First step toward an implementation of the Tower Number Field Sieve: Murphy-alpha function for polynomial selection

Aurore Guillevic

Inria Nancy, Caramba team

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Séminaire C2, Paris

Joint work with Shashank Singh, IISER Bhopal, India

Plan

Introduction: Discrete logarithm and NFS

Key sizes for DL-based crypto

Pairings

Key-sizes for pairing-based crypto

Future work

Asymmetric cryptography

Factorization (RSA cryptosystem)

Discrete logarithm problem (use in Diffie-Hellman, etc)

Given a finite cyclic group $(\mathbf{G}, \cdot)$, a generator g and $h \in \mathbf{G}$, compute x s.t. $h = g^x$.

$\rightarrow$ can we invert the exponentiation function $(g, x) \mapsto g^x$?

Common choice of $\mathbf{G}$:

- prime finite field $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ (1976)
- characteristic 2 field $\mathbb{F}_{2^n}$ ($\approx$ 1979)
- elliptic curve $E(\mathbb{F}_p)$ (1985)

Discrete log problem

How fast can we invert the exponentiation function $(g, x) \mapsto g^x$?

- $g \in \mathbf{G}$ generator, $\exists$ always a preimage $x \in \{1, \ldots, \#\mathbf{G}\}$
- naive search, try them all: $\#\mathbf{G}$ tests
- $O(\sqrt{\#G})$ algorithms
  - Shanks baby-step-giant-step (BSGS): $O(\sqrt{\#\mathbf{G}})$, deterministic
  - random walk in $\mathbf{G}$, cycle path finding algorithm in a connected graph (Floyd) $\rightarrow$ Pollard: $O(\sqrt{\#\mathbf{G}})$, probabilistic (the cycle path encodes the answer)
  - parallel search (parallel Pollard, Kangarous)
- independent search in each distinct subgroup + CRT (Pohlig-Hellman)

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$\rightarrow$ Choose $\mathbf{G}$ of large prime order (no subgroup)

$\rightarrow$ complexity of inverting exponentiation in $O(\sqrt{\#G})$

$\rightarrow$ security level 128 bits means $\sqrt{\#G} \geq 2^{128} \rightarrow \#G \geq 2^{256}$
analogy with symmetric crypto, keylength 128 bits (16 bytes)

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better way?
$\rightarrow$ Use additional structure of **G** if any.

Discrete log problem when $\mathbf{G} = (\mathbb{Z}/p\mathbb{Z})^*$

Index calculus algorithm [Western–Miller 68, Adleman 79], prequel of the Number Field Sieve algorithm (NFS)

- p prime, $(p-1)/2$ prime, $\mathbf{G} = (\mathbb{Z}/p\mathbb{Z})^*$, gen. g, target h
- get many multiplicative relations in $\mathbf{G}$
  $g^t = g_1^{e_1} g_2^{e_2} \cdots g_i^{e_i} \pmod{p}$, $g, g_1, g_2, \ldots, g_i \in \mathbf{G}$
- find a relation $h = g_1^{e_1'} g_2^{e_2'} \cdots g_i^{e_i'} \pmod{p}$
- take logarithm: linear relations

$$\begin{aligned} t &= e_1 \log_g g_1 + e_2 \log_g g_2 + \ldots + e_i \log_g g_i \pmod{p-1} \\ &\vdots \\ \log_g h &= e_1' \log_g g_1 + e_2' \log_g g_2 + \ldots + e_i' \log_g g_i \pmod{p-1} \end{aligned}$$

- solve a linear system
- get $x = \log_g h$

Index calculus in $(\mathbb{Z}/p\mathbb{Z})^*$: example

$p = 1109$, $r = (p-1)/4 = 277$ prime
Smoothness bound $B = 13$
$\mathcal{F}_{13} = \{2, 3, 5, 7, 11, 13\}$ small primes up to B, $i = \#\mathcal{F}$
B-smooth integer: $n = \prod_{p_i \leq B} p_i^{e_i}$, p_i prime
is g^s smooth? $1 \leq s \leq 72$ is enough

$$\begin{aligned} g^{1} &= 2 = 2 \\ g^{13} &= 429 = 3 \cdot 11 \cdot 13 \\ g^{16} &= 105 = 3 \cdot 5 \cdot 7 \\ g^{21} &= 33 = 3 \cdot 11 \\ g^{44} &= 1029 = 3 \cdot 7^3 \\ g^{72} &= 325 = 5^2 \cdot 13 \end{aligned} \quad \rightarrow \quad \begin{array}{c} \begin{array}{cccccc} 2 & 3 & 5 & 7 & 11 & 13 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 \end{bmatrix} \end{array} \cdot \boldsymbol{x} = \begin{bmatrix} 1 \\ 13 \\ 16 \\ 21 \\ 44 \\ 72 \end{bmatrix}$$

$\boldsymbol{x} = [1, 219, 40, 34, 79, 269] \bmod 277$

$\rightarrow \log_g 7 = 34 \bmod 277$, that is, $(g^{34})^4 = 7^4$
$g^{34} = 7u$ and $u^4 = 1$

Index calculus in $(\mathbb{Z}/p\mathbb{Z})^*$: example

$\boldsymbol{x} = [1, 219, 40, 34, 79, 269] \bmod 277$

subgroup of order 4: $g_4 = g^{(p-1)/4}$

$\{1, g_4, g_4^2, g_4^3\} = \{1, 354, 1108, 755\}$

Pohlig-Hellman:

$$\begin{aligned}
3/g^{219} &= & &= 1 \Rightarrow \log_g 3 &&= &&= 219\\
5/g^{40} &= 1108 &&= -1 \Rightarrow \log_g 5 &&= 40 + (p-1)/2 &&= 594\\
7/g^{34} &= 354 &&= g_4 \Rightarrow \log_g 7 &&= 34 + (p-1)/4 &&= 311\\
11/g^{79} &= 755 &&= g_4^3 \Rightarrow \log_g 11 &&= 79 + 3(p-1)/4 &&= 910\\
13/g^{269} &= 755 &&= g_4^3 \Rightarrow \log_g 13 &&= 269 + 3(p-1)/4 &&= 1100
\end{aligned}$$

$\boldsymbol{v} = [1, 219, 594, 311, 910, 1100] \bmod p-1$

Target $h = 777$

$g^{10} \cdot 777 = 495 = 3^2 \cdot 5 \cdot 11 \bmod p$

$\log_2 777 = -10 + 2\log_g 3 + \log_g 5 + \log_g 11 = 824 \bmod p-1$

$g^{824} = 777$

Index calculus in $(\mathbb{Z}/p\mathbb{Z})^*$: example

Trick

Multiplicative relations over the **integers**

$g_1, g_2, \ldots, g_i \longleftrightarrow$ small prime integers

Smooth integers $n = \prod_{p_i \leq B} p_i^{e_i}$ are quite common $\rightarrow$ it works

Complexity $e^{\sqrt{(2+o(1))(\log p)(\log \log p)}}$ (Pomerance 87)

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Improvements in the 80's, 90's:

- Sieve (faster relation collection)
- Smaller integers to factor
- Multiplicative relations in **number fields**
- Better **sparse linear algebra**
- Independent targets h

Number Field: Toy example with $\mathbb{Z}[i]$

1985: ElGamal, DL in GF(p^2) with two quadratic number fields
1986: Coppersmith–Odlyzko–Schroeppel, DL in GF(p)
reduce further the size of the integers to factor
If $p = 1 \bmod 4$, $\exists\ U, V$ s.t. $p = U^2 + V^2$
and $|U|, |V| < \sqrt{p}$
$U/V \equiv m \mod p$ and $m^2 + 1 = 0 \mod p$

Define a map from $\mathbb{Z}[i]$ to $\mathbb{Z}/p\mathbb{Z}$

$$\begin{array}{rcl} \phi\colon \mathbb{Z}[i] & \to & \mathbb{Z}/p\mathbb{Z} \\ i & \mapsto & m \bmod p \text{ where } m = U/V,\ m^2+1 = 0 \bmod p \end{array}$$

ring homomorphism $\phi(a + bi) = a + bm$

$$\phi(\underbrace{a+bi}_{\substack{\text{factor in}\\ \mathbb{Z}[i]}}) = a + bm = (a + b\ \underbrace{U/V}_{=m}) = (\underbrace{aV + bU}_{\text{factor in } \mathbb{Z}})V^{-1} \mod p$$

Example in $\mathbb{Z}[i]$

$p = 1109 = 1 \bmod 4$, $r = (p-1)/4 = 277$ prime
$p = 22^2 + 25^2$
$\max(|a|, |b|) = A = 20$, $B = 13$ smoothness bound

Rational side

$\mathcal{F}_{\text{rat}} = \{2, 3, 5, 7, 11, 13\}$ primes up to B

Algebraic side: think about the complex number in $\mathbb{C}$

$-i(1+i)^2 = 2$, $(2+i)(2-i) = 5$, $(2+3i)(2-3i) = 13$

All primes $p = 1 \bmod 4$, $p > 2$

- can be written as a sum of two squares $p = a^2 + b^2$
- factor into two conjugate Gaussian integers $(a+ib)(a-ib)$

Units: $i^2 = -1$

$\mathcal{F}_{\text{alg}} = \{1+i, 2+i, 2-i, 2+3i, 2-3i\}$
"primes" of norm up to B
$\mathcal{U}_{\text{alg}} = \{-1, i, -i\}$ Units

Example in $\mathbb{Z}[i]$

$p = 1109$

$(a, b) = (-4, 7)$,
$\mathrm{Norm}(-4 + 7i) = (-4)^2 + 7^2 = 65 = 5 \cdot 13$
In $\mathbb{Z}[i]$,

- $5 = (2 + i)(2 - i)$
- $13 = (2 + 3i)(2 - 3i)$

Then,

$\rightarrow$ $(2 \pm i)(2 \pm 3i)$ has norm 65

$\rightarrow$ $\pm i(2 \pm i)(2 \pm 3i) = (-4 + 7i)$

We obtain $i(2 - i)(2 + 3i) = -4 + 7i$

Example in $\mathbb{Z}[i]$

$a+bi$	$aV+bU=$ factor in $\mathbb{Z}$	a^2+b^2	factor in $\mathbb{Z}[i]$
$-17+19i$	$-7=-7$	$650=2\cdot 5^2\cdot 13$	$i(1+i)(2+i)^2(2-3i)$
$-11+2i$	$-231=-3\cdot 7\cdot 11$	$125=5^3$	$i(2+i)^3$
$-6+17i$	$224=2^5\cdot 7$	$325=5^2\cdot 13$	$(2+i)^2(2+3i)$
$-4+7i$	$54=2\cdot 3^3$	$65=5\cdot 13$	$i(2-i)(2+3i)$
$-3+4i$	$13=13$	$25=5^2$	$-(2-i)^2$
$-2+i$	$-28=-2^2\cdot 7$	$5=5$	$-(2-i)$
$-2+3i$	$16=2^4$	$13=13$	$-(2-3i)$
$-2+11i$	$192=2^6\cdot 3$	$125=5^3$	$-(2-i)^3$
$-1+i$	$-3=-3$	$2=2$	$i(1+i)$
i	$22=2\cdot 11$	$1=1$	i
$1+3i$	$91=7\cdot 13$	$10=2\cdot 5$	$(1+i)(2+i)$
$1+5i$	$135=3^3\cdot 5$	$26=2\cdot 13$	$i(1+i)(2-3i)$
$2+i$	$72=2^3\cdot 3^2$	$5=5$	$(2+i)$
$5+i$	$147=3\cdot 7^2$	$26=2\cdot 13$	$-i(1+i)(2+3i)$

Example in $\mathbb{Z}[i]$: Matrix

Build the matrix of relations:

- one row per (a, b) pair s.t. both norms are smooth
- one column per prime of $\mathcal{F}_{\text{rat}}$
- one column for $1/V$
- one column per prime ideal of $\mathcal{F}_{\text{alg}}$
- one column per unit $(-1, i)$
- store the exponents

Example in $\mathbb{Z}[i]$

$$M = \begin{array}{c} \begin{array}{cccccccccccccc} 2 & 3 & 5 & 7 & 11 & 13 & \frac{1}{V} & -1 & i & 1+i & 2+i & 2-i & 2+3i & 2-3i \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 3 & 0 & 0 & 0 \\ 5 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 1 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 2 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 6 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 3 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Example in $\mathbb{Z}[i]$

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
2 & 3 & 5 & 7 & 11 & 13 & \frac{1}{V} & -1 & i & 1+i & 2+i & 2-i & 2+3i & 2-3i
\end{array} \\
M = \begin{bmatrix}
 & & & & & & & 1 & 2 & & & & & \\
 & & & 1 & & & 1 & 1 & 1 & 1 & 2 & & & 1 \\
 & 1 & & 1 & 1 & & 1 & 1 & 1 & & 3 & & & \\
5 & & & 1 & & & 1 & & & & 2 & & 1 & \\
1 & 3 & & & & & 1 & & 1 & & & 1 & 1 & \\
 & & & & & 1 & 1 & 1 & & & & 2 & & \\
2 & & & 1 & & & 1 & & & & & 1 & & \\
4 & & & & & & 1 & 1 & & & & & & 1 \\
6 & 1 & & & & & 1 & 1 & & & & 3 & & \\
 & 1 & & & & & 1 & 1 & 1 & 1 & & & & \\
1 & & & & 1 & & 1 & & 1 & & & & & \\
 & & & 1 & & 1 & 1 & & & 1 & 1 & & & \\
 & 3 & 1 & & & & 1 & & 1 & 1 & & & & 1 \\
3 & 2 & & & & & 1 & & & & 1 & & & \\
 & 1 & & 2 & & & 1 & 1 & 1 & 1 & & & 1 &
\end{bmatrix}
\end{array}
$$

Example in $\mathbb{Z}[i]$

$$
\begin{array}{c}
\begin{array}{cccccccccccccc}
2 & 3 & 5 & 7 & 11 & 13 & \frac{1}{V} & -1 & i & 1+i & 2+i & 2-i & 2+3i & 2-3i
\end{array} \\
M = \begin{bmatrix}
 & & & & & & & -1 & -2 & & & & & \\
 & & & 1 & & & 1 & -1 & -1 & -1 & -2 & & & -1 \\
 & 1 & & 1 & 1 & & 1 & -1 & -1 & & -3 & & & \\
5 & & & 1 & & & 1 & & & & -2 & & -1 & \\
1 & 3 & & & & & 1 & & -1 & & & -1 & -1 & \\
 & & & & & 1 & 1 & -1 & & & & -2 & & \\
2 & & & 1 & & & 1 & & & & & -1 & & \\
4 & & & & & & 1 & -1 & & & & & & -1 \\
6 & 1 & & & & & 1 & -1 & & & & -3 & & \\
 & 1 & & & & & 1 & -1 & -1 & -1 & & & & \\
1 & & & & 1 & & 1 & & -1 & & & & & \\
 & & & 1 & & 1 & 1 & & & -1 & -1 & & & \\
 & 3 & 1 & & & & 1 & & -1 & -1 & & & & -1 \\
3 & 2 & & & & & 1 & & & & -1 & & & \\
 & 1 & & 2 & & & 1 & -1 & -1 & -1 & & & -1 &
\end{bmatrix}
\end{array}
$$

Example in $\mathbb{Z}[i]$

Right kernel $M \cdot \boldsymbol{x} = 0 \bmod (p-1)/4 = 277$:

$$\boldsymbol{x} = (\underbrace{1, 219, 40, 34, 79, 269}_{\text{rational side}}, \underbrace{197}_{1/V}, \underbrace{0, 0}_{\text{units}}, \underbrace{139, 84, 233, 68, 201}_{\text{algebraic side}})$$

Logarithms (in some basis)

Rational side: logarithms of $\{2, 3, 5, 7, 11, 13\}$

$\to \log x_i / \log 2$

$\boldsymbol{x} = [1, 219, 40, 34, 79, 269] \bmod 277$

$\to$ order 4 subgroup

$\boldsymbol{v} = [1, 219, 594, 311, 910, 1100] \bmod p-1$

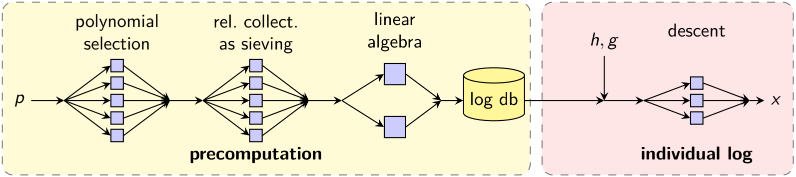
Target 314, generator $g = 2$

$314 = -20/7 \bmod p = -2^2 \cdot 5/7$

$$\begin{aligned} \log_g 314 &= \log_g -1 + 2\log_g 2 + \log_g 5 - \log_g 7 \\ &= (p-1)/2 + 2 + 594 - 311 = 839 \bmod p-1 \end{aligned}$$

$2^{839} = 314 \bmod p$

Number Field Sieve today



Graph N. Heninger

- NFS: Gordon 93, improvements Schirokauer 93
- polynomial selection Joux–Lercier 03
- Franke–Kleinjung 08 sieve, ECM factorization H. Lenstra 87
- block Lanczos, Wiedemann 86 sparse linear algebra
- Joux–Lercier 03 descent, early-abort strategy Pomerance 82

Latest DL record computation: 768-bit $\mathbb{F}_p$

Kleinjung, Diem, A. Lenstra, Priplata, Stahlke, Eurocrypt'2017.
$p = \lfloor 2^{766} \times \pi \rfloor + 62762$ prime, 768 bits, 232 decimal digits, $p =$

1219344858334286932696341909195796109526657386154251328029
2736561757668709803065055845773891258608267152015472257940
7293588325886803643328721799472154219914818284150580043314
8410869683590659346847659519108393837414567892730579162319

$(p-1)/2$ prime

$$f(x) = 140x^4 + 34x^3 + 86x^2 + 5x - 55$$

$$\begin{aligned} g(x) = {} & 370863403886416141150505523919527677231932618184100095924x^3 \\ & -1937981312833038778565617469829395544065255938015920309679x^2 \\ & -2175832936269478997875774411283330276175410950047347364 15x \\ & +2772607304003495228904226184734981485287061150033379351 50 \end{aligned}$$

Enumerate ($\sim 10^{12}$) all $f(x)$ s.t. $|f_i| \leqslant 165$
By construction, $|g_i| \approx p^{1/4}$

Latest DL record computation: 768-bit $\mathbb{F}_p$

$\gcd(f, g) = 1$ in $\mathbb{Q}[x]$
$\exists$ root m s.t. $f(m) = g(m) = 0 \pmod{p}$, $m =$

4290295629231970357488936064013995423387122927373167219112
8794979019508571426956110520280493413148710512618823586632
1484497413188392653246206774027756646444183240629650904112
110269916261074281303302883725258878464313312196475775222

Multiplicative relations: for all $|a_i| \leq A \approx 2^{32}$, $\gcd(a_0, a_1) = 1$

- factors $\mathrm{Norm}_f = \mathrm{Resultant}(f, a_0 + a_1 x) \approx 130$ bits, 39 dd
- factors $\mathrm{Norm}_g = \mathrm{Resultant}(g, a_0 + a_1 x) \approx 290$ bits, 87 dd

Linear algebra: square sparse matrix of $23.5 \cdot 10^6$ rows
Total time: 5300 core-years on Intel Xeon E5-2660 2.2GHz

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Complexity and key-sizes for cryptography

[Lenstra-Verheul'01] gives RSA key-sizes
Security estimates use

- asymptotic complexity of the best known algorithm (here NFS)
- latest record computation (now 768-bit)
- extrapolation

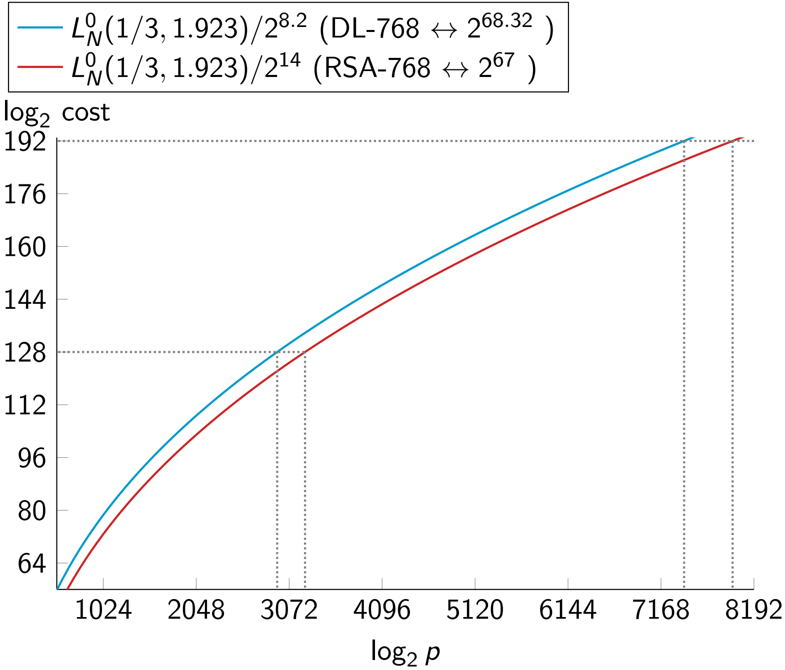
Complexity

Subexponential asymptotic complexity:

$$L_{p^n}(\alpha, c) = e^{(c+o(1))(\log p^n)^\alpha(\log\log p^n)^{1-\alpha}}$$

- $\alpha = 1$: exponential
- $\alpha = 0$: polynomial
- $0 < \alpha < 1$: sub-exponential (including NFS)

1. polynomial selection (precomp., 5% to 10% of total time)
2. relation collection $L_{p^n}(1/3, c)$
3. linear algebra $L_{p^n}(1/3, c)$
4. individual discrete log computation $L_{p^n}(1/3, c' < c)$


$L_N^0(1/3, 1.923)/2^{8.2}$ (DL-768 $\leftrightarrow$ $2^{68.32}$)
$L_N^0(1/3, 1.923)/2^{14}$ (RSA-768 $\leftrightarrow$ 2^{67})
$\log_2$ cost
192
176
160
144
128
112
96
80
64
1024
2048
3072
4096
5120
6144
7168
8192
$\log_2 p$

Key length

- `keylength.com`
- France: ANSSI RGS B

RSA modulus and prime fields for DL: 3072 to 3200 bits
sub-exponential complexity to invert DL in $\mathbb{F}_p$

Elliptic curves: over prime field of 256 bits (much smaller)
exponential cpx. to invert DL in $E(\mathbb{F}_p)$

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Why finite fields in 2019?

because old crypto in $\mathbb{F}_p$ is still in use
cpx $= L_p(1/3, 1.923)$ since 1993: very-well known
because of pairings: $\mathbb{F}_{p^n}$ since 2000

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Cryptographic pairing: black-box properties

$(\mathbf{G}_1, +), (\mathbf{G}_2, +), (\mathbf{G}_T, \cdot)$ three cyclic groups of large prime order r
Bilinear Pairing: map $e : \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$

1. bilinear: $e(P_1 + P_2,\ Q) = e(P_1, Q) \cdot e(P_2, Q)$,
   $e(P, Q_1 + Q_2) = e(P, Q_1) \cdot e(P, Q_2)$
2. non-degenerate: $e(g_1, g_2) \neq 1$ for $\langle g_1 \rangle = \mathbf{G}_1$, $\langle g_2 \rangle = \mathbf{G}_2$
3. efficiently computable.

Mostly used in practice:

$$e([a]P, [b]Q) = e([b]P, [a]Q) = e(P, Q)^{ab} \ .$$

$\rightsquigarrow$ Many applications in asymmetric cryptography.

Examples of application

- 1984: idea of identity-based encryption (IBE) by Shamir
- 1999: first practical identity-based cryptosystem of Sakai-Ohgishi-Kasahara
- 2000: constructive pairings, Joux's tri-partite key-exchange
- 2001: IBE of Boneh-Franklin, short signatures Boneh-Lynn-Shacham

...

- Broadcast encryption, re-keying
- agregate signatures
- zero-knowledge (ZK) proofs
  - non-interactive ZK proofs (NIZK)
  - ZK-SNARK (Z-cash)

Bilinear Pairings

Rely on

- Discrete Log Problem (DLP):
  given $g, h \in \mathbf{G}$, compute x s.t. $g^x = h$
- Diffie-Hellman Problem (DHP):
  given $g, g^a, g^b \in \mathbf{G}$, compute g^{ab}
- bilinear DLP and DHP
- pairing inversion problem

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \; : \; E(\mathbb{F}_{p^n})[r] \times E(\mathbb{F}_{p^n})[r] \longrightarrow \mathbb{F}_{p^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Pairing-based cryptography

Weil or Tate pairing on an elliptic curve

Discrete logarithm problem with one more dimension.

$$e \ : \ E(\mathbb{F}_{p^n})[r] \times E(\mathbb{F}_{p^n})[r] \longrightarrow \mathbb{F}_{p^n}^*, \quad e([a]P, [b]Q) = e(P, Q)^{ab}$$

Attacks

Pairing-based cryptography

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Attacks

- inversion of e : hard problem (exponential)
- discrete logarithm computation in $E(\mathbb{F}_p)$: hard problem (exponential, in $O(\sqrt{r})$)
- discrete logarithm computation in $\mathbb{F}_{p^n}^*$: **easier, subexponential** $\rightarrow$ take a large enough field

Pairing-friendly curves are special

$r \mid p^n - 1$, $\mathbf{G}_T \subset \mathbb{F}_{p^n}$, n is minimal : **embedding degree**
Tate Pairing: $e : \ \mathbf{G}_1 \times \mathbf{G}_2 \to \mathbf{G}_T$
When n is small, the curve is *pairing-friendly*.
This is very rare: usually $\log n \sim \log r$ ([Balasubramanian Koblitz]).

$\mathbf{G}_T \subset p^n$	p^2, p^6	p^3, p^4, p^6	p^{12}	p^{16}	p^{18}	p^{24}
Curve	super-singular	MNT	BN BLS12	KSS16	KSS18	BLS24

MNT, $n = 6$:
$p(x) = 4x^2 + 1$, $\#E(\mathbb{F}_p) = r(x) = x^2 \mp 2x + 1$
BN, $n = 12$:
$p(x) = 36x^4 + 36x^3 + 24x^2 + 6x + 1$
$r(x) = 36x^4 + 36x^3 + 18x^2 + 6x + 1$

Plan

Introduction: Discrete logarithm and NFS

Key sizes for DL-based crypto

Pairings

Key-sizes for pairing-based crypto

Future work

Discrete Log in $\mathbb{F}_{p^n}$

$\mathbb{F}_{p^n}$ much less investigated than $\mathbb{F}_p$ or integer factorization.
Much better results in pairing-related fields

Discrete Log in $\mathbb{F}_{p^n}$

$\mathbb{F}_{p^n}$ much less investigated than $\mathbb{F}_p$ or integer factorization.
Much better results in pairing-related fields

- Special NFS in $\mathbb{F}_{p^n}$: Joux–Pierrot 2013
- Tower NFS (TNFS): Barbulescu Gaudry Kleinjung 2015
- Extended Tower NFS: Kim–Barbulescu, Kim–Jeong, Sarkar–Singh 2016
- Tower of number fields

Use more structure: subfields

Special Tower NFS

$\mathbb{F}_{p^6}$, subfield $\mathbb{F}_{p^2}$ defined by $y^2 + 1$
Idea: $a + bx$ in NFS $\rightarrow$ $(a_0 + a_1 i) + (b_0 + b_1 i)x$ in TNFS
Integers to factor are **much smaller**

- factors integer $\mathrm{Norm}_f = \mathrm{Res}(\mathrm{Res}(\mathbf{a} + \mathbf{b}x, f_y(x)), y^2 + 1)$
- factors integer $\mathrm{Norm}_g = \mathrm{Res}(\mathrm{Res}(\mathbf{a} + \mathbf{b}x, g_y(x)), y^2 + 1)$

Res = resultant of polynomials

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Res = resultant of polynomials

Index calculus in the 80's: implemented *before* complexity known
TNFS: complexity known, no implementation

Complexities

large characteristic $p = L_{p^n}(\alpha),\ \alpha > 2/3$:	
$(64/9)^{1/3} \simeq 1.923$	NFS
special p:	
$(32/9)^{1/3} \simeq 1.526$	SNFS

medium characteristic $p = L_{p^n}(\alpha),\ 1/3 < \alpha < 2/3$:	
$(96/9)^{1/3} \simeq 2.201$	prime n NFS-HD (Conjugation)
$(48/9)^{1/3} \simeq 1.747$	composite n,
	best case of TNFS: when parameters fit perfectly
special p:	
$(64/9)^{1/3} \simeq 1.923$	NFS-HD+Joux–Pierrot'13
$(32/9)^{1/3} \simeq 1.526$	composite n, best case of STNFS

Estimating key sizes for DL in $\mathbb{F}_{p^n}$

- Latest variants of TNFS (Kim–Barbulescu, Kim–Jeong) seem most promising for $\mathbb{F}_{p^n}$ where n is composite
- We need record computations if we want to extrapolate from asymptotic complexities
- The asymptotic complexities do not correspond to a fixed n, but to a ratio between n and p

Simulation of STNFS: why?

- upper bound on the norms
- (heuristic) upper bound on the running-time of STNFS
- bound is not tight: running-time could be much faster
- security is over-estimated

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Possible solution:

- remove combinatorial factor from the bound
- smaller norms, faster STNFS, lower security
- much larger key-sizes
- bad for practical applications: larger keys are required

Example BN curves, targeted 128-bit security level:
p was 256 bits before STNFS
Now p from 384 to 512 bits
But we don't want to use too large p for nothing.

Largest record computations in $\mathbb{F}_{p^n}$ with NFS[1]

Finite field	Size of p^n	Cost: CPU days	Authors	sieving dim
$\mathbb{F}_{p^{12}}$	203	11	[HAKT13]	7
$\mathbb{F}_{p^6}$	422	9,520	[GGMT17]	3
$\mathbb{F}_{p^5}$	324	386	[GGM17]	3
$\mathbb{F}_{p^4}$	392	510	[BGGM15b]	2
$\mathbb{F}_{p^3}$	593	8,400	[GGM16]	2
$\mathbb{F}_{p^2}$	595	175	[BGGM15a]	2
$\mathbb{F}_p$	768	1,935,825	[KDLPS17]	2

None used TNFS, only NFS and NFS-HD were implemented.

[1] Data extracted from DiscreteLogDB by L.Grémy

Ranking polynomials: Murphy's α and E

B. A. Murphy, 1999

- $\alpha(f)$: bias in smoothness between norms and integers
  $\alpha(f), \alpha(g) < 0$ wanted
- $E(f, g, B_f, B_g, \text{area})$: estimation of the yield of polynomials
  How many relations would (f, g) produce?
- Rank many (f_i, g_i), choose the best pair

Generalization to the TNFS setting:

- $\alpha(h, f)$, $\alpha(h, g)$
  SageMath & Magma code, generalization from `cado-nfs` α
  (Bai, Hanrot, Thomé, Zimmermann)
- Monte-Carlo simulation for Murphy's E

Simulation without sieving

space: $\mathcal{S} = \{\sum a_{0i} y^i + (\sum a_{1i} y^i)x,\ |a_{ji}| < A\}$
Variants:

- compute $\alpha(h, f), \alpha(h, g)$ (w.r.t. subfield) **bias in smoothness**
- select polys f, g with negative bias $\alpha(f), \alpha(g)$ if possible
- **Monte-Carlo** simulation with 10^6 points in $\mathcal{S}$ taken at random. For each point:
  1. compute its algebraic norm N_f, N_g in each number field
  2. smoothness probability with Dickman-ρ
- Average smoothness probability over the subset of points
  $\rightarrow$ estimation of the total number of possible relations in $\mathcal{S}$
  $\rightarrow$ **Murphy's E for TNFS**

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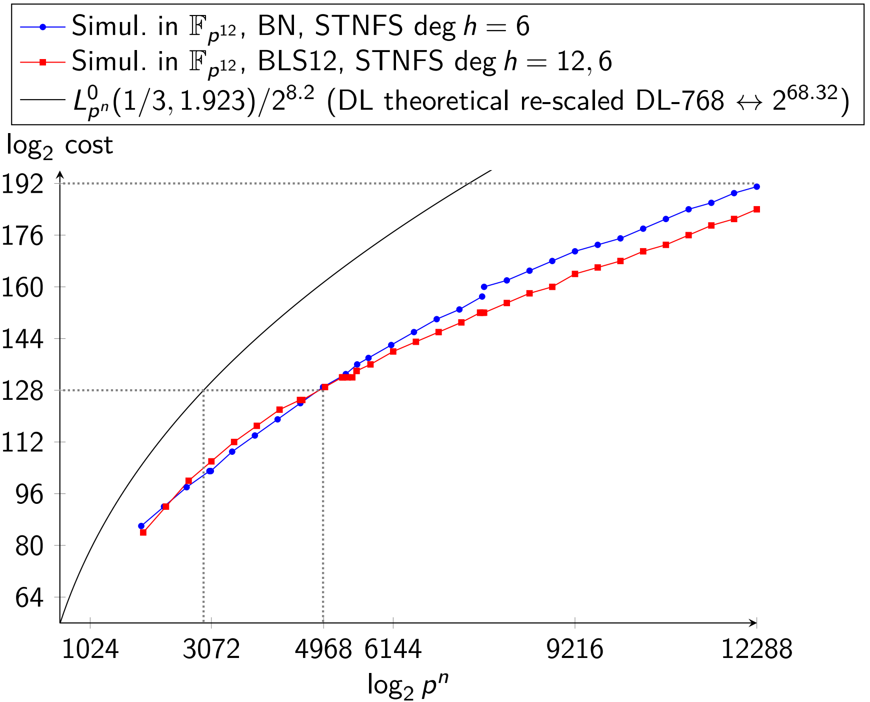
- Average smoothness probability over the subset of points
  $\rightarrow$ estimation of the total number of possible relations in $\mathcal{S}$
  $\rightarrow$ **Murphy's E for TNFS**
- dichotomy to approach the best balanced parameters: smoothness bound B, coefficient bound A.
  $\rightarrow$ refinement of Barbulescu–Duquesne technique

Example : Barreto-Naehrig curve, p 254 bits

$p = 36s^4 + 36s^3 + 24s^2 + 6s + 1$ where $s = -(2^{62} + 2^{55} + 1)$
$f = 36x^8 + 36yx^6 + 24y^2x^4 + 6y^3x^2 + y^4$
$g = x^2 + sy = x^2 + 4647714815446351873y$
$B = 2000$

h	$1/\zeta_{K_h}(2)$	$\alpha(h, f, B)$	$\alpha(h, g, B)$	$\alpha_f + \alpha_g$
$y^6 + y^5 - y^2 - y - 1$	0.953	2.042	2.479	4.521
$y^6 - y^4 + y^3 + y^2 - 1$	0.917	1.288	1.740	3.028
$y^6 + y^3 + y^2 - y - 1$	0.917	2.419	2.876	5.295
$y^6 + y^5 - y^3 + y - 1$	0.909	0.278	2.357	2.636
$y^6 + y^5 + y^4 + y^3 + y^2 + y - 1$	0.883	2.341	2.033	4.374
$y^6 + y^4 + y^3 + y - 1$	0.867	0.899	2.526	3.425
$y^6 + y^4 + y^2 + y + 1$	0.836	1.955	1.141	3.095
$y^6 + y^5 + y^2 - y + 1$	0.763	0.891	1.264	2.155
$y^6 + y^5 - y^4 + y^3 + y^2 + y - 1$	0.756	0.956	1.177	2.133
$y^6 + y^5 + y - 1$	0.736	1.925	2.108	4.032
$y^6 + y^5 + y^3 - y^2 + y - 1$	0.732	1.729	2.099	3.828
$y^6 + y^3 + y - 1$	0.728	-0.250	1.191	0.941
$y^6 + y^3 - y + 1$	0.720	1.605	1.348	2.952
$y^6 + y^3 + y^2 + 1$	0.718	1.151	1.294	2.445
$y^6 - y^4 + y^3 - y^2 - y - 1$	0.710	0.406	2.278	2.684
$y^6 + y^5 - y^3 + y^2 - y + 1$	0.697	1.572	0.818	2.390
$y^6 + y^4 + y + 1$	0.679	1.319	1.683	3.002


Simul. in $\mathbb{F}_{p^{12}}$, BN, STNFS deg $h = 6$
Simul. in $\mathbb{F}_{p^{12}}$, BLS12, STNFS deg $h = 12, 6$
$L^0_{p^n}(1/3, 1.923)/2^{8.2}$ (DL theoretical re-scaled DL-768 $\leftrightarrow 2^{68.32}$)
$\log_2$ cost
192
176
160
144
128
112
96
80
64
1024
3072
4968
6144
9216
12288
$\log_2 p^n$

Numerical example: BLS12-446 bits

$p(x) = (x-1)^2(x^4 - x^2 + 1)/3 + x$
$r(x) = x^4 - x^2 + 1$
$s = -(2^{74} + 2^{73} + 2^{63} + 2^{57} + 2^{50} + 2^{17} + 1)$
seed with `enumerate_sparse_T.sage` [G. Masson Thomé]
`https://gitlab.inria.fr/smasson/cocks-pinch-variant`
$p = p(s)$ of 446 bits, twist-secure subgroup-secure curve
p^k 5352 bits

$h = Y^6 - Y^4 + Y^3 - Y + 1$
$f_y = X^{12} - 2yX^{10} + 2y^3X^6 + y^5X^2 + y^4 - y^3 + y - 1$
$g_y = X^2 - uy = X^2 + 28343567510342708887553y$

$A = 968,\ B = 2^{68.2}$
Estimated cost: $\approx 2^{132}$

Key size for pairings

$\mathbb{F}_{p^n}$, curve	cost DL 2^{128}		cost DL 2^{192}	
	$\log_2 p$	$\log_2 p^n$	$\log_2 p$	$\log_2 p^n$
$\mathbb{F}_p$	3072–3200		7400–8000	
$\mathbb{F}_{p^6}$, MNT	640–672	3840–4032	≈ 1536	≈ 9216
$\mathbb{F}_{p^{12}}$, BN	416–448	4992–5376	≈ 1024	≈ 12288
$\mathbb{F}_{p^{12}}$, BLS	416–448	4992–5376	≈ 1120	≈ 13440
$\mathbb{F}_{p^{16}}$, KSS	330	5280	≈ 768	≈ 12288
$\mathbb{F}_{p^{18}}$, KSS	348	6264	≈ 640	≈ 11520

Plan

Future work

- automatic tool (currently developed in Python/SageMath)
- Compare Special-TNFS, TNFS and SNFS
- $a_0 + a_1 x \to$ consider $a_0 + a_1 x + a_2 x^2$, $a_i = a_{i0} + a_{i1} y + \dots$
- Estimate the proportion of duplicate relations due to units (2%, 20%, 60%?)
- How to sieve very efficiently in even dimension 4 to 24 to avoid costly factorization in the relation collection?
- Record computation in $\mathbb{F}_{p^6}$

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