Exam on Miller's Lectures, Sample

This part of the exam is intended to last 90 minutes. It has a total of 10 points, and each question is valued at 2 points. Solutions can be written in English or French. In either case, please write clearly.

Exercise 1 (An I-proof) Give an **I**-proof of the following sequent.

$$p \vee \neg p \vdash ((p \supset q) \supset p) \supset p$$

Exercise 2 (A small and a big proof) Let $a_1, a_2, a_3, \ldots$ be a sequence of propositional symbols. Let D_i be the propositional Horn clause $D_i = (\wedge_{j=1}^{i-1} a_j) \supset a_i$. Thus, $D_1 = a_1$, $D_2 = a_1 \supset a_2$, $D_3 = (a_1 \wedge a_2) \supset a_3$, etc. Clearly, the sequent

$$D_1, D_2, \ldots, D_n \vdash a_n$$

is provable. Using the proof system for classical logic (Section 4.1), find two cut-free proofs for this sequent. One should have size at-most quadratic in n and one should have size exponential in n (*size* counts the number of occurrences of inference rules). [Hint: consider bottom-up versus top-down search strategies.]

Exercise 3 (Positive) Linear logic connectives can be divided into the *positive* connectives, namely, $\mathbf{1}$, $\mathbf{0}$, $\otimes$, $\oplus$, $\exists$ and the *negative* connectives, namely, $\bot$, $\top$, $⅋$, $\&$, $\forall$. Let B and C be two formulas for which $B \equiv\, !B$ and $C \equiv\, !C$. Show that the following equivalences hold for the positive connectives.

$$\mathbf{1} \equiv\, !\mathbf{1} \qquad \mathbf{0} \equiv\, !\mathbf{0} \qquad B \otimes C \equiv\, !(B \otimes C) \qquad B \oplus C \equiv\, !(B \oplus C) \qquad \exists x.B \equiv\, !\exists x.B$$

(Recall that $B \equiv C$ means $\vdash (B \multimap C) \,\&\, (C \multimap B)$ in linear logic.)

Exercise 4 (No notconnected) Represent the finite graph $G = (N, E)$, with nodes N and edges $E \subseteq N \times N$, as the set of atomic formulas

$$\mathcal{G} = \{node(x) \mid x \in N\} \cup \{edge(x, y) \mid \langle x, y\rangle \in E\}.$$

Argue why it is impossible to write a logic program $\mathcal{P}$ in first-order hereditary Harrop formulas that specifies the predicate $nc(x, y)$ such that for all $x, y \in N$, x and y are *not* connected by a path in the graph G if and only if the sequent $\mathcal{G}, \mathcal{P} \vdash nc(x, y)$ is provable.

Exercise 5 *Consider representing the finite graph $G = (N, E)$, with nodes N and edges $E \subseteq N \times N$, as the set of formulas*

$$\mathcal{G} = \{node(x) \mid x \in N\} \cup \{!\, edge(x, y) \mid \langle x, y\rangle \in E\}$$

(note the use of $!$). Consider the logic program $\mathcal{P}$ that consists of the following three formulas.

$$\forall u[connected \circ\!- (node(u) \otimes (nd(u) \Rightarrow loop))].$$

$$loop\ .$$

$$\forall u, v[loop \circ\!- (nd(u) \otimes edge(u, v) \otimes node(v) \otimes (nd(v) \Rightarrow loop))].$$

Show that the sequent $\mathcal{G}, \mathcal{P} \vdash connected$ *is provable in linear logic (Lolli) if and only if the graph G is connected.*