SANDPILE GROUP ON THE GRAPH $\mathcal{D}_n$ OF THE DIHEDRAL GROUP

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Abstract. In this paper we study the structure of the Abelian sandpile group on the Cayley graph $\mathcal{D}_n$ of the dihedral group $D_n =< a, b \mid a^n = b^2 = (ab)^2 = 1 >$. We prove that the Smith normal form of the sandpile group is not cyclic as one can generally expect but is always the direct product of two or three cyclic groups. We conclude by considering some particular cases.

1. Introduction

The Abelian sandpile model has been introduced by Bak, Tang and Wiesenfeld in [2]. It has been widely studied as one of the simplest models that shows Self-organized criticality (SOC) [11, 1, 3, 12]. Its underlying Abelian structure was discovered by Dhar [9] and Creutz [8]. In particular, the order of the group is precisely the number of spanning trees of the graph [10], and two principal bijections exist [4].

A sandpile model could be seen as a cellular automaton on a rooted graph $\mathcal{G}$ whose cells are the vertices $\mathcal{V}(\mathcal{G})$ of $\mathcal{G}$ and each cell contains a certain number of grains of sand. The transitions of the automaton are given by the following "toppling rule": each cell i containing at least as many grains as its degree, transfers a grain of sand to each of its neighbors j. After a toppling of the vertex i, the number of grains in this cell decreases by its degree, while the number of those of its neighbors increases by 1. The root r does not topple and could be considered as collecting all the grains leaving the system. The *sandpile group* on $\mathcal{G}$ is the quotient of $\mathbb{Z}^{\mathcal{V}(\mathcal{G})}$ by the subgroup generated by the $|\mathcal{V}(\mathcal{G})| - 1$ elements expressing the toppling rules (that is, if $i \neq r$ is a vertex of degree d_i and $(j_k)_{1 \leq k \leq d_i}$ are the neighbors of i, a generator of this subgroup is $\Delta_i := d_i x_i - \sum_{1 \leq k \leq d_i} x_{j_k}$), and the element x_r.

The aim of this work is to characterize the structure of the Abelian sandpile group on the graph $\mathcal{D}_n$, for any $n \geq 3$. As shown in Figure 1, the graph $\mathcal{D}_n$ is the non oriented Cayley representation for the dihedral group D_n of symmetries for an n-sided regular polygon, with the presentation

$$< \ a, b \mid a^n = b^2 = (ab)^2 = 1 \ > .$$

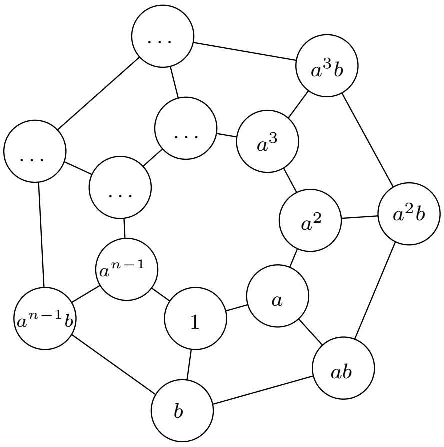

Figure 1. Graph $\mathcal{D}_n$

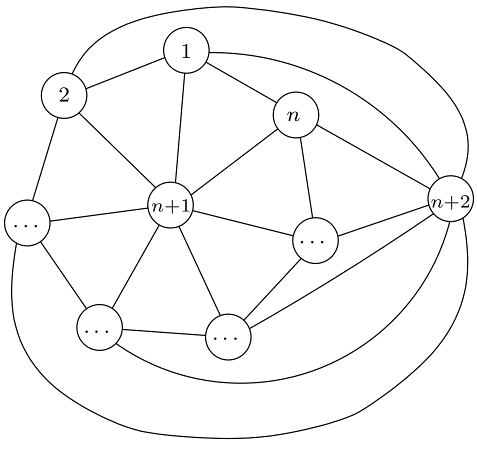

Figure 2. Dual graph of $\mathcal{D}_n$

Date: May 23th, 2003.