HEIGHT ARROW MODEL

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Abstract. We study in this article the characteristics of the so-called Height-Arrow Model (HAM), introduced by physicists as an extension of the Abelian Sandpile Model and the Eulerian Walker. We show that recurrent configurations of this model form an Abelian group and that classical algorithms such as the recurrence criterion or the burning algorithm for the ASM could be extended to the HAM.

Résumé. Dans cet article nous étudions le modèle dit hauteur-orientation introduit par des physiciens comme une généralisation du modèle du Tas de Sable Abélien et du Marcheur Eulérien. Nous montrons que les configurations récurrentes du système forment un groupe Abélien dont le cardinal est lié aux arbres couvrants du graphe sous-jacent. De plus, nous généralisons quelques algorithmes classiques connus pour le modèle du Tas de Sable Abélien comme le critère de récurrence ou l'algorithme de mise à feu.

Introduction

Bak, Tang and Wiesenfeld [1] introduced in 1987 a simple model based on a cellular automaton which depicted the critical behaviour of self-organized systems. This model has been extensively studied by physicists [2], [3], and combinatorists [4], [5], [6], [7].

This system presents two different aspects:

- A dynamical approach. Starting from a given configuration, we let the system evolve and the series of configurations it reaches under the action of the evolution rules describes its dynamic.
- The second aspect was pointed out by Dhar, Ruelle, Sen and Verma [3]. The space of recurrent configurations -i.e. those which can appear after a long evolution of the system- is an Abelian group.

The Eulerian Walkers Model (EWM) was introduced by Priezzhev, Dhar and al. in [8]. This model shares with the Abelian Sandpile Model (ASM) the Abelian group property. Both models involve a rooted map G, where some particles (also called grains or walkers sometimes) could be put on every vertex except the sink. Then the system evolves according to a toppling rule. At the end of [8] a general model is proposed which generalizes EWM and ASM. This paper is a detailled analysis of this model called Height Arrow Model (HAM).

In the first part we give basic definitions of the model and the underlying structure of combinatorial map. Then, we study the configurations of the system and show that some of them, called recurrent ones, are closely related to the recurrent of the ASM. In the last part we study the group associated to each of the different model and point out correlations between them.

1. Definition of the model

1.1. **Configurations of the HAM.** The HAM is usually described with respect to an order for the edges around a vertex. Hence, the natural structure to define the HAM is a combinatorial map. This is the embedding of a graph on a surface.

Definition 1. *A* combinatorial map *on a (finite) set B is a pair of permutations (σ, α) of B such that:*

1. *α is a fixed-point free involution,*
2. *the group $< \sigma, \alpha >$ generated by the two permutations is transitive.*