AVALANCHE POLYNOMIALS OF SOME FAMILIES OF GRAPHS

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Abstract. We study the abelian sandpile model on different families of graphs. We introduced the avalanche polynomial which enumerates the size of the avalanches triggered by the addition of a particle on a recurrent configuration. This polynomial is calculated for several families of graphs. In the case of the complete graph, the result involves some known result on Parking functions [12, 11].

Bak, Tang and Wiesenfeld [2] introduced 15 years ago the concept of self organized criticality which allowed to describe a large variety of physical systems like earthquakes [5, 14], forest fires and even some fluctuations in the stock market [1]. One version of this concept is the sandpile cellular automaton model which uses a 2 dimensional lattice; in the sites of this lattice particles are added giving rise to a toppling when their number in a site exceeds a given bound. A toppling on a site may be followed by the toppling on one or more of its neighbors and this sequence of topplings is called an avalanche. Many authors have studied the distribution of the sizes (the number of topplings performed) of the avalanches for this model showing that they obey to power-laws [10, 7, 13].

The sandpile model was also considered by combinatorists as a game on a graph called the chip firing game [3, 4]. Relationships between the structure of the graph and the recurrent configurations of the physical model were pointed out [6, 9].

Experiments on the distribution of sizes of the avalanches were considered only for the 2 dimensional lattice and for some classes of regular graphs. Very little is known for arbitrary graphs [8]. In this paper, a polynomial, encoding the avalanche sizes obtained by adding a particle to a site in a recurrent configuration, is associated to a graph. We determine this polynomial for various families of graphs.

These families are the trees, the cycles, the complete graphs and the lollypop graphs. For these families of graphs the power law observed for the 2 dimensional grid is no more satisfied. The computation of the avalanche polynomial of the complete graph uses a bijection between recurrent configurations of this graph and the so-called parking functions. This computation allows the determination of the avalanche distributions on the lollypop graphs which shows out the existence of peaks also observed in some other families of regular graphs.

1. Recurrent configurations of the sandpile model

In this section we recall the main results on the sandpile model which are useful in this paper.

In what follows $G = (X, E)$ is a connected multigraph with $n + 1$ vertices:

$$X = \{x_1, x_2, \ldots, x_n, x_{n+1}\}$$

vertex x_{n+1} is distinguished and called the *sink*. A *configuration* of the sandpile model in this graph is a sequence of n integers

$$u = (u_1, u_2, \ldots, u_n).$$

For a configuration u, the integer u_i will be considered as a number of *particles* placed on the vertex x_i.

A configuration is *stable* if the integers u_i satisfy $0 \leq u_i < d_i$, where d_i is the degree of the vertex x_i. In a configuration which is not stable a vertex x_i with $u_i \geq d_i$ may perform a *toppling* giving a new configuration v such that $v_i = u_i - d_i$ and $v_j = u_j + \varepsilon_{i,j}$, where $\varepsilon_{i,j}$ denotes the number of edges between vertices x_i and x_j in the multigraph G.

We will write the toppling of vertex x_i by: