\text{pos}} \qquad \underline{\exists x.C}_{\text{pos}} = \exists x.\underline{C}_{\text{pos}}$$

In order to define the double-negation translation that corresponds to this polarization, $(\cdot)^{\text{pos1}}$, we consider the following two functions defined via mutual recursion: this time, F is a syntactic variable denoting any classical formula that is not a top-level universal quantifier.

$$(\forall x.C)^{\text{pos1}} = \forall x.(C)^{\text{pos1}} \qquad F^{\text{pos1}} = \neg_q \neg_q F^{\text{pos2}} \qquad (B \wedge C)^{\text{pos2}} = B^{\text{pos2}} \wedge^+ C^{\text{pos2}}$$

$$(B \vee C)^{\text{pos2}} = B^{\text{pos2}} \vee^+ C^{\text{pos2}} \qquad (\forall x.C)^{\text{pos2}} = \forall x.C^{\text{pos1}} \quad A^{\text{pos2}} = A \quad (A^{\perp})^{\text{pos2}} = \neg_q A$$

6 Conclusion and future work

We have presented a systematic way for generating old and new double-negation (and CPS) translations. The decomposition through the hosting function is beneficial not only for the translation of formulas but even more so for the structure of proofs.

One can also see our method as a way to make the notion of double-negation translation more precise. Typical requirements for a function $(\cdot)^*$ to be called a double-negation translation can be taken to be the following:

- $\Gamma \vdash A$ classically, implies $\Gamma^* \vdash A^*$ intuitionistically,
- and, $\Gamma \vdash (A \leftrightarrow A^*)$ classically.

Gaspar [7] has recently shown that it is possible to have two translations $(\cdot)^1$ and $(\cdot)^2$ satisfy these conditions although $A^1 \leftrightarrow A^2$ does not hold intuitionistically. It is not clear at the moment whether Gaspar's specially crafted double-negation translations can be represented as polarizations or whether the class of translations generated by polarizations is resistant to this intuitionistic non-equivalence phenomenon.

Avigad has shown [1] how classical proofs of A can be embedded in intuitionistic logic as proofs of the Gödel-Gentzen translation of A *efficiently*, which is similar to the spirit of our Theorem 1.

In the future, we would like to address the question of whether the hosting function allows preserving the process of cut-elimination intact. If so, what we obtain might be seen as a restatement of Plotkin's Simulation Theorem [9] for CPS translations that would use polarization instead of CPS translation of formulas, and directly work in the classical sequent calculus instead of making a detour in intuitionistic logic which has more facilities than needed to analyze CPS translated proofs (Remark 1). It is known that even advanced forms of CPS translations like the one-pass translation Danvy and Nielsen [5] significantly modify the structure of source terms during translation, while our hosting function from LKF to LJF is structure preserving.

Espírito Santo, Matthes, Nakazawa, and Pinto [10], also consider a factorization of CPS translations into a monadic translation and a single instantiation mapping. The mapping has the same role as our hosting function, as it is sufficient to capture both the call-by-value and the call-by-name CPS translations. The paper also studies the cut-elimination process under translation for the $\bar{\lambda}\mu\tilde{\mu}$ sequent calculus and proves a simulation result.

One application of this work involves the design of proof checker for classical and intuitionistic logic. If one is given a proof checker for intuitionistic logic, is it possible to use it to check classical logic proofs. At first glance, it would seem that double-negation translations might be useful for translating classical theorem into intuitionistic theorems: one would then need to find ways for "proof evidence" within a classical proof to be lifted to the intuitionistic logic setting. The results of this paper suggest that if proof checking is based on polarized classical and intuitionistic logics [3], then double-negation translations are not needed: instead, polarized classical logic formulas can be hosted directly within the polarized intuitionistic proof checker (this has indeed been demonstrated in the recent paper [4]).

Acknowledgments. This work has been funded by the ERC Advanced Grant ProofCert.

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