The Deuring Correspondence in isogeny-based cryptography: SQISign and new isogeny problems.

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Yoo et al. "A post-quantum digital signature scheme based on supersingular isogenies"

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De Feo and Galbraith "SeaSign: Compact isogeny signatures from class group actions"

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- [BKV19] CSI-FiSh: Based on CSIDH + precomp. $\Rightarrow$ bad scaling, similar to SeaSign with improved efficiency and sizes.

Beullens, Kleinjung, and Vercauteren "CSI-FiSh: Efficient isogeny based signatures through class group computations"

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The Deuring Correspondence

Quaternion Algebra, Orders and Ideals

The **Quaternion algebra** $H(a, b)$ is

$$H(a, b) = \mathbb{Q} + i\mathbb{Q} + j\mathbb{Q} + k\mathbb{Q} \text{ with } i^2 = a, j^2 = b$$

[2] similary for the **right order** $\mathcal{O}_R(I)$

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The (maximal) **left order**[2] $\mathcal{O}_L(I)$ of an *ideal* is

$$\mathcal{O}_L(I) = \{\alpha \in H(a,b), \alpha I \subset I\}$$

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$$\mathrm{End}(E_0) = \langle 1, \iota, \frac{\iota + \pi}{2}, \frac{1 + \iota\pi}{2} \rangle \cong \langle 1, i, \frac{i + j}{2}, \frac{1 + k}{2} \rangle$$

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A new security problem?

Supersingular ℓ-Isogeny Problem: Given a prime p and two supersingular curves E_1 and E_2 over $\mathbb{F}_{p^2}$, compute an ℓ^e-isogeny $\varphi : E_1 \to E_2$ for $e \in \mathbb{N}^\star$.

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[Koh+14]: *heuristic polynomial* time algorithm `KLPT` for quaternion path problem.

Kohel et al. "On the quaternion ℓ-isogeny path problem"

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[Eis+18; Wes22]: use `KLPT` to prove *polynomial* time reduction from supersingular ℓ-isogeny problem to :

Endomorphism Ring Problem: Given a *supersingular elliptic curve E* over $\mathbb{F}_{p^2}$, compute its endomorphism ring.

Eisenträger et al. "Supersingular Isogeny Graphs and Endomorphism Rings: Reductions and Solutions" and Wesolowski "The supersingular isogeny path and endomorphism ring problems are equivalent"

Proof of Knowledge of Endomorphism Ring

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SQISign contributions:

- A new generic `KLPT` algorithm to reach high soundness.
- New algorithmic tools to make the scheme **practical**.

Galbraith, Petit, and Silva "Identification Protocols and Signature Schemes Based on Supersingular Isogeny Problems"

GPS Identification Scheme

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$$\begin{tikzcd} E_0 \arrow[d, dashed, "\tau"'] \\ E_A \end{tikzcd}$$

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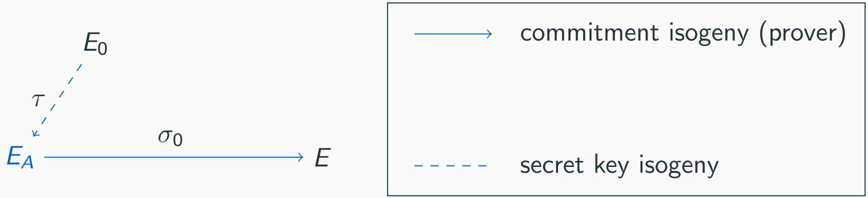

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$$\begin{tikzcd}
& E_0 \arrow[dl, dashed, "\tau"'] \arrow[dr, "\sigma_1\ (\texttt{KLPT})"] & \\
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$\longrightarrow$	commitment isogeny (prover)
$\dashrightarrow$	secret key isogeny

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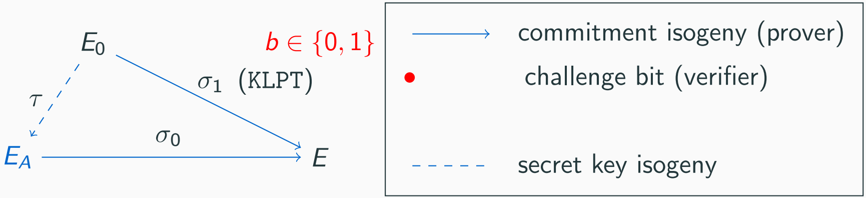

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$b \in \{0, 1\}$

- commitment isogeny (prover)
- challenge bit (verifier)
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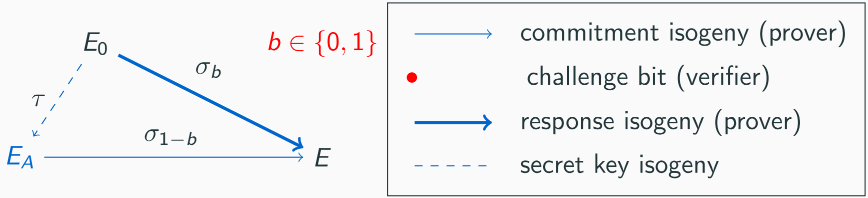

Repeat this λ times to reach 2^{λ}-bits of soundness.

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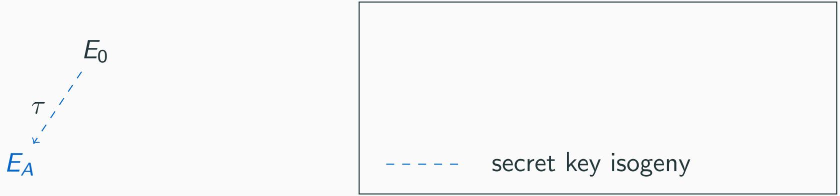

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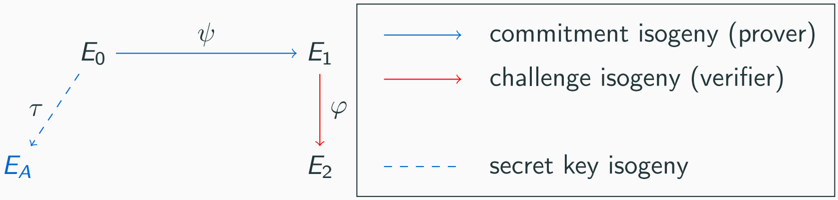

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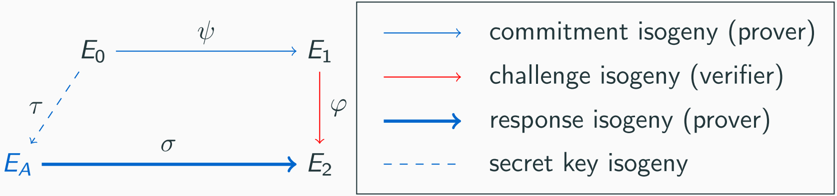

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commitment isogeny (prover)

challenge isogeny (verifier)

response isogeny (prover)

secret key isogeny

Probability to cheat without knowledge of $\mathrm{End}(E_A)$: $O(\frac{1}{\deg \varphi})$.

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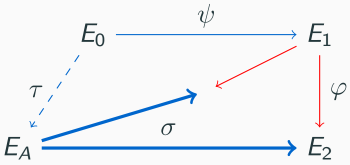

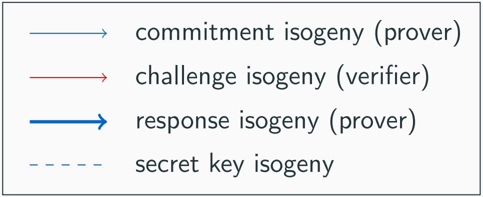

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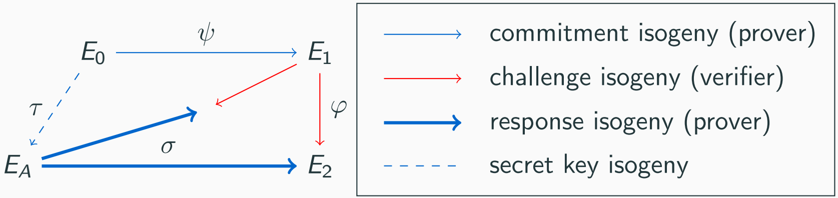

Smooth Endomorphism Problem: Given a *supersingular elliptic curve* E over $\mathbb{F}_{p^2}$, compute a non-trivial endomorphism $\theta \in End(E)$ of *smooth* norm.

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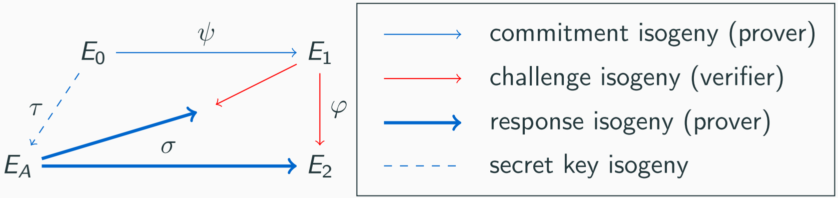

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[Eis+18]: prove *heuristic polynomial* reduction to the **Endomorphism Ring Problem**.

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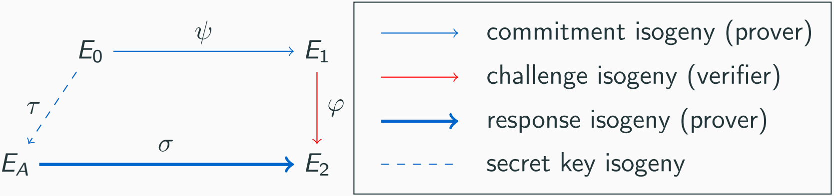

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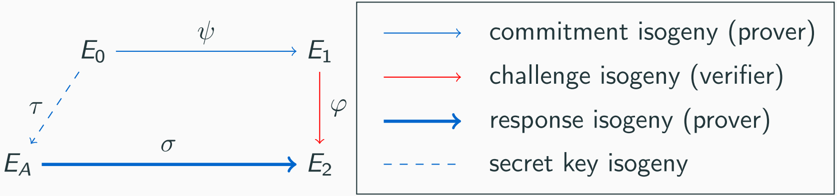

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Solution from [Koh+14]: σ reveal a path to E_0.

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- $- - - - -$ secret key isogeny

Show that σ is a random isogeny $\Rightarrow$ depends on the alg. to compute σ.

Solution from [Koh+14]: σ reveal a path to E_0.

We propose a new `SigningKLPT` algorithm.

A New Security Assumption

Lemma: Fix D as σ's degree. There exists $\mathcal{P}_{\deg(\tau)}$ a set of isogenies of degree D such that:

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 & E_0 \arrow[r, "\iota"] \arrow[dl, dashed, "\tau"'] & E_1 \\
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ZK reduces to the distinguishing problem between:

1. σ is uniformly random isogeny of degree D;
2. σ is uniformly random in $[\tau]_* \mathcal{P}_{\mathsf{deg}(\tau)}$.

$\mathcal{P}_{\mathsf{deg}(\tau)}$ can be computed from $\deg(\tau)$ only and has exponential size.

The effective Deuring Correspondence: algorithmic challenges

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- Take D powersmooth $\to E[D]$ in $\sim$ small extension ([GPS17]).
- Take $D = \ell^f$ and split σ in smaller isogenies of degree ℓ^e and apply `IdealToIsogeny` for each (**SQISign**).

New Pb: for generic E of known $\mathrm{End}(E)$, hard to evaluate $\mathrm{End}(E)$...

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We found a 256 bits prime p with $e = 33, f = 1000$ and 2^{13}-smooth integer of 395 bits:

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T = \ & 5^{21} \cdot 7^2 \cdot 11 \cdot 31 \cdot 83 \cdot 107 \cdot 137 \cdot 751 \cdot 827 \cdot 3691 \cdot 4019 \cdot 6983 \\
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Bottleneck of the signature: $\Theta(f/e)$ T-isogeny computations .

What now?

Conclusion and Important Problems

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- "A New Isogeny Representation and Applications to Cryptography", **L** (preprint).