Towards a Theory of Proofs of Classical Logic

Habilitation à diriger des recherches

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Vers une théorie des preuves pour la logique classique

Die Mathematiker sind eine Art Franzosen: Redet man zu ihnen, so übersetzen sie es in ihre Sprache, und dann ist es alsbald etwas anderes.

Johann Wolfgang von Goethe, Maximen und Reflexionen

Les questions *"Qu'est-ce qu'une preuve ?"* et *"Quand deux preuves sont-elles identiques ?"* sont fondamentales pour la théorie de la preuve. Mais pour la logique classique propositionnelle — la logique la plus répandue — nous n'avons pas encore de réponse satisfaisante.

C'est embarrassant non seulement pour la théorie de la preuve, mais aussi pour l'informatique, où la logique classique joue un rôle majeur dans le raisonnement automatique et dans la programmation logique. De même, l'architecture des processeurs est fondée sur la logique classique. Tous les domaines dans lesquels la recherche de preuve est employée peuvent bénéficier d'une meilleure compréhension de la notion de preuve en logique classique, et le célèbre problème NP-vs-coNP peut être réduit à la question de savoir s'il existe une preuve courte (c'est-à-dire, de taille polynomiale) pour chaque tautologie booléenne [CR79].

Normalement, les preuves sont étudiées comme des objets syntaxiques au sein de systèmes déductifs (par exemple, les tableaux, le calcul des séquents, la résolution, ...). Ici, nous prenons le point de vue que ces objets syntaxiques (également connus sous le nom d'arbres de preuve) doivent être considérés comme des représentations concrètes des objets abstraits que sont les preuves, et qu'un tel objet abstrait peut être représenté par un arbre en résolution ou dans le calcul des séquents.

Le thème principal de ce travail est d'améliorer notre compréhension des objets abstraits que sont les preuves, et cela se fera sous trois angles différents, étudiés dans les trois parties de ce mémoire : l'algèbre abstraite (chapitre 2), la combinatoire (chapitres 3 et 4), et la complexité (chapitre 5).

0.1 Catégories des preuves

Lambek [Lam68, Lam69] déjà observait qu'un traitement algébrique des preuves peut être fourni par la théorie des catégories. Pour cela, il est nécessaire d'accepter les postulats suivants sur les preuves :

- pour chaque preuve f de conclusion B et de prémisse A (notée $f\colon A \to B$) et pour chaque preuve g de conclusion C et de prémisse B (notée $g\colon B \to C$) il existe une unique preuve $g \circ f$ de conclusion C et de prémisse A (notée $g \circ f\colon A \to C$),
- cette composition des preuves est associative,
- pour chaque formule A il existe une preuve identité $1_A\colon A \to A$ telle que pour tout $f\colon A \to B$ on a $f \circ 1_A = f = 1_B \circ f$.

Sous ces hypothèses, les preuves sont les flèches d'une catégorie dont les objets sont les formules de la logique. Il ne reste alors plus qu'à fournir les axiomes adéquats pour la "catégorie des preuves".

Il semble que de tels axiomes soient particulièrement difficiles à trouver dans le cas de la logique classique [Hyl02, Hyl04, BHRU06]. Pour la logique intuitionniste, Prawitz [Pra71] a proposé la notion de *normalisation des preuves* pour l'identification des preuves. On a vite découvert que cette notion d'identité coïncidait avec la notion d'identité determinée par les axiomes d'une catégorie cartésienne fermée (voir par exemple [LS86]). En fait, on peut dire que les preuves de la logique intuitionniste sont les flèches de la catégorie (bi-)cartésienne fermée libre générée par l'ensemble des variables propositionnelles. Une autre manière de représenter les flèches de cette catégorie est d'utiliser les termes du λ-calcul simplement typé : la composition des flèches est la normalisation des termes. Cette observation est bien connue, sous le nom de correspondance de Curry-Howard [How80] (voir aussi [Sta82, Sim95]).

Dans le cas de la logique linéaire, on a une telle relation entre les preuves et les flèches des catégories étoile-autonomes [Bar79, Laf88, See89]. Dans le calcul des séquents pour la logique linéaire, deux preuves sont alors identifiées lorsque l'on peut transformer l'une en l'autre par des permutations triviales de règles [Laf95b]. Pour la logique linéaire multiplicative, cela coïncide avec les identifications induites par les axiomes d'une catégorie étoile-autonome [Blu93, SL04]. Par conséquent, nous pouvons dire qu'une preuve en logique linéaire multiplicative est une flèche de la catégorie étoile-autonome libre générée par les variables propositionnelles [BCST96, LS06, Hug05].

Mais pour la logique classique, il n'existe pas une telle catégorie des preuves qui soit bien acceptée. La raison principale en est que si nous partons d'une catégorie cartésienne fermée et que nous ajoutons une négation involutive (soit un isomorphisme naturel entre A et la double négation de A), nous obtenons l'effondrement dans une algèbre de Boole, c'est-à-dire que toutes les preuves $f, g\colon A \to B$ sont identifiées. Pour chaque formule il y aurait donc au plus une preuve (voir par exemple [LS86, p.67] ou l'appendice de [Gir91] pour plus de détails). De la même manière, partant d'une catégorie étoile-autonome, ajouter des transformations naturelles $A \to A \wedge A$ et $A \to \mathbf{t}$, c'est-à-dire les preuves des règles d'affaiblissement et de contraction, induit l'effondrement.

Il existe plusieurs possibilités pour gérer ce problème. De toute évidence, nous devons abandonner certaines des équations que nous aimerions utiliser sur les preuves de la

logique classique. Mais lesquelles faire disparaître ? Il y a essentiellement trois approches différentes, chacune ayant ses avantages et ses inconvénients.

(i) La première dit que les axiomes de la catégorie cartésienne fermée sont indispensables, et sacrifie au lieu de cela la dualité entre $\wedge$ et $\vee$. La motivation de cette approche est que le système de preuve pour la logique classique peut maintenant être considéré comme une extension du λ-calcul, et que la notion de normalisation ne change pas. On obtient alors un calcul sur les termes de preuve, nommément le $\lambda\mu$-calcul de Parigot [Par92] et ses variantes (par exemple, [CH00]) ainsi qu'une sémantique dénotationnelle [Gir91]. Il y a dans ce cadre une théorie des catégories [Sel01] et des réseaux de preuve [Lau99, Lau03], fondée sur celle des réseaux pour la logique linéaire multiplicative exponentielle (MELL).

(ii) La deuxième approche considère la symétrie parfaite entre $\wedge$ et $\vee$ comme un aspect essentiel de la logique booléenne, qui ne peut pas être supprimé. Par conséquent, les axiomes des catégories cartésiennes fermées et la relation étroite avec le λ-calcul doivent être sacrifiés. Plus précisément, la conjonction $\wedge$ n'est plus un produit cartésien, mais un simple produit tensoriel. Ainsi, la structure cartésienne fermée est remplacée par une structure étoile-autonome. Toutefois, l'axiomatisation précise est beaucoup moins claire que dans la première approche (voir [FP04c, LS05a, McK05a, Str07b, Lam07]).

(iii) La troisième approche maintient la symétrie parfaite entre $\wedge$ et $\vee$, ainsi que les propriétés du produit cartésien pour $\wedge$. Ce qui doit être abandonné est alors la propriété de clôture, c'est-à-dire qu'il n'y a plus de bijection entre les des preuves de

$$A \vdash B \Rightarrow C \qquad \text{et} \qquad A \wedge B \vdash C \quad ,$$

Cette approche est étudiée dans [DP04, CS09].

Dans ce mémoire, nous nous concentrons sur l'approche (ii), qui sera développée en détail dans le chapitre 2 avec une attention particulière pour la flèche *médial*

$$\mathsf{m}_{A,B,C,D} \colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D) \tag{1}$$

qui est inspirée par l'*inférence profonde* (le système SKS [BT01]) pour la logique classique. Certains des résultats de ce chapitre sont maintenant présentés.

Définition 0.1.1. Une B1-*catégorie* est une catégorie étoile-autonome dans laquelle chaque objet A possède un $\vee$-monoïde commutatif et un $\wedge$-comonoïde cocommutatif, c'est-à-dire qu'il y a des flèches $\nabla_A \colon A \vee A \to A$, $\amalg^A \colon \mathbf{f} \to A$, $\Delta_A \colon A \to A \wedge A$ et $\Pi^A \colon A \to \mathbf{t}$ telles que

$$\begin{tikzcd}
A \vee [A \vee A] \arrow[r, "A \vee \nabla_A"] \arrow[d, "\check{\alpha}_{A,A,A}"'] & A \vee A \arrow[dr, "\nabla_A"] & \\
[A \vee A] \vee A \arrow[r, "\nabla_A \vee A"'] & A \vee A \arrow[r, "\nabla_A"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \vee A \arrow[d, "\check{\sigma}_{A,A}"'] \arrow[dr, "\nabla_A"] & \\
A \vee A \arrow[r, "\nabla_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \vee \mathbf{f} \arrow[d, "A \vee \amalg^A"'] \arrow[dr, "\check{\varrho}_A"] & \\
A \vee A \arrow[r, "\nabla_A"'] & A
\end{tikzcd}
\tag{2}$$

et

$$\begin{tikzcd}[column sep=small] & A\wedge A \arrow[r, "\Delta_A\wedge A"] & (A\wedge A)\wedge A \arrow[dd, "\hat{\alpha}^{-1}_{A,A,A}"] \\ A \arrow[ur, "\Delta_A"] \arrow[dr, "\Delta_A"'] & & \\ & A\wedge A \arrow[r, "A\wedge\Delta_A"'] & A\wedge(A\wedge A) \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & A\wedge A \arrow[dd, "\hat{\sigma}^{-1}_{A,A}"] \\ A \arrow[ur, "\Delta_A"] \arrow[dr, "\Delta_A"'] & \\ & A\wedge A \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & A\wedge A \arrow[dd, "A\wedge\Pi^A"] \\ A \arrow[ur, "\Delta_A"] \arrow[dr, "\hat{\varrho}^{-1}_A"'] & \\ & A\wedge\mathbf{t} \end{tikzcd} \tag{3}$$

commute.

Définition 0.1.2. Une B1-catégorie est dite *uniquement mixée* si $\Pi^{\mathbf{f}} = \Pi^{\mathbf{t}}$.

Dans une B1-catégorie uniquement mixée il y a une unique flèche canonique $\mathsf{mix}_{A,B}\colon A\wedge B \to A\vee B$ telle que

$$\begin{tikzcd} A\wedge B \arrow[r, "\mathsf{mix}_{A,B}"] \arrow[d, "\hat{\sigma}_{A,B}"'] & A\vee B \arrow[d, "\check{\sigma}_{A,B}"] \\ B\wedge A \arrow[r, "\mathsf{mix}_{B,A}"'] & B\vee A \end{tikzcd} \tag{mix-$\hat{\sigma}$}$$

et

$$\begin{tikzcd} A\wedge(B\wedge C) \arrow[r, "A\wedge\mathsf{mix}_{B,C}"] \arrow[d, "\hat{\alpha}_{A,B,C}"'] & A\wedge[B\vee C] \arrow[r, "\mathsf{mix}_{A,B\vee C}"] \arrow[d, "\mathsf{s}_{A,B,C}"] & A\vee[B\vee C] \arrow[d, "\check{\alpha}_{A,B,C}"] \\ (A\wedge B)\wedge C \arrow[r, "\mathsf{mix}_{A\wedge B,C}"'] & (A\wedge B)\vee C \arrow[r, "\mathsf{mix}_{A,B}\vee C"'] & [A\vee B]\vee C \end{tikzcd} \tag{mix-$\hat{\alpha}$}$$

commute.

La naturalité du mix, c'est-à-dire la commutativité de

$$\begin{tikzcd} A\wedge B \arrow[r, "\mathsf{mix}_{A,B}"] \arrow[d, "f\wedge g"'] & A\vee B \arrow[d, "f\vee g"] \\ C\wedge D \arrow[r, "\mathsf{mix}_{C,D}"'] & C\vee D \end{tikzcd} \tag{4}$$

pour toutes flèches $f\colon A\to C$ et $g\colon B\to D$, détermine une flèche $f \between g\colon A\wedge B\to C\vee D$. Ensuite, pour chaque $f,g\colon A\to B$ on peut définir

$$f+g = \nabla_B\circ(f\between g)\circ\Delta_A\colon A\to B \quad .$$

Il résulte de la (co)-associativité et de la (co)-commutativité de Δ et ∇, avec la naturalité de mix, que l'opération $+$ sur les flèches est associative et commutative. Cela nous donne pour $\mathrm{Hom}(A,B)$ une structure de semi-groupe commutatif.

Notez que généralement $(f+g)h$ n'est pas égal à $fh+gh$.

Définition 0.1.3. Soit $\mathscr{C}$ une B1-catégorie uniquement mixée. Alors $\mathscr{C}$ est dite *idempotente* si pour tout A et B, le semi-groupe sur $\mathrm{Hom}(A,B)$ est idempotent, c'est-à-dire que pour chaque $f\colon A\to B$ nous avons $f+f=f$.

Dans une B1-catégorie idempotente le semi-groupe sur $\mathrm{Hom}(A,B)$ est en fait un semi-treillis, défini tel que $f\le g$ si et seulement si $f+g=g$.

Définition 0.1.4. Une B2-*catégorie* est une B1-catégorie qui vérifie les équations

$$\Pi^{\mathbf{t}} = 1_{\mathbf{t}} \colon \mathbf{t} \to \mathbf{t} \tag{B2a}$$

et

$$\begin{tikzcd} & A \wedge B \arrow[dl, "\Delta_A \wedge \Delta_B"'] \arrow[dr, "\Delta_{A \wedge B}"] & \\ A \wedge A \wedge B \wedge B \arrow[rr, "A \wedge \hat{\sigma}_{A,B} \wedge B"'] & & A \wedge B \wedge A \wedge B \end{tikzcd} \tag{B2c}$$

pour tous les objets A et B.

Le théorème suivant résume les propriétés des B2-catégories.

Théorème 0.1.1. *Dans une* B2-*catégorie, les flèches* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Δ_A, Π^A, $\Pi^B_{A\mathbb{I}}$, *et* $\Pi^A_{\mathbb{I}B}$, *sont toutes des morphismes de* $\wedge$*-comonoïdes, et les morphismes de* $\wedge$*-comonoïdes sont stables par* $\wedge$. *Dualement, les flèches* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, ∇_A, $\amalg^A$, $\amalg^B_{A\mathbb{I}}$, *et* $\amalg^A_{\mathbb{I}B}$, *sont toutes des morphismes de* $\vee$*-monoïdes, et les morphismes de* $\vee$*-monoïdes sont stables par* $\vee$.

Définition 0.1.5. On dit qu'une B2-catégorie $\mathscr{C}$ est *médialisée* si pour tous les objets A, B, C, et D il existe une flèche *médial* $\mathsf{m}_{A,B,C,D} \colon (A \wedge B) \vee (C \wedge D) \to [A \vee C] \wedge [B \vee D]$ avec les propriétés suivantes :

- elle est naturelle en A, B, C et D,
- elle est auto-duale, c'est-à-dire que

$$\begin{tikzcd} \overline{[A \vee C] \wedge [B \vee D]} \arrow[r, "\overline{\mathsf{m}_{A,B,C,D}}"] \arrow[d, "\cong"'] & \overline{(A \wedge B) \vee (C \wedge D)} \arrow[d, "\cong"] \\ (\bar{D} \wedge \bar{B}) \vee (\bar{C} \wedge \bar{A}) \arrow[r, "\mathsf{m}_{\bar{D},\bar{B},\bar{C},\bar{A}}"'] & [\bar{D} \vee \bar{C}] \wedge [\bar{B} \vee \bar{A}] \end{tikzcd} \tag{5}$$

  commute, où les flèches verticales sont les isomorphismes canoniques induits par la définition des catégories étoile-autonomes,
- et elle vérifie l'équation

$$\begin{tikzcd} & A \vee B \arrow[dl, "\Delta_A \vee \Delta_B"'] \arrow[dr, "\Delta_{A \vee B}"] & \\ (A \wedge A) \vee (B \wedge B) \arrow[rr, "\mathsf{m}_{A,A,B,B}"'] & & [A \vee B] \wedge [A \vee B] \end{tikzcd} \tag{B3c}$$

  pour tous les objets A et B.

L'équation suivante est une conséquence de (B3c) et de l'auto-dualité du médial.

$$\begin{tikzcd} (A \wedge B) \vee (A \wedge B) \arrow[rr, "\mathsf{m}_{A,B,A,B}"] \arrow[dr, "\nabla_{A \wedge B}"'] & & [A \vee A] \wedge [B \vee B] \arrow[dl, "\nabla_A \wedge \nabla_B"] \\ & A \wedge B & \end{tikzcd} \tag{B3c$'$}$$

Théorème 0.1.2. *Soit $\mathscr{C}$ une* B2*-catégorie médialisée. Alors*

(i) *Les flèches qui préservent la $\wedge$-comultiplication sont stables par $\vee$, et dualement, les flèches qui préservent la $\vee$-multiplication sont stables par $\wedge$.*

(ii) *Pour tous les flèches $A \xrightarrow{f} C$, $A \xrightarrow{g} D$, $B \xrightarrow{h} C$, et $B \xrightarrow{k} D$, on a*

$$[\langle f, g\rangle, \langle h, k\rangle] = \langle [f, h], [g, k]\rangle \colon A \vee B \to C \wedge D \quad .$$

(iii) *Pour tous les objets A, B, C, et D,*

$$\begin{aligned} \mathsf{m}_{A,B,C,D} &= \left[\, \left\langle \Pi^C_{A\mathbb{I}} \circ \Pi^B_{A\mathbb{I}}, \Pi^D_{B\mathbb{I}} \circ \Pi^A_{\mathbb{I}B} \right\rangle, \left\langle \Pi^A_{\mathbb{I}C} \circ \Pi^D_{C\mathbb{I}}, \Pi^B_{\mathbb{I}D} \circ \Pi^C_{\mathbb{I}D} \right\rangle \,\right] \\ &= \left\langle \left[\, \Pi^C_{A\mathbb{I}} \circ \Pi^B_{A\mathbb{I}}, \Pi^A_{\mathbb{I}C} \circ \Pi^D_{C\mathbb{I}} \,\right], \left[\, \Pi^D_{B\mathbb{I}} \circ \Pi^A_{\mathbb{I}B}, \Pi^B_{\mathbb{I}D} \circ \Pi^C_{\mathbb{I}D} \,\right] \right\rangle \end{aligned}$$

(iv) *Pour tous les objets A, B, C, et D, le diagramme suivant commute :*

$$\begin{tikzcd} & {[(A \wedge B) \vee (C \wedge D)] \wedge [(A \wedge B) \vee (C \wedge D)]} \arrow[dr, "{[\Pi^B_{A\mathbb{I}} \vee \Pi^D_{C\mathbb{I}}] \wedge [\Pi^A_{\mathbb{I}B} \vee \Pi^C_{\mathbb{I}D}]}"] & \\ (A \wedge B) \vee (C \wedge D) \arrow[ur, "\Delta_{(A\wedge B)\vee(C\wedge D)}"] \arrow[dr, "{(\Pi^C_{A\mathbb{I}} \wedge \Pi^D_{B\mathbb{I}}) \vee (\Pi^A_{\mathbb{I}C} \wedge \Pi^B_{\mathbb{I}D})}"'] & & {[A \vee C] \wedge [B \vee D]} \\ & {([A \vee C] \wedge [B \vee D]) \vee ([A \vee C] \wedge [B \vee D])} \arrow[ur, "\nabla_{[A\vee C]\wedge[B\vee D]}"'] & \end{tikzcd} \tag{6}$$

(v) *La diagonale horizontale de* (6) *est égale à $\mathsf{m}_{A,B,C,D}$.*

Définition 0.1.6. Une B2-catégorie $\mathscr{C}$ est *nullairement médialisée* s'il existe une flèche $\mathsf{n\check{m}} \colon \mathbf{t} \vee \mathbf{t} \to \mathbf{t}$ (appelée *médial nullaire*) telle que pour tous les objets A, B :

$$\begin{tikzcd} & A \vee B \arrow[dl, "\Pi^A \vee \Pi^B"'] \arrow[dr, "\Pi^{A\vee B}"] & \\ \mathbf{t} \vee \mathbf{t} \arrow[rr, "\mathsf{n\check{m}}"'] & & \mathbf{t} \end{tikzcd} \tag{B3b}$$

Définition 0.1.7. Une B4*-catégorie* est une B2-catégorie médialisée et nullairement médialisée, telle que

$$\Pi^{\mathbf{t}\vee\mathbf{t}} = \mathsf{n\check{m}} = \nabla_{\mathbf{t}} \colon \mathbf{t} \vee \mathbf{t} \to \mathbf{t} \tag{B3a}$$

et

$$\begin{tikzcd} {(A \wedge B) \vee (C \wedge D)} \arrow[r, "\hat{\sigma}_{A,B} \vee \hat{\sigma}_{C,D}"] \arrow[d, "\mathsf{m}_{A,B,C,D}"'] & {(B \wedge A) \vee (D \wedge C)} \arrow[d, "\mathsf{m}_{B,A,D,C}"] \\ {[A \vee C] \wedge [B \vee D]} \arrow[r, "\hat{\sigma}_{A\vee C, B\vee D}"'] & {[B \vee D] \wedge [A \vee C]} \end{tikzcd} \tag{m-$\hat{\sigma}$}$$

$$\begin{tikzcd} {(A \wedge (B \wedge C)) \vee (D \wedge (E \wedge F))} \arrow[r, "\hat{\alpha}_{A,B,C} \vee \hat{\alpha}_{D,E,F}"] \arrow[d, "\mathsf{m}_{A,B\wedge C,D,E\wedge F}"'] & {((A \wedge B) \wedge C) \vee ((D \wedge E) \wedge F)} \arrow[d, "\mathsf{m}_{A\wedge B,C,D\wedge E,F}"] \\ {[A \vee D] \wedge [(B \wedge C) \vee (E \wedge F)]} \arrow[d, "{[A\vee D] \wedge \mathsf{m}_{B,C,E,F}}"'] & {[(A \wedge B) \vee (D \wedge E)] \wedge [C \vee F]} \arrow[d, "\mathsf{m}_{A,B,D,E} \wedge [C\vee F]"] \\ {[A \vee D] \wedge ([B \vee E] \wedge [C \vee F])} \arrow[r, "\hat{\alpha}_{A\vee D, B\vee E, C\vee F}"'] & {([A \vee D] \wedge [B \vee E]) \wedge [C \vee F]} \end{tikzcd} \tag{m-$\hat{\alpha}$}$$

$$\begin{tikzcd}
{[(A \wedge B) \vee (C \wedge D)] \wedge E} \arrow[r, "\mathsf{s}_{A\wedge B, C\wedge D, E}"] \arrow[d, "\mathsf{m}_{A,B,C,D\wedge E}"'] & (A \wedge B) \vee (C \wedge D \wedge E) \arrow[d, "\mathsf{m}_{A,B,C,D\wedge E}"] \\
{[A \vee C] \wedge [B \vee D] \wedge E} \arrow[r, "{[A\vee C]\wedge \mathsf{s}_{B,D,E}}"'] & {[A \vee C] \wedge [B \vee (D \wedge E)]}
\end{tikzcd} \tag{m-s}$$

commutent.

Théorème 0.1.3. *Toute* B4*-catégorie est uniquement mixée, et dans toute* B4*-catégorie les morphismes de* $\vee$*-monoïdes ainsi que les morphismes de* $\wedge$*-comonoïdes cocommutatifs sont stables par* $\vee$ *et* $\wedge$*. En outre, les flèches* $\mathsf{m}_{A,B,C,D}$ *et* $\check{\mathsf{nm}}$ *et* $\hat{\mathsf{nm}}$*, ainsi que les flèches* $\hat{\alpha}_{A,B,C}$*,* $\hat{\sigma}_{A,B}$*,* $\hat{\varrho}_A$*,* $\hat{\lambda}_A$ *et* $\check{\alpha}_{A,B,C}$*,* $\check{\sigma}_{A,B}$*,* $\check{\varrho}_A$*,* $\check{\lambda}_A$*, ainsi que* $\mathsf{s}_{A,B,C}$ *et* $\mathsf{mix}_{A,B}$ *sont toutes des morphismes de* $\vee$*-monoïdes et de* $\wedge$*-comonoïdes.*

Corollaire 0.1.8. *Dans une* B4*-catégorie, le diagramme*

$$\begin{tikzcd}
A \wedge B \wedge C \wedge D \arrow[r, "A\wedge \hat{\sigma}_{B,C}\wedge D"] \arrow[d, "\mathsf{mix}_{A\wedge B, C\wedge D}"'] & A \wedge C \wedge B \wedge D \arrow[d, "\mathsf{mix}_{A,C}\wedge \mathsf{mix}_{B,D}"] \\
(A \wedge B) \vee (C \wedge D) \arrow[r, "\mathsf{m}_{A,B,C,D}"'] & {[A \vee C] \wedge [B \vee D]}
\end{tikzcd} \tag{m-mix}$$

commute.

Évidemment, il est possible de se donner de nouveaux diagrammes, tels que (m-mix), et de se demander s'ils commutent, comme par exemple le suivant, proposé par [McK05b] :

$$\begin{tikzcd}
(A \wedge \mathbf{f}) \vee (B \wedge C) \arrow[rr, "\mathsf{m}_{A,\mathbf{f},B,C}"] \arrow[d, "(A\wedge \Pi^{\mathbf{f}})\vee(B\wedge C)"'] & & {[A \vee B] \wedge [\mathbf{f} \vee C]} \arrow[d, "{[A\vee B]\wedge \check{\lambda}_C}"] \\
(A \wedge \mathbf{t}) \vee (B \wedge C) \arrow[dr, "\hat{\varrho}_A\vee(B\wedge C)"'] & & {[A \vee B] \wedge C} \arrow[dl, "\mathsf{s}_{A,B,C}"] \\
& A \vee (B \wedge C) &
\end{tikzcd} \tag{7}$$

On peut montrer que (7) est équivalent à :

$$\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[dr, "\mathsf{mix}_{A,B}\vee(C\wedge D)"] \arrow[dd, "\mathsf{m}_{A,B,C,D}"'] & \\
& A \vee B \vee (C \wedge D) \\
{[A \vee C] \wedge [B \vee D]} \arrow[ur, "\hat{\mathsf{t}}_{A,C,B,D}"'] &
\end{tikzcd} \tag{mix-m-$\hat{\mathsf{t}}$}$$

Voici deux autres exemples qui ne contiennent pas les unités :

$$
\begin{tikzcd}
{[(A \wedge B) \vee (C \wedge D)] \wedge [E \vee F]} \arrow[r, "\mathsf{m}_{A,B,C,D \wedge (E \vee F)}"] \arrow[d, "\mathsf{s}_{A \wedge B, C \wedge D, E \wedge F}"'] & {[A \vee C] \wedge [B \vee D] \wedge [E \vee F]} \arrow[d, "\check{\mathsf{t}}_{A \vee C, B \vee D, E, F}"] \\
{(A \wedge B) \vee (C \wedge D \wedge [E \vee F])} \arrow[d, "(A \wedge B) \vee \check{\mathsf{t}}_{C,D,E,F}"'] & {([A \vee C] \wedge F) \vee (E \wedge [B \vee D])} \arrow[d, "\mathsf{m}_{A \vee C, F, E, B \vee D}"] \\
{(A \wedge B) \vee (C \wedge F) \vee (E \wedge D)} \arrow[r, "\check{\mathsf{m}}^2_{A,B,C,F,E,D}"'] & {[A \vee C \vee E] \wedge [F \vee B \vee D]}
\end{tikzcd}
\tag{m-$\check{\mathsf{t}}$-s}
$$

$$
\begin{tikzcd}
& {[A' \vee A] \wedge [B' \vee B] \wedge [C' \vee C] \wedge [D' \vee D]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
{([A' \vee B'] \wedge [C' \vee D']) \vee (D \wedge C) \vee (B \wedge A)} \arrow[d, "\check{\mathsf{m}}^2_{A' \vee B', C' \vee D', D, C, B, A}"'] & & {([A' \vee A] \wedge [B' \vee C]) \vee (B \wedge D') \vee (D \vee C')} \arrow[d, "\check{\mathsf{m}}^2_{A' \vee A, B' \vee C, B, D', D, C'}"] \\
{[A' \vee B' \vee B \vee D] \wedge [D' \vee C' \vee C \vee A]} \arrow[dr, "\hat{\mathsf{t}}_{A' \vee B', B \vee D, D' \vee C', C \vee A}"'] & & {[A' \vee A \vee B \vee D] \wedge [D' \vee C' \vee B' \vee C]} \arrow[dl, "\hat{\mathsf{t}}_{A' \vee A, B \vee D, D' \vee C', B' \vee C}"] \\
& {A' \vee B' \vee ([B \vee D] \wedge [D' \vee C']) \vee C \vee A} &
\end{tikzcd}
\tag{$\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$}
$$

où p et q sont les flèches canoniques déterminées par la structure des catégories étoile-autonomes. On peut facilement démontrer la proposition suivante.

Proposition 0.1.9. *Dans toute* $\mathsf{B4}$*-catégorie l'équation* (7) *est vraie si et seulement si l'équation* (mix-m-$\hat{\mathsf{t}}$) *est vraie.*

Définition 0.1.10. Une $\mathsf{B5}$-*catégorie* est une $\mathsf{B4}$-catégorie qui vérifie les équations (mix-m-$\hat{\mathsf{t}}$), (m-$\check{\mathsf{t}}$-s), et ($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$) pour tous les objets.

Lemme 0.1.11. *Dans une* $\mathsf{B5}$*-catégorie l'équation suivante est vraie pour tous les objets*

A, A', B, B', C, C', D, et D' :

$$
\begin{tikzcd}
 & [A' \vee A] \wedge [B' \vee B] \wedge [C' \vee C] \wedge [D' \vee D] \arrow[dl, "(A' \vee A) \wedge \hat{\mathsf{t}}_{B',B,C',C} \wedge (D' \vee D)"'] \arrow[dr, "\hat{\mathsf{t}}_{A',A,B',B} \wedge \hat{\mathsf{t}}_{C',C,D',D}"] & \\
[A' \vee A] \wedge [B' \vee (B \wedge C') \vee C] \wedge [D' \vee D] \arrow[d, "\mathsf{mix}_{A' \vee A, B' \vee (B \wedge C') \vee C} \wedge [D' \vee D]"'] & & [A' \vee B' \vee (A \wedge B)] \wedge [C' \vee D' \vee (C \wedge D)] \arrow[d, "\hat{\mathsf{t}}_{A' \vee B', A \wedge B, C' \vee D', C \wedge D}"] \\
[A' \vee B' \vee (B \wedge C') \vee C \vee A] \wedge [D' \vee D] \arrow[d, "\mathsf{s}_{A' \vee B' \vee C \vee A, B \wedge C', D' \vee D}"'] & & ([A' \vee B'] \wedge [C' \vee D']) \vee (D \wedge C) \vee (B \wedge A) \arrow[d, "([A' \vee B'] \wedge [C' \vee D']) \vee \mathsf{m}_{D,C,B,A}"] \\
A' \vee B' \vee (B \wedge [D' \vee D] \wedge C') \vee C \vee A \arrow[d, "A' \vee B' \vee \check{\mathsf{t}}_{B,D',D,C'} \vee C \vee A"'] & & ([A' \vee B'] \wedge [C' \vee D']) \vee ([D \vee B] \wedge [C \vee A]) \arrow[d, "\mathsf{m}_{A' \vee B', C' \vee D', D \vee B, C \vee A}"] \\
A' \vee B' \vee (B \wedge D') \vee (D \wedge C') \vee C \vee A \arrow[dr, "A' \vee B' \vee \mathsf{m}_{B,D',D,C'} \vee C \vee A"'] & & [A' \vee B' \vee B \vee D] \wedge [D' \vee C' \vee C \vee A] \arrow[dl, "\hat{\mathsf{t}}_{A' \vee B', B \vee D, D' \vee C', C \vee A}"] \\
 & A' \vee B' \vee ([B \vee D] \wedge [D' \vee C']) \vee C \vee A &
\end{tikzcd}
$$

Définition 0.1.12. On dit qu'une $\mathsf{B1}$-catégorie est *plate* si pour tout objet A, les flèches Π^A, $\amalg^A$, Δ_A et ∇_A sont toutes des morphismes de $\vee$-monoïdes et de $\wedge$-comonoïdes.

Définition 0.1.13. Une $\mathsf{B1}$-catégorie est *contractile* si le diagramme suivant commute pour tous les objets A.

$$
\begin{tikzcd}
\mathbf{t} \arrow[r, "\check{\mathsf{i}}_A"] \arrow[dd, "\check{\mathsf{i}}_A"'] & \bar{A} \vee A \arrow[d, "\Delta_{\bar{A} \vee A}"] \\
 & (\bar{A} \vee A) \wedge (\bar{A} \vee A) \arrow[d, "\hat{\mathsf{t}}"] \\
\bar{A} \vee A & \bar{A} \vee (A \wedge \bar{A}) \vee A \arrow[l, "\bar{A} \vee \hat{\mathsf{i}}_A \vee A"]
\end{tikzcd}
\tag{8}
$$

Théorème 0.1.4. *Dans une* $\mathsf{B5}$*-catégorie plate et contractile, on a*

$$1_A + 1_A = 1_A$$

pour tous les objets A.

La démonstration utilise la commutativité des deux diagrammes suivants

$$
\begin{tikzcd}
\mathbf{t} \arrow[rrr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[dr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[ddd, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"'] & & & [\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"] \\
& [\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"'] & & [\bar{A}\vee A]\wedge[\bar{A}\vee(A\wedge\bar{A})\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\mathsf{mix}_{\bar{A}\vee A,\bar{A}\vee(A\wedge\bar{A})\vee A}"] \\
& [\bar{A}\vee A]\wedge[\bar{A}\vee(A\wedge\bar{A})\vee A] \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[dl, "\hat{\mathsf{i}}_A"] \arrow[d, "\mathsf{mix}"] & & [\bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\mathsf{s}_{\bar{A}\vee\bar{A}\vee A\vee A,A\wedge\bar{A},\bar{A}\vee A}"] \\
[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\mathsf{mix}_{\bar{A}\vee A,\bar{A}\vee A}"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[r, "\check{\mathsf{i}}_A"'] \arrow[dl, "\hat{\mathsf{i}}_A"] & & \bar{A}\vee\bar{A}\vee(A\wedge[\bar{A}\vee A]\wedge\bar{A})\vee A\vee A \arrow[d, "\check{\mathsf{t}}_{A,\bar{A},A,\bar{A}}"] \\
\bar{A}\vee\bar{A}\vee A\vee A \arrow[d, "\nabla_{\bar{A}}\vee\nabla_A"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A \arrow[l, "\hat{\mathsf{i}}_A"] \arrow[d, "\nabla_{\bar{A}}\vee\nabla_A"'] & & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee(A\wedge\bar{A})\vee A\vee A \arrow[l, "\nabla_A\wedge\bar{A}"'] \arrow[dl, "\nabla_A\wedge\nabla_{\bar{A}}"] \arrow[d, "\mathsf{m}_{A,\bar{A},A,\bar{A}}"] \\
\bar{A}\vee A & \bar{A}\vee(A\wedge\bar{A})\vee A \arrow[l, "\hat{\mathsf{i}}_A"] & & \bar{A}\vee\bar{A}\vee([A\vee A]\wedge[\bar{A}\vee\bar{A}])\vee A\vee A \arrow[l, "\nabla_{\bar{A}}\vee(\nabla_A\wedge\nabla_{\bar{A}})\vee\nabla_A"]
\end{tikzcd}
$$

et

$$
\begin{tikzcd}
\mathbf{t} \arrow[rrr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[dr, "\check{\mathsf{i}}_A"] \arrow[drrr, "\check{\mathsf{i}}_{(A\wedge A)\vee(A\wedge A)}"] \arrow[ddd, "\check{\mathsf{i}}_A"'] & & & [\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}\wedge\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"] \\
& & & [\bar{A}\vee\bar{A}\vee(A\wedge A)]\wedge[\bar{A}\vee\bar{A}\vee(A\wedge A)] \arrow[d, "\hat{\mathsf{t}}_{\bar{A}\vee\bar{A},A\wedge A,\bar{A}\vee\bar{A},A\wedge A}"] \\
& \bar{A}\vee A \arrow[d, "\Delta_{\bar{A}}\vee\Delta_A"'] & & ([\bar{A}\vee\bar{A}]\wedge[\bar{A}\vee\bar{A}])\vee(A\wedge A)\vee(A\wedge A) \arrow[dl, "(\nabla_{\bar{A}}\wedge\nabla_{\bar{A}})\vee\nabla_{A\wedge A}"'] \arrow[d, "\mathsf{m}_{A,A,A,A}"] \\
& (\bar{A}\wedge\bar{A})\vee(A\wedge A) \arrow[d, "\mathsf{m}_{\bar{A},\bar{A},A,A}"'] & & ([\bar{A}\vee\bar{A}]\wedge[\bar{A}\vee\bar{A}])\vee([A\vee A]\wedge[A\vee A]) \arrow[l, "(\nabla_{\bar{A}}\wedge\nabla_{\bar{A}})\vee(\nabla_A\wedge\nabla_A)"] \arrow[d, "\mathsf{m}_{\bar{A}\wedge\bar{A},\bar{A}\vee\bar{A},A\vee A,A\vee A}"] \\
& [\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"'] & & [\bar{A}\vee\bar{A}\vee A\vee A]\wedge[\bar{A}\vee\bar{A}\vee A\vee A] \arrow[l, "{[\nabla_{\bar{A}}\vee\nabla_A]\wedge[\nabla_{\bar{A}}\vee\nabla_A]}"] \arrow[d, "\hat{\mathsf{t}}_{\bar{A}\wedge\bar{A},A\vee A,\bar{A}\vee\bar{A},A\vee A}"] \\
\bar{A}\vee A & \bar{A}\vee(A\wedge\bar{A})\vee A \arrow[l, "\hat{\mathsf{i}}_A"] & & \bar{A}\vee\bar{A}\vee([A\vee A]\wedge[\bar{A}\vee\bar{A}])\vee A\vee A \arrow[l, "\nabla_{\bar{A}}\vee(\nabla_A\wedge\nabla_{\bar{A}})\vee\nabla_A"]
\end{tikzcd}
$$

ainsi que le Lemme 0.1.11. L'idée de cette preuve est illustrée dans la figure 1.

Corollaire 0.1.14. *Soit $\mathscr{A}$ un ensemble de variables propositionnelles et $\mathscr{C}$ la* $\mathsf{B5}$*-catégorie plate et contractile libre générée par $\mathscr{A}$. Alors $\mathscr{C}$ est idempotente.*

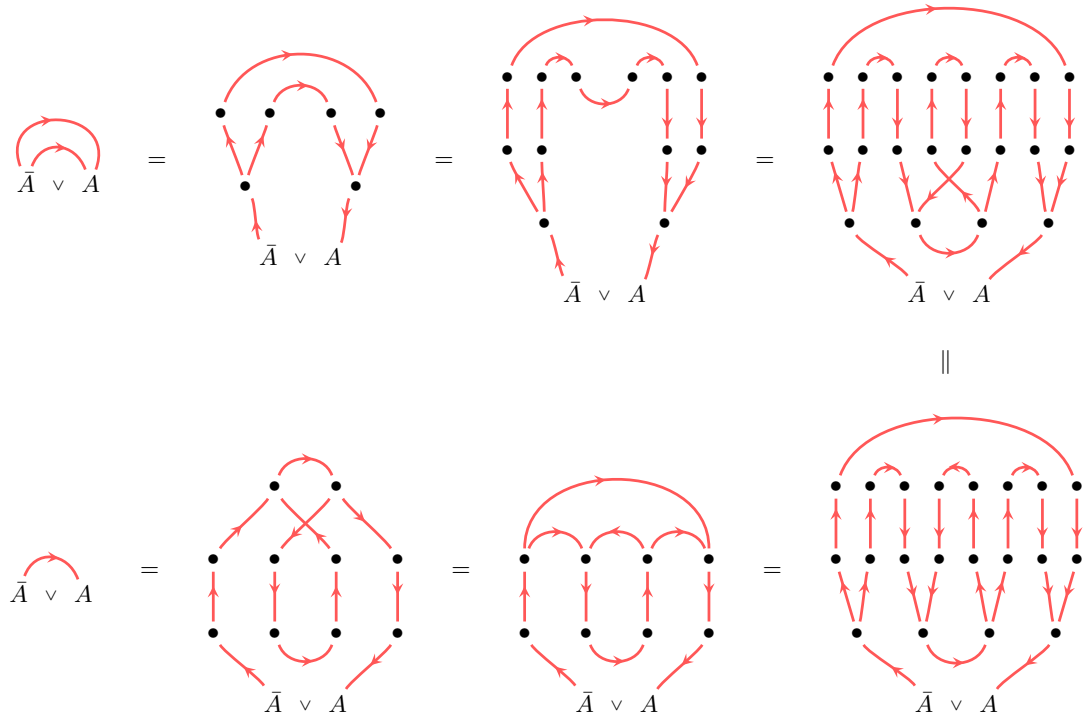

Figure 1: Idée de la démonstration du théorème 0.1.4

0.2 Notations syntaxique pour les preuves

Voyons maintenant ce qui arrive lorsque l'on aborde le problème de l'identité des preuves par l'autre bout, celui de la syntaxe. Pour expliquer le problème nous utilisons ici le calcul des séquents, mais ce qui suit s'applique aussi bien à la déduction naturelle. Il est bien connu que les problèmes commencent lorsqu'une preuve contient des coupures et doit donc être normalisée. On représente cette situation de la manière suivante :

$$\mathsf{cut}\ \frac{\begin{array}{c}\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash \bar{A}, \Delta\end{array}}{\vdash \Gamma, \Delta}$$

On emploie ici la notation monolatère pour les séquents, où une expression comme A pourrait être une formule annotée par des informations de polarité, au lieu d'une simple formule. Les noms π_1 et π_2 représentent les preuves qui ont conduit à ces séquents : ils pourraient être des arbres de preuve du calcul des séquents, des termes de preuve ou encore des réseaux de preuve. De même, l'expression $\bar{A}$ est la *négation formelle* de A; cette notation pourrait être utilisée par exemple dans un contexte de déduction naturelle comme le $\lambda\mu$-calcul [Par92], où la "coupure" ci-dessus ne serait qu'une substitution d'un terme dans

un autre. La négation $\bar{A}$ indique qu'il s'agit d'une entrée, d'une λ-variable. En tout cas, les propositions suivantes devraient être considérées comme des caractéristiques souhaitables :

1. $\bar{A}$ est la négation *logique* de A,
2. $\bar{\bar{A}}$ est *structurellement équivalente (isomorphe)* à A.

Ces symétries permettraient par exemple d'obtenir la dualité structurelle de De Morgan. Le deuxième point n'est pas valide, par exemple, lorsque la négation est un symbole introduit, comme dans le cas du calcul des séquents bilatère ou le $\lambda\mu$-calcul (pour lequel le premier point n'est pas valide non plus).

Le problème de l'élimination des coupures (ou normalisation) est contenu dans deux cas, appelés affaiblissement-affaiblissement et contraction-contraction dans [Gir91], qui ici sera nommé weak-weak et cont-cont :

$$\mathsf{cut}\,\dfrac{\mathsf{weak}\,\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, A} \qquad \mathsf{weak}\,\dfrac{\begin{array}{c}\pi_2\\ \vdash \Delta\end{array}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \qquad \text{and} \qquad \mathsf{cut}\,\dfrac{\mathsf{cont}\,\dfrac{\begin{array}{c}\pi_1\\ \Gamma, A, A\end{array}}{\vdash \Gamma, A} \qquad \mathsf{cont}\,\dfrac{\begin{array}{c}\pi_2\\ \vdash \bar{A}, \bar{A}, \Delta\end{array}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad .$$

Il est bien connu [GLT89, Gir91] que les deux réductions ne peuvent être atteintes sans choisir entre les deux côtés, et que le résultat est très dépendant de ce choix.

La façon la plus standard de résoudre ce dilemme consiste à introduire une asymétrie dans le système (si elle n'est pas déjà là), à l'aide d'informations de polarité sur les formules, et en les utilisant lors du choix. Historiquement, les premières approches par polarisation ont été étroitement liées à la dualité prémisse-conclusion dont nous avons parlé. Une raison pour cela est qu'elles sont issues de traductions par double négation de la logique classique dans la logique intuitionniste. Si une preuve classique peut être transformée en une preuve intuitionniste, alors il y aura dans chaque séquent de cette preuve une formule spéciale : la conclusion du séquent intuitionniste correspondant. Cette approche correspond au point (i) mentionné ci-dessus, comme cela se fait, par exemple, dans le $\lambda\mu$-calcul. L'asymétrie gauche-droite est également le fondement de la sémantique des jeux de Coquand [Coq95], où elle se traduit par la présence de deux joueurs. Dans [Gir91] Girard présente le système LC, où les séquents ont *au plus* une formule spéciale. Non seulement y a-t-il autant de formules positives que négatives — en nombre arbitraire — mais en plus il y a un *bénitier*, qui est vide ou contient *une seule* formule, qui doit être positive. Puis, lorsqu'un choix doit être fait pendant la normalisation, la présence ou l'absence de formule positive dans le bénitier est utilisée en complément de l'information de polarité.

Cette direction de recherche a été extrêmement fructueuse. Elle a engendré une analyse systématique des traductions de la logique classique dans la logique linéaire [DJS97]. En outre, l'approche de LC aux polarités a été étendu à la formulation de la logique polarisée LLP [Lau02]. Celle-ci a l'avantage d'offrir une théorie des réseaux plus simple (par exemple, en ce qui concerne les boîtes) et de produire des traductions particulièrement claires de logiques plus traditionnelles. Cette nouvelle syntaxe de réseaux de preuve a été utilisée pour représenter les preuves de LC [Lau02] et le $\lambda\mu$-calcul [Lau03].

$$\mathsf{id}\ \frac{}{\{\,a \frown \bar{a}\,\} \rhd a, \bar{a}} \qquad \mathbf{t}\ \frac{}{\{\,\mathbf{t} \frown \mathbf{t}\,\} \rhd \mathbf{t}} \qquad \vee\ \frac{P \rhd A, B, \Gamma}{P \rhd A \vee B, \Gamma} \qquad \wedge\ \frac{P \rhd \Gamma, A \quad Q \rhd B, \Delta}{P \oplus Q \rhd \Gamma, A \wedge B, \Delta}$$

$$\mathsf{weak}\ \frac{P \rhd \Gamma}{P \rhd A, \Gamma} \qquad \mathsf{cont}\ \frac{P \rhd A, A, \Gamma}{P' \rhd A, \Gamma} \qquad \mathsf{exch}\ \frac{P \rhd \Gamma, A, B, \Delta}{P \rhd \Gamma, B, A, \Delta} \qquad \mathsf{mix}\ \frac{P \rhd \Gamma \quad Q \rhd \Delta}{P \oplus Q \rhd \Gamma, \Delta}$$

Figure 2: Traduction du calcul des séquents dans les réseaux

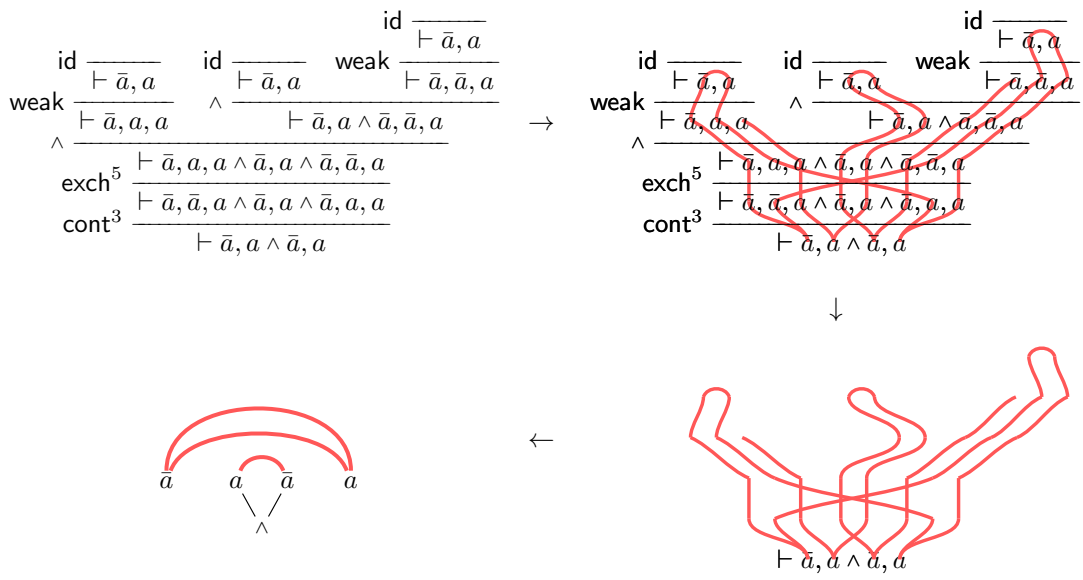

Figure 3: Du calcul des séquents aux ℕ-réseaux

Revenons maintenant aux problèmes weak-weak et cont-cont. Il est bien connu qu'une manière de résoudre weak-weak est d'ajouter une règle mix dans le système :

$$\mathsf{mix}\ \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta} \quad .$$

En ce qui concerne le problème cont-cont, le formalisme du calcul des structures [GS01, BT01] a permis l'émergence d'une nouvelle solution : grâce à l'utilisation de l'inférence profonde, une preuve peut toujours être transformée en une autre dont les contractions sont uniquement appliquées à des *formules atomiques*. Ceci est obtenu par l'utilisation de la règle de médial, qui est une règle d'inférence profonde réalisant la flèche médial (1) :

$$\mathsf{m}\ \frac{K\{(A \wedge B) \vee (C \wedge D)\}}{K\{[A \vee C] \wedge [B \vee D]\}} \tag{9}$$

Dans le chapitre 3 nous exploitons ces deux idées pour construire plusieurs systèmes d'invariants de preuve pour la logique classique, qui sont à la frontière entre la syntaxe

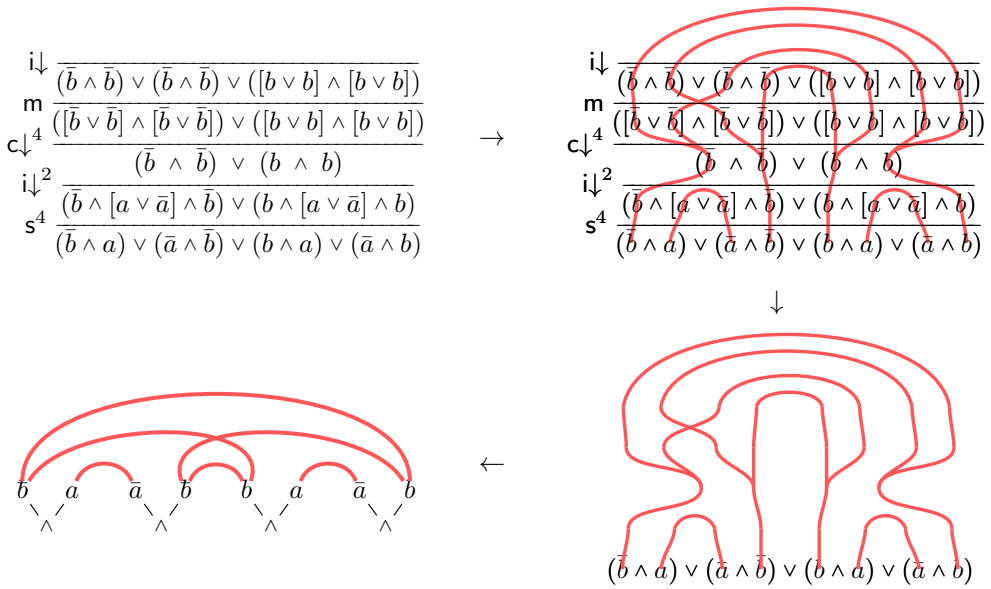

Figure 4: Du calcul des structures aux $\mathbb{N}$-réseaux

$$\mathsf{ai}{\downarrow}\,\frac{K\{\mathsf{t}\}}{K\{a \vee \bar{a}\}} \qquad \mathsf{s}\,\frac{K\{A \wedge [B \vee C]\}}{K\{(A \wedge B) \vee C\}} \qquad \mathsf{ai}{\uparrow}\,\frac{K\{a \wedge \bar{a}\}}{K\{\mathsf{f}\}}$$

$$\mathsf{aw}{\downarrow}\,\frac{K\{\mathsf{f}\}}{K\{a\}} \qquad \mathsf{m}\,\frac{K\{(A \wedge B) \vee (C \wedge D)\}}{K\{[A \vee C] \wedge [B \vee D]\}} \qquad \mathsf{aw}{\uparrow}\,\frac{K\{a\}}{K\{\mathsf{t}\}}$$

$$\mathsf{ac}{\downarrow}\,\frac{K\{a \vee a\}}{K\{a\}} \qquad \mathsf{nm}{\downarrow}\,\frac{K\{\mathsf{f}\}}{K\{\mathsf{f} \wedge \mathsf{f}\}} \qquad \mathsf{nm}{\uparrow}\,\frac{K\{\mathsf{t} \vee \mathsf{t}\}}{K\{\mathsf{t}\}} \qquad \mathsf{ac}{\uparrow}\,\frac{K\{a\}}{K\{a \wedge a\}}$$

$$\alpha{\downarrow}\,\frac{K\{A \vee [B \vee C]\}}{K\{[A \vee B] \vee C\}} \qquad \sigma{\downarrow}\,\frac{K\{A \vee B\}}{K\{B \vee A\}} \qquad \sigma{\uparrow}\,\frac{K\{A \wedge B\}}{K\{B \wedge A\}} \qquad \alpha{\uparrow}\,\frac{K\{A \wedge (B \wedge C)\}}{K\{(A \wedge B) \wedge C\}}$$

$$\mathsf{f}{\downarrow}\,\frac{K\{A\}}{K\{A \vee \mathsf{f}\}} \qquad \mathsf{t}{\uparrow}\,\frac{K\{A \vee \mathsf{f}\}}{K\{A\}} \qquad \mathsf{f}{\uparrow}\,\frac{K\{\mathsf{t} \wedge A\}}{K\{A\}} \qquad \mathsf{t}{\downarrow}\,\frac{K\{A\}}{K\{\mathsf{t} \wedge A\}}$$

Figure 5: Les règles d'inférence du système SKS

et la sémantique, et qui possèdent à la fois les caractéristiques souhaitables mentionnées ci-dessus et quelques-unes des caractéristiques principales des réseaux de preuve. L'idée de base est illustrée dans les figures 3 et 4. Formellement, la traduction du calcul des séquents dans les réseaux de preuve est faite comme indiqué dans la figure 2.

Les dérivations dans le système d'inférence profonde SKS, présenté en figure 5, sont traduites en réseaux avec coupures en attribuant à chaque règle d'inférence un *réseau de*

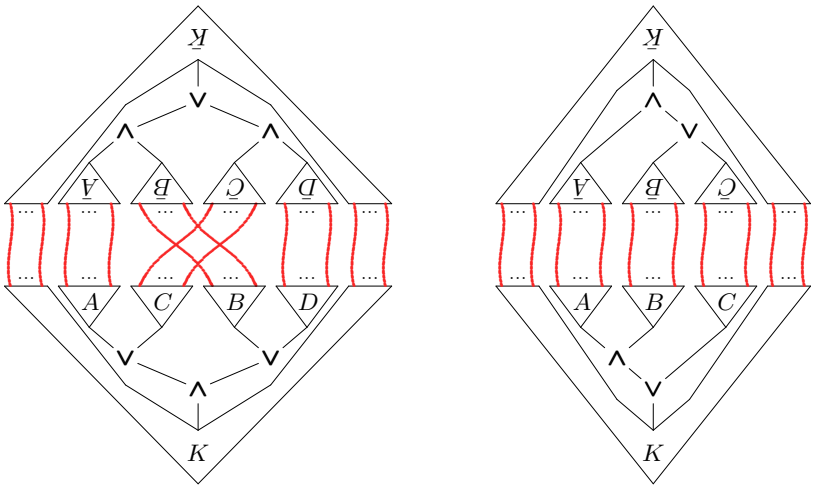

Figure 6: La forme des m-réseaux et des s-réseaux

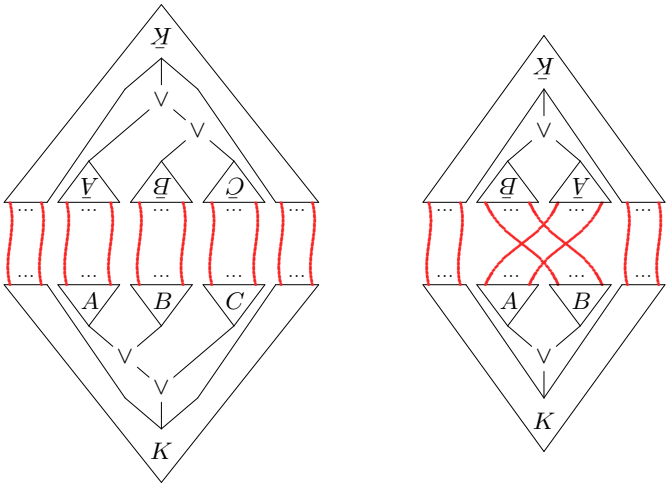

Figure 7: La forme des $\alpha\downarrow$-réseaux et des $\sigma\downarrow$-réseaux

règle :

Les figures 6–10 montrent les réseaux de règle pour les règles du système SKS (voir figure 5). Nous pouvons utiliser les coupures pour connecter ces réseaux afin d'obtenir un *réseau de dérivation*, comme indiqué dans la figure 11.

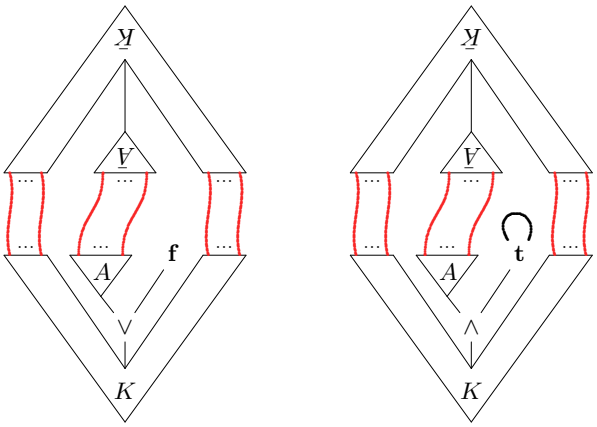

Figure 8: La forme des **f**↓-réseaux et des **t**↓-réseaux

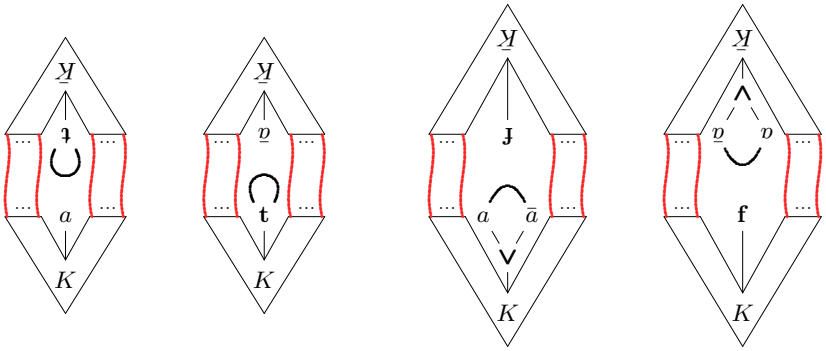

Figure 9: La forme des réseaux pour aw↓, aw↑, ai↓, et ai↑

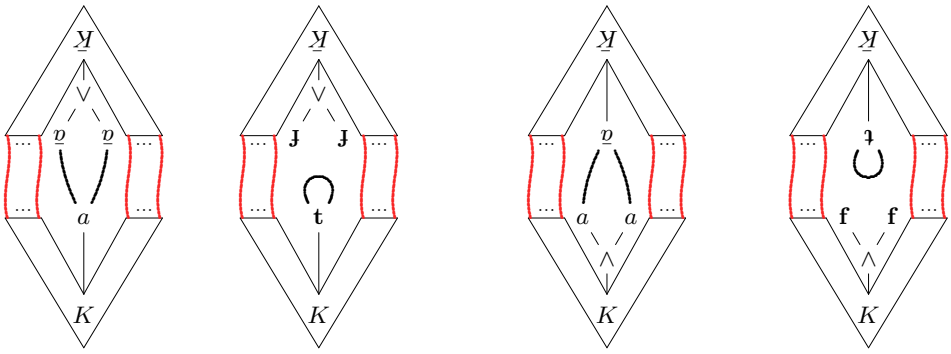

Figure 10: La forme des réseaux pour ac↓, ac↑, nm↓, et nm↑

0.3 Taille des preuves

Comme précédemment dit, la règle de coupure joue un rôle central en théorie de la preuve. En fait, on peut distinguer deux sortes de preuves : les preuves *avec coupures* et les preuves *sans coupure*. Dans un système de preuve bien conçu, il est toujours possible de convertir une preuve avec coupures en une preuve sans coupure. En logique classique propositionnelle, cela se fait au prix d'une augmentation exponentielle de la taille de la preuve (voir par exemple [TS00]). Les coupures sont généralement comprises comme "l'utilisation de lemmes auxiliaires à l'intérieur de la preuve", et l'outil principal pour étudier la règle de coupure

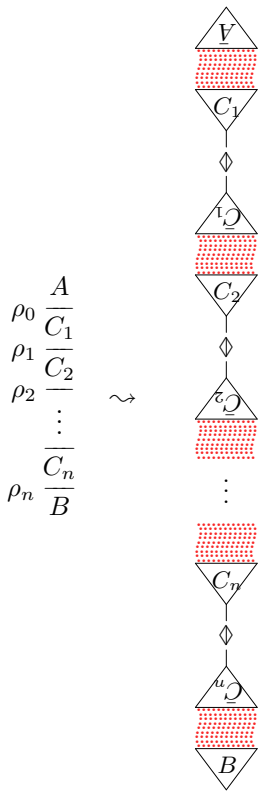

Figure 11: Des dérivations aux réseaux de dérivation

et son élimination est le calcul de séquents de Gentzen [Gen34].

Toutefois, pour l'étude de la complexité des preuves (en logique classique propositionnelle) on fait la distinction entre deux autres types de systèmes de preuve : les *systèmes de Frege* et les *systèmes de Frege augmentés* [CR79]. Sans entrer dans le détail, un système de Frege est constitué d'un ensemble d'axiomes et du *modus ponens*, et dans une système de Frege augmenté on peut en outre utiliser des "abréviations", c'est-à-dire des variables propositionnelles fraîches abrégant des formules arbitraires figurant dans la preuve. Évidemment, toute preuve dans un système de Frege augmenté peut être converti en une preuve sans augmentation, en remplaçant systématiquement toutes les abréviations par la formule qu'elles représentent, au prix d'une augmentation exponentielle de la taille de la preuve.

Les deux classifications des systèmes de preuve ne sont généralement pas étudiés ensemble. En réalité, tout système de Frege contient des coupures en raison de la présence du modus ponens. Par conséquent, il n'existe pas de "système de Frege sans coupure", ou de "système de Frege augmenté sans coupure". De même, il n'y a pas de "système de Gentzen augmenté", parce qu'il n'y a pas de sens à parler des abréviations dans le calcul de séquents sans coupure, où les formules sont décomposées par leur connecteur principal lors de la recherche de preuve.[1] Cela peut se résumer par la classification des systèmes de preuve

[1] L'augmentation discutée ici ne doit pas être confondue avec la notion de "définition" du calcul des

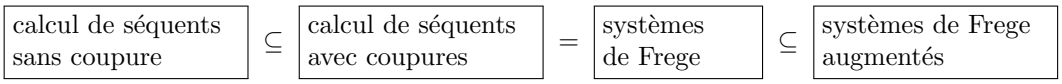

Figure 12: Classification des systèmes de preuve

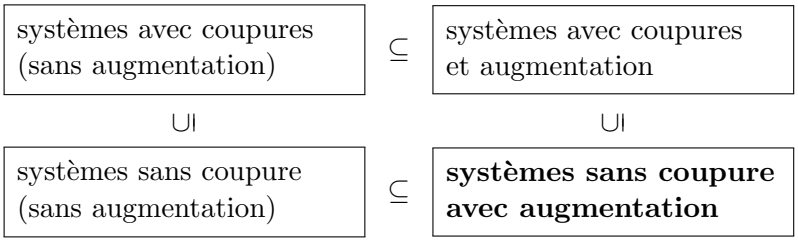

Figure 13: Classification affinée des systèmes de preuve

donnée en figure 12, où $\mathsf{S}_1 \subseteq \mathsf{S}_2$ signifie que S_2 contient S_1, et donc que S_2 *p-simule*[2] S_1. Nous écrivons également $\mathsf{S}_1 = \mathsf{S}_2$ pour signifier que S_1 et S_2 se p-simulent mutuellement, c'est-à-dire sont *p-équivalents*.

Dans le chapitre 5 nous présentons un système déductif dans lequel l'augmentation est indépendante de la coupure, c'est-à-dire que nous pouvons maintenant étudier des systèmes sans coupure mais avec l'augmentation. La figure 13 montre l'affinage de la classification des systèmes de preuve. Pour y parvenir, nous utilisons un système d'inférence profonde, ce qui permet de réunir les avantages des systèmes de Frege et des systèmes de Gentzen.

En plus des systèmes de Frege augmentés, on étudie les *systèmes de Frege avec substitution*. Dans [CR79] Cook et Reckhow ont montré que les systèmes de Frege avec substitution peuvent p-simuler les systèmes de Frege augmentés, et Krajícek et Pudlák [KP89] ont montré que les systèmes de Frege augmentés peuvent p-simuler les systèmes de Frege avec substitution. Dans [BG09] Bruscoli et Guglielmi étudient l'augmentation et la substitution dans le cadre de l'inférence profonde et montrent leur p-équivalence. Toutefois, dans cet article, la substitution en inférence profonde est plus faible que la substitution dans les systèmes de Frege, et la preuve de p-équivalence dans [BG09] s'appuie sur le résultat de Krajícek et Pudlák [KP89].

Dans le chapitre 5, je vais proposer une autre définition de l'augmentation et de la substitution dans le cadre de l'inférence profonde, et montrer leur p-équivalence ainsi que la p-équivalence de l'augmentation et de la substitution dans les systèmes de Frege, donnant ainsi une autre preuve des résultats de Cook et Reckhow [CR79] et de Krajícek et Pudlák [KP89].

séquents LKDe [BHL+06], dans lequel l'abréviation peut se produire dans le *séquent conclusion* de la preuve.

[2] Un système de preuve S_2 p-simule un système de preuve S_1 s'il y a une fonction PTIME f telle que pour chaque preuve π dans S_1, $f(\pi)$ est une preuve de S_2 de même conclusion que π.

1

Introduction

The questions *"What is a proof?"* and *"When are two proofs the same?"* are fundamental for proof theory. But for the most prominent logic, Boolean (or classical) propositional logic, we still have no satisfactory answers.

This is embarrassing not only for proof theory itself, but also for computer science, where classical propositional logic plays a major role in automated reasoning and logic programming. Also the design and verification of hardware is based on classical Boolean logic. Every area in which proof search is employed can benefit from a better understanding of the concept of proof in classical logic, and the famous NP-versus-coNP problem can be reduced to the question whether there is a short (i.e., polynomial size) proof for every Boolean tautology [CR79].

Usually proofs are studied as syntactic objects within some deductive system (e.g., tableaux, sequent calculus, resolution, ...). Here we take the point of view that these syntactic objects (also known as proof trees) should be considered as concrete representations of certain abstract proof objects, and that such an abstract proof object can be represented by a resolution proof tree as well as by a sequent calculus proof tree, or even by several different sequent calculus proof trees.

The main theme of this work is to get a grasp on these abstract proof objects, and this will be done from three different perspectives, studied in the three parts of this thesis: abstract algebra (Chapter 2), combinatorics (Chapters 3 and 4), and complexity (Chapter 5).

1.1 Categories of Proofs

Already Lambek [Lam68, Lam69] observed that an algebraic treatment of proofs can be provided by category theory. For this, it is necessary to accept the following postulates about proofs:

- for every proof f of conclusion B from hypothesis A (denoted by $f\colon A \to B$) and every proof g of conclusion C from hypothesis B (denoted by $g\colon B \to C$) there is a uniquely defined composite proof $g \circ f$ of conclusion C from hypothesis A (denoted by $g \circ f\colon A \to C$),
- this composition of proofs is associative,

- for each formula A there is an identity proof $1_A \colon A \to A$ such that for $f \colon A \to B$ we have $f \circ 1_A = f = 1_B \circ f$.

Under these assumptions the proofs are the arrows in a category whose objects are the formulas of the logic. What remains is to provide the right axioms for the "category of proofs".

It seems that finding these axioms is particularly difficult for the case of classical logic [Hyl02, Hyl04, BHRU06]. For intuitionistic logic, Prawitz [Pra71] proposed the notion of *proof normalization* for identifying proofs. It was soon discovered that this notion of identity coincides with the notion of identity that results from the axioms of a Cartesian closed category (see, e.g., [LS86]). In fact, one can say that the proofs of intuitionistic logic are the arrows in the free (bi-)Cartesian closed category generated by the set of propositional variables. An alternative way of representing the arrows in that category is via terms in the simply-typed λ-calculus: arrow composition is normalization of terms. This observation is well-known as the Curry-Howard-correspondence [How80] (see also [Sta82, Sim95]).

In the case of linear logic, the relation to *-autonomous categories [Bar79] was noticed immediately after its discovery [Laf88, See89]. In the sequent calculus, linear logic proofs are identified when they can be transformed into each other via "trivial" rule permutations [Laf95b]. For multiplicative linear logic this coincides with the proof identifications induced by the axioms of a *-autonomous category [Blu93, SL04]. Therefore, we can safely say that a proof in multiplicative linear logic is an arrow in the free *-autonomous category generated by the propositional variables [BCST96, LS06, Hug05].

But for classical logic no such well-accepted category of proofs exists. The main reason is that if we start from a Cartesian closed category and add an involutive negation (i.e., a natural isomorphism between A and the double-negation of A), we get the collapse into a Boolean algebra, i.e., any two proofs $f, g \colon A \to B$ are identified. For every formula there would be at most one proof (see, e.g., [LS86, p.67] or the appendix of [Gir91] for details). Alternatively, starting from a *-autonomous category and adding natural transformations $A \to A \wedge A$ and $A \to \mathbf{t}$, i.e., the proofs for weakening and contraction, yields the same collapse.

There are several possibilities to cope with this problem. Clearly, we have to drop some of the equations that we would like to hold between proofs in classical logic. But which ones should go? There are now essentially three different approaches, and all three have their advantages and disadvantages.

(i) The first says that the axioms of Cartesian closed categories are essential and cannot be dispensed with. Instead, one sacrifices the duality between $\wedge$ and $\vee$. The motivation for this approach is that a proof system for classical logic can now be seen as an extension of the λ-calculus and the notion of normalization does not change. One has term calculi for proofs, namely Parigot's $\lambda\mu$-calculus [Par92] and its many variants (e.g., [CH00]) as well as a denotational semantics [Gir91]. An important aspect is the computational meaning in terms of continuations [Thi97, SR98]. There is a well explored category theoretical axiomatization [Sel01], and, of course, a theory of proof nets [Lau99, Lau03], which is based on the proof nets for multiplicative exponential linear logic (MELL).

(ii) The second approach considers the perfect symmetry between $\wedge$ and $\vee$ to be an essential facet of Boolean logic, that cannot be dispensed with. Consequently, the

axioms of Cartesian closed categories and the close relation to the λ-calculus have to be sacrificed. More precisely, the conjunction $\wedge$ is no longer a Cartesian product, but merely a tensor-product. Thus, the Cartesian closed structure is replaced by a star-autonomous structure, as it it known from linear logic. However, the precise category theoretical axiomatization is much less clear than in the first approach (see [FP04c, LS05a, McK05a, Str07b, Lam07]).

(iii) The third approach keeps the perfect symmetry between $\wedge$ and $\vee$, as well as the Cartesian product property for $\wedge$. What has to be dropped is the property of being closed, i.e., there is no longer a bijection between the proofs of

$$A \vdash B \Rightarrow C \qquad \text{and} \qquad A \wedge B \vdash C \quad ,$$

which means we lose currying. This approach is studied in [DP04, CS09].

In this thesis, we focus on approach (ii), which will be developed in detail in Chapter 2, with special focus on the so-called *medial map*

$$\mathsf{m}_{A,B,C,D} \colon (A \wedge B) \vee (C \wedge D) \to (A \vee C) \wedge (B \vee D) \tag{1.1}$$

which is inspired by the *deep inference* system SKS [BT01] for classical logic.

1.2 Syntactic Denotations for Proofs

Let us now see what happens when the problem of proof identity is approached from the other end, that of syntax. For explaining the problem we use here the sequent calculus, but what follows applies also to natural deduction. It is well known that problems begin when a proof contains cuts and has to be normalized. Let us represent this situation in the following manner

$$\mathsf{cut}\ \frac{\begin{array}{c}\pi_1\\ \vdash \Gamma, A\end{array} \qquad \begin{array}{c}\pi_2\\ \vdash \bar{A}, \Delta\end{array}}{\vdash \Gamma, \Delta}$$

For the sake of generality, let us here use the one-sided notation for sequents, where an expression like A could be a formula with some polarity information added, instead of just a formula. The π_1 and π_2 represent the proofs that led to the sequents: they could be sequent calculus trees, proof terms or proof nets. Similarly, the expression $\bar{A}$ is a *formal negation* for A; this notation could be used for instance in a natural deduction context like the $\lambda\mu$-calculus [Par92], where the "cut" inference above would just be a substitution of a term into another, the negated $\bar{A}$ meaning that it is on the side of the input/premises/λ-variables. But in any case, the following should be considered as desirable features:

1. $\bar{A}$ is the *logical* negation of A,
2. $\bar{\bar{A}}$ is *structurally equivalent (isomorphic)* to A.

These symmetries would allow things like structural De Morgan duals. The second feature will not happen, for example, when negation is an introduced symbol, as in the case for two-sided sequent calculi or the $\lambda\mu$-calculus (for which the first feature does not hold either).

The problem of cut-elimination (or normalization) is encapsulated in two cases, called weakening-weakening and contraction-contraction in [Gir91], which hereafters will be written as weak-weak and cont-cont:

$$\mathsf{cut}\,\dfrac{\mathsf{weak}\,\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma\end{array}}{\vdash \Gamma, A} \qquad \mathsf{weak}\,\dfrac{\begin{array}{c}\pi_2\\ \vdash \Delta\end{array}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad \text{and} \quad \mathsf{cut}\,\dfrac{\mathsf{cont}\,\dfrac{\begin{array}{c}\pi_1\\ \Gamma, A, A\end{array}}{\vdash \Gamma, A} \qquad \mathsf{cont}\,\dfrac{\begin{array}{c}\pi_2\\ \vdash \bar{A}, \bar{A}, \Delta\end{array}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta} \quad . \tag{1.2}$$

It is well known [GLT89, Gir91] that both reductions cannot be achieved without choosing a side, and that the outcome is very much dependent on that choice.

The most standard way to resolve these dilemmas is to introduce asymmetry in the system (if it is not already there), by the means of polarity information on the formulas, and using this to dictate the choices. Historically, the first approaches to polarization were closely aligned on the premise/conclusion duality we have mentioned. One reason for this is that they were derived from double-negation style translations of classical logic into intuitionistic logic. If a classical proof can be turned into an intuitionistic one, then in every sequent in the history of that proof there will be a special formula: the conclusion of the corresponding intuitionistic sequent. This corresponds to approach (i) mentioned above, as done, for example, in the $\lambda\mu$-calculus. The left-right asymmetry is also at the bottom of Coquand's game semantics [Coq95], where it translates as the two players. In [Gir91] Girard presented System LC, where the sequents have *at most* one special formula. Not only are there both positive and negative formulas—this time an arbitrary number of each—but in addition there is a *stoup*, which is either empty or contains *one* formula, which has to be positive. Then when a choice has to be made at normalization time, the presence or absence of the positive formula in the stoup is used in addition to the polarity information.

This direction of research has been extremely fruitful. It has given a systematic analysis of translations of classical logic into linear logic [DJS97]. Moreover LC's approach to polarities was extended to the formulation of polarized logic LLP [Lau02]. It has the advantage of a much simpler theory of proof nets (e.g., for boxes) and produces particularly perspicuous translations of more traditional logics. This new proof net syntax has been used to represent proofs in LC [Lau02] and the $\lambda\mu$-calculus [Lau03].

Let us come back to the weak-weak and cont-cont problems. It is well-known that one way of solving weak-weak is to permit the system to have a mix rule:

$$\mathsf{mix}\,\dfrac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta} \quad .$$

As for the cont-cont problem, the proof formalism of the calculus of structures [GS01, BT01] has permitted the emergence of a novel solution: through the use of deep inference a proof

cut-free sequent systems	$\subseteq$	sequent systems with cut	$=$	Frege systems	$\subseteq$	extended Frege systems

Figure 1.1: Classification of proof systems

can always be transformed into one whose only contractions are made on *atomic formulas*. This is achieved by the use of the medial rule, which is a deep inference rule incorporating the medial map (1.1):

$$\mathsf{m}\,\frac{K\{(A \wedge B) \vee (C \wedge D)\}}{K\{[A \vee C] \wedge [B \vee D]\}} \tag{1.3}$$

In Chapter 3 we exploit these two ideas to construct several systems of proof invariants for classical logic, which are at the border of syntax and semantics, and which possess both desirable features mentioned above, as well as some of the main features of proof nets. But they lack one important feature that one expects from proof nets: a correctness criterion. Very roughly speaking, a correctness criterion provides a polynomial algorithm deciding if a given invariant does indeed correspond to a correct proof. Chapter 4 of this thesis contains some preliminary results that hopefully will eventually lead to such a criterion for the invariants presented in Chapter 3.

1.3 Size of Proofs

A said before, the cut-rule plays a central role in proof theory. In fact, one distinguishes between two kinds of proofs: those *with cut* and those *without cut*. In a well-designed proof system, it is always possible to convert a proof with cuts into a cut-free proof. In classical propositional logic this comes at the cost of an exponential increase of the size of the proof (see, e.g., [TS00]). The cuts are usually understood as "the use of auxiliary lemmas inside the proof", and the main tool for investigating the cut and its elimination from a proof is Gentzen's sequent calculus [Gen34].

However, for studying proof complexity (for classical propositional logic) one essentially distinguish between two other kinds of proof systems: *Frege systems* and *extended Frege systems* [CR79]. Roughly speaking, a Frege-system consists of a set of axioms and *modus ponens*, and in an extended Frege-system one can additionally use "abbreviations", i.e., fresh propositional variables abbreviating arbitrary formulas appearing in the proof. Clearly, any extended Frege-proof can be converted into a Frege-proof by systematically replacing the abbreviations by the formulas they abbreviate, at the cost of an exponential increase of the size of the proof.

The two proof classifications are usually not studied together. In fact, every Frege-system contains cut because of the presence of modus ponens. Hence, there is no such thing as a "cut-free Frege system", or a "cut-free extended Frege-system". Similarly, there are no "extended Gentzen systems", because it does not make sense to speak of abbreviations in the cut-free sequent calculus, where formulas are decomposed along their main connectives during proof search.[3] This can be summarized by the classification of proof systems shown

[3]The extension discussed here should not be confused with the notion of "definition" in the sequent calculus LKDe [BHL+06], in which the abbreviation may occur in the endsequent of the proof.

systems with cut (without extension)	$\subseteq$	systems with cut and extension
$\cup\!\!\!\mid$		$\cup\!\!\!\mid$
cut-free systems (without extension)	$\subseteq$	**cut-free systems with extension**

Figure 1.2: Refined classification of proof systems

in Figure 1.1, where $\mathsf{S}_1 \subseteq \mathsf{S}_2$ means that S_2 includes S_1, and therefore S_2 *p-simulates*[4] S_1.

In Chapter 5 we provide a deductive system in which extension is independent from the cut, i.e., we can now study cut-free systems with extension. Figure 1.2 shows the refined classification of proof systems. For achieving this, we use a deep inference system, which allows to bring together the advantages of both, Frege systems and Gentzen systems.

Besides extended Frege systems, one also studies *Frege systems with substitution*. Cook and Reckhow have shown in [CR79] that Frege systems with substitution p-simulate extended Frege systems, and Krajícek and Pudlák [KP89] have shown that extended Frege systems p-simulate Frege systems with substitution. In [BG09], Bruscoli and Guglielmi study extension and substitution inside deep inference and show their p-equivalence. However, in that paper, substitution in deep inference is weaker than substitution in Frege systems, and the proof of p-equivalence in [BG09] relies on the result of Krajícek and Pudlák [KP89].

In Chapter 5, I will propose an alternative way of incorporating extension and substitution inside deep inference, show their p-equivalence, as well as the p-equivalence to extension and substitution in Frege systems, and thus give an alternative proof of the results by Cook and Reckhow [CR79] and Krajícek and Pudlák [KP89].

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[4] A proof system S_2 p-simulates a proof system S_1 if there is a PTIME-function f such that for every proof π in S_1 we have that $f(\pi)$ is a proof in S_2 that has the same conclusion as π.

2

On the Algebra of Proofs in Classical Logic

There have already been several accounts for a proof theory for classical logic based on the axioms of Cartesian closed categories. The first were probably Parigot's $\lambda\mu$-calculus [Par92] and Girard's LC [Gir91]. The work on polarized proof nets by Laurent [Lau99, Lau03] shows that there is in fact not much difference between the two. Later, the category-theoretic axiomatisations underlying this proof theory has been investigated and the close relationship to continuations [Thi97, SR98] has been established, culminating in Selinger's *control categories* [Sel01]. However, by sticking to the axioms of Cartesian closed categories, one has to sacrifice the perfect symmetry of Boolean logic.

In this thesis, I will go the opposite way. In the attempt of going from a Boolean algebra to a Boolean category I insist on keeping the symmetry between $\wedge$ and $\vee$. By doing so, we have to leave the realm of Cartesian closed categories. That this is very well possible has recently been shown by several authors [DP04, FP04c, LS05a], where the first falls into approach (iii) mentioned in Section 1.1, and the other two fall into approach (ii) mentioned in Section 1.1. However, the fact that all these proposals considerably differ from each other suggests that there might be no canonical way of giving a categorical axiomatisation for proofs in classical logic.

In order to understand this situation, I will in this chapter propose a series (with increasing strength) of possible axiomatisations for Boolean categories. They all have in common that they are built on the structure of star-autonomous categories in which every object is equipped with a monoid and an comonoid structure. While introducing axioms, I will also show their consequences.

The results of this chapter have already been published in [Str06, Str07b], and are partially based on [LS05a].

2.1 What is a Boolean Category ?

A Boolean category should be for categories, what a Boolean algebra is for posets. This leads to the following definition:

Definition 2.1.1. We say a category $\mathscr{C}$ is a $\mathsf{B0}$-*category* if there is a Boolean algebra $\mathscr{B}$ and a mapping $F\colon \mathscr{C} \to \mathscr{B}$ from objects of $\mathscr{C}$ to elements of $\mathscr{B}$, such that for all objects A and B in $\mathscr{C}$, we have $F(A) \leq F(B)$ in $\mathscr{B}$ if and only if there is an arrow $f\colon A \to B$ in $\mathscr{C}$.

Alternatively, we could say that a $\mathsf{B0}$-category is a category whose image under the forgetful functor from the category of categories to the category of posets is a Boolean algebra. From the proof-theoretic point of view one should have that there is a proof from A to B if and only if $A \Rightarrow B$ is a valid implication. However, from the algebraic point of view there are many models, including the category **Rel** of sets and relations, as well as the models constructed in [Lam07], which have a map between *any* two objects A and B. Note that these models are not ruled out by Definition 2.1.1 because there is the trivial one-element Boolean algebra. In any case, we can make the following (trivial) observation.

Observation 2.1.2. In a $\mathsf{B0}$-category, we can for any pair of objects A and B, provide objects $A \wedge B$ and $A \vee B$ and $\bar{A}$, and there are objects $\mathbf{t}$ and $\mathbf{f}$, such that there are maps

$$\begin{aligned}
\hat{\alpha}_{A,B,C}&\colon A \wedge (B \wedge C) \to (A \wedge B) \wedge C & \check{\alpha}_{A,B,C}&\colon A \vee [B \vee C] \to [A \vee B] \vee C \\
\hat{\sigma}_{A,B}&\colon A \wedge B \to B \wedge A & \check{\sigma}_{A,B}&\colon A \vee B \to B \vee A \\
\hat{\varrho}_{A}&\colon A \wedge \mathbf{t} \to A & \check{\varrho}_{A}&\colon A \vee \mathbf{f} \to A \\
\hat{\lambda}_{A}&\colon \mathbf{t} \wedge A \to A & \check{\lambda}_{A}&\colon \mathbf{f} \vee A \to A \\
\hat{\mathsf{i}}_{A}&\colon A \wedge \bar{A} \to \mathbf{f} & \check{\mathsf{i}}_{A}&\colon \mathbf{t} \to \bar{A} \vee A
\end{aligned} \tag{2.1}$$

$$\mathsf{s}_{A,B,C}\colon [A \vee B] \wedge C \to A \vee (B \wedge C)$$
$$\mathsf{m}_{A,B,C,D}\colon (A \wedge B) \vee (C \wedge D) \to [A \vee C] \wedge [B \vee D]$$

$$\begin{aligned}
\Delta_A&\colon A \to A \wedge A & \nabla_A&\colon A \vee A \to A \\
\Pi^A&\colon A \to \mathbf{t} & \amalg^A&\colon \mathbf{f} \to A
\end{aligned}$$

for all objects A, B, C, and D. This can easily be shown by verifying that all of them correspond to valid implications in Boolean logic. Conversely, a category in which every arrow can be given as a composite of the ones given above by using only the operations of $\wedge$, $\vee$, and the usual arrow composition, is a $\mathsf{B0}$-category. This is a consequence of the completeness of system SKS [BT01], which is a deep inference deductive system for Boolean logic incorporating the maps in (2.1) as inference rules.

Notation 2.1.3. Throughout this thesis, we will use the following convention: The symbols $\wedge$ and $\vee$ denote classical *and* and *or*, respectively. I use parentheses $(\ldots)$ around expressions whose main connective is $\wedge$, and $[\ldots]$ around expressions whose main connective is $\vee$. This is notational redundancy, but makes large expressions with many $\wedge$ and $\vee$ easier to read. The symbol $\Rightarrow$ will denote classical implication, and $\Leftrightarrow$ classical equivalence.

Note that Definition 2.1.1 is neither enlightening nor useful. It is necessary to add some additional structure in order to obtain a "nicely behaved" theory of Boolean categories. However, as already mentioned in the introduction, the naive approach of adding structure, namely adding the structure of a bi-Cartesian closed category (also called Heyting category)

with an involutive negation leads to collapse: Every Boolean category in that strong sense is a Boolean algebra. The hom-sets are either singletons or empty. This observation has first been made by André Joyal, and the proof can be found, for example, in [LS86], page 67. For the sake of completeness, we repeat the argument here: First, recall that in a Cartesian closed category, we have, among other properties, (i) binary products, that we (following Notation 2.1.3) denote by $\wedge$, (ii) a terminal object $\mathbf{t}$ with the property that $\mathbf{t} \wedge A \cong A$ for all objects A, and (iii) a natural bijection between the maps $f\colon A \wedge B \to C$ and $f^*\colon A \to B \Rightarrow C$, where $B \Rightarrow C$ denotes the exponential of B and C. Going from f to f^* is also known as *currying*. Adding an involutive negation means adding a contravariant endofunctor $\overline{(-)}$ such that there is a natural bijection between maps $f\colon A \to B$ and $\bar{f}\colon \bar{B} \to \bar{A}$. It also means that there is an initial object $\mathbf{f} = \bar{\mathbf{t}}$. Hence, we have in particular for all objects A and B, that

$$\mathrm{Hom}(A, B) \cong \mathrm{Hom}(\mathbf{t} \wedge A, B) \cong \mathrm{Hom}(\mathbf{t}, A \Rightarrow B) \cong \mathrm{Hom}(\overline{A \Rightarrow B}, \mathbf{f}) \quad . \tag{2.2}$$

Now observe that whenever we have an object X such that the two projections

$$\pi_1, \pi_2\colon X \wedge X \to X$$

are equal, then for all objects Y, any two maps $f, g\colon Y \to X$ are equal, because

$$f = \pi_1 \circ \langle f, g\rangle = \pi_2 \circ \langle f, g\rangle = g \quad . \tag{2.3}$$

Now note that since $\mathbf{f}$ is initial, there is exactly one map $\mathbf{f} \to \mathbf{f} \Rightarrow \mathbf{f}$, hence, by uncurrying there is exactly one map $\mathbf{f} \wedge \mathbf{f} \to \mathbf{f}$. Therefore, for every Y, there is at most one map $Y \to \mathbf{f}$. Thus, from (2.2) we get that for all A, B, the set $\mathrm{Hom}(A, B)$ is either singleton or empty.

Recapitulating the situation, we have here two extremes of Boolean categories: no structure and too much structure. Neither of them is very interesting, neither for proof theory nor for category theory. But there is a whole universe between the two, which we will start to investigate now. On our path, we will stick to (2.2) and carefully avoid to have (2.3). This is what makes our approach different from control categories [Sel01], in which the equation $f = \pi_1 \circ \langle f, g\rangle$ holds, but the rightmost bijection in (2.2) is absent.

2.2 Star-Autonomous Categories

Let us stress the fact that in a plain B0-category there is no relation between the maps listed in (2.1). In particular, there is no functoriality of $\vee$ and $\wedge$, no naturality of $\hat{\alpha}$, $\hat{\sigma}$, ..., and no de Morgan duality. Adding this structure means exactly adding the structure of a star-autonomous category [Bar79].

Since we are working in classical logic, we will here use the symbols $\wedge, \vee, \mathbf{t}, \mathbf{f}$ for the usual $\otimes, \parr, 1, \perp$.

Definition 2.2.1. A B0-category $\mathscr{C}$ is *symmetric $\wedge$-monoidal* if the operation $- \wedge - : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is a bifunctor and the maps $\hat{\alpha}_{A,B,C}, \hat{\sigma}_{A,B}, \hat{\varrho}_A, \hat{\lambda}_A$ in (2.1) are natural isomorphisms

that obey the following equations:

$$\begin{tikzcd}
A \wedge (B \wedge (C \wedge D)) \arrow[rr, "A \wedge \hat{\alpha}_{B,C,D}"] \arrow[d, "\hat{\alpha}_{A,B,C \wedge D}"'] & & A \wedge ((B \wedge C) \wedge D) \arrow[d, "\hat{\alpha}_{A,B \wedge C,D}"] \\
(A \wedge B) \wedge (C \wedge D) \arrow[dr, "\hat{\alpha}_{A \wedge B,C,D}"'] & & (A \wedge (B \wedge C)) \wedge D \arrow[dl, "\hat{\alpha}_{A,B,C \wedge D}"] \\
& ((A \wedge B) \wedge C) \wedge D &
\end{tikzcd}$$

$$\begin{tikzcd}
A \wedge (B \wedge C) \arrow[r, "A \wedge \hat{\sigma}_{B,C}"] \arrow[d, "\hat{\alpha}_{A,B,C}"'] & A \wedge (C \wedge B) \arrow[d, "\hat{\alpha}_{A,C,B}"] \\
(A \wedge B) \wedge C \arrow[d, "\hat{\sigma}_{A \wedge B,C}"'] & (A \wedge C) \wedge B \arrow[d, "\hat{\sigma}_{A,C} \wedge B"] \\
C \wedge (A \wedge B) \arrow[r, "\hat{\alpha}_{C,A,B}"'] & (C \wedge A) \wedge B
\end{tikzcd}$$

$$\begin{tikzcd}
A \wedge (\mathbf{t} \wedge B) \arrow[rr, "\hat{\alpha}_{A,\mathbf{t},B}"] \arrow[dr, "A \wedge \hat{\lambda}_B"'] & & (A \wedge \mathbf{t}) \wedge B \arrow[dl, "\hat{\varrho}_A \wedge B"] \\
& A \wedge B &
\end{tikzcd}
\qquad
\begin{tikzcd}
A \wedge B \arrow[rr, "\hat{\sigma}_{A,B}"] \arrow[dr, "1_{A \wedge B}"'] & & B \wedge A \arrow[dl, "\hat{\sigma}_{B,A}"] \\
& A \wedge B &
\end{tikzcd}$$

The notion of *symmetric* $\vee$*-monoidal* is defined in a similar way.

An important property of symmetric monoidal categories is the coherence theorem [Mac63], which says that every diagram containing only natural isomorphisms built out of $\hat{\alpha}$, $\hat{\sigma}$, $\hat{\varrho}$, $\hat{\lambda}$, and the identity 1 via $\wedge$ and $\circ$ must commute (for details, see [Mac71] and [Kel64]).[5]

As a consequence of the coherence theorem, we can omit certain parentheses to ease the reading. For example, we will write $A \wedge B \wedge C \wedge D$ for $(A \wedge B) \wedge (C \wedge D)$ as well as for $A \wedge ((B \wedge C) \wedge D)$. This can be done because there is a uniquely defined "coherence isomorphism" between any two of these objects.

Let us now turn our attention to a very important feature of Boolean logic: the duality between $\wedge$ and $\vee$. We can safely say that it is reasonable to ask for this duality also in a Boolean category. That means, we are asking for $\bar{\bar{A}} \cong A$ and $\overline{A \wedge B} \cong \bar{A} \vee \bar{B}$. At the same time we ask for the possibility of transposition (or currying): The proofs of $A \wedge B \to C$ are in one-to-one correspondence with the proofs of $A \to \bar{B} \vee C$. This is exactly what makes a monoidal category star-autonomous.

Definition 2.2.2. A B0-category $\mathscr{C}$ is *star-autonomous* if it is symmetric $\wedge$-monoidal and is equipped with a contravariant functor $\overline{(-)} : \mathscr{C} \to \mathscr{C}$, such that $\overline{\overline{(-)}} : \mathscr{C} \to \mathscr{C}$ is a natural

[5] In [Kel64], Kelly provides some simplifications to MacLane's conditions in [Mac63]. For example, the equations $\hat{\varrho}_{\mathbf{t}} = \hat{\lambda}_{\mathbf{t}} : \mathbf{t} \wedge \mathbf{t} \to \mathbf{t}$ and $\hat{\varrho}_A \circ \hat{\sigma}_{\mathbf{t},A} = \hat{\lambda}_A : \mathbf{t} \wedge A \to A$ follow from the ones in Definition 2.2.1.

isomorphism and such that for any three objects A, B, C there is a natural bijection

$$\mathrm{Hom}_{\mathscr{C}}(A \wedge B, C) \quad \cong \quad \mathrm{Hom}_{\mathscr{C}}(A, \bar{B} \vee C) \quad . \tag{$\star$}$$

where the bifunctor $- \vee -$ is defined via $A \vee B = \overline{\bar{B} \wedge \bar{A}}$.[6] We also define $\mathbf{f} = \bar{\mathbf{t}}$.

Clearly, if a B0-category $\mathscr{C}$ is *star-autonomous*, then it is also $\vee$-monoidal with $\check{\alpha}_{A,B,C} = \overline{\hat{\alpha}_{\bar{C},\bar{B},\bar{A}}}$ and $\check{\sigma}_{A,B} = \overline{\hat{\sigma}_{\bar{B},\bar{A}}}$ and $\check{\varrho}_A = \overline{\hat{\lambda}_{\bar{A}}}$ and $\check{\lambda}_A = \overline{\hat{\varrho}_{\bar{A}}}$.

Note that this definition for star-autonomous categories is not the original one, but it is not difficult to show the equivalence, and this was already done in [Bar79]. For further information, see also [BW99, Bar91, Hug05, LS06].

Let us continue with stating some well-known facts about star-autonomous categories (for proofs of these facts, see e.g. [LS06]). Via the bijection ($\star$) we can assign to every map $f\colon A \wedge B \to C$ a map $g\colon A \to \bar{B} \vee C$, and vice versa. We say that f and g are *transposes* of each other if they determine each other via ($\star$). We will use the term "transpose" in a very general sense: given objects A, B, C, D, E such that $D \cong A \wedge B$ and $E \cong \bar{B} \vee C$, then any $f\colon D \to C$ uniquely determines a $g\colon A \to E$, and vice versa. Also in that general case we will say that f and g are transposes of each other. For example, $\hat{\lambda}_A\colon \mathbf{t} \wedge A \to A$ and $\check{\varrho}_A\colon A \to A \vee \mathbf{f}$ are transposes of each other, and another way of transposing them yields the maps

$$\check{\iota}_A\colon \mathbf{t} \to \bar{A} \vee A \qquad \text{and} \qquad \hat{\iota}_A\colon A \wedge \bar{A} \to \mathbf{f} \quad .$$

If we have $f\colon A \wedge B \to C$ and $b\colon B' \to B$, then

$$A \wedge B' \xrightarrow{A \wedge b} A \wedge B \xrightarrow{f} C \quad \text{is transpose of} \quad A \xrightarrow{g} \bar{B} \vee C \xrightarrow{\bar{b} \vee C} \overline{B'} \vee C \tag{2.4}$$

where g is transpose of f.

Let us now transpose the identity $1_{B \vee C}\colon B \vee C \to B \vee C$. This yields the *evaluation map* $\mathsf{eval}_{B,C}\colon [B \vee C] \wedge \bar{C} \to B$. Taking the $\wedge$ of this with $1_A\colon A \to A$ and transposing back determines a map $\mathsf{s}_{A,B,C}\colon A \wedge [B \vee C] \to (A \wedge B) \vee C$ that is natural in all three arguments, and that we call the *switch map* [Gug07, BT01].[7] In a similar fashion we obtain maps $[A \vee B] \wedge C \to A \vee (B \wedge C)$ and $A \wedge [B \vee C] \to B \vee (A \wedge C)$ and $[A \vee B] \wedge C \to (A \wedge C) \vee B$. Alternatively these maps can be obtained from s by composing with $\hat{\sigma}$ and $\check{\sigma}$. For this reason we will use the term "switch" for all of them, and denote them by $\mathsf{s}_{A,B,C}$ if it is clear from context which one is meant, as for example in the two diagrams

$$\begin{tikzcd}
{[A \vee B] \wedge [C \vee D]} \arrow[r, "\mathsf{s}_{A,B,C \vee D}"] \arrow[d, "\mathsf{s}_{A \vee B,C,D}"'] & A \vee (B \wedge [C \vee D]) \arrow[d, "A \vee \mathsf{s}_{B,C,D}"] \\
({[A \vee B] \wedge C}) \vee D \arrow[r, "\mathsf{s}_{A,B,C} \vee D"'] & A \vee (B \wedge C) \vee D
\end{tikzcd} \tag{2.5}$$

[6] Although we live in the commutative world, we invert the order of the arguments when taking the negation.

[7] To category theorists it is probably better known under the names *weak distributivity* [HdP93, CS97b] or *linear distributivity*. However, strictly speaking, it is not a form of distributivity. An alternative is the name *dissociativity* [DP04].

and

$$\begin{tikzcd} A \wedge [B \vee C] \wedge D \arrow[r, "A \wedge \mathsf{s}_{B,C,D}"] \arrow[d, "\mathsf{s}_{A,B,C} \wedge D"'] & A \wedge [B \vee (C \wedge D)] \arrow[d, "\mathsf{s}_{A,B,C \wedge D}"] \\ [(A \wedge B) \vee C] \wedge D \arrow[r, "\mathsf{s}_{A \wedge B,C,D}"'] & (A \wedge B) \vee (C \wedge D) \end{tikzcd} \tag{2.6}$$

which commute in any star-autonomous category. Sometimes we will denote the map defined by the diagonal of (2.5) by $\hat{\mathsf{t}}_{A,B,C,D} \colon [A \vee B] \wedge [C \vee D] \to A \vee (B \wedge C) \vee D$, called the *tensor map*[8] and the one of (2.6) by $\check{\mathsf{t}}_{A,B,C,D} \colon A \wedge [B \vee C] \wedge D \to (A \wedge B) \vee (C \wedge D)$, called the *cotensor map*.

Note that the switch map is self-dual, while the two maps $\hat{\mathsf{t}}$ and $\check{\mathsf{t}}$ are dual to each other, i.e.,

$$\begin{tikzcd} \overline{(A \wedge B) \vee C} \arrow[r, "\overline{\mathsf{s}_{A,B,C}}"] \arrow[d, "\cong"'] & \overline{A \wedge [B \vee C]} \arrow[d, "\cong"] \\ \bar{C} \wedge [\bar{B} \vee \bar{A}] \arrow[r, "\mathsf{s}_{\bar{C},\bar{B},\bar{A}}"'] & (\bar{C} \wedge \bar{B}) \vee \bar{A} \end{tikzcd} \tag{2.7}$$

and

$$\begin{tikzcd} \overline{(A \wedge B) \vee (C \wedge D)} \arrow[r, "\overline{\check{\mathsf{t}}_{A,B,C,D}}"] \arrow[d, "\cong"'] & \overline{A \wedge [B \vee C] \wedge D} \arrow[d, "\cong"] \\ [\bar{D} \vee \bar{C}] \wedge [\bar{B} \vee \bar{A}] \arrow[r, "\hat{\mathsf{t}}_{\bar{D},\bar{C},\bar{B},\bar{A}}"'] & \bar{D} \vee (\bar{C} \wedge \bar{B}) \vee \bar{A} \end{tikzcd} \tag{2.8}$$

where the vertical maps are the canonical isomorphisms determined by the star-autonomous structure. Another property of switch that we will use later is the commutativity of the following diagrams:

$$\begin{tikzcd} {[A \vee B] \wedge \mathbf{t}} \arrow[dr, "\hat{\varrho}_{A \vee B}"] \arrow[dd, "\mathsf{s}_{A,B,\mathbf{t}}"'] & \\ & A \vee B \\ A \vee (B \wedge \mathbf{t}) \arrow[ur, "A \vee \hat{\varrho}_B"'] & \end{tikzcd} \qquad \begin{tikzcd} & {[\mathbf{f} \vee A] \wedge B} \arrow[dd, "\mathsf{s}_{\mathbf{f},A,B}"] \\ A \wedge B \arrow[ur, "\check{\lambda}_A^{-1} \wedge B"] \arrow[dr, "\check{\lambda}_{A \vee B}^{-1}"'] & \\ & \mathbf{f} \vee (A \wedge B) \end{tikzcd} \tag{2.9}$$

2.3 Some remarks on mix

In this section we will recall what it means for a star-autonomous category to have mix. Although most of the material of this section can also be found in [CS97a], [FP04a], [DP04], and [Lam07], we give here a complete survey since the main result, Corollary 2.3.3, is rather crucial for the following sections. This corollary essentially says that the mix-rule in the sequent calculus

$$\mathsf{mix}\ \frac{\vdash \Gamma \qquad \vdash \Delta}{\vdash \Gamma, \Delta}$$

[8] This map describes precisely the tensor rule in the sequent system for linear logic.

is a consequence of the fact that false implies true. Although this is not a very deep result, it might be surprising for logicians that a property of sequents (if two sequents can be proved independently, then they can be proved together) which does not involve any units comes out of an algebraic property concerning only the units.

Theorem 2.3.1. *Let $\mathscr{C}$ be a star-autonomous category and $e\colon \mathbf{f} \to \mathbf{t}$ be a map in $\mathscr{C}$. Then*

$$\begin{tikzcd} \mathbf{f} \wedge \mathbf{f} \arrow[r, "e\wedge\mathbf{f}"] \arrow[d, "\mathbf{f}\wedge e"'] & \mathbf{t} \wedge \mathbf{f} \arrow[d, "\hat{\lambda}_{\mathbf{f}}"] \\ \mathbf{f} \wedge \mathbf{t} \arrow[r, "\hat{\varrho}_{\mathbf{f}}"'] & \mathbf{f} \end{tikzcd} \tag{2.10}$$

if and only if

$$\begin{tikzcd} \mathbf{t} \arrow[r, "\check{\lambda}_{\mathbf{t}}^{-1}"] \arrow[d, "\check{\varrho}_{\mathbf{t}}^{-1}"'] & \mathbf{f} \vee \mathbf{t} \arrow[d, "e\vee\mathbf{t}"] \\ \mathbf{t} \vee \mathbf{f} \arrow[r, "\mathbf{t}\vee e"'] & \mathbf{t} \vee \mathbf{t} \end{tikzcd} \tag{2.11}$$

if and only if

$$\begin{tikzcd} A \wedge B \arrow[r, "A\wedge\check{\lambda}_B^{-1}"] \arrow[d, "\check{\varrho}_A^{-1}\wedge B"'] & A \wedge [\mathbf{f} \vee B] \arrow[r, "\mathsf{s}_{A,\mathbf{f},B}"] & (A \wedge \mathbf{f}) \vee B \arrow[d, "(A\wedge e)\vee B"] \\ [A \vee \mathbf{f}] \wedge B \arrow[d, "\mathsf{s}_{A,\mathbf{f},B}"'] & & (A \wedge \mathbf{t}) \vee B \arrow[d, "\hat{\varrho}_A\vee B"] \\ A \vee (\mathbf{f} \wedge B) \arrow[r, "A\vee(e\wedge B)"'] & A \vee (\mathbf{t} \wedge B) \arrow[r, "A\vee\hat{\lambda}_B"'] & A \vee B \end{tikzcd} \tag{2.12}$$

for all objects A and B.

Proof. First we show that (2.10) implies (2.12). For this, chase

$$
\begin{tikzcd}
 & & & A \wedge B \arrow[dl, "\check{\varrho}_A^{-1}"'] \arrow[dr, "\check{\lambda}_B^{-1}"] & & & \\
 & & [A \vee \mathbf{f}] \wedge B \arrow[dl, "\mathsf{s}"'] \arrow[dr, "\check{\lambda}_B^{-1}"] & & A \wedge [\mathbf{f} \vee B] \arrow[dl, "\check{\varrho}_A^{-1}"'] \arrow[dr, "\mathsf{s}"] & & \\
 & A \vee (\mathbf{f} \wedge B) \arrow[dl, "e"'] \arrow[dr, "\check{\lambda}_B^{-1}"] & & [A \vee \mathbf{f}] \wedge [\mathbf{f} \vee B] \arrow[dl, "\mathsf{s}"'] \arrow[dr, "\mathsf{s}"] & & (A \wedge \mathbf{f}) \vee B \arrow[dl, "\check{\varrho}_A^{-1}"'] \arrow[dr, "e"] & \\
A \vee (\mathbf{t} \wedge B) \arrow[dd, "\hat{\lambda}_B"'] \arrow[dr, "\check{\lambda}_B^{-1}"] & & A \vee (\mathbf{f} \wedge [\mathbf{f} \vee B]) \arrow[dl, "e"'] \arrow[dr, "\mathsf{s}"] & & ([A \vee \mathbf{f}] \wedge \mathbf{f}) \vee B \arrow[dl, "\mathsf{s}"'] \arrow[dr, "e"] & & (A \wedge \mathbf{t}) \vee B \arrow[dl, "\check{\varrho}_A^{-1}"'] \arrow[dd, "\hat{\varrho}_A"] \\
 & A \vee (\mathbf{t} \wedge [\mathbf{f} \vee B]) \arrow[dd, "\hat{\lambda}_{\mathbf{f} \vee B}"'] \arrow[dr, "\mathsf{s}"] & & A \vee (\mathbf{f} \wedge \mathbf{f}) \vee B \arrow[dl, "e"'] \arrow[dr, "e"] & & ([A \vee \mathbf{f}] \wedge \mathbf{t}) \vee B \arrow[dl, "\mathsf{s}"'] \arrow[dd, "\hat{\varrho}_{A \vee \mathbf{f}}"] & \\
 & & A \vee (\mathbf{t} \wedge \mathbf{f}) \vee B \arrow[dl, "\hat{\lambda}_{\mathbf{f}}"'] & & A \vee (\mathbf{f} \wedge \mathbf{t}) \vee B \arrow[dr, "\hat{\varrho}_{\mathbf{f}}"] & & \\
 & A \vee \mathbf{f} \vee B \arrow[rr, equal] \arrow[dr, "\check{\lambda}_B"] & & & A \vee \mathbf{f} \vee B \arrow[dl, "\check{\varrho}_A"'] & & \\
A \vee B \arrow[ur, "\check{\lambda}_B^{-1}"] \arrow[rr, equal] & & & A \vee B \arrow[rr, equal] & & & A \vee B \arrow[ul, "\check{\varrho}_A^{-1}"']
\end{tikzcd}
\tag{2.13}
$$

The big triangle at the center is an application of (2.10). The two little triangles next to it are (variations of) (2.9), and the triangles at the bottom are trivial. The topmost square is functoriality of $\wedge$, the square in the center is (2.5), and all other squares commute because of naturality of s, $\hat{\lambda}$, $\hat{\varrho}$, $\check{\lambda}$, and $\check{\varrho}$. Now observe that (2.12) commutes if and only if

$$
\begin{tikzcd}
A \wedge B \arrow[r, "A \wedge \check{\lambda}_B^{-1}"] \arrow[d, "\check{\varrho}_A^{-1} \wedge B"'] & A \wedge [\mathbf{f} \vee B] \arrow[r, "A \wedge [e \vee B]"] & A \wedge [\mathbf{t} \vee B] \arrow[d, "\mathsf{s}_{A,\mathbf{t},B}"] \\
[A \vee \mathbf{f}] \wedge B \arrow[d, "{[A \vee e] \wedge B}"'] & & (A \wedge \mathbf{t}) \vee B \arrow[d, "\hat{\varrho}_A \vee B"] \\
[A \vee \mathbf{t}] \wedge B \arrow[r, "\mathsf{s}_{A,\mathbf{t},B}"'] & A \vee (\mathbf{t} \wedge B) \arrow[r, "A \vee \hat{\lambda}_B"'] & A \vee B
\end{tikzcd}
\tag{2.14}
$$

commutes (because of naturality of switch), and that the diagonals of (2.12) and (2.14) are the same map $\mathsf{mix}_{A,B} \colon A \wedge B \to A \vee B$. Note that by the dual of (2.13) we get that (2.11) implies (2.14). Therefore we also get that (2.11) implies (2.12). Now we show that (2.14)

implies (2.11). We will do this by showing that

$$\begin{tikzcd}
 & \mathbf{t} \arrow[dl, "\check{\varrho}_{\mathbf{t}}^{-1}"'] \arrow[d, "\hat{\varrho}_{\mathbf{t}}^{-1}=\hat{\lambda}_{\mathbf{t}}^{-1}"] \arrow[dr, "\check{\lambda}_{\mathbf{t}}^{-1}"] & \\
\mathbf{t} \vee \mathbf{f} \arrow[dr, "\mathbf{t}\vee e"'] & \mathbf{t} \wedge \mathbf{t} \arrow[d, "\mathsf{mix}_{\mathbf{t},\mathbf{t}}"] & \mathbf{f} \vee \mathbf{t} \arrow[dl, "e\vee\mathbf{t}"] \\
 & \mathbf{t} \vee \mathbf{t} &
\end{tikzcd} \tag{2.15}$$

commutes. For this, consider

$$\begin{tikzcd}
\mathbf{t} \arrow[dr, "\hat{\varrho}_{\mathbf{t}}^{-1}"] \arrow[dd, equal] & & \\
 & \mathbf{t} \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_{\mathbf{t}}"'] \arrow[dd, "\check{\varrho}_{\mathbf{t}}^{-1}\wedge\mathbf{t}"] & \\
\mathbf{t} \arrow[dd, "\check{\varrho}_{\mathbf{t}}^{-1}"'] & & \\
 & [\mathbf{t} \vee \mathbf{f}] \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_{\mathbf{t}\vee\mathbf{f}}"'] \arrow[dd, "{[\mathbf{t}\vee e]\wedge\mathbf{t}}"] & \\
\mathbf{t} \vee \mathbf{f} \arrow[dd, "\mathbf{t}\vee e"'] & & \\
 & [\mathbf{t} \vee \mathbf{t}] \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_{\mathbf{t}\vee\mathbf{t}}"'] \arrow[dd, "\mathsf{s}_{\mathbf{t},\mathbf{t},\mathbf{t}}"] & \\
\mathbf{t} \vee \mathbf{t} \arrow[dd, equal] & & \\
 & \mathbf{t} \vee (\mathbf{t} \wedge \mathbf{t}) \arrow[dl, "\mathbf{t}\vee\hat{\lambda}_{\mathbf{t}}"] \arrow[ul, "\mathbf{t}\vee\hat{\varrho}_{\mathbf{t}}"] & \\
\mathbf{t} \vee \mathbf{t} & &
\end{tikzcd} \tag{2.16}$$

which says that the left triangle in (2.15) commutes because the right down path in (2.16) is exactly the lower left path in (2.14). Similarly we obtain the commutativity of the right triangle in (2.15). In the same way we show that (2.12) implies (2.10), which completes the proof. □

Therefore, in a star-autonomous category every map $e\colon \mathbf{f} \to \mathbf{t}$ obeying (2.10) uniquely determines a map $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ which is natural in A and B. It can be shown that this *mix map* goes well with the twist, associativity, and switch maps:

Proposition 2.3.2. *The map* $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ *obtained from* (2.12) *is natural in both arguments and obeys the equations*

$$\begin{tikzcd}
A \wedge B \arrow[r, "\mathsf{mix}_{A,B}"] \arrow[d, "\hat{\sigma}_{A,B}"'] & A \vee B \arrow[d, "\check{\sigma}_{A,B}"] \\
B \wedge A \arrow[r, "\mathsf{mix}_{B,A}"'] & B \vee A
\end{tikzcd} \tag{mix-$\hat{\sigma}$}$$

and

$$\begin{tikzcd}
A \wedge (B \wedge C) \arrow[r, "A \wedge \mathsf{mix}_{B,C}"] \arrow[d, "\hat{\alpha}_{A,B,C}"'] & A \wedge [B \vee C] \arrow[r, "\mathsf{mix}_{A,B \vee C}"] \arrow[d, "\mathsf{s}_{A,B,C}"] & A \vee [B \vee C] \arrow[d, "\check{\alpha}_{A,B,C}"] \\
(A \wedge B) \wedge C \arrow[r, "\mathsf{mix}_{A \wedge B,C}"'] & (A \wedge B) \vee C \arrow[r, "\mathsf{mix}_{A,B} \vee C"'] & [A \vee B] \vee C
\end{tikzcd} \qquad (\mathsf{mix}\text{-}\hat{\alpha})$$

Proof. Naturality of mix follows immediately from the naturality of switch. Equation (mix-$\hat{\sigma}$) follows immediately from the definition of switch, and (mix-$\hat{\alpha}$) can be shown with a similar diagram as (2.13). □

Corollary 2.3.3. *In a star-autonomous category there is a one-to-one correspondence between the maps* $e\colon \mathbf{f} \to \mathbf{t}$ *obeying* (2.10) *and the natural transformations* $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ *obeying* (mix-$\hat{\sigma}$) *and* (mix-$\hat{\alpha}$).

Proof. Whenever we have a map $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ for all A and B, we can form the map

$$e\colon \mathbf{f} \xrightarrow{\hat{\varrho}_{\mathbf{f}}^{-1}} \mathbf{f} \wedge \mathbf{t} \xrightarrow{\mathsf{mix}_{\mathbf{f},\mathbf{t}}} \mathbf{f} \vee \mathbf{t} \xrightarrow{\check{\lambda}_{\mathbf{t}}} \mathbf{t} \tag{2.17}$$

One can now easily show that naturality of mix, as well as (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$) are exactly what is needed to let the map $e\colon \mathbf{f} \to \mathbf{t}$ defined in (2.17) obey equation (2.10). We leave the details to the reader. Hint: Show that both maps of (2.10) are equal to

$$\mathbf{f} \wedge \mathbf{f} \xrightarrow{\mathsf{mix}_{\mathbf{f},\mathbf{f}}} \mathbf{f} \vee \mathbf{f} \xrightarrow{\check{\lambda}_{\mathbf{f}} = \check{\varrho}_{\mathbf{f}}} \mathbf{f} \quad .$$

It remains to show that plugging the map of (2.17) into (2.12) gives back the same natural transformation $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ we started from. Similarly, plugging in the the mix defined via (2.12) into (2.17) gives back the same map $e\colon \mathbf{f} \to \mathbf{t}$ that has been plugged into (2.12). Again, we leave the details to the reader. □

Note that a star-autonomous category can have many different maps $e\colon \mathbf{f} \to \mathbf{t}$ with the property of Theorem 2.3.1, each of them defining its own natural mix obeying (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$).

2.4 $\vee$-Monoids and $\wedge$-comonoids

The structure investigated so far is exactly the same as for proofs in linear logic (with or without mix). For classical logic, we need to provide algebraic structure for the maps $\nabla_A\colon A \vee A \to A$ and $\Pi^A\colon \mathbf{f} \to A$, as well as $\Delta_A\colon A \to A \wedge A$ and $\Pi^A\colon A \to \mathbf{t}$, which are listed in (2.1). This is done via monoids and comonoids.

Definition 2.4.1. A B0-category has *commutative ∨-monoids* if it is symmetric ∨-monoidal and for every object A, the maps ∇_A and $\amalg^A$ obey the equations

$$\begin{tikzcd}
A \vee [A \vee A] \arrow[r, "A \vee \nabla_A"] \arrow[d, "\check{\alpha}_{A,A,A}"'] & A \vee A \arrow[dr, "\nabla_A"] & \\
[A \vee A] \vee A \arrow[r, "\nabla_A \vee A"'] & A \vee A \arrow[r, "\nabla_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \vee A \arrow[d, "\check{\sigma}_{A,A}"'] \arrow[dr, "\nabla_A"] & \\
A \vee A \arrow[r, "\nabla_A"'] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \vee \mathbf{f} \arrow[d, "A \vee \amalg^A"'] \arrow[dr, "\check{\varrho}_A"] & \\
A \vee A \arrow[r, "\nabla_A"'] & A
\end{tikzcd}
\tag{2.18}$$

Dually, we say that a B0-category has *cocommutative ∧-comonoids* if it is symmetric ∧-monoidal and for every object A, the maps Δ_A and Π^A obey the equations

$$\begin{tikzcd}
& A \wedge A \arrow[r, "\Delta_A \wedge A"] & (A \wedge A) \wedge A \arrow[d, "\hat{\alpha}^{-1}_{A,A,A}"] \\
A \arrow[ur, "\Delta_A"] \arrow[dr, "\Delta_A"'] & & \\
& A \wedge A \arrow[r, "A \wedge \Delta_A"'] & A \wedge (A \wedge A)
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \wedge A \arrow[d, "\hat{\sigma}^{-1}_{A,A}"] \\
A \arrow[ur, "\Delta_A"] \arrow[dr, "\Delta_A"'] & \\
& A \wedge A
\end{tikzcd}
\qquad
\begin{tikzcd}
& A \wedge A \arrow[d, "A \wedge \Pi^A"] \\
A \arrow[ur, "\Delta_A"] \arrow[dr, "\hat{\varrho}^{-1}_A"'] & \\
& A \wedge \mathbf{t}
\end{tikzcd}
\tag{2.19}$$

Remark 2.4.2. The (co)associativity of the maps Δ_A and ∇_A allows us to use the notation $\Delta^2_A \colon A \to A \wedge A \wedge A$ and $\nabla^2_A \colon A \vee A \vee A \to A$.

Proposition 2.4.3. *Let $\mathscr{C}$ be a category with commutative ∨-monoids, and let*

$$\begin{tikzcd}
A \vee \mathbf{f} \arrow[d, "A \vee f"'] \arrow[dr, "\check{\varrho}_A"] & \\
A \vee A \arrow[r, "\nabla_A"'] & A
\end{tikzcd}$$

commute for some $f \colon \mathbf{f} \to A$. Then $f = \amalg^A$.

Proof. This is a well-known fact from algebra: in a monoid the unit is uniquely defined. Written as diagram, the standard proof looks as follows:

$$\begin{tikzcd}
& & \mathbf{f} \arrow[dll, equal] \arrow[dr, "\check{\lambda}^{-1}_{\mathbf{f}} = \check{\varrho}^{-1}_{\mathbf{f}}"] \arrow[drr, equal] & & \\
\mathbf{f} \arrow[dd, equal] & & \mathbf{f} \vee \mathbf{f} \arrow[ll, "\check{\lambda}_{\mathbf{f}}"] \arrow[rr, "\check{\varrho}_{\mathbf{f}}"'] \arrow[dl, "\mathbf{f} \vee f"'] \arrow[dr, "\amalg^A \vee \mathbf{f}"] & & \mathbf{f} \arrow[dd, equal] \\
& \mathbf{f} \vee A \arrow[r, "\amalg^A \vee A"] \arrow[ddr, "\check{\lambda}_A"'] & A \vee A \arrow[d, "\nabla_A"] & A \vee \mathbf{f} \arrow[l, "A \vee f"'] \arrow[ddl, "\check{\varrho}_A"] & \\
\mathbf{f} \arrow[rr, "f"'] & & A & & \mathbf{f} \arrow[ll, "\amalg^A"]
\end{tikzcd}$$

Note that in the same way it follows that the counit in a comonoid is uniquely defined. □

Although the operations $\wedge$ and $\vee$ are *not* the product and coproduct in the category-theoretic sense, we use the notation:

$$\langle f, g\rangle = (f \wedge g) \circ \Delta_A \colon A \to C \wedge D \quad \text{and} \quad [f, h] = \nabla_C \circ [f \vee h] \colon A \vee B \to C \tag{2.20}$$

where $f \colon A \to C$ and $g \colon A \to D$ and $h \colon B \to C$ are arbitrary maps.

Another helpful notation (see [LS05a]) is the following:

$$\begin{aligned} \Pi^B_{A\mathbb{I}} &= \hat{\varrho}_A \circ (A \wedge \Pi^B) \colon A \wedge B \to A & \Pi^A_{\mathbb{I}B} &= \hat{\lambda}_B \circ (\Pi^A \wedge B) \colon A \wedge B \to B \\ \amalg^B_{A\mathbb{I}} &= [A \vee \amalg^B] \circ \check{\varrho}_A^{-1} \colon A \to A \vee B & \amalg^A_{\mathbb{I}B} &= [\amalg^A \vee B] \circ \check{\lambda}_B^{-1} \colon B \to A \vee B \end{aligned} \tag{2.21}$$

Note that

$$\nabla_A \circ \amalg^A_{\mathbb{I}A} = 1_A = \nabla_A \circ \amalg^A_{A\mathbb{I}} \qquad \text{and} \qquad \Pi^A_{\mathbb{I}A} \circ \Delta_A = 1_A = \Pi^A_{A\mathbb{I}} \circ \Delta_A \tag{2.22}$$

Definition 2.4.4. Let $f \colon A \to B$ be a map in a $\mathsf{B0}$-category with commutative $\vee$-monoids and cocommutative $\wedge$-comonoids. Consider the following four diagrams:

$$\begin{tikzcd} A \vee A \arrow[r, "f \vee f"] \arrow[d, "\nabla_A"'] & B \vee B \arrow[d, "\nabla_B"] \\ A \arrow[r, "f"'] & B \end{tikzcd} \qquad \begin{tikzcd} & \mathbf{f} \arrow[dl, "\amalg^A"'] \arrow[dr, "\amalg^B"] & \\ A \arrow[rr, "f"'] & & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "\Pi^A"'] & & B \arrow[dl, "\Pi^B"] \\ & \mathbf{t} & \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\Delta_A"'] & B \arrow[d, "\Delta_B"] \\ A \wedge A \arrow[r, "f \wedge f"'] & B \wedge B \end{tikzcd}$$

We say that

- f *preserves the* $\vee$*-multiplication* if the left square commutes,
- f *preserves the* $\vee$*-unit* if the left triangle commutes,
- f *preserves the* $\wedge$*-counit* if the right triangle commutes,
- f *preserves the* $\wedge$*-comultiplication* if the right square commutes,
- f is an $\vee$*-monoid morphism* if the two left diagrams commute,
- f is an $\wedge$*-comonoid morphism* if the two right diagrams commute,
- f is a *quasientropy* if both triangles commute,
- f is *clonable* if both squares commute,
- f is *strong* if all four diagrams commute.

Definition 2.4.5. A $\mathsf{B1}$-*category* is a $\mathsf{B0}$-category that is star-autonomous and has cocommutative $\wedge$-comonoids.

Clearly, a $\mathsf{B1}$-category also has commutative $\vee$-monoids with ∇ dual to Δ, and $\amalg$ dual to Π.

Remark 2.4.6. Definition 2.4.5 exhibits a "creative tension" between algebra and proof theory. From the algebraic point of view one should add the phrase "and all isomorphisms preserve the $\wedge$-comonoid structure" because in a semantics of proofs this will probably be inevitable. But here we do not assume it from the beginning, but systematically give conditions that will ensure it in the end (cf. Theorem 2.6.19 and Remark 2.6.20). From the

proof-theoretic view point this is more interesting because when seen syntactically, these conditions are more primitive. The reason is that in syntax the morphisms (i.e., proofs) come after the objects (i.e., formulas), and the formulas can always be decomposed into subformulas, whereas in semantics we have no access to the outermost connective. Furthermore, forcing all isomorphisms to preserve the ∧-comonoid structure can cause identifications of proofs that might not necessarily be wanted by every proof theorist.

Remark 2.4.7. For each object A in a $\mathsf{B1}$-category $\mathscr{C}$, the identity map $1_A\colon A \to A$ is strong, and all kinds of maps defined in Definition 2.4.4 are closed under composition. Therefore, each kind defines a wide subcategory (i.e., a subcategory that has all objects) of $\mathscr{C}$, e.g., the wide subcategory of quasientropies, or the wide subcategory of ∨-monoid morphisms.

In a $\mathsf{B1}$-category we have two canonical maps $\mathbf{f} \to \mathbf{t}$, namely $\amalg^{\mathbf{f}}$ and $\amalg^{\mathbf{t}}$. Because of the ∧-comonoid structure on $\mathbf{f}$ and the ∨-monoid structure on $\mathbf{t}$, we have

$$\begin{tikzcd}
\mathbf{f} \vee \mathbf{t} \arrow[r, "\amalg^{\mathbf{t}} \vee \mathbf{t}"] \arrow[dr, "\check{\lambda}_{\mathbf{t}}"'] & \mathbf{t} \vee \mathbf{t} \arrow[d, "\nabla_{\mathbf{t}}"] & \mathbf{t} \vee \mathbf{f} \arrow[l, "\mathbf{t} \vee \amalg^{\mathbf{t}}"'] \arrow[dl, "\check{\varrho}_{\mathbf{t}}"] \\
& \mathbf{t} &
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{t} \wedge \mathbf{f} & \mathbf{f} \wedge \mathbf{f} \arrow[l, "\amalg^{\mathbf{f}} \wedge \mathbf{f}"'] \arrow[r, "\mathbf{f} \wedge \amalg^{\mathbf{f}}"] & \mathbf{f} \wedge \mathbf{t} \\
& \mathbf{f} \arrow[ul, "\hat{\lambda}_{\mathbf{f}}^{-1}"] \arrow[u, "\Delta_{\mathbf{f}}"] \arrow[ur, "\hat{\varrho}_{\mathbf{f}}^{-1}"'] &
\end{tikzcd}$$

(which even hold if the (co)monoids are not (co)commutative.) Since $\check{\lambda}_{\mathbf{t}}$, $\check{\varrho}_{\mathbf{t}}$, $\hat{\lambda}_{\mathbf{f}}$, and $\hat{\varrho}_{\mathbf{f}}$ are isomorphisms, we immediately can conclude that the following two diagrams commute (cf. [FP04a]):

$$\begin{tikzcd}
\mathbf{t} \arrow[r, "\check{\lambda}_{\mathbf{t}}^{-1}"] \arrow[d, "\check{\varrho}_{\mathbf{t}}^{-1}"'] & \mathbf{f} \vee \mathbf{t} \arrow[d, "\amalg^{\mathbf{t}} \vee \mathbf{t}"] \\
\mathbf{t} \vee \mathbf{f} \arrow[r, "\mathbf{t} \vee \amalg^{\mathbf{t}}"'] & \mathbf{t} \vee \mathbf{t}
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\mathbf{f} \wedge \mathbf{f} \arrow[r, "\mathbf{f} \wedge \amalg^{\mathbf{f}}"] \arrow[d, "\amalg^{\mathbf{f}} \wedge \mathbf{f}"'] & \mathbf{f} \wedge \mathbf{t} \arrow[d, "\hat{\varrho}_{\mathbf{f}}"] \\
\mathbf{t} \wedge \mathbf{f} \arrow[r, "\hat{\lambda}_{\mathbf{f}}"'] & \mathbf{f}
\end{tikzcd}$$

By Section 2.3, this gives us two different mix maps $A \wedge B \to A \vee B$, and motivates the following definition:

Definition 2.4.8. A $\mathsf{B1}$-category is called *single-mixed* if $\amalg^{\mathbf{f}} = \amalg^{\mathbf{t}}$.

In a single-mixed $\mathsf{B1}$-category we have, as the name says, a single canonical mix map $\mathsf{mix}_{A,B}\colon A \wedge B \to A \vee B$ obeying (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$). The naturality of mix, i.e., the commutativity of

$$\begin{tikzcd}
A \wedge B \arrow[r, "\mathsf{mix}_{A,B}"] \arrow[d, "f \wedge g"'] & A \vee B \arrow[d, "f \vee g"] \\
C \wedge D \arrow[r, "\mathsf{mix}_{C,D}"'] & C \vee D
\end{tikzcd} \tag{2.23}$$

for all maps $f\colon A \to C$ and $g\colon B \to D$, uniquely determines a map $f \between g\colon A \wedge B \to C \vee D$. Then, for every $f, g\colon A \to B$ we can define

$$f + g = \nabla_B \circ (f \between g) \circ \Delta_A \colon A \to B \quad .$$

It follows from (co)-associativity and (co)-commutativity of Δ and ∇, along with naturality of mix, that the operation $+$ on maps is associative and commutative. This gives us for $\mathrm{Hom}(A, B)$ a commutative semigroup structure.

Note that in general the semigroup structure on the Hom-sets is not an enrichment, e.g., $(f + g)h$ is in general not the same as $fh + gh$.

Definition 2.4.9. Let $\mathscr{C}$ be a single-mixed $\mathsf{B1}$-category. Then $\mathscr{C}$ is called *idempotent* if for every A and B, the semigroup on $\mathrm{Hom}(A, B)$ is idempotent, i.e., for every $f\colon A \to B$ we have $f + f = f$.

In an idempotent $\mathsf{B1}$-category the semigroup structure on $\mathrm{Hom}(A, B)$ is in fact a sup-semilattice structure, given by $f \leq g$ iff $f + g = g$.

One can argue that the structure of $\mathsf{B1}$-categories is in some sense the minimum of algebraic structure that a Boolean category should have: star-autonomous categories provide the right structure for linear logic proofs, and the $\vee$-monoids and $\wedge$-comonoids seem to be exactly what is needed to "model contraction and weakening" in classical logic. There are certainly reasons to argue against that, since it is by no means God-given that the proofs in classical logic obey the bijection $(\star)$ nor that "contraction is associative". But let us, for the time being, assume that proofs in classical logic form a $\mathsf{B1}$-category. Then it is desirable that there is some more structure. This can be, for example, an agreement between the $\wedge$-monoidal structure (Definition 2.2.1) and the $\wedge$-comonoid structure (Definition 2.4.1), or, a more sophisticated condition like the commutativity of the diagram

$$\begin{tikzcd}
{[(A \wedge B) \vee (A \wedge B)] \wedge [A \vee B]} \arrow[r, "\nabla_{A\wedge B} \wedge [A \vee B]"] \arrow[d, "\mathsf{s}_{A\wedge B, A\wedge B, A\vee B}"'] & A \wedge B \wedge [A \vee B] \arrow[d, "\cong"] \\
(A \wedge B) \vee (A \wedge B \wedge [A \vee B]) \arrow[d, "\cong"'] & A \wedge [B \vee A] \wedge B \arrow[d, "\check{\mathsf{t}}_{A,B,A,B}"] \\
(A \wedge B) \vee (A \wedge [B \vee A] \wedge B) \arrow[d, "(A\wedge B) \vee \check{\mathsf{t}}_{A,B,A,B}"'] & (A \wedge B) \vee (A \wedge B) \arrow[d, "\nabla_{A\wedge B}"] \\
(A \wedge B) \vee (A \wedge B) \vee (A \wedge B) \arrow[r, "\nabla^2_{A\wedge B}"'] & A \wedge B
\end{tikzcd} \tag{2.24}$$

for all objects A and B. We now start to add the axioms for this.

Proposition 2.4.10. *Let $\mathscr{C}$ be a $\mathsf{B1}$-category in which the equation*

$$\Pi^{\mathbf{t}} = 1_{\mathbf{t}}\colon \mathbf{t} \to \mathbf{t} \tag{B2a}$$

holds. Then we have that

(i) $\Delta_{\mathbf{t}} = \hat{\varrho}_{\mathbf{t}}^{-1}\colon \mathbf{t} \to \mathbf{t} \wedge \mathbf{t}$

(ii) *For all objects A, the map Π^A is a $\wedge$-comonoid morphism.*

Proof. The equation $\Delta_{\mathbf{t}} = \hat{\varrho}_{\mathbf{t}}^{-1}$ follows immediately from $\Pi^{\mathbf{t}} = 1_{\mathbf{t}}$ and the definition of ∧-comonoids. That Π^A preserves the ∧-counit is trivial and that it preserves the ∧-comultiplication follows from

$$\begin{tikzcd}
A \arrow[rr, "\Pi^A"] \arrow[dr, "\hat{\varrho}_A^{-1}"] \arrow[dd, "\Delta_A"'] & & \mathbf{t} \arrow[dd, "\Delta_{\mathbf{t}}=\hat{\varrho}_{\mathbf{t}}^{-1}"] \\
& A \wedge \mathbf{t} \arrow[dr, "\Pi^A \wedge \mathbf{t}"] & \\
A \wedge A \arrow[ur, "A \wedge \Pi^A"] \arrow[rr, "\Pi^A \wedge \Pi^A"'] & & \mathbf{t} \wedge \mathbf{t}
\end{tikzcd}$$

where the left triangle is the definition of ∧-comonoids, the lower triangle is functoriality of ∧ and the big "triangle" is naturality of $\hat{\varrho}$. □

Lemma 2.4.11. *If a* B1*-category is single-mixed and obeys* (B2a)*, then*

$$1_{\mathbf{t}} + 1_{\mathbf{t}} = 1_{\mathbf{t}} \qquad \textit{and} \qquad 1_{\mathbf{f}} + 1_{\mathbf{f}} = 1_{\mathbf{f}} \tag{2.25}$$

Proof. First, we show that

$$\Pi^{\mathbf{f}}_{[\![\mathbf{f}} = \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \colon \mathbf{f} \wedge \mathbf{f} \to \mathbf{f} \tag{2.26}$$

This is done by chasing the diagram

$$\begin{tikzcd}
\mathbf{f} \wedge \mathbf{f} \arrow[r, "\check{\lambda}_{\mathbf{f}}^{-1} \wedge \mathbf{f}"] \arrow[d, "\Pi^{\mathbf{f}} \wedge \mathbf{f}"'] & {[\mathbf{f} \vee \mathbf{f}] \wedge \mathbf{f}} \arrow[d, "{[\mathbf{f} \vee \Pi^{\mathbf{f}}] \wedge \mathbf{f}}"] \\
\mathbf{t} \wedge \mathbf{f} \arrow[r, "\check{\lambda}_{\mathbf{t}}^{-1} \wedge \mathbf{f}"] \arrow[dr, "\check{\lambda}_{\mathbf{t} \wedge \mathbf{f}}^{-1}"] \arrow[dd, equal] & {[\mathbf{f} \vee \mathbf{t}] \wedge \mathbf{f}} \arrow[d, "\mathsf{s}_{\mathbf{f},\mathbf{t},\mathbf{f}}"] \\
& \mathbf{f} \vee (\mathbf{t} \wedge \mathbf{f}) \arrow[dl, "\check{\lambda}_{\mathbf{t} \wedge \mathbf{f}}"'] \arrow[d, "\mathbf{f} \vee \hat{\lambda}_{\mathbf{f}}"] \\
\mathbf{t} \wedge \mathbf{f} \arrow[d, "\hat{\lambda}_{\mathbf{f}}"'] & \mathbf{f} \vee \mathbf{f} \arrow[dl, "\check{\lambda}_{\mathbf{f}}"'] \arrow[d, "\nabla_{\mathbf{f}}"] \\
\mathbf{f} \arrow[r, equal] & \mathbf{f}
\end{tikzcd} \tag{2.27}$$

The right-down path is $\nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}}$ and the left down path is $\Pi^{\mathbf{f}}_{[\![\mathbf{f}}$. The two squares commute because of naturality of $\check{\lambda}$, the upper triangle holds because (2.9), the big triangle in the center is trivial, and that the lower triangle commutes follows from (the dual of) Proposition 2.4.10 (i). Now we can proceed:

$$1_{\mathbf{f}} = \Pi^{\mathbf{f}}_{[\![\mathbf{f}} \circ \Delta_{\mathbf{f}} = \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \circ \Delta_{\mathbf{f}} = 1_{\mathbf{f}} + 1_{\mathbf{f}}$$

The equation $1_{\mathbf{t}} = 1_{\mathbf{t}} + 1_{\mathbf{t}}$ follows by duality. □

Remark 2.4.12. The proof nets and categorical axioms presented in [LS05b, LS05a] do not have proper units, but only "weak units" (see [LS05b, LS05a] for details). In that setting, Lemma 2.4.11 (which is a consequence of having proper units) does not hold.

Proposition 2.4.13. *In a* B1*-category that is single-mixed and obeys* (B2a)*, we have*

$$f + \Pi^A \;=\; f \tag{2.28}$$

for all maps $f\colon A \to \mathbf{t}$*. Dually, we have*

$$g + \Pi^B \;=\; g \tag{2.29}$$

for all maps $g\colon \mathbf{f} \to B$*.*

Proof. Chase the diagram

$$\begin{tikzcd}
A \arrow[r, "\Delta_A"] \arrow[d, equal] & A \wedge A \arrow[d, "A \wedge \Pi^A"] \\
A \arrow[d, "f"'] & A \wedge \mathbf{t} \arrow[l, "\hat{\varrho}_A"'] \arrow[d, "f \wedge \mathbf{t}"] \\
\mathbf{t} \arrow[dd, equal] \arrow[dr, "\Delta_\mathbf{t}"'] & \mathbf{t} \wedge \mathbf{t} \arrow[l, "\hat{\varrho}_\mathbf{t}"'] \arrow[d, equal] \\
 & \mathbf{t} \wedge \mathbf{t} \arrow[d, "\mathsf{mix}_{\mathbf{t},\mathbf{t}}"] \\
\mathbf{t} & \mathbf{t} \vee \mathbf{t} \arrow[l, "\nabla_\mathbf{t}"]
\end{tikzcd} \tag{2.30}$$

The first square is the comonoid equation, the second one is naturality of $\hat{\varrho}$, the triangle commutes because of Proposition 2.4.10 (i), and the lower quadrangle is (2.25). □

Proposition 2.4.14. *In a* B1*-category obeying* (B2a)*, the equation*

$$\begin{tikzcd}
 & A \wedge B \arrow[dl, "\Pi^A \wedge \Pi^B"'] \arrow[dr, "\Pi^{A \wedge B}"] & \\
\mathbf{t} \wedge \mathbf{t} \arrow[rr, "\hat{\varrho}_\mathbf{t}"'] & & \mathbf{t}
\end{tikzcd} \tag{B2b}$$

holds if and only if

(i) $\Pi^{\mathbf{t} \wedge \mathbf{t}} = \hat{\varrho}_\mathbf{t}\colon \mathbf{t} \wedge \mathbf{t} \to \mathbf{t}$ *and*

(ii) *the maps that preserve the* $\wedge$*-counit are closed under* $\wedge$*.*

Proof. We see that (i) follows from (B2a) and (B2b) by plugging in $\mathbf{t}$ for A and B in (B2b). That (ii) holds follows from

$$\begin{tikzcd}
A \wedge B \arrow[rr, "f \wedge g"] \arrow[dr, "\Pi^A \wedge \Pi^B"] \arrow[ddr, "\Pi^{A \wedge B}"'] & & C \wedge D \arrow[dl, "\Pi^C \wedge \Pi^D"'] \arrow[ddl, "\Pi^{C \wedge D}"] \\
& \mathbf{t} \wedge \mathbf{t} \arrow[d, "\hat{\varrho}_{\mathbf{t}}"] & \\
& \mathbf{t} &
\end{tikzcd} \tag{2.31}$$

where $f\colon A \to B$ and $g\colon C \to D$ are maps that preserve the $\wedge$-counit. Conversely, it follows from (ii) and Proposition 2.4.10 that $\Pi^A \wedge \Pi^B$ preserves the $\wedge$-counit. With (i) this yields (B2b). □

Proposition 2.4.15. *In a* B1*-category obeying* (B2a) *and* (B2b) *the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Π^A, $\Pi^A_{B\mathbb{I}}$, *and* $\Pi^A_{\mathbb{I}B}$ *all preserve the* $\wedge$*-counit. And dually, the maps* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, Π^A, $\Pi^A_{B\mathbb{I}}$, *and* $\Pi^A_{\mathbb{I}B}$ *all preserve the* $\vee$*-unit.*

Proof. We show the case for $\hat{\sigma}_{A,B}$:

$$\begin{tikzcd}
A \wedge B \arrow[rrr, "\hat{\sigma}_{A,B}"] \arrow[dr, "\Pi^A \wedge \Pi^B"] \arrow[ddr, "\Pi^{A \wedge B}"'] & & & B \wedge A \arrow[dl, "\Pi^B \wedge \Pi^A"'] \arrow[ddl, "\Pi^{B \wedge A}"] \\
& \mathbf{t} \wedge \mathbf{t} \arrow[r, "\hat{\sigma}_{\mathbf{t},\mathbf{t}}"] \arrow[dr, "\hat{\varrho}_{\mathbf{t}}"] & \mathbf{t} \wedge \mathbf{t} \arrow[dl, "\hat{\varrho}_{\mathbf{t}}"'] & \\
& & \mathbf{t} &
\end{tikzcd}$$

The quadrangle in naturality of $\hat{\sigma}$ and the commutativity of triangle in the center is a consequence of the coherence theorem for monoidal categories. The two slim triangles are just (B2b). The cases for $\hat{\alpha}_{A,B,C}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$ are similar. For Π^A, it follows directly from (B2a) and for $\Pi^A_{B\mathbb{I}}$ and $\Pi^A_{\mathbb{I}B}$ from Proposition 2.4.14 (ii) and from (2.21). □

Proposition 2.4.16. *If a* B1*-category obeys* (B2a) *and the equation*

$$\begin{tikzcd}
& A \wedge B \arrow[dl, "\Delta_A \wedge \Delta_B"'] \arrow[dr, "\Delta_{A \wedge B}"] & \\
A \wedge A \wedge B \wedge B \arrow[rr, "A \wedge \hat{\sigma}_{A,B} \wedge B"'] & & A \wedge B \wedge A \wedge B
\end{tikzcd} \tag{B2c}$$

then

(i) *also the equation* (B2b) *holds,*

(ii) *for every* A*, the map* Δ_A *is a* $\wedge$*-comonoid morphism, and*

(iii) *the maps that preserve the* $\wedge$*-comultiplication are closed under* $\wedge$*.*

Proof. (i) For showing that (B2b) holds, consider the diagram

$$\begin{tikzcd}
A \wedge B \arrow[rrr, equal] \arrow[dr, "\Delta_A \wedge \Delta_B"] \arrow[ddd, "\Delta_{A\wedge B}"'] & & & A \wedge B \arrow[dl, "\hat{\varrho}_A^{-1} \wedge \hat{\varrho}_B^{-1}"'] \arrow[ddd, "\hat{\varrho}_{A\wedge B}^{-1}"] \\
& A \wedge A \wedge B \wedge B \arrow[r, "A \wedge \Pi^A \wedge B \wedge \Pi^B"] \arrow[dl, "A \wedge \hat{\sigma}_{A,B} \wedge B"'] & A \wedge \mathbf{t} \wedge B \wedge \mathbf{t} \arrow[d, "A \wedge \hat{\sigma}_{\mathbf{t},B} \wedge \mathbf{t}"] & \\
A \wedge B \wedge A \wedge B \arrow[rr, "A \wedge B \wedge \Pi^A \wedge \Pi^B"'] & & A \wedge B \wedge \mathbf{t} \wedge \mathbf{t} \arrow[r, "A \wedge B \wedge \hat{\varrho}_{\mathbf{t}}"'] & A \wedge B \wedge \mathbf{t}
\end{tikzcd}$$

The triangle on the left is (B2c), the upper quadrangle is the comonoid equation, the lower quadrangle is naturality of $\hat{\sigma}$ and the quadrangle on the right commutes because of the coherence in monoidal categories. The outer square says that $\hat{\varrho}_{\mathbf{t}} \circ (\Pi^A \wedge \Pi^B)$ is $\wedge$-counit for $\Delta_{A\wedge B}$. By Proposition 2.4.3 (uniqueness of units) it must therefore be equal to $\Pi^{A\wedge B}$.
(ii) That Δ_A preserves the $\wedge$-comultiplication follows from

$$\begin{tikzcd}
A \arrow[rr, "\Delta_A"] \arrow[dd, "\Delta_A"'] & & A \wedge A \arrow[dl, "\Delta_A \wedge \Delta_A"'] \arrow[dd, "\Delta_{A\wedge A}"] \\
& A \wedge A \wedge A \wedge A \arrow[dr, "A \wedge \hat{\sigma}_{A,A} \wedge A"] & \\
A \wedge A \arrow[rr, "\Delta_A \wedge \Delta_A"'] & & A \wedge A \wedge A \wedge A
\end{tikzcd} \tag{2.32}$$

where the pentagon commutes because of the coassociativity and cocommutativity of $\Delta_A \colon A \to A \wedge A$. For showing that Δ_A preserves the $\wedge$-counit, consider the diagram

$$\begin{tikzcd}
A \arrow[rr, "\Delta_A"] \arrow[dd, "\Pi^A"'] & & A \wedge A \arrow[dl, "\Pi^A \wedge \Pi^A"'] \arrow[dd, "\Pi^{A\wedge A}"] \\
& \mathbf{t} \wedge \mathbf{t} \arrow[dr, "\hat{\varrho}_{\mathbf{t}}"] & \\
\mathbf{t} \arrow[ur, "\Delta_{\mathbf{t}}"] \arrow[rr, equal] & & \mathbf{t}
\end{tikzcd}$$

The big and the lower triangle commute by Proposition 2.4.10, and the left triangle is (B2b) which has been shown before. For (iii) chase

$$\begin{tikzcd}
A \wedge B \arrow[rrr, "f \wedge g"] \arrow[dr, "\Delta_A \wedge \Delta_B"] \arrow[ddd, "\Delta_{A\wedge B}"'] & & & C \wedge D \arrow[dl, "\Delta_C \wedge \Delta_D"'] \arrow[ddd, "\Delta_{C\wedge D}"] \\
& A \wedge A \wedge B \wedge B \arrow[r, "f \wedge f \wedge g \wedge g"] \arrow[dl, "A \wedge \hat{\sigma}_{A,B} \wedge B"'] & C \wedge C \wedge D \wedge D \arrow[dr, "C \wedge \hat{\sigma}_{C,D} \wedge D"] & \\
A \wedge B \wedge A \wedge B \arrow[rrr, "f \wedge g \wedge f \wedge g"'] & & & C \wedge D \wedge C \wedge D
\end{tikzcd}$$

where $f \colon A \to B$ and $g \colon C \to D$ are maps preserving the $\wedge$-comultiplication. $\square$

Proposition 2.4.17. *In a* B1*-category obeying* (B2a) *and* (B2c) *the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Π^A, $\Pi^B_{A\mathbb{I}}$, *and* $\Pi^A_{\mathbb{I}B}$ *all preserve the* ∧*-comultiplication. Dually, the maps* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, $\amalg^A$, $\amalg^B_{A\mathbb{I}}$, *and* $\amalg^A_{\mathbb{I}B}$ *all preserve the* ∨*-multiplication.*

Proof. Again, we show the case only for $\hat{\sigma}$:

$$\begin{tikzcd}
A \wedge B \arrow[rrr, "\hat{\sigma}_{A,B}"] \arrow[dr, "\Delta_A \wedge \Delta_B"] \arrow[ddd, "\Delta_{A\wedge B}"'] & & & B \wedge A \arrow[dl, "\Delta_B \wedge \Delta_A"'] \arrow[ddd, "\Delta_{B\wedge A}"] \\
& A \wedge A \wedge B \wedge B \arrow[r, "\hat{\sigma}_{A\wedge A, B\wedge B}"] \arrow[dl, "A\wedge\hat{\sigma}_{A,B}\wedge B"'] & B \wedge B \wedge A \wedge A \arrow[dr, "B\wedge\hat{\sigma}_{B,A}\wedge A"] & \\
A \wedge B \wedge A \wedge B \arrow[rrr, "\hat{\sigma}_{A,B}\wedge\hat{\sigma}_{A,B}"'] & & & B \wedge A \wedge B \wedge A
\end{tikzcd}$$

The two triangles are (B2c), the upper square is naturality of $\hat{\sigma}$ and the lower square commutes because of coherence in monoidal categories. For $\hat{\alpha}$, $\hat{\varrho}$, and $\hat{\lambda}$ the situation is similar. For Π^A it has been shown already in Proposition 2.4.10, and for $\Pi^B_{A\mathbb{I}}$, and $\Pi^A_{\mathbb{I}B}$ it follows from Proposition 2.4.16. □

Propositions 2.4.10–2.4.16 give rise to the following definition:

Definition 2.4.18. A B2*-category* is a B1-category which obeys equations (B2a) and (B2c) for all objects A and B.

The following theorem summarizes the properties of B2-categories.

Theorem 2.4.19. *In a* B2*-category, the maps* $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$, Δ_A, Π^A, $\Pi^B_{A\mathbb{I}}$, *and* $\Pi^A_{\mathbb{I}B}$, *all are* ∧*-comonoid morphisms, and the* ∧*-comonoid morphisms are closed under* ∧*. Dually, the maps* $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$, ∇_A, $\amalg^A$, $\amalg^B_{A\mathbb{I}}$, *and* $\amalg^A_{\mathbb{I}B}$, *all are* ∨*-monoid morphisms, and the* ∨*-monoid morphisms are closed under* ∨*.*

Proof. Propositions 2.4.10, 2.4.14, 2.4.15, 2.4.16, and 2.4.17. □

Proposition 2.4.20. *Let* $f\colon A \to C$ *and* $g\colon A \to D$ *and* $h\colon B \to C$ *and* $a\colon A' \to A$ *and* $b\colon B' \to B$ *and* $c\colon C \to C'$ *and* $d\colon D \to D'$ *be maps for some objects* A, B, C, D, A', B', C', D' *in a* B2*-category. Diagrammatically:*

$$\begin{tikzcd}
A' \arrow[r, "a"] & A \arrow[r, "f"] \arrow[dr, "g"] & C \arrow[r, "c"] & C' \\
B' \arrow[r, "b"] & B \arrow[ur, "h"] & D \arrow[r, "d"] & D'
\end{tikzcd}$$

Then we have:

(i) $(c \wedge d) \circ \langle f, g\rangle = \langle c \circ f, d \circ g\rangle$.

(ii) *If* a *preserves the comultiplication, then* $\langle f, g\rangle \circ a = \langle f \circ a, g \circ a\rangle$.

(iii) *If* g *preserves the counit, then* $\Pi^C_{\mathbb{I}D} \circ \langle f, g\rangle = f$.

If f *preserves the counit, then* $\Pi^D_{C\mathbb{I}} \circ \langle f, g\rangle = g$.

(iv) $\langle \Pi^D_{C\mathbb{I}}, \Pi^C_{\mathbb{I}D} \rangle = 1_{C\wedge D}$.

Dually, we also have:

(i) $[f,h] \circ (a \vee b) = [f \circ a, h \circ b]$.

(ii) *If c preserves the multiplication, then* $c \circ [f,h] = [c \circ f, c \circ h]$.

(iii) *If h preserves the unit, then* $[f,h] \circ \Pi^B_{A\mathbb{I}} = f$.

If f preserves the unit, then $[f,h] \circ \Pi^A_{\mathbb{I}B} = h$.

(iv) $[\Pi^B_{A\mathbb{I}}, \Pi^A_{\mathbb{I}B}] = 1_{A\vee B}$.

Proof. Straightforward calculation. Note that (i)–(iii) hold already in a B1-category, only for (iv) is the equation (B2c) needed. □

As observed before, if a B1-category is single-mixed then $\mathrm{Hom}(A,B)$ carries a semigroup structure. If we additionally have the structure of a B2-category, then the bijection $(\star)$ of Definition 2.2.2 preserves this semigroup structure:

Proposition 2.4.21. *In a single-mixed* B2*-category the bijection* $(\star)$ *is a semigroup isomorphism.*

Proof. Let $f,g\colon A\wedge B \to C$ be two maps for some objects A, B, and C, and let $f',g'\colon A \to \bar{B}\vee C$ be their transposes. We have to show that $f'+g'$ is the transpose of $f+g$. First note, that in any star-autonomous category the map

$$A\wedge A\wedge B\wedge B \xrightarrow{A\wedge\hat{\sigma}_{A,B}\wedge B} A\wedge B\wedge A\wedge B \xrightarrow{f\wedge g} C\wedge C$$

is a transpose of

$$A\wedge A \xrightarrow{f'\wedge g'} [\bar{B}\vee C]\wedge[\bar{B}\vee C] \xrightarrow{\hat{\mathsf{t}}} \bar{B}\vee\bar{B}\vee(C\wedge C)$$

where $\hat{\mathsf{t}}$ is the canonical map obtained from two switches, cf. (2.5). Now, by definition, $f+g$ is the map

$$\begin{tikzcd}
A\wedge B \arrow[r, "\Delta_{A\wedge B}"] \arrow[dr, "\Delta_A\wedge\Delta_B"'] & A\wedge B\wedge A\wedge B \arrow[r, "f\wedge g"] & C\wedge C \arrow[r, "\mathsf{mix}_{C,C}"] & C\vee C \arrow[r, "\nabla_C"] & C \\
& A\wedge A\wedge B\wedge B \arrow[u, "A\wedge\hat{\sigma}_{A,B}\wedge B"'] & & &
\end{tikzcd}$$

By (2.4) and what has been said above, the transpose of the lower path is the outermost path of the following:

$$\begin{tikzcd}
A \arrow[r, "\Delta_A"] & A\wedge A \arrow[r, "f'\wedge g'"] & [\bar{B}\vee C]\wedge[\bar{B}\vee C] \arrow[r, "\hat{\mathsf{t}}"] \arrow[d, "\mathsf{mix}_{\bar{B}\vee C,\bar{B}\vee C}"'] & \bar{B}\vee\bar{B}\vee(C\wedge C) \arrow[d, "\bar{B}\vee\bar{B}\vee\mathsf{mix}_{C,C}"] \\
& & \bar{B}\vee C\vee\bar{B}\vee C \arrow[r, "\bar{B}\vee\hat{\sigma}_{C,\bar{B}}\vee C"] \arrow[dr, "\nabla_{\bar{B}\vee C}"'] & \bar{B}\vee\bar{B}\vee C\vee C \arrow[d, "\nabla_{\bar{B}}\vee\nabla_C"] \\
& & & \bar{B}\vee C
\end{tikzcd}$$

The innermost path is by definition $f' + g'$. The square commutes because of (mix-$\hat{\sigma}$) and (mix-$\hat{\alpha}$), and the triangle is the dual of ($\mathsf{B2c}$). □

2.5 Order enrichment

In [FP04c], Führman and Pym equipped $\mathsf{B2}$-categories with an order enrichment, such that the proof identifications induced by the axioms are exactly the same as the proof identifications made by Gentzen's sequent calculus LK [Gen34], modulo "trivial rule permutations" (see [Laf95b, Rob03]), and such that $f \preccurlyeq g$ if g is obtained from f via cut elimination (which is not confluent in LK).

Definition 2.5.1. A $\mathsf{B2}$-category is called an LK-*category* if for every A, B, the set $\mathrm{Hom}(A, B)$ is equipped with a partial order structure $\preccurlyeq$ such that

(i) the arrow composition $\circ$, as well as the bifunctors $\wedge$ and $\vee$ are monotone in both arguments,

(ii) for every map $f\colon A \to B$ we have

$$\Pi^B \circ f \preccurlyeq \Pi^A \tag{LK-Π}$$

$$\Delta_B \circ f \preccurlyeq (f \wedge f) \circ \Delta_A \tag{LK-Δ}$$

(iii) and the bijection ($\star$) of Definition 2.2.2 is an order isomorphism for $\preccurlyeq$.

Although in [FP04c, FP04b] Führmann and Pym use the term "classical category", we use here the term LK-categories because—as worked out in detail in [FP04c]—they provide a category-theoretic axiomatisation of sequent calculus proofs in Gentzen's system LK [Gen34]. However, it should be clear that LK-categories are only one particular example of a wide range of possible category-theoretic axiomatisations of proofs in classical logic.

Remark 2.5.2. Note that in [FP04c] Führmann and Pym give a different definition for LK-categories. Since they start from a weakly distributive category [CS97b] instead of a star-autonomous one, they do not have immediate access to transposition. For this reason, they have to give a larger set of inequalities, defining the order $\preccurlyeq$:

$$\begin{array}{ll}
\Delta_B \circ f \preccurlyeq (f \wedge f) \circ \Delta_A & f \circ \nabla_A \preccurlyeq \nabla_B \circ [f \vee f] \\
\Pi^B \circ f \preccurlyeq \Pi^A & f \circ \Pi^A \preccurlyeq \Pi^B \\
A \vee \Delta_B \preccurlyeq [\nabla_A \vee (B \wedge B)] \circ \hat{\mathsf{t}} \circ \Delta_{A \vee B} & A \wedge \nabla_B \preccurlyeq \Delta_{A \wedge B} \circ \check{\mathsf{t}} \circ (\Delta_A \wedge [B \vee B]) \\
A \vee \Pi^B \preccurlyeq [\Pi^A \vee \mathbf{t}] \circ \check{\lambda}_{\mathbf{t}}^{-1} \circ \Pi^{A \vee B} & A \wedge \Pi^B \preccurlyeq \Pi^{A \wedge B} \circ \hat{\lambda}_{\mathbf{f}} \circ (\Pi^A \wedge \mathbf{f})
\end{array} \tag{FP}$$

where $f\colon A \to B$ is an arbitrary map and $\hat{\mathsf{t}}\colon [A \vee B] \wedge [A \vee B] \to A \vee A \vee (B \wedge B)$ and $\check{\mathsf{t}}\colon A \wedge A \wedge [B \vee B] \to (A \wedge B) \vee (A \wedge B)$ are the tensor and cotensor map, cf. (2.5) and (2.6). One can now easily show that both definitions are equivalent: Clearly the inequations on the right in (FP) are just transposes of the ones on the left. The two top ones on the left are just (LK-Π) and (LK-Δ), and the two bottom ones follow as follows. If we transpose $A \vee B \xrightarrow{A \vee \Delta_B} A \vee (B \wedge B)$ we get the map

$$\bar{A} \wedge [A \vee B] \xrightarrow{\mathsf{eval}} B \xrightarrow{\Delta_B} B \wedge B$$

By (LK-Δ), this is smaller or equal to

$$\bar{A} \wedge [A \vee B] \xrightarrow{\Delta_{\bar{A}\wedge[A\vee B]}} \bar{A} \wedge [A \vee B] \wedge \bar{A} \wedge [A \vee B] \xrightarrow{\mathsf{eval}\wedge\mathsf{eval}} B \wedge B$$

By (B2c) this is the same map as

$$\bar{A} \wedge [A \vee B] \xrightarrow{\Delta_{\bar{A}}\wedge\Delta_{A\vee B}} \bar{A} \wedge \bar{A} \wedge [A \vee B] \wedge [A \vee B] \xrightarrow{\cong} \bar{A} \wedge [A \vee B] \wedge \bar{A} \wedge [A \vee B] \xrightarrow{\mathsf{eval}\wedge\mathsf{eval}} B \wedge B$$

Transposing back yields

$$A \vee B \xrightarrow{\Delta_{A\vee B}} [A \vee B] \wedge [A \vee B] \xrightarrow{\hat{\mathsf{t}}} A \vee A \vee (B \wedge B) \xrightarrow{\nabla_A} A \vee (B \wedge B)$$

This shows the third inequation on the left in (FP). For the last one, we proceed similarly: Transposing $A \vee B \xrightarrow{A\vee\Pi^B} A \vee \mathbf{t}$ yields

$$\bar{A} \wedge [A \vee B] \xrightarrow{\mathsf{eval}} B \xrightarrow{\Pi^B} \mathbf{t}$$

which is by (LK-Π) smaller or equal to

$$\bar{A} \wedge [A \vee B] \xrightarrow{\Pi^{\bar{A}\wedge[A\vee B]}} \mathbf{t}$$

which is by (B2b) and (2.21) the same as

$$\bar{A} \wedge [A \vee B] \xrightarrow{\Pi^{\bar{A}}_{\mathbb{I}A\vee B}} A \vee B \xrightarrow{\Pi^{A\vee B}} \mathbf{t}$$

If transpose back, we get

$$A \vee B \xrightarrow{\Pi^{A\vee B}} \mathbf{t} \xrightarrow{\Pi^{A}_{\mathbb{I}\mathbf{t}}} A \vee \mathbf{t}$$

as desired. We do not show here the other direction because it is rather tedious: It is almost literally the same as the proof for showing that any weakly distributive category with negation is a star-autonomous category (see [CS97b, BCST96]).

The following theorem states the main properties of LK-categories. It has first been observed and proved by Führmann and Pym in [FP04a].

Theorem 2.5.3. *Every* LK*-category is single-mixed and idempotent. Furthermore, for all maps $f, g\colon A \to B$, we have $f \leq g$ iff $g \preccurlyeq f$.*

Proof. Because of (B2a) and (LK-Π) we have that $\Pi^{\mathbf{t}} = 1_{\mathbf{t}} \circ \Pi^{\mathbf{t}} = \Pi^{\mathbf{t}} \circ \Pi^{\mathbf{t}} \preccurlyeq \Pi^{\mathbf{f}}$. By duality, we also get $\Pi^{\mathbf{f}} \preccurlyeq \Pi^{\mathbf{t}}$. Therefore $\Pi^{\mathbf{f}} = \Pi^{\mathbf{t}}$, i.e., the category is single-mixed. Next, we show that $f + g \preccurlyeq f$ for all maps $f, g\colon A \to B$. For this, note that

$$A \wedge B \xrightarrow{\mathsf{mix}_{A,B}} A \vee B \quad \preccurlyeq \quad A \wedge B \xrightarrow{\Pi^{A}_{\mathbb{I}B}} B \xrightarrow{\Pi^{A}_{\mathbb{I}B}} A \vee B$$

because these are the transposes of

$$A \wedge \bar{A} \xrightarrow{\hat{\mathsf{i}}_A} \mathbf{f} \xrightarrow{\Pi^{\mathbf{f}}} \mathbf{t} \xrightarrow{\check{\mathsf{i}}_B} B \vee \bar{B} \quad \preccurlyeq \quad A \wedge \bar{A} \xrightarrow{\Pi^{A\wedge\bar{A}}} \mathbf{t} \xrightarrow{\check{\mathsf{i}}_B} B \vee \bar{B}$$

Now we can proceed as follows:

$$\begin{aligned}
f+g &= \nabla_B \circ [f \vee g] \circ \mathsf{mix}_{A,A} \circ \Delta_A \\
&\preccurlyeq \nabla_B \circ [f \vee g] \circ \amalg^A_{\mathbb{I}A} \circ \Pi^A_{\mathbb{I}A} \circ \Delta_A \\
&= \nabla_B \circ [f \vee g] \circ \amalg^A_{\mathbb{I}A} \circ 1_A \\
&= \nabla_B \circ [f \vee B] \circ [A \vee g] \circ [A \vee \amalg^A] \circ \check{\varrho}_A \\
&= \nabla_B \circ [f \vee B] \circ [A \vee g \circ \amalg^A] \circ \check{\varrho}_A \\
&\preccurlyeq \nabla_B \circ [f \vee B] \circ [A \vee \amalg^B] \circ \check{\varrho}_A \\
&= \nabla_B \circ \amalg^B_{\mathbb{I}B} \circ f \\
&= f
\end{aligned}$$

Similarly, we get $f + g \preccurlyeq g$. Now we show that $f \preccurlyeq f + f$ for $f\colon A \to B$. Let $\hat{f}\colon A \wedge \bar{B} \to \mathbf{f}$ be the transpose of f. Then we have

$$\begin{aligned}
\hat{f} &= 1_{\mathbf{f}} \circ \hat{f} \\
&= (1_{\mathbf{f}} + 1_{\mathbf{f}}) \circ \hat{f} \\
&= \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \circ \Delta_{\mathbf{f}} \circ \hat{f} \\
&\preccurlyeq \nabla_{\mathbf{f}} \circ \mathsf{mix}_{\mathbf{f},\mathbf{f}} \circ (\hat{f} \wedge \hat{f}) \circ \Delta_{A \wedge \bar{B}} \\
&= \hat{f} + \hat{f} \\
&= \widehat{f + f}
\end{aligned}$$

The second equation is Lemma 2.4.11, the third one is the definition of $+$, the fourth one is (LK-Δ), the fifth again the definition of $+$, and the last equation uses Proposition 2.4.21. By transposing back, we get $f \preccurlyeq f + f$. From this together with $f + f \preccurlyeq f$ we get idempotency. For showing that $f \leq g$ iff $g \preccurlyeq f$, we need to show that $g \preccurlyeq f$ iff $f + g = g$. Since $f + g \preccurlyeq f$, we have that $f + g = g$ implies $g \preccurlyeq f$. Now suppose $g \preccurlyeq f$. Then we have $g = g + g \preccurlyeq f + g$. This finishes the proof since $f + g \preccurlyeq g$ has been shown already. □

Note that the converse is not necessarily true. Not every single-mixed idempotent B2-category is an LK-category. Nonetheless, because of Proposition 2.4.13, in every single-mixed idempotent B2-category we have for every $f\colon A \to B$ that $\Pi^B \circ f + \Pi^A = \Pi^B \circ f$, and hence $\Pi^A \leq \Pi^B \circ f$ which is exactly (LK-Π). However, the inequality (LK-Δ) does not follow from idempotency. One can easily construct countermodels along the lines of [Str05] (see also Section 3.6).

2.6 The medial map and the nullary medial map

That LK-categories are idempotent means that they are already at the degenerate end of the spectrum of Boolean categories (cf. also the discussion at the end of Section 3.6). On the other hand, B2-categories have (apart from Theorem 2.4.19) very little structure. The question that arises now is therefore: how can we add additional structure to B2-categories without getting too much collapse? In particular, can we extend the structure such that all the maps mentioned in Theorem 2.4.19 become $\vee$-monoid morphisms *and* $\wedge$-comonoid morphisms? This is where *medial* enters the scene.

Definition 2.6.1. We say, a B2-category $\mathscr{C}$ *has medial* if for all objects A, B, C, and D there is a map $\mathsf{m}_{A,B,C,D}\colon (A\wedge B)\vee(C\wedge D)\to[A\vee C]\wedge[B\vee D]$ with the following properties:

- it is natural in A, B, C and D,
- it is self-dual, i.e.,

$$\begin{tikzcd}
\overline{[A\vee C]\wedge[B\vee D]} \arrow[r, "\overline{\mathsf{m}_{A,B,C,D}}"] \arrow[d, "\cong"'] & \overline{(A\wedge B)\vee(C\wedge D)} \arrow[d, "\cong"] \\
(\bar{D}\wedge\bar{B})\vee(\bar{C}\wedge\bar{A}) \arrow[r, "\mathsf{m}_{\bar{D},\bar{B},\bar{C},\bar{A}}"'] & [\bar{D}\vee\bar{C}]\wedge[\bar{B}\vee\bar{A}]
\end{tikzcd} \tag{2.33}$$

  commutes, where the vertical maps are the canonical isomorphisms induced by Definition 2.2.2,

- and it obeys the equation

$$\begin{tikzcd}
& A\vee B \arrow[dl, "\Delta_A\vee\Delta_B"'] \arrow[dr, "\Delta_{A\vee B}"] & \\
(A\wedge A)\vee(B\wedge B) \arrow[rr, "\mathsf{m}_{A,A,B,B}"'] & & [A\vee B]\wedge[A\vee B]
\end{tikzcd} \tag{B3c}$$

  for all objects A and B.

The following equation is a consequence of (B3c) and the self-duality of medial.

$$\begin{tikzcd}
(A\wedge B)\vee(A\wedge B) \arrow[rr, "\mathsf{m}_{A,B,A,B}"] \arrow[dr, "\nabla_{A\wedge B}"'] & & [A\vee A]\wedge[B\vee B] \arrow[dl, "\nabla_A\wedge\nabla_B"] \\
& A\wedge B &
\end{tikzcd} \tag{B3c$'$}$$

Theorem 2.6.2. *Let $\mathscr{C}$ be a* B2*-category that has medial. Then*

(i) *The maps that preserve the $\wedge$-comultiplication are closed under $\vee$, and dually, the maps that preserve the $\vee$-multiplication are closed under $\wedge$.*

(ii) *For all maps $A\xrightarrow{f}C$, $A\xrightarrow{g}D$, $B\xrightarrow{h}C$, and $B\xrightarrow{k}D$, we have that*

$$[\langle f,g\rangle,\langle h,k\rangle]=\langle[f,h],[g,k]\rangle\colon A\vee B\to C\wedge D \quad .$$

(iii) *For all objects A, B, C, and D,*

$$\begin{aligned}
\mathsf{m}_{A,B,C,D} &= \left[\left\langle \Pi^C_{A\mathbb{I}}\circ\Pi^B_{A\mathbb{I}},\Pi^D_{B\mathbb{I}}\circ\Pi^A_{\mathbb{I}B}\right\rangle,\left\langle \Pi^A_{\mathbb{I}C}\circ\Pi^D_{C\mathbb{I}},\Pi^B_{\mathbb{I}D}\circ\Pi^C_{\mathbb{I}D}\right\rangle\right]\\
&= \left\langle\left[\Pi^C_{A\mathbb{I}}\circ\Pi^B_{A\mathbb{I}},\Pi^A_{\mathbb{I}C}\circ\Pi^D_{C\mathbb{I}}\right],\left[\Pi^D_{B\mathbb{I}}\circ\Pi^A_{\mathbb{I}B},\Pi^B_{\mathbb{I}D}\circ\Pi^C_{\mathbb{I}D}\right]\right\rangle
\end{aligned}$$

(iv) *For all objects A, B, C, and D, the following diagram commutes:*

$$\begin{tikzcd}
 & {[(A\wedge B)\vee(C\wedge D)]\wedge[(A\wedge B)\vee(C\wedge D)]} \arrow[dr, "{[\Pi^B_{A[\!]}\vee\Pi^D_{C[\!]}]\wedge[\Pi^A_{[\!]B}\vee\Pi^C_{[\!]D}]}"] & \\
(A\wedge B)\vee(C\wedge D) \arrow[ur, "\Delta_{(A\wedge B)\vee(C\wedge D)}"] \arrow[dr, "{(\Pi^C_{A[\!]}\wedge\Pi^D_{B[\!]})\vee(\Pi^A_{[\!]C}\wedge\Pi^B_{[\!]D})}"'] & & {[A\vee C]\wedge[B\vee D]} \\
 & {([A\vee C]\wedge[B\vee D])\vee([A\vee C]\wedge[B\vee D])} \arrow[ur, "\nabla_{[A\vee C]\wedge[B\vee D]}"'] &
\end{tikzcd} \tag{2.34}$$

(v) *The horizontal diagonal of* (2.34) *is equal to* $\mathsf{m}_{A,B,C,D}$.

Proof. For (i), chase the following (compare with the proof of Proposition 2.4.16 (iii))

$$\begin{tikzcd}
A\vee B \arrow[rrr, "f\vee g"] \arrow[dr, "\Delta_A\vee\Delta_B"] \arrow[ddd, "\Delta_{A\vee B}"'] & & & C\vee D \arrow[dl, "\Delta_C\vee\Delta_D"'] \arrow[ddd, "\Delta_{C\vee D}"] \\
 & (A\wedge A)\vee(B\wedge B) \arrow[r, "{(f\wedge f)\vee(g\wedge g)}"] \arrow[dl, "\mathsf{m}_{A,A,B,B}"] & (C\wedge C)\vee(D\wedge D) \arrow[dr, "\mathsf{m}_{C,C,D,D}"'] & \\
{[A\vee B]\wedge[A\vee B]} \arrow[rrr, "{[f\vee g]\wedge[f\vee g]}"'] & & & {[C\vee D]\wedge[C\vee D]}
\end{tikzcd}$$

For (ii) chase the diagram

$$\begin{tikzcd}
A\vee B \arrow[rrr, "{[\langle f,g\rangle,\langle h,k\rangle]}"] \arrow[dr, "\Delta_A\vee\Delta_B"] \arrow[ddd, equal] \arrow[drrr, "{\langle f,g\rangle\vee\langle h,k\rangle}"'] & & & C\wedge D \arrow[ddd, equal] \\
 & (A\wedge A)\vee(B\wedge B) \arrow[r, "{(f\wedge g)\vee(h\wedge k)}"'] \arrow[d, "\mathsf{m}_{A,A,B,B}"] & (C\wedge D)\vee(C\wedge D) \arrow[d, "\mathsf{m}_{C,D,C,D}"] \arrow[dr, "\nabla_{C\wedge D}"] & \\
A\vee B \arrow[dr, "\Delta_{A\vee B}"'] \arrow[ddd, equal] & {[A\vee B]\wedge[A\vee B]} \arrow[r, "{[f\vee h]\wedge[g\vee k]}"] \arrow[drrr, "{[f,h]\wedge[g,k]}"] & {[C\vee C]\wedge[D\vee D]} \arrow[r, "\nabla_C\wedge\nabla_D"'] & C\wedge D \arrow[ddd, equal] \\
 & & & \\
A\vee B \arrow[rrr, "{\langle[f,h],[g,k]\rangle}"'] & & & C\wedge D
\end{tikzcd}$$

where the square in the center is naturality of medial, the two small triangles are (B3c) and (B3c′). The big triangles are just (2.20). Note the importance of naturality of medial

in the two diagrams above. Let us now continue with (iv) and (v), which are proved by

$$\begin{tikzcd}
(A\wedge B)\vee(C\wedge D) \arrow[rr, "\Delta_{(A\wedge B)\vee(C\wedge D)}"] \arrow[dr, "\Delta_{A\wedge B}\vee\Delta_{C\wedge D}"] \arrow[dd, "(\Delta_A\wedge\Delta_B)\vee(\Delta_C\wedge\Delta_D)"'] & & [(A\wedge B)\vee(C\wedge D)]\wedge[(A\wedge B)\vee(C\wedge D)] \arrow[dd, "{[\Pi^B_{A\,[\!]}\vee\Pi^D_{C\,[\!]}]\wedge[\Pi^A_{[\!]\,B}\vee\Pi^C_{[\!]\,D}]}"] \\
& (A\wedge B\wedge A\wedge B)\vee(C\wedge D\wedge C\wedge D) \arrow[ur, "\mathsf{m}"] \arrow[dd, "{(\Pi^B_{A\,[\!]}\wedge\Pi^A_{[\!]\,B})\vee(\Pi^D_{C\,[\!]}\wedge\Pi^C_{[\!]\,D})}"] & \\
(A\wedge A\wedge B\wedge B)\vee(C\wedge C\wedge D\wedge D) \arrow[ur, "\cong"] \arrow[dr, "{(\Pi^A_{A\,[\!]}\wedge\Pi^B_{[\!]\,B})\vee(\Pi^C_{C\,[\!]}\wedge\Pi^D_{[\!]\,D})}"'] & & [A\vee C]\wedge[B\vee D] \\
& (A\wedge B)\vee(C\wedge D) \arrow[ur, "\mathsf{m}"'] &
\end{tikzcd}$$

and (2.22) and the self-duality of medial. It remains to show (iii). For this consider

$$\begin{tikzcd}
(A\wedge B)\vee(C\wedge D) \arrow[r, "\Delta_{(A\wedge B)\vee(C\wedge D)}"] \arrow[d, "\mathsf{m}_{A,B,C,D}"'] & [(A\wedge B)\vee(C\wedge D)]\wedge[(A\wedge B)\vee(C\wedge D)] \arrow[dl, "{[\Pi^B_{A\,[\!]}\vee\Pi^D_{C\,[\!]}]\wedge[\Pi^A_{[\!]\,B}\vee\Pi^C_{[\!]\,D}]}"] \arrow[d, "{(\Pi^C_{A\,[\!]}\circ\Pi^B_{A\,[\!]}\vee\Pi^A_{[\!]\,C}\circ\Pi^D_{C\,[\!]})\wedge(\Pi^D_{B\,[\!]}\circ\Pi^A_{[\!]\,B}\vee\Pi^B_{[\!]\,D}\circ\Pi^C_{[\!]\,D})}"] \\
[A\vee C]\wedge[B\vee D] \arrow[r, "{[\Pi^C_{A\,[\!]}\vee\Pi^A_{[\!]\,C}]\wedge[\Pi^D_{B\,[\!]}\vee\Pi^B_{[\!]\,D}]}"'] \arrow[d, "{[\Pi^A_{[\!]\,A}\vee\Pi^C_{C\,[\!]}]\wedge[\Pi^B_{[\!]\,B}\vee\Pi^D_{D\,[\!]}]}"'] & [A\vee C\vee A\vee C]\wedge[B\vee D\vee B\vee D] \arrow[d, "\nabla_{A\vee C}\wedge\nabla_{B\vee D}"] \\
[A\vee A\vee C\vee C]\wedge[B\vee B\vee D\vee D] \arrow[ur, "{[A\vee\check{\sigma}_{A,C}\vee C]\wedge[B\vee\check{\sigma}_{B,D}\vee D]}"] \arrow[r, "{[\nabla_A\vee\nabla_C]\wedge[\nabla_B\vee\nabla_D]}"'] & [A\vee C]\wedge[B\vee D]
\end{tikzcd}$$

The topmost triangle is (v), the two middle ones are trivial, and the bottommost triangle is (B2c) twice. Note that the first-right-then-down path is

$$\left\langle \left[\, \Pi^C_{A\,[\!]}\circ\Pi^B_{A\,[\!]}, \Pi^A_{[\!]\,C}\circ\Pi^D_{C\,[\!]} \,\right], \left[\, \Pi^D_{B\,[\!]}\circ\Pi^A_{[\!]\,B}, \Pi^B_{[\!]\,D}\circ\Pi^C_{[\!]\,D} \,\right] \right\rangle$$

by definition, and the first-down-then-right path is $\mathsf{m}_{A,B,C,D}$ because of (2.22). We get (iii) by self-duality of medial. □

Remark 2.6.3. Because of (iii) and (v) in Theorem 2.6.2, we could obtain a *weak medial map* by adding (iv) or (ii) as axiom to a B2-category. This weak medial map would be self-dual. By also adding Theorem 2.6.2 (i) as axiom, we could even recover equations (B3c) and (B3c′), as the following diagram shows:

$$\begin{tikzcd}
A\vee B \arrow[r, "\Delta_{A\vee B}"] \arrow[d, "\Delta_A\vee\Delta_B"'] & [A\vee B]\wedge[A\vee B] \arrow[r, equal] \arrow[dr, "{[\Delta_A\vee\Delta_B]\wedge[\Delta_A\vee\Delta_B]}"] & [A\vee B]\wedge[A\vee B] \\
(A\wedge A)\vee(B\wedge B) \arrow[rr, "\Delta_{(A\wedge A)\vee(B\wedge B)}"'] & & [(A\wedge A)\vee(B\wedge B)]\wedge[(A\wedge A)\vee(B\wedge B)] \arrow[u, "{[\Pi^A_{A\,[\!]}\vee\Pi^B_{B\,[\!]}]\wedge[\Pi^A_{[\!]\,A}\vee\Pi^B_{[\!]\,B}]}"']
\end{tikzcd}$$

where the left square says that $\Delta_A \vee \Delta_B$ preserves the $\wedge$-comultiplication. However, by doing this, we would *not* get naturality of medial, which is crucial for algebraic as well as for proof-theoretic reasons.

Definition 2.6.4. A B2-category $\mathscr{C}$ *has nullary medial* if there is a map $\check{\mathsf{nm}}\colon \mathbf{t} \vee \mathbf{t} \to \mathbf{t}$ (called the *nullary medial map*) such that for all objects A, B, the following holds:

$$\begin{tikzcd} & A \vee B \arrow[dl, "\Pi^A \vee \Pi^B"'] \arrow[dr, "\Pi^{A \vee B}"] & \\ \mathbf{t} \vee \mathbf{t} \arrow[rr, "\check{\mathsf{nm}}"'] & & \mathbf{t} \end{tikzcd} \tag{B3b}$$

Clearly, if a a B2-category has nullary medial, then $\check{\mathsf{nm}} = \Pi^{\mathbf{t} \vee \mathbf{t}}$. This can be seen by plugging in $\mathbf{t}$ for A and B in (B3b). By duality $\Pi^{\mathbf{f} \wedge \mathbf{f}} = \hat{\mathsf{nm}}\colon \mathbf{f} \to \mathbf{f} \wedge \mathbf{f}$ (the *nullary comedial map*) obeys the dual of (B3b).

Proposition 2.6.5. *In a* B2*-category* $\mathscr{C}$ *that has nullary medial, we have that*

(i) *The maps that preserve the* $\wedge$*-counit are closed under* $\vee$*, and dually, the maps that preserve the* $\vee$*-unit are closed under* $\wedge$*.*

(ii) *For all objects* A, B, C*, the map* $\mathsf{s}_{A,B,C}$ *is a quasientropy.*

Proof. For showing the first statement, replace in (2.31) every $\wedge$ by an $\vee$, and $\hat{\varrho}_{\mathbf{t}}$ by $\check{\mathsf{nm}}$. The second statement is shown by

$$\begin{tikzcd} [A \vee B] \wedge C \arrow[rr, "\mathsf{s}_{A,B,C}"] \arrow[d, "{[\Pi^A \vee \Pi^B] \wedge \Pi^C}"'] & & A \vee (B \wedge C) \arrow[d, "\Pi^A \vee (\Pi^B \wedge \Pi^C)"] \\ [\mathbf{t} \vee \mathbf{t}] \wedge \mathbf{t} \arrow[rr, "\mathsf{s}_{\mathbf{t},\mathbf{t},\mathbf{t}}"] \arrow[d, "\check{\mathsf{nm}} \wedge \mathbf{t}"'] \arrow[dr, "\hat{\varrho}_{\mathbf{t} \vee \mathbf{t}}"] & & \mathbf{t} \vee (\mathbf{t} \wedge \mathbf{t}) \arrow[dl, "\mathbf{t} \vee \hat{\varrho}_{\mathbf{t}}"] \\ \mathbf{t} \wedge \mathbf{t} \arrow[dr, "\hat{\varrho}_{\mathbf{t}}"'] & \mathbf{t} \vee \mathbf{t} \arrow[d, "\check{\mathsf{nm}}"] & \\ & \mathbf{t} & \end{tikzcd}$$

where the left down-path is $\Pi^{[A \vee B] \wedge C}$ and the right down-path is $\Pi^{A \vee (B \wedge C)}$ (because of (B2b) and (B3b)). The two squares are naturality of s and $\hat{\varrho}$, and the triangle at the center is just (2.9). Hence, switch preserves the $\wedge$-counit, and by duality also the $\vee$-unit. $\square$

Proposition 2.6.6. *Let* $\mathscr{C}$ *be a* B2*-category with medial and nullary medial. Then* $\mathscr{C}$ *obeys the equation*

$$\begin{tikzcd} (A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t}) \arrow[r, "\mathsf{m}_{A,\mathbf{t},B,\mathbf{t}}"] \arrow[d, "\hat{\varrho}_A \vee \hat{\varrho}_B"'] & [A \vee B] \wedge [\mathbf{t} \vee \mathbf{t}] \arrow[d, "{[A \vee B] \wedge \check{\mathsf{nm}}}"] \\ A \vee B \arrow[r, "\hat{\varrho}^{-1}_{A \vee B}"'] & [A \vee B] \wedge \mathbf{t} \end{tikzcd} \tag{m-$\hat{\varrho}$}$$

Proof. Chase

$$
\begin{tikzcd}
(A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t}) \arrow[rr, "\mathsf{m}_{A,\mathbf{t},B,\mathbf{t}}"] \arrow[dr, "\Delta_{(A\wedge\mathbf{t})\vee(B\wedge\mathbf{t})}"] \arrow[dd, "\hat{\varrho}_A \vee \hat{\varrho}_B"'] & & [A \vee B] \wedge [\mathbf{t} \vee \mathbf{t}] \arrow[dd, "{[A\vee B]\wedge\mathsf{n\check{m}}}"] \\
& [(A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t})] \wedge [(A \wedge \mathbf{t}) \vee (B \wedge \mathbf{t})] \arrow[ur, "{[\Pi^{\mathbf{t}}_{A[\,]} \vee \Pi^{\mathbf{t}}_{B[\,]}]\wedge[\Pi^{A}_{[\,]\mathbf{t}} \vee \Pi^{B}_{[\,]\mathbf{t}}]}"] \arrow[d, "{[\hat{\varrho}_A \vee \hat{\varrho}_B]\wedge[\hat{\varrho}_A \vee \hat{\varrho}_B]}"'] & \\
& [A \vee B] \wedge [A \vee B] \arrow[uur, "{[A\vee B]\wedge[\Pi^A \vee \Pi^B]}"] \arrow[dr, "{[A\vee B]\wedge\Pi^{A\vee B}}"] & \\
A \vee B \arrow[ur, "\Delta_{A\vee B}"] \arrow[rr, "\hat{\varrho}^{-1}_{A\vee B}"'] & & [A \vee B] \wedge \mathbf{t}
\end{tikzcd}
$$

The upper triangle is Theorem 2.6.2 (v), the lower triangle is the comonoid equation, the left square says that $\hat{\varrho}_A \vee \hat{\varrho}_B$ preserves the $\wedge$-comultiplication (Theorems 2.4.19 and 2.6.2 (i)), the triangle on the right is (B3b), and the triangle in the middle commutes because $\Pi^{\mathbf{t}}_{A[\,]} = \hat{\varrho}_A$ and $\Pi^{A}_{[\,]\mathbf{t}} = \Pi^A \circ \hat{\varrho}_A$, where the latter equation holds because of (2.21) and naturality of $\hat{\varrho}$. $\square$

Proposition 2.6.7. *In a* B2*-category with medial and nullary medial the following are equivalent:*

(i) *We have*

$$\Pi^{\mathbf{t}\vee\mathbf{t}} = \mathsf{n\check{m}} = \nabla_{\mathbf{t}} \colon \mathbf{t} \vee \mathbf{t} \to \mathbf{t} \tag{B3a}$$

(ii) *For all objects A, the map $\hat{\varrho}_A$ preserves the $\vee$-multiplication.*

Proof. Chasing the diagram

$$
\begin{tikzcd}
(A \wedge \mathbf{t}) \vee (A \wedge \mathbf{t}) \arrow[rrr, "\hat{\varrho}_A \vee \hat{\varrho}_A"] \arrow[dr, "\mathsf{m}_{A,\mathbf{t},B,\mathbf{t}}"] \arrow[ddd, "\nabla_{A\wedge\mathbf{t}}"'] & & & A \vee A \arrow[ddd, "\nabla_A"] \\
& [A \vee A] \wedge [\mathbf{t} \vee \mathbf{t}] \arrow[r, "{[A\vee A]\wedge\mathsf{n\check{m}}}"] \arrow[d, "\nabla_A \wedge [\mathbf{t}\vee\mathbf{t}]"] \arrow[ddl, "\nabla_A \wedge \nabla_{\mathbf{t}}"'] & [A \vee A] \wedge \mathbf{t} \arrow[ur, "\hat{\varrho}_{A\vee A}"] \arrow[d, "\nabla_A \wedge \mathbf{t}"] & \\
& A \wedge [\mathbf{t} \vee \mathbf{t}] \arrow[r, "A\wedge\mathsf{n\check{m}}"'] \arrow[dl, "A\wedge\nabla_{\mathbf{t}}"] & A \wedge \mathbf{t} \arrow[dr, "\hat{\varrho}_A"] & \\
A \wedge \mathbf{t} \arrow[rrr, "\hat{\varrho}_A"'] & & & A
\end{tikzcd}
$$

shows that in the presence of medial, nullary medial, and (B3a) the map $\hat{\varrho}_A$ preserves the $\vee$-multiplication. Note that in that diagram the uppermost square is (m-$\hat{\varrho}$) from the previous proposition. The lowermost square commutes because of (B3a), and the big left triangle is

(B3c′). Conversely, consider the diagram

$$\begin{tikzcd}
\mathbf{t}\vee\mathbf{t} \arrow[rrr, "\Delta_{\mathbf{t}\vee\mathbf{t}}"] \arrow[dr, "\Delta_{\mathbf{t}}\vee\Delta_{\mathbf{t}}=\hat{\varrho}_{\mathbf{t}}^{-1}\vee\hat{\varrho}_{\mathbf{t}}^{-1}"] \arrow[ddd, equal] \arrow[ddr, "\Pi^{\mathbf{t}}_{\mathbf{t}[]}\vee\Pi^{\mathbf{t}}_{[]\mathbf{t}}"'] & & & [\mathbf{t}\vee\mathbf{t}]\wedge[\mathbf{t}\vee\mathbf{t}] \arrow[ddd, "1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}}"] \\
& (\mathbf{t}\wedge\mathbf{t})\vee(\mathbf{t}\wedge\mathbf{t}) \arrow[urr, "\mathsf{m}_{\mathbf{t},\mathbf{t},\mathbf{t},\mathbf{t}}"] \arrow[r, "p"'] \arrow[ddr, "(\Pi^{\mathbf{t}}_{\mathbf{t}[]}\wedge\mathbf{t})\vee(\Pi^{\mathbf{t}}_{[]\mathbf{t}}\wedge\mathbf{t})"] & ([\mathbf{t}\vee\mathbf{t}]\wedge[\mathbf{t}\vee\mathbf{t}])\vee([\mathbf{t}\vee\mathbf{t}]\wedge[\mathbf{t}\vee\mathbf{t}]) \arrow[ur, "\nabla_{[\mathbf{t}\vee\mathbf{t}]\wedge[\mathbf{t}\vee\mathbf{t}]}"'] \arrow[d, "(1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}})\vee(1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}})"] & \\
& \mathbf{t}\vee\mathbf{t}\vee\mathbf{t}\vee\mathbf{t} \arrow[r, "\hat{\varrho}^{-1}_{\mathbf{t}\vee\mathbf{t}}\vee\hat{\varrho}^{-1}_{\mathbf{t}\vee\mathbf{t}}"'] \arrow[dl, "\nabla_{\mathbf{t}\vee\mathbf{t}}"'] & ([\mathbf{t}\vee\mathbf{t}]\wedge\mathbf{t})\vee([\mathbf{t}\vee\mathbf{t}]\wedge\mathbf{t}) \arrow[dr, "\nabla_{[\mathbf{t}\vee\mathbf{t}]\wedge\mathbf{t}}"] & \\
\mathbf{t}\vee\mathbf{t} \arrow[rrr, "\hat{\varrho}^{-1}_{\mathbf{t}\vee\mathbf{t}}"'] & & & [\mathbf{t}\vee\mathbf{t}]\wedge\mathbf{t}
\end{tikzcd}$$

where $p = [\Pi^{\mathbf{t}}_{\mathbf{t}[]}\vee\Pi^{\mathbf{t}}_{\mathbf{t}[]}]\wedge[\Pi^{\mathbf{t}}_{[]\mathbf{t}}\vee\Pi^{\mathbf{t}}_{[]\mathbf{t}}]$. The upper two triangles are (B3c) and Theorem 2.6.2 (v). The left triangle commutes because of Proposition 2.4.20 (iv), and the triangle at the center is the monoid equation. The triangle-shaped square is the naturality of $\hat{\varrho}$, and the rightmost square commutes because $1_{\mathbf{t}\vee\mathbf{t}}\wedge\nabla_{\mathbf{t}}$ preserves the $\vee$-multiplication, which follows from (the dual of) Proposition 2.4.16 (ii) and Theorem 2.6.2 (i). Finally, the lower square commutes because we assumed that $\hat{\varrho}_A$ preserved the $\vee$-multiplication. Note that the commutativity of the outer square says that $\nabla_{\mathbf{t}}$ is unit for $\Delta_{\mathbf{t}\vee\mathbf{t}}$. Therefore, by Proposition 2.4.3, we can conclude that $\mathsf{n}\check{\mathsf{m}} = \Pi^{\mathbf{t}\vee\mathbf{t}} = \nabla_{\mathbf{t}}$. □

Definition 2.6.8. A B3-*category* is a B2-category that obeys (B3a) and has medial and nullary medial.

Corollary 2.6.9. *In a* B3-*category, the maps* $\hat{\varrho}_A$, $\hat{\lambda}_A$, $\check{\varrho}_A$, *and* $\check{\lambda}_A$ *are clonable for all objects* A, *i.e, they preserve both the* $\vee$-*multiplication and the* $\wedge$-*comultiplication.*

Proof. Theorem 2.4.19 and Proposition 2.6.7 suffice to show that $\hat{\varrho}_A$ is clonable. For $\hat{\lambda}_A$ it is similar, and for $\check{\varrho}_A$ and $\check{\lambda}_A$ it follows by duality. □

It has first been observed by Lamarche [Lam07] that the presence of a natural and self-dual map $\mathsf{m}_{A,B,C,D}\colon (A\wedge B)\vee(C\wedge D)\to[A\vee C]\wedge[B\vee D]$ in a star-autonomous category induces two canonical maps $\mathsf{e}_1,\mathsf{e}_2\colon \mathbf{f}\to\mathbf{t}$, namely

$$\mathsf{e}_1\colon \mathbf{f}\xrightarrow{\check{\lambda}^{-1}_{\mathbf{f}}}\mathbf{f}\vee\mathbf{f}\xrightarrow{\hat{\varrho}^{-1}_{\mathbf{f}}\vee\hat{\lambda}^{-1}_{\mathbf{f}}}(\mathbf{f}\wedge\mathbf{t})\vee(\mathbf{t}\wedge\mathbf{f})\xrightarrow{\mathsf{m}_{\mathbf{f},\mathbf{t},\mathbf{t},\mathbf{f}}}[\mathbf{f}\vee\mathbf{t}]\wedge[\mathbf{t}\vee\mathbf{f}]\xrightarrow{\check{\lambda}_{\mathbf{t}}\wedge\check{\varrho}_{\mathbf{t}}}\mathbf{t}\wedge\mathbf{t}\xrightarrow{\hat{\varrho}_{\mathbf{t}}}\mathbf{t}$$

and

$$\mathsf{e}_2\colon \mathbf{f}\xrightarrow{\check{\lambda}^{-1}_{\mathbf{f}}}\mathbf{f}\vee\mathbf{f}\xrightarrow{\hat{\lambda}^{-1}_{\mathbf{f}}\vee\hat{\varrho}^{-1}_{\mathbf{f}}}(\mathbf{t}\wedge\mathbf{f})\vee(\mathbf{f}\wedge\mathbf{t})\xrightarrow{\mathsf{m}_{\mathbf{t},\mathbf{f},\mathbf{f},\mathbf{t}}}[\mathbf{t}\vee\mathbf{f}]\wedge[\mathbf{f}\vee\mathbf{t}]\xrightarrow{\check{\varrho}_{\mathbf{t}}\wedge\check{\lambda}_{\mathbf{t}}}\mathbf{t}\wedge\mathbf{t}\xrightarrow{\hat{\varrho}_{\mathbf{t}}}\mathbf{t}$$

which are both self-dual (while $\Pi^{\mathbf{f}}$ and $\Pi^{\mathbf{t}}$ are dual to each other). By adding sufficient structure one can enforce that $\mathsf{e}_1=\mathsf{e}_2$ and that this map has the properties of Theorem 2.3.1. In [Lam07], Lamarche shows how this can be done without the $\wedge$-comonoid and $\vee$-monoid structure for every object by using equation (m-$\hat{\sigma}$) that we will introduce in

Proposition 2.6.13. In our case the structure of a $\mathsf{B2}$-category is sufficient to obtain that $\mathsf{e}_1 = \mathsf{e}_2$. But for letting this map have the properties of Theorem 2.3.1, as it is the case with $\Pi^{\mathbf{f}}$ and $\amalg^{\mathbf{t}}$, we need all the structure of a $\mathsf{B3}$-category. Then we have the following:

Theorem 2.6.10. *In a* $\mathsf{B3}$*-category we have* $\Pi^{\mathbf{f}} = \mathsf{e}_1 = \mathsf{e}_2 = \amalg^{\mathbf{t}}$*, i.e., every* $\mathsf{B3}$*-category is single-mixed.*

Proof. We will first show that $\Pi^{\mathbf{f}} = \mathsf{e}_1$. For this, note that

$$\begin{aligned}
&(\check{\lambda}_{\mathbf{t}} \wedge \check{\varrho}_{\mathbf{t}}) \circ \mathsf{m}_{\mathbf{f},\mathbf{t},\mathbf{t},\mathbf{f}} \circ [\hat{\varrho}_{\mathbf{f}}^{-1} \vee \hat{\lambda}_{\mathbf{f}}^{-1}] \\
= \;& (\check{\lambda}_{\mathbf{t}} \wedge \check{\varrho}_{\mathbf{t}}) \circ \left\langle \left[\amalg^{\mathbf{t}}_{\mathbf{f}\,\mathbb{I}} \circ \Pi^{\mathbf{t}}_{\mathbf{f}\,\mathbb{I}}, \amalg^{\mathbf{f}}_{\mathbb{I}\,\mathbf{t}} \circ \Pi^{\mathbf{f}}_{\mathbf{t}\,\mathbb{I}} \right], \left[\amalg^{\mathbf{f}}_{\mathbf{t}\,\mathbb{I}} \circ \Pi^{\mathbf{f}}_{\mathbb{I}\,\mathbf{t}}, \amalg^{\mathbf{t}}_{\mathbb{I}\,\mathbf{f}} \circ \Pi^{\mathbf{t}}_{\mathbb{I}\,\mathbf{f}} \right] \right\rangle \circ [\hat{\varrho}_{\mathbf{f}}^{-1} \vee \hat{\lambda}_{\mathbf{f}}^{-1}] \\
= \;& \left\langle \left[\check{\lambda}_{\mathbf{t}} \circ \amalg^{\mathbf{t}}_{\mathbf{f}\,\mathbb{I}} \circ \Pi^{\mathbf{t}}_{\mathbf{f}\,\mathbb{I}} \circ \hat{\varrho}_{\mathbf{f}}^{-1}, \check{\lambda}_{\mathbf{t}} \circ \amalg^{\mathbf{f}}_{\mathbb{I}\,\mathbf{t}} \circ \Pi^{\mathbf{f}}_{\mathbf{t}\,\mathbb{I}} \circ \hat{\lambda}_{\mathbf{f}}^{-1} \right], \left[\check{\varrho}_{\mathbf{t}} \circ \amalg^{\mathbf{f}}_{\mathbf{t}\,\mathbb{I}} \circ \Pi^{\mathbf{f}}_{\mathbb{I}\,\mathbf{t}} \circ \hat{\varrho}_{\mathbf{f}}^{-1}, \check{\varrho}_{\mathbf{t}} \circ \amalg^{\mathbf{t}}_{\mathbb{I}\,\mathbf{f}} \circ \Pi^{\mathbf{t}}_{\mathbb{I}\,\mathbf{f}} \circ \hat{\lambda}_{\mathbf{f}}^{-1} \right] \right\rangle \\
= \;& \left\langle \left[\amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \right], \left[\Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \right] \right\rangle
\end{aligned}$$

The first equation is an application of Theorem 2.6.2 (iii), the second one uses Proposition 2.4.20 together with the fact that $\hat{\varrho}_{\mathbf{f}}$ and $\hat{\lambda}_{\mathbf{f}}$ preserve the $\wedge$-comultiplication (Theorem 2.4.19) and that these maps are closed under $\vee$ (Theorem 2.6.2 (i)). The third equation is an easy calculation, involving (2.20) and the naturality of $\hat{\varrho}$ and $\hat{\lambda}$. Before we proceed, notice that:

$$\amalg^{\mathbf{t}}_{\mathbb{I}\,\mathbf{t}} = \hat{\lambda}_{\mathbf{t}} = \hat{\varrho}_{\mathbf{t}} = \Pi^{\mathbf{t}}_{\mathbf{t}\,\mathbb{I}} \colon \mathbf{t} \wedge \mathbf{t} \to \mathbf{t} \qquad \text{and} \qquad \amalg^{\mathbf{f}}_{\mathbb{I}\,\mathbf{f}} = \hat{\lambda}_{\mathbf{f}}^{-1} = \hat{\varrho}_{\mathbf{f}}^{-1} = \Pi^{\mathbf{f}}_{\mathbf{f}\,\mathbb{I}} \colon \mathbf{f} \to \mathbf{f} \vee \mathbf{f} \tag{2.35}$$

Now we have:

$$\begin{aligned}
\mathsf{e}_1 &= \hat{\varrho}_{\mathbf{t}} \circ (\check{\lambda}_{\mathbf{t}} \wedge \check{\varrho}_{\mathbf{t}}) \circ \mathsf{m}_{\mathbf{f},\mathbf{t},\mathbf{t},\mathbf{f}} \circ [\hat{\varrho}_{\mathbf{f}}^{-1} \vee \hat{\lambda}_{\mathbf{f}}^{-1}] \circ \check{\lambda}_{\mathbf{f}}^{-1} \\
&= \hat{\varrho}_{\mathbf{t}} \circ \left\langle \left[\amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \right], \left[\Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \right] \right\rangle \circ \check{\lambda}_{\mathbf{f}}^{-1} \\
&= \hat{\varrho}_{\mathbf{t}} \circ \left\langle \left[\amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \right] \circ \check{\lambda}_{\mathbf{f}}^{-1}, \left[\Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \right] \circ \check{\lambda}_{\mathbf{f}}^{-1} \right\rangle \\
&= \Pi^{\mathbf{t}}_{\mathbf{t}\,\mathbb{I}} \circ \left\langle \left[\amalg^{\mathbf{t}}, \Pi^{\mathbf{f}} \right] \circ \amalg^{\mathbf{f}}_{\mathbb{I}\,\mathbf{f}}, \left[\Pi^{\mathbf{f}}, \amalg^{\mathbf{t}} \right] \circ \amalg^{\mathbf{f}}_{\mathbf{f}\,\mathbb{I}} \right\rangle \\
&= \Pi^{\mathbf{t}}_{\mathbf{t}\,\mathbb{I}} \circ \left\langle \Pi^{\mathbf{f}}, \Pi^{\mathbf{f}} \right\rangle \\
&= \Pi^{\mathbf{f}}
\end{aligned}$$

The first two equations are just the definition of e_1 and the previous calculation. The third equation uses Proposition 2.4.20 and the fact that $\check{\lambda}_{\mathbf{f}} = \check{\varrho}_{\mathbf{f}}$ preserves the $\wedge$-comultiplication (Corollary 2.6.9). The fourth equation applies (2.35), and the last two equations are again Proposition 2.4.20, together with the fact that $\amalg^{\mathbf{t}}$ preserves the $\vee$-unit and $\Pi^{\mathbf{f}}$ preserves the $\wedge$-counit (Theorem 2.4.19). Similarly, we show that $\mathsf{e}_2 = \Pi^{\mathbf{f}}$ and dually, we obtain $\mathsf{e}_1 = \mathsf{e}_2 = \amalg^{\mathbf{t}}$. □

Theorem 2.6.11. *In a* $\mathsf{B3}$*-category, the strong maps (in fact, all types of maps defined in Definition 2.4.4) are closed under* $\wedge$ *and* $\vee$*. Furthermore, the maps* $\mathsf{m}_{A,B,C,D}$ *and* $\check{\mathsf{m}}$ *and* $\hat{\mathsf{m}}$ *are strong.*

Proof. By Propositions 2.4.14 and 2.4.16, the $\wedge$-comonoid morphisms are closed under $\wedge$, and by Proposition 2.6.5 and Theorem 2.6.2 they are closed under $\vee$. Dually, the $\vee$-monoid morphisms are closed under $\vee$ and $\wedge$, and therefore also the strong maps have this property.

Since by Theorem 2.6.2 (v), medial is $([\Pi^B_{A\mathbb{I}} \vee \Pi^D_{C\mathbb{I}}] \wedge [\Pi^A_{\mathbb{I}B} \vee \Pi^C_{\mathbb{I}D}]) \circ \Delta_{(A\wedge B)\vee(C\wedge D)}$ as well as $\nabla_{[A\vee C]\wedge[B\vee D]} \circ [(\Pi^C_{A\mathbb{I}} \wedge \Pi^D_{B\mathbb{I}}) \vee (\Pi^A_{\mathbb{I}C} \wedge \Pi^B_{\mathbb{I}D})]$, we have by Theorem 2.4.19 that it is a $\wedge$-comonoid morphism and a $\vee$-monoid morphism, and therefore strong. Since $\mathsf{n\check{m}} = \Pi^{\mathbf{t}\vee\mathbf{t}} = \nabla_{\mathbf{t}}$, we get again from Theorem 2.4.19 that it is a $\wedge$-comonoid morphism and a $\vee$-monoid morphism. Similarly for $\mathsf{n\hat{m}} = \Pi^{\mathbf{f}\wedge\mathbf{f}} = \Delta_{\mathbf{f}}$. □

Proposition 2.6.12. *In a* B3*-category the maps* $\check{\alpha}_{A,B,C}$*,* $\check{\sigma}_{A,B}$*,* $\check{\lambda}_A$*, and* $\check{\varrho}_A$ *preserve the* $\wedge$*-counit for all objects* A, B, C*. Dually, the maps* $\hat{\alpha}_{A,B,C}$*,* $\hat{\sigma}_{A,B}$*,* $\hat{\lambda}_A$*, and* $\hat{\varrho}_A$ *all preserve the* $\vee$*-unit.*

Proof. As before, the cases for $\check{\alpha}_{A,B,C}$ and $\check{\sigma}_{A,B}$ are similar. This time, we show the case for $\check{\alpha}_{A,B,C}$:

$$\begin{tikzcd}
A \vee [B \vee C] \arrow[r, "\check{\alpha}_{A,B,C}"] \arrow[d, "\Pi^A\vee[\Pi^B\vee\Pi^C]"'] & {[A \vee B] \vee C} \arrow[d, "{[\Pi^A\vee\Pi^B]\vee\Pi^C}"] \\
\mathbf{t} \vee [\mathbf{t} \vee \mathbf{t}] \arrow[r, "\check{\alpha}_{\mathbf{t},\mathbf{t},\mathbf{t}}"'] \arrow[d, "\mathbf{t}\vee\nabla_{\mathbf{t}}"'] & {[\mathbf{t} \vee \mathbf{t}] \vee \mathbf{t}} \arrow[d, "\nabla_{\mathbf{t}}\vee\mathbf{t}"] \\
\mathbf{t} \vee \mathbf{t} \arrow[dr, "\nabla_{\mathbf{t}}"'] & \mathbf{t} \vee \mathbf{t} \arrow[dl, "\nabla_{\mathbf{t}}"] \\
& \mathbf{t}
\end{tikzcd}$$

The square is naturality of $\check{\alpha}$, and the pentagon is associativity of ∇. The left down path is $\Pi^{A\vee[B\vee C]}$ and the right down path is $\Pi^{[A\vee B]\vee C}$ (because of (B3b) and (B3a)). For $\check{\varrho}_A$, chase

$$\begin{tikzcd}
A \vee \mathbf{f} \arrow[rr, "\check{\varrho}_A"] \arrow[dr, "\Pi^A\vee\mathbf{f}"] \arrow[ddr, "\Pi^A\vee\Pi^{\mathbf{f}}"] \arrow[ddd, "\Pi^{A\vee\mathbf{f}}"'] & & A \arrow[ddd, "\Pi^A"] \\
& \mathbf{t} \vee \mathbf{f} \arrow[d, "\mathbf{t}\vee\Pi^{\mathbf{t}}"] \arrow[ddr, "\check{\varrho}_{\mathbf{t}}"] & \\
& \mathbf{t} \vee \mathbf{t} \arrow[dl, "\mathsf{n\check{m}}"'] \arrow[dr, "\nabla_{\mathbf{t}}"] & \\
\mathbf{t} \arrow[rr, equal] & & \mathbf{t}
\end{tikzcd}$$

The upper right quadrangle is naturality of $\check{\varrho}$. The leftmost triangle is (B3b). The one in the center next to it commutes because of functoriality of $\vee$ and $\Pi^{\mathbf{f}} = \Pi^{\mathbf{t}}$ (Theorem 2.6.10). The lower right triangle is the monoid equation and the triangle at the bottom is (B3a). The case for $\check{\lambda}_A$ is similar. □

Proposition 2.6.13. *In a* B3*-category the following are equivalent:*

(i) *The equation*

$$\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[r, "\hat{\sigma}_{A,B} \vee \hat{\sigma}_{C,D}"] \arrow[d, "\mathsf{m}_{A,B,C,D}"'] & (B \wedge A) \vee (D \wedge C) \arrow[d, "\mathsf{m}_{B,A,D,C}"] \\
{[A \vee C] \wedge [B \vee D]} \arrow[r, "\hat{\sigma}_{A \vee C, B \vee D}"'] & {[B \vee D] \wedge [A \vee C]}
\end{tikzcd} \tag{m-$\hat{\sigma}$}$$

holds for all objects A, B, C, and D.

(ii) *The map $\hat{\sigma}_{A,B} \colon A \wedge B \to B \wedge A$ preserves the $\vee$-multiplication.*

(iii) *The map $\check{\sigma}_{A,B} \colon A \vee B \to B \vee A$ preserves the $\wedge$-comultiplication.*

(iv) *The equation*

$$\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[r, "\check{\sigma}_{A \wedge B, C \wedge D}"] \arrow[d, "\mathsf{m}_{A,B,C,D}"'] & (C \wedge D) \vee (A \wedge B) \arrow[d, "\mathsf{m}_{C,D,A,B}"] \\
{[A \vee C] \wedge [B \vee D]} \arrow[r, "\check{\sigma}_{A,C} \wedge \check{\sigma}_{B,D}"'] & {[C \vee A] \wedge [D \vee B]}
\end{tikzcd} \tag{m-$\check{\sigma}$}$$

holds for all objects A, B, C, and D.

Proof. Suppose (m-$\hat{\sigma}$) does hold. Then we have

$$\begin{tikzcd}
(A \wedge B) \vee (A \wedge B) \arrow[r, "\hat{\sigma}_{A,B} \vee \hat{\sigma}_{A,B}"] \arrow[d, "\mathsf{m}_{A,B,A,B}"'] & (B \wedge A) \vee (B \wedge A) \arrow[d, "\mathsf{m}_{B,A,B,A}"] \\
{[A \vee A] \wedge [B \vee B]} \arrow[r, "\hat{\sigma}_{A \vee A, B \vee B}"] \arrow[d, "\nabla_A \wedge \nabla_B"'] & {[B \vee B] \wedge [A \vee A]} \arrow[d, "\nabla_B \wedge \nabla_A"] \\
A \wedge B \arrow[r, "\hat{\sigma}_{A,B}"'] & B \wedge A
\end{tikzcd}$$

which together with ($\mathsf{B3c}'$) says that $\hat{\sigma}_{A,B}$ preserves the $\vee$-multiplication. Conversely, we have

$$\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[rr, "\hat{\sigma}_{A,B} \vee \hat{\sigma}_{C,D}"] \arrow[d, "(\mathrm{II}^C_{A[\!]} \wedge \mathrm{II}^D_{B[\!]}) \vee (\mathrm{II}^A_{[\!]C} \wedge \mathrm{II}^B_{[\!]D})"'] & & (B \wedge A) \vee (D \wedge C) \arrow[d, "(\mathrm{II}^D_{B[\!]} \wedge \mathrm{II}^C_{A[\!]}) \vee (\mathrm{II}^B_{[\!]D} \wedge \mathrm{II}^A_{[\!]C})"] \\
{([A \vee C] \wedge [B \vee D]) \vee ([A \vee C] \wedge [B \vee D])} \arrow[r, "\hat{\sigma}_{A \vee C, B \vee D} \vee \hat{\sigma}_{A \vee C, B \vee D}"] \arrow[d, "\nabla_{[A \vee C] \wedge [B \vee D]}"'] & {([B \vee D] \wedge [A \vee C]) \vee ([B \vee D] \wedge [A \vee C])} \arrow[d, "\nabla_{[B \vee D] \wedge [A \vee C]}"] \\
{[A \vee C] \wedge [B \vee D]} \arrow[r, "\hat{\sigma}_{A \vee C, B \vee D}"'] & {[B \vee D] \wedge [A \vee C]}
\end{tikzcd}$$

The upper square is naturality of $\hat{\sigma}$, and the lower square says that $\hat{\sigma}_{A,B}$ preserves the $\vee$-multiplication. Together with Theorem 2.6.2 (v), this is (m-$\hat{\sigma}$). Hence (i) and (ii) are equivalent. The other equivalences follow because of duality. □

Proposition 2.6.14. *In a* $\mathsf{B3}$*-category the following are equivalent:*

(i) *The equation*

$$\begin{tikzcd}
(A \wedge (B \wedge C)) \vee (D \wedge (E \wedge F)) \arrow[r, "\hat{\alpha}_{A,B,C} \vee \hat{\alpha}_{D,E,F}"] \arrow[d, "\mathsf{m}_{A,B\wedge C,D,E\wedge F}"'] & ((A \wedge B) \wedge C) \vee ((D \wedge E) \wedge F) \arrow[d, "\mathsf{m}_{A\wedge B,C,D\wedge E,F}"] \\
{[A \vee D] \wedge [(B \wedge C) \vee (E \wedge F)]} \arrow[d, "{[A\vee D]\wedge \mathsf{m}_{B,C,E,F}}"'] & {[(A \wedge B) \vee (D \wedge E)] \wedge [C \vee F]} \arrow[d, "\mathsf{m}_{A,B,D,E}\wedge[C\vee F]"] \\
{[A \vee D] \wedge ([B \vee E] \wedge [C \vee F])} \arrow[r, "\hat{\alpha}_{A\vee D,B\vee E,C\vee F}"'] & {([A \vee D] \wedge [B \vee E]) \wedge [C \vee F]}
\end{tikzcd} \tag{m-$\hat{\alpha}$}$$

holds for all objects A, B, C, D, E, and F.

(ii) *The map $\hat{\alpha}_{A,B,C} \colon A \wedge (B \wedge C) \to (A \wedge B) \wedge C$ preserves the $\vee$-multiplication.*

Proof. Similar to the previous proposition. (Here the statements corresponding to (iii) and (iv) in Proposition 2.6.13 are omitted to save space, but obviously they hold accordingly.) □

Remark 2.6.15. This proposition allows us to speak of uniquely defined maps

$$\hat{\mathsf{m}}^2_{A,B,C,D,E,F} \colon (A \wedge B \wedge C) \vee (D \wedge E \wedge F) \to [A \vee D] \wedge [B \vee E] \wedge [C \vee F]$$

and dually

$$\check{\mathsf{m}}^2_{A,B,C,D,E,F} \colon (A \wedge B) \vee (C \wedge D) \vee (E \wedge F) \to [A \vee C \vee E] \wedge [B \vee D \vee F]$$

A more sophisticated and more general notation for composed variations of medial is introduced by Lamarche in [Lam07].

Proposition 2.6.16. *In a a* $\mathsf{B3}$*-category obeying* (m-$\hat{\sigma}$) *and* (m-$\hat{\alpha}$) *the following are equivalent:*

(i) *The equation*

$$\begin{tikzcd}
{[(A \wedge B) \vee (C \wedge D)] \wedge E} \arrow[r, "\mathsf{s}_{A\wedge B,C\wedge D,E}"] \arrow[d, "\mathsf{m}_{A,B,C,D}\wedge E"'] & (A \wedge B) \vee (C \wedge D \wedge E) \arrow[d, "\mathsf{m}_{A,B,C,D\wedge E}"] \\
{[A \vee C] \wedge [B \vee D] \wedge E} \arrow[r, "{[A\vee C]\wedge \mathsf{s}_{B,D,E}}"'] & {[A \vee C] \wedge [B \vee (D \wedge E)]}
\end{tikzcd} \tag{m-s}$$

holds for all objects A, B, C, D, and E.

(ii) *The map $\mathsf{s}_{A,B,C} \colon A \wedge [B \vee C] \to (A \wedge B) \vee C$ preserves the $\wedge$-comultiplication.*

Proof. First note that if the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s) hold, we can compose them to get the commutativity of diagrams like

$$\begin{tikzcd}
{[(A \wedge B) \vee (C \wedge D)] \wedge E \wedge F} \arrow[r] \arrow[d, "\mathsf{m}_{A,B,C,D}\wedge E\wedge F"'] & (A \wedge B) \vee (C \wedge E \wedge D \wedge F) \arrow[d, "\mathsf{m}_{A,B,C\wedge E,D\wedge F}"] \\
{[A \vee C] \wedge [B \vee D] \wedge E \wedge F} \arrow[r] & {[A \vee (C \wedge E)] \wedge [B \vee (D \wedge F)]}
\end{tikzcd} \tag{2.36}$$

where the horizontal maps are the canonical maps (composed of twist, associativity, and switch) that are uniquely determined by the star-autonomous structure. Now chase

$$\begin{tikzcd}
{[A \vee B] \wedge C} \arrow[rr, "\mathsf{s}_{A,B,C}"] \arrow[d, "{[\Delta_A \vee \Delta_B] \wedge \Delta_C}"'] & & A \vee (B \wedge C) \arrow[d, "\Delta_A \vee (\Delta_B \wedge \Delta_C)"] \\
{[(A \wedge A) \vee (B \wedge B)] \wedge C \wedge C} \arrow[rr, "\mathsf{s}_{A\wedge A, B\wedge B, C\wedge C}"] \arrow[d, "\mathsf{m}_{A,A,B,B} \wedge C \wedge C"'] \arrow[drr] & & (A \wedge A) \vee (B \wedge B \wedge C \wedge C) \arrow[d, "(A\wedge A) \vee (B \wedge \hat{\sigma}_{B,C} \wedge C)"] \\
{[A \vee B] \wedge [A \vee B] \wedge C \wedge C} \arrow[d, "{[A \vee B] \wedge \hat{\sigma}_{A \vee B, C} \wedge C}"'] \arrow[drr] & & (A \wedge A) \vee (B \wedge C \wedge B \wedge C) \arrow[d, "\mathsf{m}_{A,A,B\wedge C, B \wedge C}"] \\
{[A \vee B] \wedge C \wedge [A \vee B] \wedge C} \arrow[rr, "\mathsf{s}_{A,B,C} \wedge \mathsf{s}_{A,B,C}"'] & & {[A \vee (B \wedge C)] \wedge [A \vee (B \wedge C)]}
\end{tikzcd}$$

where the parallelogram is just (2.36), the upper square is naturality of switch and the two triangles are laws of star-autonomous categories. Note that, by (B2c) and (B3c), the vertical paths are just $\Delta_{[A \vee B] \wedge C}$ and $\Delta_{A \vee (B \wedge C)}$. Therefore switch preserves the $\wedge$-comultiplication. Conversely, consider the diagram

$$\begin{tikzcd}
{[(A \wedge B) \vee (C \wedge D)] \wedge E} \arrow[rr, "\mathsf{s}_{A\wedge B, C \wedge D, E}"] \arrow[d, "\Delta_{[(A\wedge B) \vee (C \wedge D)] \wedge E}"'] & & {(A \wedge B) \vee (C \wedge D \wedge E)} \arrow[d, "\Delta_{(A \wedge B) \vee (C \wedge D \wedge E)}"] \\
{[(A \wedge B) \vee (C \wedge D)] \wedge E \wedge [(A \wedge B) \vee (C \wedge D)] \wedge E} \arrow[r, "\mathsf{s} \wedge \mathsf{s}"] \arrow[d, "p"'] & {[(A \wedge B) \vee (C \wedge D \wedge E)] \wedge [(A \wedge B) \vee (C \wedge D \wedge E)]} \arrow[d, "q"] \\
{[(A \wedge \mathbf{t}) \vee (C \wedge \mathbf{t})] \wedge \mathbf{t} \wedge [(\mathbf{t} \wedge B) \vee (\mathbf{t} \wedge D)] \wedge E} \arrow[r, "\mathsf{s} \wedge \mathsf{s}"] \arrow[d, "\cong"'] & {[(A \wedge \mathbf{t}) \vee (C \wedge \mathbf{t} \wedge \mathbf{t})] \wedge [(\mathbf{t} \wedge B) \vee (\mathbf{t} \wedge D \wedge E)]} \arrow[d, "\cong"] \\
{[A \vee C] \wedge [B \vee C] \wedge E} \arrow[rr, "{[A \vee C] \wedge \mathsf{s}_{B,D,E}}"'] & & {[A \vee C] \wedge [B \vee (D \wedge E)]}
\end{tikzcd}$$

where

$$\begin{aligned}
p &= [(A \wedge \Pi^B) \vee (C \wedge \Pi^D)] \wedge \Pi^E \wedge [(\Pi^A \wedge B) \vee (\Pi^C \wedge D)] \wedge E \\
q &= [(A \wedge \Pi^B) \vee (C \wedge \Pi^D \wedge \Pi^E)] \wedge [(\Pi^A \wedge B) \vee (\Pi^C \wedge D \wedge E)]
\end{aligned}$$

Note that the left vertical map is $\mathsf{m}_{A,B,C,D} \wedge 1_E$ while the right vertical map is $\mathsf{m}_{A,B,C,D \wedge E}$. The upper square commutes because we assumed that switch preserves the $\wedge$-comultiplication, the middle one is naturality of switch, and the lower one commutes because the category is star-autonomous (the isomorphisms are just compositions of $\hat{\varrho}$ and $\hat{\lambda}$). $\square$

Definition 2.6.17. A B4-*category* is a B3-category that obeys the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s).

Remark 2.6.18. Equivalently, one can define a B4-category as a B3-category in which $\hat{\sigma}$, $\hat{\alpha}$, and s are strong. We have chosen the form of Definition (2.6.17) to show the resemblance to the work [Lam07] where the equations (m-$\hat{\sigma}$), (m-$\hat{\alpha}$), and (m-s) are also considered as primitives.

Theorem 2.6.19. *In a* $\mathsf{B4}$*-category, the maps* $\hat{\alpha}_{A,B,C}$*,* $\hat{\sigma}_{A,B}$*,* $\hat{\varrho}_A$*,* $\hat{\lambda}_A$ *and* $\check{\alpha}_{A,B,C}$*,* $\check{\sigma}_{A,B}$*,* $\check{\varrho}_A$*,* $\check{\lambda}_A$*, as well as* $\mathsf{s}_{A,B,C}$ *and* $\mathsf{mix}_{A,B}$ *are all strong.*

Proof. That $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, $\hat{\lambda}_A$ and $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, $\check{\lambda}_A$ are quasientropies follows from Theorem 2.4.19 and Proposition 2.6.12. That $\hat{\varrho}_A$, $\hat{\lambda}_A$ and $\check{\varrho}_A$, $\check{\lambda}_A$ are clonable has been said already in Corollary 2.6.9. For $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$ and $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$ this follows from Theorem 2.4.19 and Propositions 2.6.13 and 2.6.14 (and by duality). Hence, all these maps are strong. That $\mathsf{s}_{A,B,C}$ is strong follows from Proposition 2.6.5 and Proposition 2.6.16 (and self-duality of switch). For showing that $\mathsf{mix}_{A,B}$ is also strong it suffices to observe that mix is a composition of strong maps via $\circ$, $\wedge$, and $\vee$; see (2.12), Theorem 2.6.10, and Theorem 2.6.11. $\square$

Remark 2.6.20. Theorem 2.6.19 gives justification to the algebraic concern raised in Remark 2.4.6. In a $\mathsf{B4}$-category all isomorphisms that are imposed by the $\mathsf{B4}$-structure do preserve the $\vee$-monoid and $\wedge$-comonoid structure (and are therefore "proper isomorphisms"). Note that there might still be "improper isomorphisms" in a $\mathsf{B4}$-category. But these live outside the $\mathsf{B4}$-structure and are therefore not accessible to proof-theoretic investigations.

It has first been observed by Lamarche in [Lam07] that the equation (m-mix) (see below) is a consequence of the equations (m-$\hat{\alpha}$), (m-$\hat{\sigma}$), and (m-s). Due to the presence of the $\vee$-monoids and $\wedge$-comonoids, we can give here a simpler proof of that fact:

Corollary 2.6.21. *In a* $\mathsf{B4}$*-category, the diagram*

$$\begin{tikzcd} A \wedge B \wedge C \wedge D \arrow[r, "A \wedge \hat{\sigma}_{B,C} \wedge D"] \arrow[d, "\mathsf{mix}_{A \wedge B, C \wedge D}"'] & A \wedge C \wedge B \wedge D \arrow[d, "\mathsf{mix}_{A,C} \wedge \mathsf{mix}_{B,D}"] \\ (A \wedge B) \vee (C \wedge D) \arrow[r, "\mathsf{m}_{A,B,C,D}"'] & [A \vee C] \wedge [B \vee D] \end{tikzcd} \qquad (\mathsf{m}\text{-}\mathsf{mix})$$

commutes.

Proof. Chase

$$
\begin{tikzcd}
A \wedge B \wedge C \wedge D \arrow[rrr, "A \wedge \hat{\sigma}_{B,C} \wedge D"] \arrow[dr, "\Delta_A \wedge \Delta_B \wedge \Delta_C \wedge \Delta_D"'] \arrow[ddd, equal] & & & A \wedge C \wedge B \wedge D \arrow[ddl, "\Delta_A \wedge \Delta_C \wedge \Delta_B \wedge \Delta_D"] \arrow[ddd, equal] \\
& A \wedge A \wedge B \wedge B \wedge C \wedge C \wedge D \wedge D \arrow[d, "A \wedge A \wedge \hat{\sigma}_{B \wedge B, C \wedge C} \wedge D \wedge D"'] & & \\
& A \wedge A \wedge C \wedge C \wedge B \wedge B \wedge D \wedge D \arrow[d, "\cong"'] \arrow[drr, "\Pi^A_{\llbracket A} \wedge \Pi^C_{\llbracket C} \wedge \Pi^B_{B\rrbracket} \wedge \Pi^D_{D\rrbracket}"] & & \\
A \wedge B \wedge C \wedge D \arrow[r, "\Delta_{A \wedge B \wedge C \wedge D}"] \arrow[dd, "\mathsf{mix}"'] & A \wedge B \wedge C \wedge D \wedge A \wedge B \wedge C \wedge D \arrow[d, "\mathsf{mix}_{A \wedge B, C \wedge D} \wedge \mathsf{mix}_{A \wedge B, C \wedge D}"'] \arrow[rr, "(\Pi^B_{A\rrbracket} \wedge \Pi^D_{C\rrbracket}) \wedge \Pi^A_{\llbracket B} \wedge \Pi^C_{\llbracket D}"'] & & A \wedge C \wedge B \wedge D \arrow[dd, "\mathsf{mix} \wedge \mathsf{mix}"] \\
& [(A \wedge B) \vee (C \wedge D)] \wedge [(A \wedge B) \vee (C \wedge D)] \arrow[dr, "[\Pi^B_{A\rrbracket} \vee \Pi^D_{C\rrbracket}] \wedge [\Pi^A_{\llbracket B} \vee \Pi^C_{\llbracket D}]"] & & \\
(A \wedge B) \vee (C \wedge D) \arrow[ur, "\Delta_{(A \wedge B) \vee (C \wedge D)}"] \arrow[rrr, "\mathsf{m}_{A,B,C,D}"'] & & & [A \vee C] \wedge [B \vee D]
\end{tikzcd}
$$

The topmost quadrangle commutes because of naturality of $\hat{\sigma}$. The pentagon below consists of several applications of (B2c). The two triangles on the right are trivial. The quadrangle on the lower left commutes because mix preserves the $\wedge$-comultiplication, and the quadrangle on the lower right because of naturality of mix. Finally, the triangle on the bottom is Theorem 2.6.2 (v). $\square$

Obviously one can come up with more diagrams like (m-mix) or (m-$\hat{\varrho}$) and ask whether they commute, for example the following due to McKinley [McK05b]:

$$
\begin{tikzcd}
(A \wedge \mathbf{f}) \vee (B \wedge C) \arrow[rr, "\mathsf{m}_{A,\mathbf{f},B,C}"] \arrow[d, "(A \wedge \Pi^{\mathbf{f}}) \vee (B \wedge C)"'] & & [A \vee B] \wedge [\mathbf{f} \vee C] \arrow[d, "[A \vee B] \wedge \check{\lambda}_C"] \\
(A \wedge \mathbf{t}) \vee (B \wedge C) \arrow[dr, "\hat{\varrho}_A \vee (B \wedge C)"'] & & [A \vee B] \wedge C \arrow[dl, "\mathsf{s}_{A,B,C}"] \\
& A \vee (B \wedge C) &
\end{tikzcd}
\tag{2.37}
$$

It was soon discovered independently by several people that (2.37) is equivalent to:

$$
\begin{tikzcd}
(A \wedge B) \vee (C \wedge D) \arrow[d, "\mathsf{m}_{A,B,C,D}"'] \arrow[dr, "\mathsf{mix}_{A,B} \vee (C \wedge D)"] & \\
[A \vee C] \wedge [B \vee D] \arrow[r, "\hat{\mathsf{t}}_{A,C,B,D}"'] & A \vee B \vee (C \wedge D)
\end{tikzcd}
\tag{mix-m-$\hat{\mathsf{t}}$}
$$

Here are two other examples that do not contain the units:

$$\begin{tikzcd}
{[(A\wedge B)\vee(C\wedge D)]\wedge[E\vee F]} \arrow[r, "\mathsf{m}_{A,B,C,D\wedge(E\vee F)}"] \arrow[d, "\mathsf{s}_{A\wedge B,C\wedge D,E\wedge F}"'] & {[A\vee C]\wedge[B\vee D]\wedge[E\vee F]} \arrow[d, "\check{\mathsf{t}}_{A\vee C,B\vee D,E,F}"] \\
{(A\wedge B)\vee(C\wedge D\wedge[E\vee F])} \arrow[d, "(A\wedge B)\vee\check{\mathsf{t}}_{C,D,E,F}"'] & {([A\vee C]\wedge F)\vee(E\wedge[B\vee D])} \arrow[d, "\mathsf{m}_{A\vee C,F,E,B\vee D}"] \\
{(A\wedge B)\vee(C\wedge F)\vee(E\wedge D)} \arrow[r, "\check{\mathsf{m}}^2_{A,B,C,F,E,D}"'] & {[A\vee C\vee E]\wedge[F\vee B\vee D]}
\end{tikzcd} \tag{m-$\check{\mathsf{t}}$-s}$$

$$\begin{tikzcd}
& {[A'\vee A]\wedge[B'\vee B]\wedge[C'\vee C]\wedge[D'\vee D]} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
{([A'\vee B']\wedge[C'\vee D'])\vee(D\wedge C)\vee(B\wedge A)} \arrow[d, "\check{\mathsf{m}}^2_{A'\vee B',C'\vee D',D,C,B,A}"'] & & {([A'\vee A]\wedge[B'\vee C])\vee(B\wedge D')\vee(D\vee C')} \arrow[d, "\check{\mathsf{m}}^2_{A'\vee A,B'\vee C,B,D',D,C'}"] \\
{[A'\vee B'\vee B\vee D]\wedge[D'\vee C'\vee C\vee A]} \arrow[dr, "\hat{\mathsf{t}}_{A'\vee B',B\vee D,D'\vee C',C\vee A}"'] & & {[A'\vee A\vee B\vee D]\wedge[D'\vee C'\vee B'\vee C]} \arrow[dl, "\hat{\mathsf{t}}_{A'\vee A,B\vee D,D'\vee C',B'\vee C}"] \\
& {A'\vee B'\vee([B\vee D]\wedge[D'\vee C'])\vee C\vee A} &
\end{tikzcd} \tag{$\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$}$$

where p and q are the canonical maps (composed of several switches, twists, and associativity) that are determined by the star-autonomous structure. We can easily show the following proposition.

Proposition 2.6.22. *In every* B4*-category*

(i) *the equation* (2.37) *holds if and only if equation* (mix-m-$\hat{\mathsf{t}}$) *holds, and*

(ii) *the equation* (2.24) *holds if and only if equation* (m-$\check{\mathsf{t}}$-s) *holds.*

Proof. This is not needed later, and I leave the proof as an exercise to the reader. □

Definition 2.6.23. A B5*-category* is a B4-category that obeys equations (mix-m-$\hat{\mathsf{t}}$), (m-$\check{\mathsf{t}}$-s), and ($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$) for all objects.

The motivation for this definition is the following lemma which will be needed in the next section.

Lemma 2.6.24. *In a* B5*-category the following equation holds for all objects A, A', B, B',*

C, C', D, and D':

$$
\begin{tikzcd}
 & {[A' \vee A] \wedge [B' \vee B] \wedge [C' \vee C] \wedge [D' \vee D]} \arrow[dl, "{(A' \vee A) \wedge \hat{\mathsf{t}}_{B',B,C',C} \wedge (D' \vee D)}"'] \arrow[dr, "{\hat{\mathsf{t}}_{A',A,B',B} \wedge \hat{\mathsf{t}}_{C',C,D',D}}"] & \\
{[A' \vee A] \wedge [B' \vee (B \wedge C') \vee C] \wedge [D' \vee D]} \arrow[d, "{\mathsf{mix}_{A' \vee A, B' \vee (B \wedge C') \vee C} \wedge [D' \vee D]}"'] & & {[A' \vee B' \vee (A \wedge B)] \wedge [C' \vee D' \vee (C \wedge D)]} \arrow[d, "{\hat{\mathsf{t}}_{A' \vee B', A \wedge B, C' \vee D', C \wedge D}}"] \\
{[A' \vee B' \vee (B \wedge C') \vee C \vee A] \wedge [D' \vee D]} \arrow[d, "{\mathsf{s}_{A' \vee B' \vee C \vee A, B \wedge C', D' \vee D}}"'] & & {([A' \vee B'] \wedge [C' \vee D']) \vee (D \wedge C) \vee (B \wedge A)} \arrow[d, "{([A' \vee B'] \wedge [C' \vee D']) \vee \mathsf{m}_{D,C,B,A}}"] \\
{A' \vee B' \vee (B \wedge [D' \vee D] \wedge C') \vee C \vee A} \arrow[d, "{A' \vee B' \vee \check{\mathsf{t}}_{B,D',D,C'} \vee C \vee A}"'] & & {([A' \vee B'] \wedge [C' \vee D']) \vee ([D \vee B] \wedge [C \vee A])} \arrow[d, "{\mathsf{m}_{A' \vee B', C' \vee D', D \vee B, C \vee A}}"] \\
{A' \vee B' \vee (B \wedge D') \vee (D \wedge C') \vee C \vee A} \arrow[dr, "{A' \vee B' \vee \mathsf{m}_{B,D',D,C'} \vee C \vee A}"'] & & {[A' \vee B' \vee B \vee D] \wedge [D' \vee C' \vee C \vee A]} \arrow[dl, "{\hat{\mathsf{t}}_{A' \vee B', B \vee D, D' \vee C', C \vee A}}"] \\
 & {A' \vee B' \vee ([B \vee D] \wedge [D' \vee C']) \vee C \vee A} &
\end{tikzcd}
$$

Proof. Chase the following diagram:

$$
\begin{tikzcd}
{[A' \vee A] \wedge [B' \vee (B \wedge C') \vee C] \wedge [D' \vee D]} \arrow[dd, "\mathsf{mix}"'] \arrow[dr, "\mathsf{s}"] & {[A' \vee A] \wedge [B' \vee B] \wedge [C' \vee C] \wedge [D' \vee D]} \arrow[l, "\hat{\mathsf{t}}"'] \arrow[r, "\hat{\mathsf{t}} \wedge \hat{\mathsf{t}}"] & {[A' \vee B' \vee (A \wedge B)] \wedge [C' \vee D' \vee (C \wedge D)]} \arrow[dd, "\hat{\mathsf{t}}"] \\
 & {[([A' \vee A] \wedge [B' \vee C]) \vee (B \wedge C')] \wedge [D' \vee D]} \arrow[r, "\mathsf{s}"] \arrow[dl, "\mathsf{mix}"'] \arrow[dd, "\mathsf{m}"] & {([A' \vee A] \wedge [B' \vee C]) \vee (B \wedge C' \wedge [D' \vee D])} \arrow[dd, "\check{\mathsf{t}}"'] \\
{[A' \vee B' \vee (B \wedge C') \vee C \vee A] \wedge [D' \vee D]} \arrow[dd, "\mathsf{s}"'] & & {([A' \vee B'] \wedge [C' \vee D']) \vee (D \wedge C) \vee (B \wedge A)} \arrow[dd, "\mathsf{m}"] \\
 & {[A' \vee A \vee B] \wedge [B' \vee C \vee C'] \wedge [D' \vee D]} \arrow[ul, "\hat{\mathsf{t}}"] \arrow[dd, "\check{\mathsf{t}}"] & {([A' \vee A] \wedge [B' \vee C]) \vee (B \wedge D') \vee (C' \wedge D)} \arrow[dd, "\check{\mathsf{m}}^2"'] \\
{A' \vee B' \vee (B \wedge [D' \vee D] \wedge C') \vee C \vee A} \arrow[dd, "\check{\mathsf{t}}"'] & & {([A' \vee B'] \wedge [C' \vee D']) \vee ([B \vee D] \wedge (C \wedge A))} \arrow[dd, "\mathsf{m}"] \\
 & {([A' \vee A \vee B] \wedge D') \vee (D \wedge [B' \vee C \vee C'])} \arrow[r, "\mathsf{m}"] \arrow[dl, "\mathsf{s} \vee \mathsf{s}"'] & {[A' \vee A \vee B \vee D] \wedge [D' \vee B' \vee C \vee C']} \arrow[dl, "\hat{\mathsf{t}}"] \\
{A' \vee B' \vee (B \wedge D') \vee (D \wedge C') \vee C \vee A} \arrow[r, "\mathsf{m}"'] & {A' \vee B' \vee ([B \vee D] \wedge [D' \vee C']) \vee C \vee A} & {[A' \vee B' \vee B \vee D] \wedge [D' \vee C' \vee C \vee A]} \arrow[l, "\hat{\mathsf{t}}"]
\end{tikzcd}
$$

The little triangle in the upper left commutes because of (mix-$\hat{\alpha}$). The little triangle below it is just (mix-m-$\hat{\mathsf{t}}$), and the pentagon below commutes because of the coherence in star-autonomous categories[9] [BCST96, LS06]. The big square in the center is (m-$\check{\mathsf{t}}$-s) and the

[9] It even commutes in the setting of weakly distributive categories.

small parallelogram at the bottom is just two applications of (m-s) plugged together, and the big horse-shoe shape on the left is ($\check{\mathsf{m}}^2$-s-$\check{\mathsf{m}}^2$). □

2.7 Beyond medial

The definition of monoidal categories settles how the maps $\hat{\alpha}_{A,B,C}$, $\hat{\sigma}_{A,B}$, $\hat{\varrho}_A$, and $\hat{\lambda}_A$ behave with respect to each other, and how the maps $\check{\alpha}_{A,B,C}$, $\check{\sigma}_{A,B}$, $\check{\varrho}_A$, and $\check{\lambda}_A$ behave with respect to each other. The notion of star-autonomous category then settles via the bijection ($\star$) how the two monoidal structures interact. Then, the structure of a B1-category adds $\vee$-monoids and $\wedge$-comonoids, and the structure of B2-categories allows the $\vee$-monoidal structure to go well with the $\vee$-monoids and the $\wedge$-monoidal structure to go well with the $\wedge$-comonoids. Finally, the structure of B4-categories ensures that *both* monoidal structures go well with the $\vee$-monoids *and* the $\wedge$-comonoids.

However, what has been neglected so far is how the $\vee$-monoids and the $\wedge$-comonoids go along with each other. Recall that in any B2-category the maps ∇ and $\amalg$ preserve the $\vee$-monoid structure and the maps Δ and Π preserve the $\wedge$-comonoid structure (Theorem 2.6.19).

2.7.1. We have the following possibilities:

(i) The maps Π and $\amalg$ are quasientropies.

(ii) The maps Π and $\amalg$ are clonable.

(iii) The maps Δ and ∇ are quasientropies.

(iv) The maps Δ and ∇ are clonable.

Condition (i) says in particular that the following diagram commutes

$$\begin{tikzcd} & \mathbf{f} \arrow[dl, "\amalg^A"'] \arrow[dr, "\Pi^{\mathbf{f}}"] & \\ A \arrow[rr, "\Pi^A"'] & & \mathbf{t} \end{tikzcd} \tag{2.38}$$

Every B1-category obeying (B2a) and (2.38) is not only single-mixed but also for every object A the composition $\mathbf{f} \overset{\amalg^A}{\to} A \overset{\Pi^A}{\to} \mathbf{t}$ yields the same result. In [LS05a] the equation (2.38) was used as basic axiom, and the mix map was constructed from that without the use proper units, as we did in Section 2.3 (see also Remark 2.4.12).

The next observation to make is that (ii) and (iii) of 2.7.1 are equivalent, provided (B3b) and (B3a) are present:

Proposition 2.7.2. *In a* B2*-category with nullary medial and* (B3a) *the following are equivalent for every object A:*

(i) *The map Π^A preserves the $\vee$-multiplication.*

(ii) *The map ∇_A preserves the $\wedge$-counit.*

(iii) *The map $\amalg^{\bar{A}}$ preserves the $\wedge$-comultiplication.*

(iv) *The map* $\Delta_{\bar{A}}$ *preserves the* $\vee$*-unit.*

Proof. The equivalence of (ii) and (i) follows from

$$\begin{tikzcd}
A \vee A \arrow[r, "\nabla_A"] \arrow[d, "\Pi^A \vee \Pi^A"'] \arrow[dr, "\Pi^{A \vee A}"] & A \arrow[d, "\Pi^A"] \\
\mathbf{t} \vee \mathbf{t} \arrow[r, "\nabla_{\mathbf{t}}"'] & \mathbf{t}
\end{tikzcd}$$

The lower triangle is (B3b) together with (B3a). The upper triangle is (ii), and the square is (i). The other equivalences follow by duality. □

Condition 2.7.1 (iv) exhibits yet another example of a "creative tension" between algebra and proof theory. From the viewpoint of algebra, it makes perfect sense to demand that the $\vee$-monoid structure and the $\wedge$-comonoid structure be compatible with each other, i.e., that 2.7.1 (i)–(iv) do all hold (see [Lam07]). However, from the proof-theoretic point of view it is reasonable to make some fine distinctions: We have to keep in mind that in the sequent calculus it is the "contraction-contraction-case"

$$\mathsf{cut}\,\dfrac{\mathsf{cont}\,\dfrac{\begin{array}{c}\pi_1\\ \vdash \Gamma, A, A\end{array}}{\vdash \Gamma, A} \qquad \mathsf{cont}\,\dfrac{\begin{array}{c}\pi_2\\ \vdash \bar{A}, \bar{A}, \Delta\end{array}}{\vdash \bar{A}, \Delta}}{\vdash \Gamma, \Delta}$$

which spoils the confluence of cut elimination and which causes the exponential blow-up of the size of the proof. This questions 2.7.1 (iv), i.e., the commutativity of the diagram

$$\begin{tikzcd}
A \vee A \arrow[r, "\nabla_A"] \arrow[d, "\Delta_A \vee \Delta_A"'] & A \arrow[d, "\Delta_A"] \\
(A \wedge A) \vee (A \wedge A) \arrow[r, "\nabla_{A \wedge A}"'] & A \wedge A
\end{tikzcd} \tag{2.39}$$

motivates the distinction made in the following definition.

Definition 2.7.3. We say a B1-category is *weakly flat* if for every object A, the maps Π^A and $\amalg^A$ are strong and the maps Δ_A and ∇_A are quasientropies (i.e., 2.7.1 (i)–(iii) hold), and it is *flat* if for every object A, the maps Π^A, $\amalg^A$, Δ_A and ∇_A are all strong (i.e., all of 2.7.1 (i)–(iv) do hold).

Corollary 2.7.4. *A* B3*-category is weakly flat, if and only if* Π^A *is a* $\vee$*-monoid morphism for every object* A*.*

To understand the next (and final) axiom of this chapter, recall that in every star-autonomous category we have

$$\begin{tikzcd} \mathbf{t} \arrow[r, "\check{\iota}_A \wedge \check{\iota}_A"] \arrow[d, "\check{\iota}_A"'] & {[\bar{A} \vee A] \wedge [\bar{A} \vee A]} \arrow[d, "\hat{\mathsf{t}}"] \\ \bar{A} \vee A & \bar{A} \vee (A \wedge \bar{A}) \vee A \arrow[l, "\bar{A} \vee \hat{\iota}_A \vee A"] \end{tikzcd} \tag{2.40}$$

and that this equation is the reason why the cut elimination for multiplicative linear logic (proof nets as well as sequent calculus) works so well. The motivation for the following definition is to obtain something similar for classical logic (cf. [LS05a]).

Definition 2.7.5. A $\mathsf{B1}$-category is *contractible* if the following diagram commutes for all objects A.

$$\begin{tikzcd} \mathbf{t} \arrow[r, "\check{\iota}_A"] \arrow[dd, "\check{\iota}_A"'] & \bar{A} \vee A \arrow[d, "\Delta_{\bar{A} \vee A}"] \\ & {[\bar{A} \vee A] \wedge [\bar{A} \vee A]} \arrow[d, "\hat{\mathsf{t}}"] \\ \bar{A} \vee A & \bar{A} \vee (A \wedge \bar{A}) \vee A \arrow[l, "\bar{A} \vee \hat{\iota}_A \vee A"] \end{tikzcd} \tag{2.41}$$

The following theorem states the most surprising result of this chapter. It explains the deep reasons why the cut elimination for the proof nets of [LS05b], that I will discuss in the next chapter, is not confluent in the general case. It also shows that the combination of equations (2.39) and (2.41) together with the $\mathsf{B5}$-structure leads to a certain collapse, which can be compared to the collapse made by an LK-category. Nonetheless, even with this collapse we can find reasonable models for proofs of classical proposional logic, as it is shown in the next chapter.

Theorem 2.7.6. *In a* $\mathsf{B5}$*-category that is flat and contractible, we have*

$$1_A + 1_A = 1_A$$

for all objects A*.*

Proof. We proceed by showing that $\check{\iota}_A + \check{\iota}_A = \check{\iota}_A \colon \mathbf{t} \to \bar{A} \vee A$ for all objects A. From this the result follows by Proposition 2.4.21. Note that in particular we have that $\check{\iota}_A + \check{\iota}_A$ is the map

$$\mathbf{t} \xrightarrow{\check{\iota}_{A \wedge A}} \bar{A} \vee \bar{A} \vee (A \wedge A) \xrightarrow{\bar{A} \vee \bar{A} \vee \mathsf{mix}_{A,A}} \bar{A} \vee \bar{A} \vee A \vee A \xrightarrow{\nabla_{\bar{A}} \vee \nabla_A} \bar{A} \vee A$$

which is (because of (mix-$\hat{\alpha}$) and the star-autonomous structure) the same as the left-most

down path in the following diagram.

$$
\begin{tikzcd}
\mathbf{t} \arrow[rr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[dr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[ddd, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"'] & & [\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"] \\
& [\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"'] & [\bar{A}\vee A]\wedge[\bar{A}\vee(A\wedge\bar{A})\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\mathsf{mix}_{\bar{A}\vee A,\bar{A}\vee(A\wedge\bar{A})\vee A}"] \\
& [\bar{A}\vee A]\wedge[\bar{A}\vee(A\wedge\bar{A})\vee A] \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[dl, "\hat{\mathsf{i}}_A"] \arrow[d, "\mathsf{mix}"] & [\bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\mathsf{s}_{\bar{A}\vee\bar{A}\vee A\vee A,A\wedge\bar{A},\bar{A}\vee A}"] \\
[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\mathsf{mix}_{\bar{A}\vee A,\bar{A}\vee A}"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A \arrow[ur, "\check{\mathsf{i}}_A"] \arrow[r, "\check{\mathsf{i}}_A"'] \arrow[dl, "\hat{\mathsf{i}}_A"] & \bar{A}\vee\bar{A}\vee(A\wedge[\bar{A}\vee A]\wedge\bar{A})\vee A\vee A \arrow[d, "\check{\mathsf{t}}_{A,\bar{A},A,\bar{A}}"] \\
\bar{A}\vee\bar{A}\vee A\vee A \arrow[d, "\nabla_{\bar{A}}\vee\nabla_A"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee A\vee A \arrow[l, "\hat{\mathsf{i}}_A"] \arrow[d, "\nabla_{\bar{A}}\vee\nabla_A"'] & \bar{A}\vee\bar{A}\vee(A\wedge\bar{A})\vee(A\wedge\bar{A})\vee A\vee A \arrow[l, "\nabla_{A\wedge\bar{A}}"'] \arrow[dl, "\nabla_A\wedge\nabla_{\bar{A}}"] \arrow[d, "\mathsf{m}_{A,\bar{A},A,\bar{A}}"] \\
\bar{A}\vee A & \bar{A}\vee(A\wedge\bar{A})\vee A \arrow[l, "\hat{\mathsf{i}}_A"] & \bar{A}\vee\bar{A}\vee([A\vee A]\wedge[\bar{A}\vee\bar{A}])\vee A\vee A \arrow[l, "\nabla_{\bar{A}}\vee(\nabla_A\wedge\nabla_{\bar{A}})\vee\nabla_A"]
\end{tikzcd}
$$

The upper triangle commutes because of functoriality of $\wedge$, the square in the lower left corner because of functoriality of $\vee$, and the parallelograms because of naturality of mix and $\hat{\mathsf{t}}$. The quadrangle in the upper left commutes because of (2.40), and the little triangle in the right center is just (2.9) together with naturality of switch. The pentagon below it is just the dual of (2.41), and the two little triangles at the lower right corner are ($\mathsf{B3c}'$) and functoriality of $\vee$. Therefore, this diagram gives us a complicated way of writing just $\check{\mathsf{i}}_A + \check{\mathsf{i}}_A$. Similarly, the next diagram gives us a complicated way of writing $\check{\mathsf{i}}_A$:

$$
\begin{tikzcd}
\mathbf{t} \arrow[rr, "\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A\wedge\check{\mathsf{i}}_A"] \arrow[dr, "\check{\mathsf{i}}_A"] \arrow[drr, "\check{\mathsf{i}}_{(A\wedge A)\vee(A\wedge A)}"] \arrow[ddd, "\check{\mathsf{i}}_A"'] & & [\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}\wedge\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"] \\
& & [\bar{A}\vee\bar{A}\vee(A\wedge A)]\wedge[\bar{A}\vee\bar{A}\vee(A\wedge A)] \arrow[d, "\hat{\mathsf{t}}_{\bar{A}\vee\bar{A},A\wedge A,\bar{A}\vee\bar{A},A\wedge A}"] \\
& \bar{A}\vee A \arrow[d, "\Delta_{\bar{A}}\vee\Delta_A"] & ([\bar{A}\vee\bar{A}]\wedge[\bar{A}\vee\bar{A}])\vee(A\wedge A)\vee(A\wedge A) \arrow[dl, "(\nabla_{\bar{A}}\wedge\nabla_{\bar{A}})\vee\nabla_{A\wedge A}"'] \arrow[d, "\mathsf{m}_{A,A,A,A}"] \\
& (\bar{A}\wedge\bar{A})\vee(A\wedge A) \arrow[d, "\mathsf{m}_{\bar{A},\bar{A},A,A}"'] & ([\bar{A}\vee\bar{A}]\wedge[\bar{A}\vee\bar{A}])\vee([A\vee A]\wedge[A\vee A]) \arrow[l, "(\nabla_{\bar{A}}\wedge\nabla_{\bar{A}})\vee(\nabla_A\wedge\nabla_A)"] \arrow[d, "\mathsf{m}_{\bar{A}\wedge\bar{A},\bar{A}\vee\bar{A},A\vee A,A\vee A}"] \\
& [\bar{A}\vee A]\wedge[\bar{A}\vee A] \arrow[d, "\hat{\mathsf{t}}_{\bar{A},A,\bar{A},A}"] & [\bar{A}\vee\bar{A}\vee A\vee A]\wedge[\bar{A}\vee\bar{A}\vee A\vee A] \arrow[l, "{[\nabla_{\bar{A}}\vee\nabla_A]\wedge[\nabla_{\bar{A}}\vee\nabla_A]}"] \arrow[d, "\hat{\mathsf{t}}_{\bar{A}\wedge\bar{A},A\vee A,\bar{A}\vee\bar{A},A\vee A}"] \\
\bar{A}\vee A & \bar{A}\vee(A\wedge\bar{A})\vee A \arrow[l, "\hat{\mathsf{i}}_A"] & \bar{A}\vee\bar{A}\vee([A\vee A]\wedge[\bar{A}\vee\bar{A}])\vee A\vee A \arrow[l, "\nabla_{\bar{A}}\vee(\nabla_A\wedge\nabla_{\bar{A}})\vee\nabla_A"]
\end{tikzcd}
$$

Here the big upper right "triangle" commutes because of the star-autonomous structure. The irregular quadrangle in the center is a transposed version of (2.39), the little triangle below it is ($\mathsf{B3c}'$), the two squares at the bottom are naturality of m and $\hat{\mathsf{t}}$, and the left-most part of the diagram commutes because of (2.41) and ($\mathsf{B3c}$). Finally, we apply Lemma 2.6.24 to paste the two diagrams together, which yields $\check{\mathsf{i}}_A + \check{\mathsf{i}}_A = \check{\mathsf{i}}_A$ as desired. □

In the next chapter, in Figure 3.15 (on page 72), I use a concrete model, based on proof nets, to visualize the basic idea of this proof. In that figure, the first four equations express the idea behind the first big diagram in the proof of Theorem 2.7.6, and the last three equations in Figure 3.15 express the idea of the second diagram.

Corollary 2.7.7. *Let $\mathscr{A}$ be a set of propositional variables and let $\mathscr{C}$ be the free flat and contractible* $\mathsf{B5}$*-category generated by $\mathscr{A}$. Then $\mathscr{C}$ is idempotent.*

This shows that with the axioms of $\mathsf{B5}$-categories that are at the same time flat and contractible we reach a certain unwanted degeneration because idempotency does not allow us to address issues of proof complexity that should be taken into account in a theory of proofs of classical logic. Thus, we are again in the situation that some axioms have to go. But which ones?

3

Some Combinatorial Invariants of Proofs in Classical Logic

In the previous chapter, I tried to be as abstract as possible, and in this chapter I am trying to be as concrete as possible. I will present a series of "graph-like" invariants of proofs. The basic idea is in all cases to follow the "flow" of the atom occurrences in the proofs, similar to what happens in *coherence graphs* [KM71] or *logical flow graphs* [Bus91]. The first invariant that I present, called $\mathbb{B}$-*nets* is a variation of Andrews' *matings* [And76] and Bibel's *matrix proofs* [Bib81]. Since $\mathbb{B}$-nets are too rough for modeling proofs in classical logic because they make too many identifications, I will show several refinements: $\mathbb{N}$-nets, C-nets, and atomic flows. They reduce the number of proof identifications by keeping more information in the data structure. I will also show how these invariants can be extracted from formal deductive proofs. However, we have a correctness criterion only for $\mathbb{B}$-nets (which is unfortunately exponential). This means that we do not know what is the right amount of information about the proofs to keep.

This chapter is based on the publications [LS05b, Str05, Str09, GGS10, Str07b].

3.1 Cut free nets for classical propositional logic

We let $\mathscr{A} = \{a, b, \dots\}$ be an arbitrary set of atoms, and $\bar{\mathscr{A}} = \{\bar{a}, \bar{b}, \dots\}$ the set of their duals. Then the set of *formulas* is defined as follows:

$$\mathscr{F} ::= \mathscr{A} \mid \bar{\mathscr{A}} \mid \mathbf{t} \mid \mathbf{f} \mid \mathscr{F} \wedge \mathscr{F} \mid \mathscr{F} \vee \mathscr{F} \quad .$$

The elements of the set $\{\mathbf{t}, \mathbf{f}\}$ are called *units*. We will use A, B, ... to denote formulas, and we follow the same notational conventions as in the previous chapter. *Sequents*, denoted by Γ, Δ, ..., are finite lists of formulas, separated by comma.

In the following, we will consider formulas as binary trees (and sequents as forests), whose leaves are decorated by elements of $\mathscr{A} \cup \bar{\mathscr{A}} \cup \{\mathbf{t}, \mathbf{f}\}$, and whose inner nodes are decorated by $\wedge$ or $\vee$. Given a formula A or a sequent Γ, we write $\mathscr{L}(A)$ or $\mathscr{L}(\Gamma)$, respectively, to denote its set of leaves.

For defining our proof nets, we start with a commutative semiring of *weights* $(W, 0, 1, +, \cdot)$. That is, $(W, 0, +)$ is a commutative monoid structure, $(W, 1, \cdot)$ is another commutative monoid structure, and we have the usual distributivity laws $x \cdot (y + z) = (x \cdot y) + (x \cdot z)$ and $0 \cdot x = 0$. This abstraction layer is just there to ensure a uniform treatment for the two cases that we will encounter in practice, namely $W = \mathbb{N}$ (the semiring of the natural numbers with the usual operations), and $W = \mathbb{B} = \{0, 1\}$ (the Boolean semiring, where addition is disjunction, i.e., $1 + 1 = 1$, and multiplication is conjunction). There are two additional algebraic properties that we will need:

$$v + w = 0 \quad \text{implies} \quad v = w = 0 \tag{3.1}$$

$$v \cdot w = 0 \quad \text{implies} \quad \text{either } v = 0 \text{ or } w = 0 \quad . \tag{3.2}$$

These are obviously true in both concrete cases. No other structure or property on $\mathbb{B}$ and $\mathbb{N}$ is needed, and thus other choices for W can be made. They all give sound semantics, but completeness (or sequentialization) is another matter.

Definition 3.1.1. Given W and a sequent Γ, a *W-linking for* Γ is a function $P\colon \mathscr{L}(\Gamma) \times \mathscr{L}(\Gamma) \to W$ which is symmetrical (i.e., $P(x, y) = P(y, x)$ always), and such that whenever $P(x, y) \neq 0$ then either

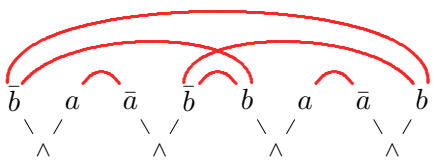
- $x = y$ and the leaf x is decorated by $\mathbf{t}$ and $P(x, x) = 1$, or
- $x \neq y$ and one of x and y is decorated by an atom a and the other by its dual $\bar{a}$.

A *W-pre-proof-net*[10] (or shortly *W-prenet*) consists of a sequent Γ and a linking P for it. It will be denoted by $P \rhd \Gamma$. If $W = \mathbb{B}$ we say it is a *simple pre-proof-net* (or shortly *simple prenet*).

In what follows, we will simply say prenet, if no semiring W is specified.

Remark 3.1.2. In [LS05b] the definition of prenet is different from Definition 3.1.1 because of a different treatment of the units. There, the resulting categories have only weak units (see also [LS05a]), whereas Definition 3.1.1 ensures proper units, as presented in Chapter 2 of this thesis.

If we choose $W = \mathbb{B}$, a linking is just an ordinary, undirected graph structure on $\mathscr{L}(\Gamma)$, with an edge between x and y when $P(x, y) = 1$. An example is



(3.3)

To save space that example can also be written as

$$\{\, \overset{\frown}{\bar{b}_1\ b_5}\,,\ \overset{\frown}{\bar{b}_1\ b_8}\,,\ \overset{\frown}{\bar{b}_4\ b_5}\,,\ \overset{\frown}{\bar{b}_4\ b_8}\,,\ \overset{\frown}{a_2\ \bar{a}_3}\,,\ \overset{\frown}{a_6\ \bar{a}_7}\,\} \rhd \bar{b}_1 \wedge a_2, \bar{a}_3 \wedge \bar{b}_4, b_5 \wedge a_6, \bar{a}_7 \wedge b_8\ ,$$

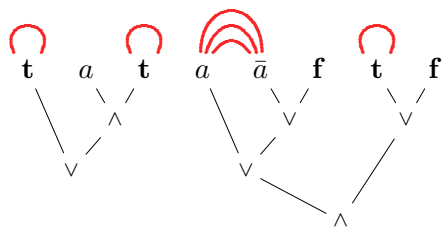
where the set of linked pairs is written explicitly in front of the sequent. Here we use the indices only to distinguish different atom occurrences (i.e., a_3 and a_6 are not different atoms but different occurrences of the same atom a).

[10] What I call *pre-proof-net* here is in the literature (on linear logic) also called a *proof structure*.

For more general cases of W, we can consider the edges to be decorated with elements of $W \setminus \{0\}$. When $P(x,y) \neq 0$ we say that x and y are *linked*. Here is an example of an $\mathbb{N}$-prenet:

$$\{\, \mathbf{t}_1 \overset{\frown}{} \mathbf{t}_1 \,,\, \mathbf{t}_3 \overset{\frown}{} \mathbf{t}_3 \,,\, a_4 \overset{3}{\frown} \bar{a}_5 \,,\, \mathbf{t}_7 \overset{\frown}{} \mathbf{t}_7 \,\} \rhd \mathbf{t}_1 \vee (a_2 \wedge \mathbf{t}_3), (a_4 \vee (\bar{a}_5 \vee \mathbf{f}_6)) \wedge (\mathbf{t}_7 \vee \mathbf{f}_8) \,. \tag{3.4}$$

As before, no link means that the value is 0. Furthermore, we use the convention that a link without value means that the value of the linking is 1. When drawing $\mathbb{N}$-prenets as graphs (i.e., the sequent forest plus linking), we will draw n links between two leaves x and y, if $P(x,y) = n$. Although this might be a little cumbersome, I find it more intuitive. Example (3.4) is then written as



. (3.5)

Now for some more notation: let $P \rhd \Gamma$ be a prenet and $L \subseteq \mathscr{L}(\Gamma)$ an arbitrary subset of leaves. There is always a $P|_L \colon L \times L \to W$ which is obtained by restricting P on the smaller domain. But L also determines a subforest $\Gamma|_L$ of Γ, in the manner that all elements of L are leaves of $\Gamma|_L$, and an inner node s of Γ is in $\Gamma|_L$ if *one or two* of its children is in $\Gamma|_L$. Thus $\Gamma|_L$ is something a bit more general than a "subsequent" or "sequent of subformulas", since some of the connectors are allowed to be unary, although still labeled by $\wedge$ and $\vee$. Let $\Gamma' = \Gamma|_L$. Then not only is Γ' determined by L, but the converse is also true: $L = \mathscr{L}(\Gamma')$. We will say that $P|_L \rhd \Gamma'$ is a *sub-prenet* of $P \rhd \Gamma$, although it is not *strictu sensu* a prenet. Since this sub-prenet is entirely determined by Γ', we can also write it as $P|_{\Gamma'} \rhd \Gamma'$ without mentioning L any further.

On the set of W-linkings we define the following operations. Let $P \colon L \times L \to W$ and $Q \colon M \times M \to W$ be two W-linkings.

- If $L = M$ we define the *sum* $P + Q$ to be the pointwise sum $(P+Q)(x,y) = P(x,y) + Q(x,y)$. When $W = \mathbb{B}$ this is just the union of the two graphs.
- If $L \subseteq M$ we define the *extension* $P{\uparrow}^M$ *of* P *to* M as the following binary function on M:
$$P{\uparrow}^M\,(x,y) = \begin{cases} P(x,y) & \text{if } x,y \in L \\ 1 & \text{if } x = y \text{ and } x \notin L \text{ and } x \text{ is decorated by } \mathbf{t} \\ 0 & \text{otherwise} \end{cases}$$
Most of the times we will write $P{\uparrow}^M$ simply as P.
- If L and M are disjoint, we define the *disjoint sum* $P \oplus Q$ on the disjoint union[11] $L \uplus M$ as $P \oplus Q = P{\uparrow}^{L \uplus M} + Q{\uparrow}^{L \uplus M}$.

The $\oplus$ operation is needed in the next section for translating sequent proofs into W-prenets.

[11] If L and M are not actually disjoint, we can always rename their elements to ensure that they are.

$$\mathsf{id}\ \frac{}{\{\, a \frown \bar{a} \,\} \rhd a, \bar{a}} \qquad \mathbf{t}\ \frac{}{\{\, \mathbf{t} \frown \mathbf{t} \,\} \rhd \mathbf{t}} \qquad \vee\ \frac{P \rhd A, B, \Gamma}{P \rhd A \vee B, \Gamma} \qquad \wedge\ \frac{P \rhd \Gamma, A \quad Q \rhd B, \Delta}{P \oplus Q \rhd \Gamma, A \wedge B, \Delta}$$

$$\mathsf{weak}\ \frac{P \rhd \Gamma}{P \rhd A, \Gamma} \qquad \mathsf{cont}\ \frac{P \rhd A, A, \Gamma}{P' \rhd A, \Gamma} \qquad \mathsf{exch}\ \frac{P \rhd \Gamma, A, B, \Delta}{P \rhd \Gamma, B, A, \Delta} \qquad \mathsf{mix}\ \frac{P \rhd \Gamma \quad Q \rhd \Delta}{P \oplus Q \rhd \Gamma, \Delta}$$

Figure 3.1: Translation of cut free sequent calculus proofs into prenets

Definition 3.1.3. A *conjunctive resolution* of a prenet $P \rhd \Gamma$ is a sub-prenet $P|_{\Gamma'} \rhd \Gamma'$ where Γ' has been obtained by deleting one child subformula for every conjunction node of Γ (i.e., in $P|_{\Gamma'} \rhd \Gamma'$ every $\wedge$-node is unary).

Definition 3.1.4. A W-prenet $P \rhd \Gamma$ is said to be *correct* if for every one of its conjunctive resolutions $P|_{\Gamma'} \rhd \Gamma'$ the W-linking $P|_{\Gamma'}$ is not the zero function. A *W-proof-net* (or shortly *W-net*) is a correct W-prenet. A correct $\mathbb{B}$-prenet is also called a *simple net*.

Both examples shown so far are correct: (3.3) is a $\mathbb{B}$-net as well as an $\mathbb{N}$-net; (3.5) is an $\mathbb{N}$-net. Notice that the definition of correctness does not take the exact values of the weights into account, only the presence ($P(x, y) \neq 0$) or absence ($P(x, y) = 0$) of a link. Notice also that correctness is a monotone property because of Axiom (3.1): if $P \rhd \Gamma$ is correct then $P + Q \rhd \Gamma$ is also correct.

The terms "linking" and "resolution", as well as the $\rhd$-notation, have been lifted directly from the work on multiplicative additive (MALL) proof nets of [HvG03], and we use them in the same way, for the same reasons. In fact, there is a remarkable similarity between the MALL correctness criterion therein—when restricted to the purely additive case—and ours, which is essentially the same as Andrews' [And76] and Bibel's [Bib81].

3.2 Sequentialization

Figure 3.1 shows how sequent proofs are mapped into prenets. The sequent system we use contains the multiplicative versions of the $\wedge$- and $\vee$-rules, the usual axioms for identity (reduced to atoms) and truth, as well as the rules of exchange, weakening, contraction, and mix. We call that system CL (for Classical Logic).

Note that the mix-rule is not strictly necessary from the point of view of provability. But, although we do not get more tautologies by including it in the system, we get *more proofs*. These new proofs are needed in order to make cut elimination confluent. It is only one more example of the extreme usefulness of completions in mathematics: adding stuff can make your life simpler.

Let us now explain the translation from CL sequent proofs to prenets.

Translation 3.2.1. This is done inductively on the size of the proof according to the rules shown in Figure 3.1. In the two rules for disjunction and exchange, nothing happens to the linking. In the case of weakening we apply the extension operation to P (i.e., nothing changes, except that we add loops to all $\mathbf{t}$ inside A). In the mix and $\wedge$ rules we get

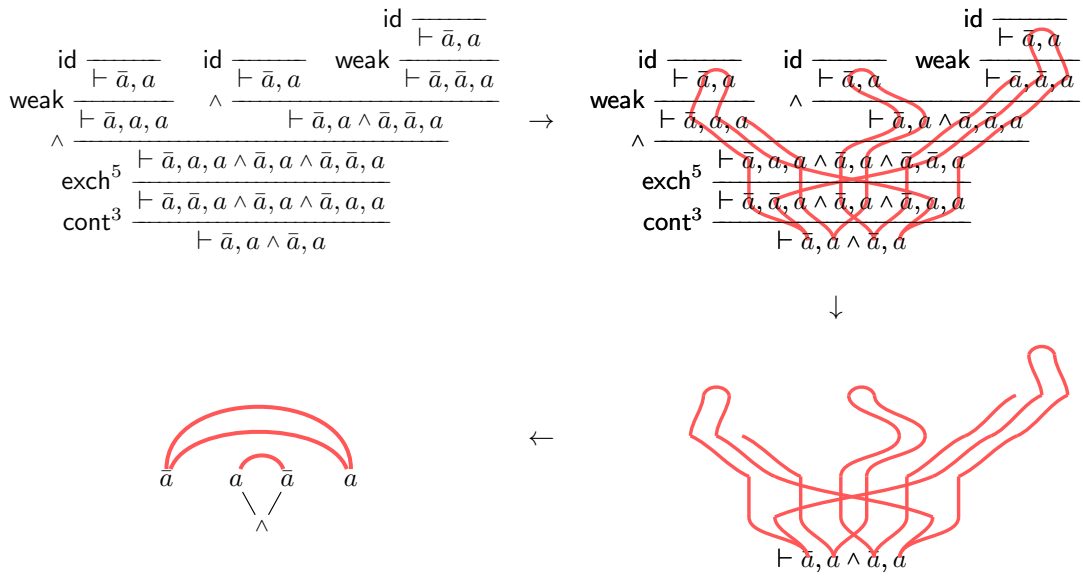


Figure 3.2: From sequent calculus to ℕ-prenets

the linking of the conclusion by forming the disjoint sum of the linkings of the premises. Therefore, the only rule that deserves further explanation is contraction. Consider the two sequents $\Delta' = A, \Gamma$ and $\Delta = A, A, \Gamma$, where Δ is obtained from Δ' by duplicating A. Let $p\colon \mathscr{L}(\Delta) \to \mathscr{L}(\Delta')$ be the function that identifies the two occurrences of A. We see in particular that for any leaf $x \in \mathscr{L}(\Delta')$, the inverse image $p^{-1}\{x\}$ has either one or two elements. Given $P \rhd \Delta$ let us define the linking P' for Δ' as

$$P'(x,y) \quad = \sum_{\substack{z \in p^{-1}\{x\} \\ w \in p^{-1}\{y\}}} P(z,w) \quad .$$

Remark 3.2.2. If $W = \mathbb{N}$, then there is a more intuitive way of translating a sequent proof into a prenet, by simply drawing the logical flow graph [Bus91] (restricted to atoms) and "pulling the edges" as indicated in Figure 3.2. To save space we did in the figure several steps in one. For example cont^3 stands for three applications of the contraction rule.

Theorem 3.2.3 (Soundness)**.** *Given any semiring W of weights and a sequent proof in* CL*, the construction in 3.2.1 yields a correct W-prenet.*

Proof. The proof is an easy induction on the structure of the CL-derivation. Notice that we need condition (3.1) for the contraction rule, but for the time being we do not need (3.2). □

We say a prenet is *sequentializable* if it can be obtained from a sequent calculus proof via this translation. Theorem 3.2.3 says that every sequentializable W-prenet is a W-net. The converse holds only for $\mathbb{B}$-nets.

Theorem 3.2.4 (Sequentialization)**.** *Every $\mathbb{B}$-net is sequentializable in* CL*.*

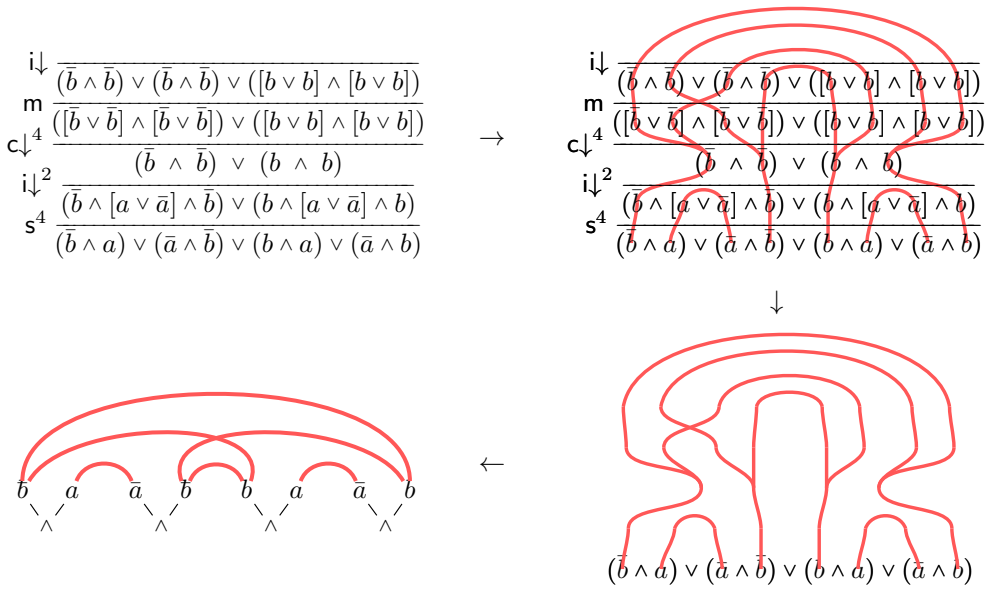


Figure 3.3: From the calculus of structures to $\mathbb{N}$-nets

Proof. We proceed by induction on the size of the sequent (i.e., the number of $\wedge$-nodes, $\vee$-nodes, and leaves in the sequent forest). Consider a $\mathbb{B}$-net $P \rhd \Delta$. We have the following cases:

- If Δ contains a formula $A \vee B$, then we can apply the $\vee$-rule, and proceed by induction hypothesis.
- If Δ contains a formula $A \wedge B$, i.e., $\Delta = \Gamma, A \wedge B$, then we can form the three $\mathbb{B}$-nets $P' \rhd \Gamma, A$ and $P'' \rhd \Gamma, B$ and $P \rhd \Gamma, A, B$, where $P' = P|_{\Gamma,A}$ and $P'' = P|_{\Gamma,B}$. All three of them are correct. Therefore, we can apply the induction hypothesis to them. Now we apply the $\wedge$-rule twice to get $P' \oplus P'' \oplus P \rhd \Gamma, \Gamma, \Gamma, A \wedge B, A \wedge B$. To finally get $P \rhd \Gamma, A \wedge B$, we only need a sufficient number of contractions (and exchanges). Let me make two remarks about that case:
  - If $P \rhd \Gamma, A \wedge B$ contains no link between A and B, i.e., $P|_{A,B} = P|_A \oplus P|_B$, then we do not need the $\mathbb{B}$-net $P \rhd \Gamma, A, B$, and can instead proceed by a single use of the $\wedge$-rule, followed by contractions.
  - This is the only case where the fact that $W = \mathbb{B}$ is needed.
- If $\Delta = \Gamma, A$, such that $P = P|_\Gamma$, i.e., the formula A does not take part in the linking P, then we can apply the weakening rule and proceed by induction hypothesis.
- The only remaining case is where all formulas in Δ are atoms, negated atoms, or units. Then the sequent proof is obtained by a sufficient number of instances of the axioms id, t, and the rules cont, exch, and mix. $\square$

Note that the particular sequent system CL is only needed for obtaining the completeness, i.e., sequentialization, of $\mathbb{B}$-nets. For obtaining the soundness result, any sequent system for classical propositional logic (with the identity axiom reduced to atomic form)

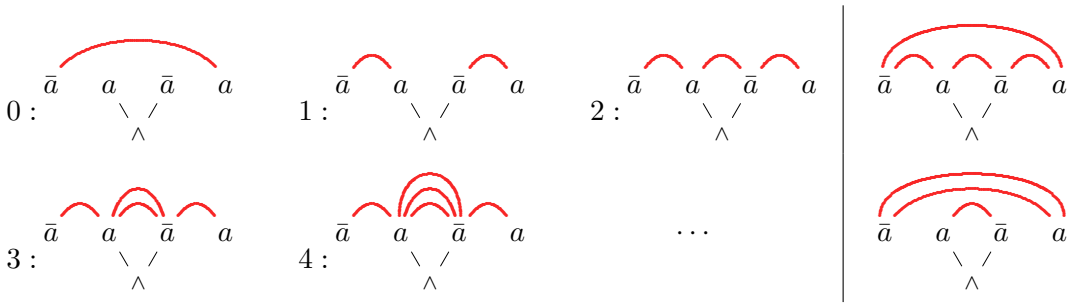


Figure 3.4: **Left:** Church numerals as $\mathbb{N}$-nets **Right:** "Fake" Church numerals

can be used. Moreover, this is not restricted to sequent calculus. We can also start from resolution proofs (as done in [And76]), tableau proofs, Hilbert style proofs, etc.

Whereas $\mathbb{B}$-nets only take into account *whether* a certain axiom link is used in the proof, $\mathbb{N}$-nets also count *how often* it is used. Therefore, $\mathbb{N}$-nets can be used for investigating certain complexity issues related to the size of proofs, e.g., the exponential blow-up usually related to cut elimination. This is not visible for $\mathbb{B}$-nets, where the size of a proof net is always quadratic in the size of the sequent. It should not come as a surprise that finding a fully complete correctness criterion for $\mathbb{N}$-nets is much harder. One reason is the close connection to the NP vs. co-NP problem [CR79]. Moreover, there are correct $\mathbb{N}$-nets for which no corresponding sequent proof exists (for example (3.3) seen as an $\mathbb{N}$-net), but which can be represented in other formalisms, for example the calculus of structures [GS01, BT01], as shown in Figure 3.3. We will come back to the deductive system used in that figure in Section 3.5.

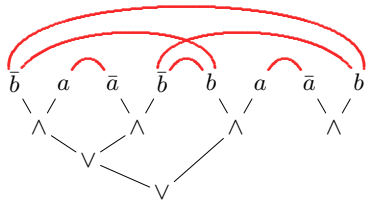
To give some more examples, consider the sequent $\vdash \bar{a}, a \wedge \bar{a}, a$. This is equivalent to the formula $(a \Rightarrow a) \Rightarrow (a \Rightarrow a)$ modulo some applications of associativity and commutativity (here $\Rightarrow$ stands for implication). Hence, the proofs of that sequent can be used to encode the Church numerals. Figure 3.4 shows on the left the encodings of the numbers 0 to 4 as $\mathbb{N}$-nets. Observe that using $\mathbb{B}$-nets, we can distinguish only the numbers 0, 1, and 2, because all numbers ≥ 2 are collapsed. Note that there are also proofs of that sequent that do not encode a numeral. There are two examples on the right of Figure 3.4. The top one is obtained by simply mixing together the two proofs 0 and 2. One of the arguments for not having the mix rule in a system is that it causes types (resp. formulas) to be inhabited by more terms (resp. proofs) than the intended ones. However, let me stress the (well-known) fact, that this phenomenon is by no means caused by the mix rule, as the bottom "fake" numeral in Figure 3.4 shows, which comes from the mix-free sequent proof shown in Figure 3.2.

3.3 Nets with cuts

A *cut* is a formula $A \Diamond \bar{A}$, where $\Diamond$ is called the *cut connective*, and where the function $\overline{(-)}$ is defined on formulas as follows (with a trivial abuse of notation):

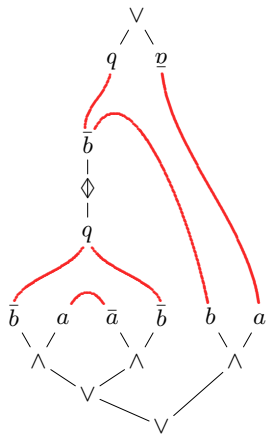
$$\bar{a} = \bar{a}\,, \quad \bar{\bar{a}} = a\,, \quad \bar{\mathbf{t}} = \mathbf{f}\,, \quad \bar{\mathbf{f}} = \mathbf{t}\,, \quad \overline{(A \wedge B)} = \bar{A} \vee \bar{B}\,, \quad \overline{(A \vee B)} = \bar{A} \wedge \bar{B}\,.$$

A *sequent with cuts* is a sequent where some of the formulas are cuts. But cuts are not allowed to occur inside formulas, i.e., all $\Diamond\!\!\!\!\!\!\,\,\,

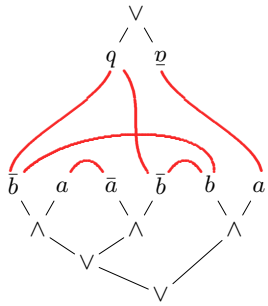


(3.8)

The two prenets in (3.7) and (3.8) can also be drawn as follows where the two formulas in Γ are at the outer ends of the picture:



and



(3.9)

The motivation for this notation is, of course, to indicate the category, whose objects are the formulas, and whose arrows are the prenets with two conclusions. However, to get such a category, we have to say how to compose arrows. This is usually done by cut elimination, which is the subject of the next section.

3.4 Cut Reduction

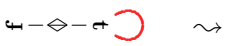
Cut elimination in W-prenets has much in common with proof nets of multiplicative linear logic. The cut-reduction step on a compound formula is exactly the same:

$$P \rhd (A \wedge B) \Diamond (\bar{A} \vee \bar{B}), \Gamma \;\rightarrow\; P \rhd A \Diamond \bar{A}, B \Diamond \bar{B}, \Gamma$$

and so it does not affect the linking itself (although we have to show it preserves correctness). As picture:

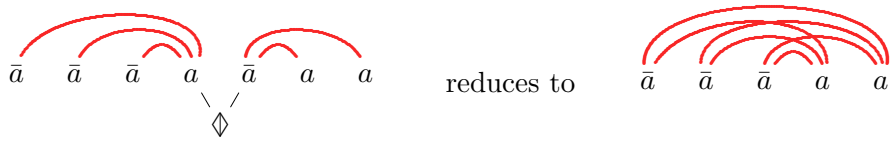
(3.10)

For saving space, the picture is put on the side. Cuts on the units can simply be removed:



(3.11)

The really interesting things happen in the atomic case. Here cut elimination means "counting paths through the cuts". Let us illustrate the idea by an example:



More generally, if some weights are different from 1, we multiply them:

$$\{\, \bar{a}_1 \overset{p}{\frown} a_4 \,,\, \bar{a}_2 \overset{q}{\frown} a_4 \,,\, \bar{a}_3 \overset{r}{\frown} a_4 \,,\, \bar{a}_5 \overset{m}{\frown} a_6 \,,\, \bar{a}_5 \overset{n}{\frown} a_7 \,\} \rhd \bar{a}_1, \bar{a}_2, \bar{a}_3, a_4 \Diamond \bar{a}_5, a_6, a_7$$
$$\to \{\, \bar{a}_1 \overset{pm}{\frown} a_6 \,,\, \bar{a}_1 \overset{pn}{\frown} a_7 \,,\, \bar{a}_2 \overset{qm}{\frown} a_6 \,,\, \bar{a}_2 \overset{qn}{\frown} a_7 \,,\, \bar{a}_3 \overset{rm}{\frown} a_6 \,,\, \bar{a}_3 \overset{rn}{\frown} a_7 \,\} \rhd \bar{a}_1, \bar{a}_2, \bar{a}_3, a_6, a_7 \,.$$

To understand certain subtleties let us consider

$$\{\, \bar{a}_1 \overset{p}{\frown} a_4 \,,\, \bar{a}_1 \overset{q}{\frown} a_2 \,,\, a_2 \overset{r}{\frown} \bar{a}_3 \,,\, \bar{a}_3 \overset{s}{\frown} a_4 \,\} \rhd \bar{a}_1, a_2 \Diamond \bar{a}_3, a_4 \quad \to \quad \{\, \bar{a}_1 \overset{z}{\frown} a_4 \,\} \rhd \bar{a}_1, a_4 \quad .$$

What is the value of z? We certainly cannot just take $q \cdot s$, we also have to add p. But the question is what happens to r, i.e., is the result $z = p + qs$ or $z = p + q \cdot (1 + r) \cdot s = p + qs + qrs$ or even $z = p + q \cdot (1 + r + r^2 + r^3 + \cdots) \cdot s$?[12] All choices lead to a sensible theory of cut elimination, but here we only treat the first, simplest case: we drop r. In [LS05a], this property is called *loop-killing*, and it corresponds to contractibility, as defined in Definition 2.7.5.

Let us now introduce some notation. Given $P \rhd \Gamma$ and $x, y, u_1, u_2, \ldots, u_n \in \mathscr{L}(\Gamma)$, with n even, we write $P(\, x \frown u_1 \cdot u_2 \frown u_3 \cdot \ldots \cdot u_n \frown y \,)$ as an abbreviation for $P(x, u_1) \cdot P(u_2, u_3) \cdot \ldots \cdot P(u_n, y)$. In addition we define

$$P(\, x \uparrow u_1 \cdot u_2 \frown u_3 \cdot \ldots \cdot u_n \uparrow y \,) \;=\; P(\, x \frown u_1 \cdot u_2 \frown u_3 \cdot \ldots \cdot u_n \frown y \,) + P(\, x \frown u_n \cdot \ldots \cdot u_3 \frown u_2 \cdot u_1 \frown y \,) \tag{3.12}$$

We now can define cut reduction formally for a single atomic cut:

$$P \rhd u \Diamond v, \Gamma \;\to\; Q \rhd \Gamma \;, \text{ where } Q(x, y) = P(x, y) + P(\, x \uparrow u \cdot v \uparrow y \,)$$

for all $x, y \in \mathscr{L}(\Gamma)$, and where u is labelled by an arbitrary atom and v by its dual. But we can go further and do *simultaneous reduction* on a set of atomic cuts:

$$P \rhd u_1 \Diamond v_1, u_2 \Diamond v_2, \ldots, u_n \Diamond v_n, \Gamma \;\to\; Q \rhd \Gamma \quad , \tag{3.13}$$

where each u_i labelled by an arbitrary atom or unit, and v_i by its dual. For defining Q, we need the following notion:
A *cut-path between x and y* in a net $P \rhd \Delta$ with $x, y \in \mathscr{L}(\Delta)$ is an expression of the form $x \frown w_1 \cdot z_1 \frown w_2 \cdot z_2 \frown w_3 \cdot \ldots \cdot z_k \frown y$ where $w_i \Diamond z_i$ are all *distinct* atomic cuts in Δ, and such that $P(\, x \uparrow w_1 \cdot z_1 \frown w_2 \cdot z_2 \frown w_3 \cdot \ldots \cdot z_k \uparrow y \,) \neq 0$. For a set S of atomic cuts in Δ, the cut-path is

[12] Notice that this is really the Geometry of Interaction's [Gir89] Execution formula.

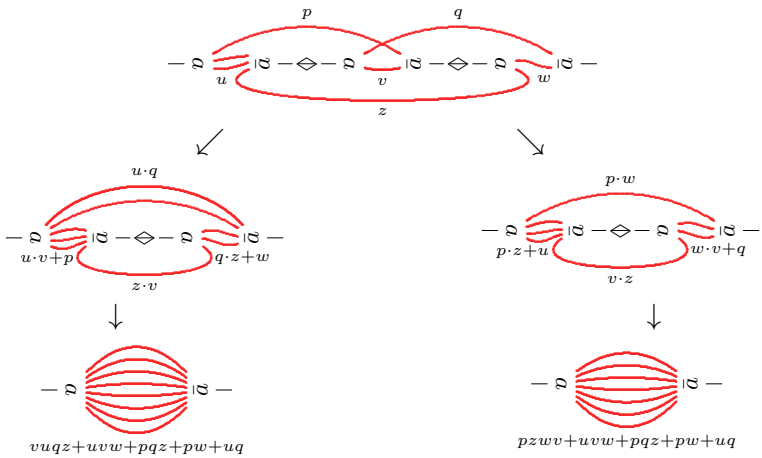


Figure 3.5: Cut reduction is not confluent for ℕ-prenets

covered by S if all the $w_i \Diamond z_i$ are in S, and it *touches* S if at least one of the $w_i \Diamond z_i$ is in S. The Q in (3.13) is now given by

$$Q(x,y) = P(x,y) + \sum_{\{x \frown w_1 \cdot z_1 \frown w_2 \cdot \ldots \cdot z_k \frown y\}} P(\, x \upharpoonleft w_1 \cdot z_1 \frown w_2 \cdot \ldots \cdot z_k \upharpoonright y\,) \quad ,$$

where the sum ranges over all cut-paths covered by $\{u_1 \Diamond v_1, u_2 \Diamond v_2, \ldots, u_n \Diamond v_n\}$. Notice that in the case of cut paths at least one of the summands in (3.12) is always zero.

Lemma 3.4.1. *Let $P \rhd \Delta$ be a W-prenet, and let $P \rhd \Delta \to P' \rhd \Delta'$. If $P \rhd \Delta$ is correct, then $P' \rhd \Delta'$ is also correct.*

The proof is an ordinary case analysis, and it is the only place where Axiom (3.2) is used. The next observation is that there is no infinite sequence $P \rhd \Gamma \to P' \rhd \Gamma' \to P'' \rhd \Gamma'' \to \cdots$, because in each reduction step the size of the sequent (i.e., the number of $\wedge$, $\vee$ and $\Diamond$-nodes) is reduced. Therefore we have:

Lemma 3.4.2. *The cut reduction relation $\to$ is terminating.*

Let us now attack the issue of confluence. Obviously we only have to consider atomic cuts; let us begin when two singleton cuts $P \rhd a_i \Diamond \bar{a}_j, a_h \Diamond \bar{a}_k, \Gamma$ are reduced. If the first cut is reduced, we get $P' \rhd a_h \Diamond \bar{a}_k, \Gamma$, where $P'(x,y) = P(x,y) + P(\, x \upharpoonleft a_i \cdot \bar{a}_j \upharpoonright y\,)$. Then reducing the second cut gives us $Q_1 \rhd \Gamma$, where $Q_1(x,y) = P'(x,y) + P'(\, x \upharpoonleft a_h \cdot \bar{a}_k \upharpoonright y\,)$. An easy computation shows that

$$\begin{aligned} Q_1(x,y) = {} & P(x,y) + P(\, x \upharpoonleft a_i \cdot \bar{a}_j \upharpoonright y\,) + P(\, x \upharpoonleft a_h \cdot \bar{a}_h \upharpoonright y\,) + P(\, x \upharpoonleft a_i \cdot \bar{a}_j \frown a_k \cdot \bar{a}_h \upharpoonright y\,) + \\ & P(\, x \upharpoonleft a_h \cdot \bar{a}_k \frown a_i \cdot \bar{a}_j \upharpoonright y\,) + P(\, x \upharpoonleft a_i \cdot \bar{a}_j \frown a_h \cdot \bar{a}_k \frown a_i \cdot \bar{a}_j \upharpoonright y\,) \quad . \end{aligned} \tag{3.14}$$

Reducing the two cuts in the opposite order yields $Q_2 \rhd \Gamma$, where

$$\begin{aligned} Q_2(x,y) = {} & P(x,y) + P(\, x \upharpoonleft a_i \cdot \bar{a}_j \upharpoonright y\,) + P(\, x \upharpoonleft a_h \cdot \bar{a}_k \upharpoonright y\,) + P(\, x \upharpoonleft a_i \cdot \bar{a}_j \frown a_k \cdot \bar{a}_h \upharpoonright y\,) + \\ & P(\, x \upharpoonleft a_h \cdot \bar{a}_k \frown a_i \cdot \bar{a}_j \upharpoonright y\,) + P(\, x \upharpoonleft a_h \cdot \bar{a}_k \frown a_i \cdot \bar{a}_j \frown a_h \cdot \bar{a}_k \upharpoonright y\,) \quad . \end{aligned} \tag{3.15}$$

We see that the last summand is different in the two results. But if we reduce both cuts simultaneously, we get $Q \rhd \Gamma$, where

$$\begin{aligned} Q(x,y) = P(x,y) + P(\, x\!\uparrow\!a_i \cdot \bar{a}_j\!\uparrow\!y\,) + P(\, x\!\uparrow\!a_h \cdot \bar{a}_k\!\uparrow\!y\,) + \\ P(\, x\!\uparrow\!a_i \cdot \bar{a}_j\,\frown\,a_k \cdot \bar{a}_h\!\uparrow\!y\,) + P(\, x\!\uparrow\!a_h \cdot \bar{a}_k\,\frown\,a_i \cdot \bar{a}_j\!\uparrow\!y\,) \quad . \end{aligned} \tag{3.16}$$

Now the troublesome summand is absent. There are also good news: In the case of $\mathbb{B}$-nets we have that $Q = Q_1 = Q_2$. The reason is that if in either Q_1 or Q_2 the last summand is 1, then at least one of the other summands is also 1. This ensures that the whole sum is 1, because of idempotency of addition. Therefore:

Lemma 3.4.3. *On $\mathbb{B}$-prenets the cut reduction relation $\to$ is locally confluent.*

Lemmas 3.4.1–3.4.3 together give us immediately:

Theorem 3.4.4. *On $\mathbb{B}$-nets with cuts the cut elimination via $\to$ is terminating and confluent, and the normal forms are cut free $\mathbb{B}$-nets.*

This theorem allows us to form the category of $\mathbb{B}$-nets, more precisely, we have two categories whose objects are the formulas. The arrows of the first category, that I call here **Pre**$\mathbb{B}$ are the cut-free $\mathbb{B}$-prenets with two conclusions, and the arrows of the second category, that I call **Net**$\mathbb{B}$, are the cut-free $\mathbb{B}$-nets with two conclusions (i.e., an arrow $A \to B$ is a (pre-)net $P \rhd \bar{A}, B$). In both categories, the arrow composition is defined by cut elimination. More precisely, the composition of $P \rhd \bar{A}, B$ (which is an arrow from A to B) and $Q \rhd \bar{B}, C$ (which goes from B to C) is obtained by elimination the cut from $P \oplus Q \rhd \bar{A}, B \Diamond \bar{B}, C$.

Theorem 3.4.5. *The category* **Net**$\mathbb{B}$ *is a flat and contractible* B5*-category.*

The proof is a straightforward exercise since the basic morphisms demanded by the definitions (see Chapter 2) are obtained by the obvious sequent proofs, and the demanded equations hold trivially.

Note that we do *not* have results like Theorem 3.4.4 or 3.4.5 for general W. Figure 3.5 shows an example for the non-confluence of cut reduction for $\mathbb{N}$-prenets.

Although we do not have confluence in general, we could for a given proof with cuts define a "most canonical" way of obtaining a cut-free proof net: do simultaneous elimination of all the cuts at once. But still, this is not enough to give us a category. In Section 3.6 we will see how $\mathbb{N}$-nets can be made into a category by adding further restrictions.

Let us now compare our cut elimination with other (syntactic) cut elimination procedures for classical propositional logic. In the case of $\mathbb{B}$-nets, the situation is quite similar to the sequent calculus: the main differences is that we do not lose any information in the weak-weak-case (shown in (1.2) on page 4), although we lose some information of a numeric nature in the cont-cont case.

In the case of $\mathbb{N}$-nets, the situation is very different. Let us use (3.6) as example, whose

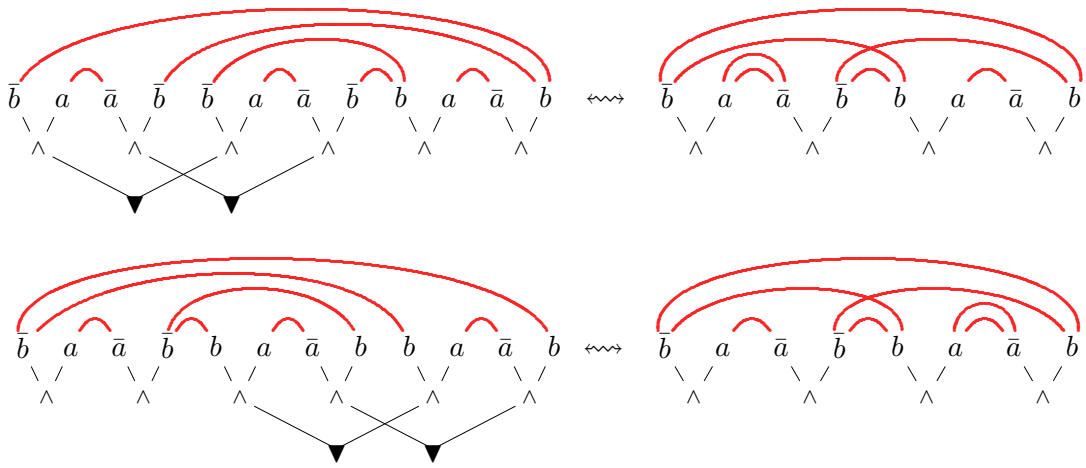


Figure 3.6: The two different results of applying *sequent calculus cut elimination* to the proof (3.6). **Left:** Written as Girard/Robinson proof-net. **Right:** Written as $\mathbb{N}$-proof-net.

corresponding sequent proof is shown below:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \bar{b}, b}\,\mathsf{id} \quad \dfrac{}{\vdash a, \bar{a}}\,\mathsf{id}}{\vdash \bar{b} \wedge a, \bar{a}, b}\,\wedge \quad \dfrac{}{\vdash \bar{b}, b}\,\mathsf{id}}{\vdash \bar{b} \wedge a, \bar{a} \wedge \bar{b}, b, b}\,\wedge}{\vdash \bar{b} \wedge a, \bar{a} \wedge \bar{b}, b}\,\mathsf{cont} \quad \dfrac{\dfrac{\dfrac{}{\vdash \bar{b}, b}\,\mathsf{id} \quad \dfrac{\dfrac{}{\vdash a, \bar{a}}\,\mathsf{id} \quad \dfrac{}{\vdash \bar{b}, b}\,\mathsf{id}}{\vdash \bar{b}, a, \bar{a} \wedge b}\,\wedge}{\vdash \bar{b}, \bar{b}, \bar{b} \wedge a, \bar{a} \wedge \bar{b}}\,\wedge}{\vdash \bar{b}, b \wedge a, \bar{a} \wedge b}\,\mathsf{cont}}{\vdash \bar{b} \wedge a, \bar{a} \wedge \bar{b}, b \wedge a, \bar{a} \wedge b}\,\mathsf{cut}$$

The sequent calculus cut elimination needs to duplicate either the right-hand side proof or the left-hand side proof. The two possible outcomes, together with their presentation as $\mathbb{N}$-nets, are shown in Figure 3.6, where the ▼ stand for contractions[13]. However, in our setting, the result of eliminating the cut in (3.6) is always (3.3), whether we are in $\mathbb{B}$-nets or in $\mathbb{N}$-nets.

Although for $\mathbb{N}$-nets the cut elimination operation does not have a close relationship to the sequent calculus, there is a good correspondence with cut elimination in the calculus of structures, when done via splitting [Gug07].

3.5 From Deep Inference Derivations to Prenets

In Section 3.2 we have seen how sequent proofs can be translated into prenets. In this section, I will do the same for deep inference derivations. Let me use the system SKS [BT01], whose rules are given in Figure 3.7. Recall that these rules can, like rewrite rules, be applied inside arbitrary contexts. For example,

$$\mathsf{ai}{\uparrow}\,\dfrac{[a \vee (c \wedge [(b \wedge \bar{b}) \vee \bar{a}])] \vee (b \wedge \bar{c})}{[a \vee (c \wedge [\mathsf{f} \vee \bar{a}])] \vee (b \wedge \bar{c})}$$

[13]This idea of using explicit contraction nodes was sketched in [Gir91] and is carried out in detail in [Rob03].

$$\mathsf{ai}{\downarrow}\,\frac{K\{\mathsf{t}\}}{K\{a \vee \bar{a}\}} \qquad \mathsf{s}\,\frac{K\{A \wedge [B \vee C]\}}{K\{(A \wedge B) \vee C\}} \qquad \mathsf{ai}{\uparrow}\,\frac{K\{a \wedge \bar{a}\}}{K\{\mathsf{f}\}}$$

$$\mathsf{aw}{\downarrow}\,\frac{K\{\mathsf{f}\}}{K\{a\}} \qquad \mathsf{m}\,\frac{K\{(A \wedge B) \vee (C \wedge D)\}}{K\{[A \vee C] \wedge [B \vee D]\}} \qquad \mathsf{aw}{\uparrow}\,\frac{K\{a\}}{K\{\mathsf{t}\}}$$

$$\mathsf{ac}{\downarrow}\,\frac{K\{a \vee a\}}{K\{a\}} \qquad \mathsf{nm}{\downarrow}\,\frac{K\{\mathsf{f}\}}{K\{\mathsf{f} \wedge \mathsf{f}\}} \qquad \mathsf{nm}{\uparrow}\,\frac{K\{\mathsf{t} \vee \mathsf{t}\}}{K\{\mathsf{t}\}} \qquad \mathsf{ac}{\uparrow}\,\frac{K\{a\}}{K\{a \wedge a\}}$$

$$\alpha{\downarrow}\,\frac{K\{A \vee [B \vee C]\}}{K\{[A \vee B] \vee C\}} \qquad \sigma{\downarrow}\,\frac{K\{A \vee B\}}{K\{B \vee A\}} \qquad \sigma{\uparrow}\,\frac{K\{A \wedge B\}}{K\{B \wedge A\}} \qquad \alpha{\uparrow}\,\frac{K\{A \wedge (B \wedge C)\}}{K\{(A \wedge B) \wedge C\}}$$

$$\mathbf{f}{\downarrow}\,\frac{K\{A\}}{K\{A \vee \mathsf{f}\}} \qquad \mathbf{t}{\uparrow}\,\frac{K\{A \vee \mathsf{f}\}}{K\{A\}} \qquad \mathbf{f}{\uparrow}\,\frac{K\{\mathsf{t} \wedge A\}}{K\{A\}} \qquad \mathbf{t}{\downarrow}\,\frac{K\{A\}}{K\{\mathsf{t} \wedge A\}}$$

Figure 3.7: The inference rules of system SKS

is a correct application of the rule $\mathsf{ai}{\uparrow}$ inside the context $[a \vee (c \wedge [\{\cdot\} \vee \bar{a}])] \vee (b \wedge \bar{c})$. Note that we allow negation only on atoms, and therefore the hole of the context never occurs in a negative position. A *derivation* $\Phi\colon A \to B$ in SKS is a rewrite path from A to B using the rules in Figure 3.7. We call A the *premise* and B the *conclusion* of Φ. A derivation is also denoted as

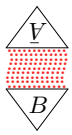
$$\begin{array}{c} A \\ \Phi \Big\| \mathsf{S} \\ B \end{array} .$$

where S is the set of inference rules used in Φ. A derivation whose premise is t is called a *proof* and is denoted by

$$\begin{array}{c} \Phi \Big\| \mathsf{S} \\ B \end{array} .$$

The translation from a derivation into prenets with cuts is done by assigning to each inference rule a *rule net*:

$$\mathsf{r}\,\frac{A}{B} \quad \rightsquigarrow$$



where the linking is subject to certain side conditions which depend on the rule r.

For the occurrences of $\mathbf{t}$ and $\mathbf{f}$ in the premise and conclusion of r there is no choice: There can never be an edge coming out of an $\mathbf{f}$, and there is always exactly one edge connecting a $\mathbf{t}$ to itself. But we have to explain how to connect propositional variables.

Figures 3.8, 3.9, 3.10, 3.11 and 3.12 show the rule nets for the rules of system SKS, as they are given in Figure 3.7. For the rules s, m, $\alpha{\downarrow}$, $\alpha{\uparrow}$, $\sigma{\downarrow}$, and $\sigma{\uparrow}$, it is intuitively

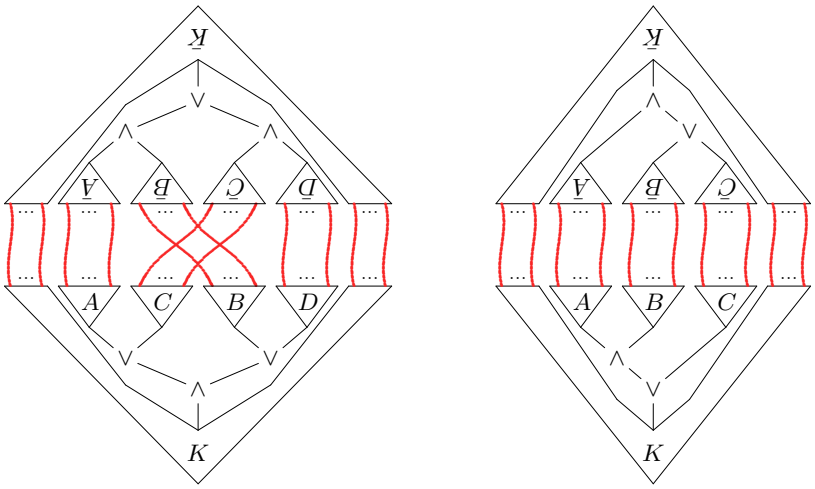


Figure 3.8: The shape of m-rule nets and s-rule nets

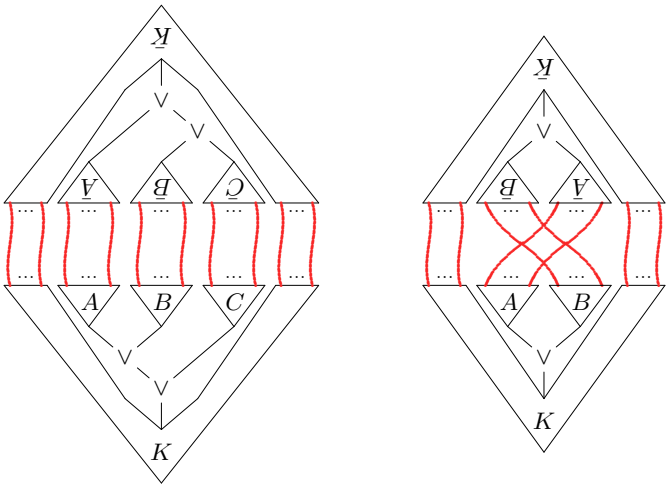


Figure 3.9: The shape of $\alpha\downarrow$-rule nets and $\sigma\downarrow$-rule nets

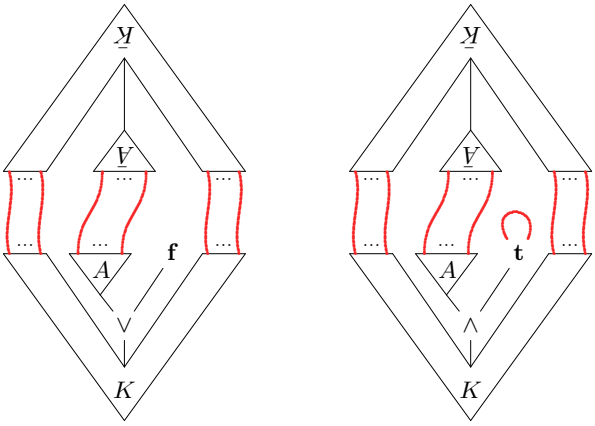


Figure 3.10: The shape of $\mathbf{f}\downarrow$-rule nets and $\mathbf{t}\downarrow$-rule nets

clear what should happen: every leaf in the premise tree is connected to its counterpart

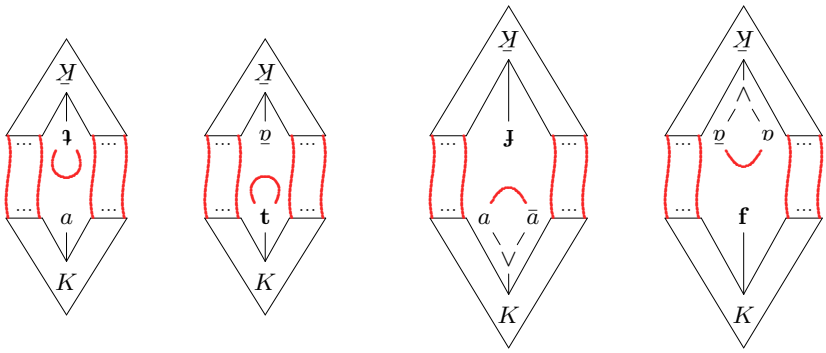


Figure 3.11: The shape of rule nets for aw↓, aw↑, ai↓, and ai↑

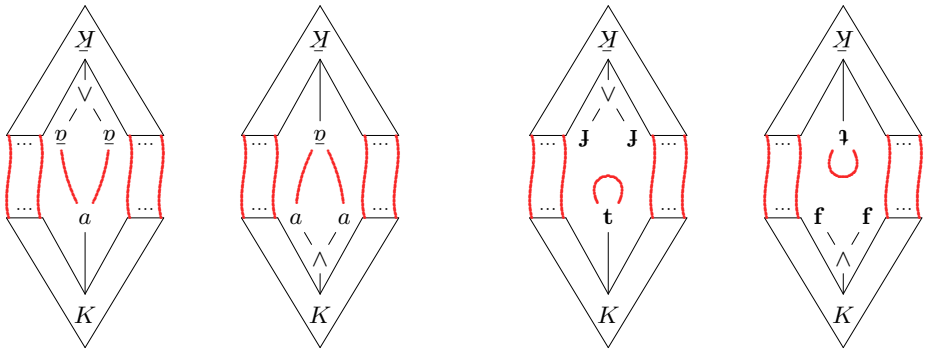


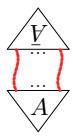
Figure 3.12: The shape of rule net for ac↓, ac↑, nm↓, and nm↑

in the conclusion via an edge in the linking; and there are no other edges. Note that the rule nets for $\alpha\downarrow$ and $\alpha\uparrow$ are identical; one written as the upside-down version of the other. The same holds for all other pairs of dual rules. For $\alpha\downarrow$, $\sigma\downarrow$, $\mathbf{f}\downarrow$, and $\mathbf{t}\downarrow$ only one version is shown (Figures 3.9 and 3.10), but for the atomic rules both versions are given (Figures 3.11 and 3.12) because it is instructive to see them next to each other. In Figure 3.12 we show the rule nets for ac↓, ac↑, nm↓, and nm↑.

We can use cuts to plug rule nets together to get *derivation nets*, as it is shown in Figure 3.13. Note that in derivation nets the "duality" between derivations

$$\begin{array}{c} A \\ \Delta \Big\| \mathsf{s} \\ B \end{array} \qquad \text{and} \qquad \begin{array}{c} \bar{B} \\ \bar{\Delta} \Big\| \mathsf{s} \\ \bar{A} \end{array}$$

disappears because both are represented by *the same* net. A derivation which contains no rules, i.e., premise and conclusion coincide, is represented by the *identity net*:



(3.17)

For eliminating the cut we can proceed as in Section 3.4. And if we choose $\mathbb{B}$ as semiring of weights, we get again the category $\mathbf{Net}\mathbb{B}$.

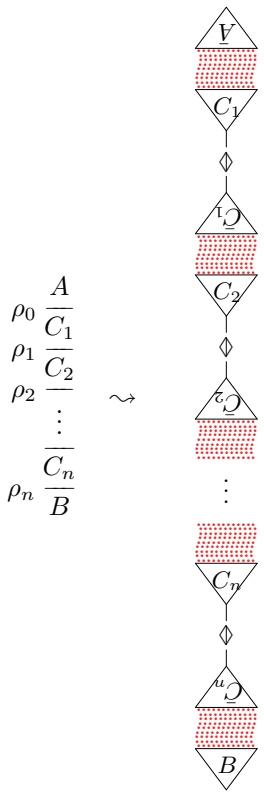
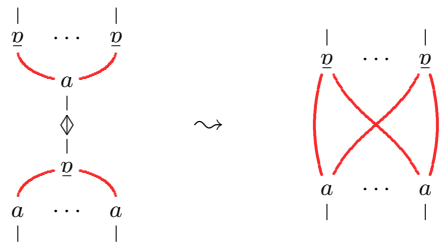


Figure 3.13: From derivations to derivation nets.

3.6 Proof Invariants Through Restricted Cut Elimination

If we do not eliminate any cut, then a derivation net, as defined in the previous section, contains the same amount of information as an SKS derivation; there is a one-to-one correspondence between the two. Thus, these derivation nets are not suited as invariants for proofs. On the other hand, if we eliminate all the cuts, with $\mathbb{N}$ as semiring of weights, we do not get reasonable invariants for proofs either, because the cut-free derivation net for a given derivation is not uniquely defined. The obtained $\mathbb{N}$-prenet depends on the chosen order for eliminating the cuts. This, in particular, means that we do not have a category of $\mathbb{N}$-prenets.

An inspection of the cut elimination process for $\mathbb{N}$-prenets shows that there is only one case which is responsible for the non-confluence: atomic cut reduction when both cut atoms are connected to many other atoms:



(3.18)

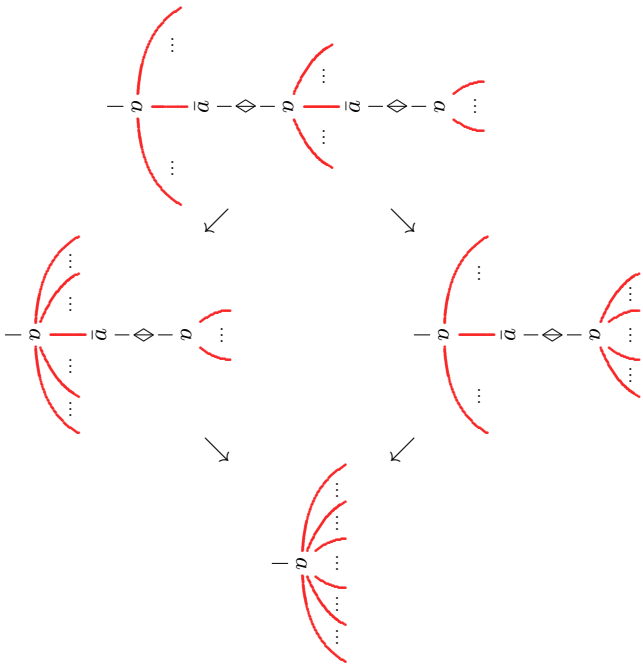
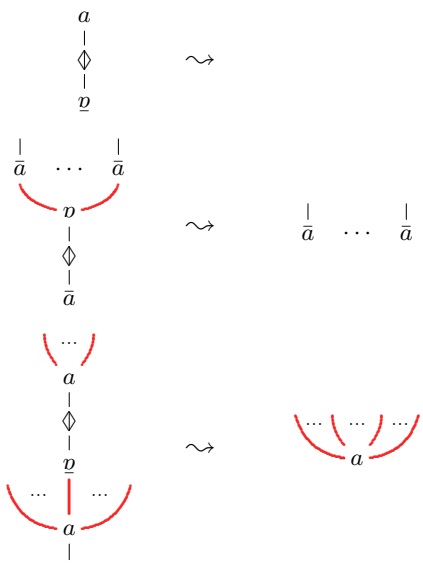


Figure 3.14: Critical peak for level-C cut reduction

For this reason, we will now forbid this reduction. We speak of *level-C cut reduction*[14] if a cut on atoms can only be reduced if at least one of the two cut atoms is connected to at most one other atom that is not participating in the cut. I.e., we allow only atomic cut reductions of the following forms:



A prenet is called *C-reduced* if no further cut reduction steps according to these restrictions are possible.

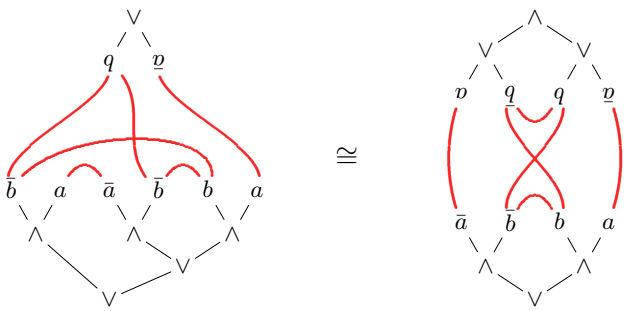
[14]The terminology comes from the terminology of bureaucracy of type A, B, and C [Gug04, Gug04, Str09].

Theorem 3.6.1. *Level-C reduction is terminating and confluent.*

Proof. Termination follows for the same reason as before. For showing confluence, it suffices to note that Figure 3.14 shows the only critical peak. □

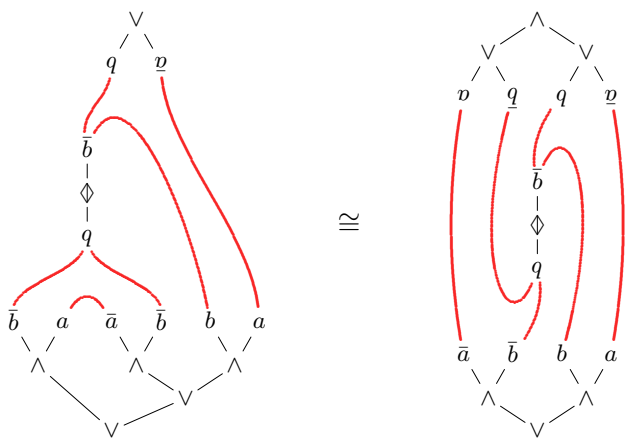
Because of this theorem, we can define two more categories that I call here **PreC** and **DeriC**. As before, in both categories the objects are the formulas. For **PreC** the arrows are the C-reduced prenets with two conclusions, and for **DeriC** we consider only those C-reduced prenets that are obtained from an SKS-derivation as described in the previous section. As expected, arrow composition is defined by level-C cut reduction.

As the category **Pre**$\mathbb{B}$, the category **PreC** is star-autonomous and obeys all the axioms needed for a B5-category. However, **PreC** is not flat and not contractible. In the same way, the category **DeriC** is not flat and not contractible. Furthermore, surprisingly the category **DeriC** is not even a B5-category, simply because it is not star-autonomous. The reason is that we do in the general case not have a natural bijection between the proofs from $A \wedge B$ to C and the proofs from B to $A \Rightarrow C$. To see this consider the example on the left below, which can easily be obtained from an SKS-derivation:



(3.19)

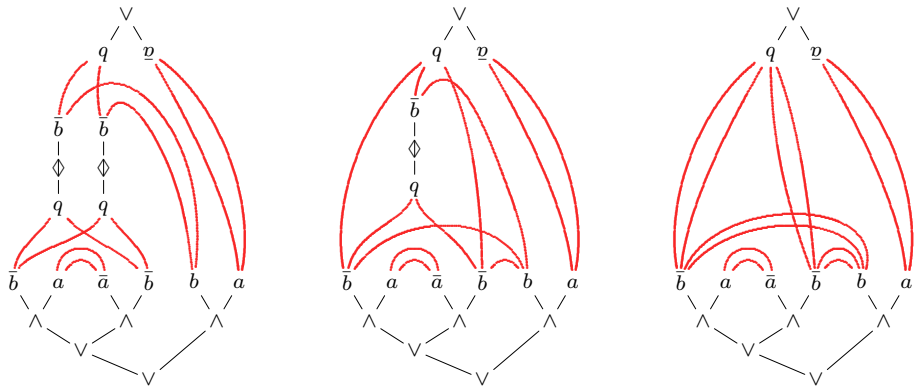
The prenet on the right above is obtained by the star-autonomous structure of **PreC**. But there is no SKS-derivation that correspond to this prenet. On the other hand, if we reintroduce the cut as in the example on the left below, we can do the same transformation as above, without leaving the category **DeriC**:



(3.20)

Both prenets in (3.20) are direct translations of SKS-derivations, as the reader can easily verify. This raises the question whether we can find a deep inference proof system S such that its C-reduced nets form a star-autonomous subcategory of **PreC**.

Even though **DeriC** is not star-autonomous, it obeys the equations discussed in Section 2.6. **PreC** and **DeriC** also carry the semigroup structure on the homsets, as defined in Section 2.4. In particular, this semigroup structure is *not* idempotent. The sum of two prenets is given by their union. This is best understood by seeing an example. Let f be the proof graph on the left in (3.9) and g the one on the right. Then we can form $f + f$, $f + g$, and $g + g$ as follows:



Furthermore, we can equip the Hom-sets of our categories with a partial order, defined by cut elimination, as indicated in Section 2.5: We say $f \leq g$ if g is obtained from f by eliminating some of the remaining cuts[15], as it is the case in our example above for f and g. Then we also have $f + f \leq f + g \leq g + g$. The important observation about the semigroup structure and this partial order structure is, that *they are independent*. Although this seems to be natural from the viewpoint of our prenets, it is not the case in the LK-*categories* of [FP04c] which are based on the proof nets in [Rob03] and the sequent calculus LK [Gen34]. In an LK-category the sum-of-proofs-semigroup structure and the cut-elimination-partial-order structure on the Hom-sets determine each other uniquely via $f \preccurlyeq g$ iff $f + g = f$ (see Section 2.5).[16] In the case of prenets this collapse is only present in the case of $W = \mathbb{B}$:

Theorem 3.6.2. *The category* $\mathbf{Net}\mathbb{B}$ *is an* LK-*category.*

Proof. Let $f, g\colon A \to B$ be two maps in $\mathbf{Net}\mathbb{B}$. Let f be given by the proof net $P \rhd \bar{A}, B$ and g be given by $Q \rhd \bar{A}, B$. Then, by what has been said above, we define $f \preccurlyeq g$ iff $Q \subseteq P$. After Theorem 3.4.5 it only remains to show that equation (LK-Δ) holds for all f. But this follows immediately from the definition of composition of prenets. □

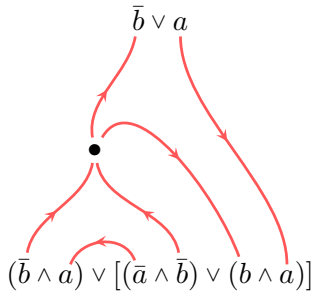
3.7 Prenets as Coherence Graphs

We can give a more compact notation of C-reduced prenets if we write the formulas not as trees but as usual strings, and replace the atomic cuts by big dots. The lost information can be recovered by simply directing the edges. Here is a simple example that corresponds

[15] Even if this process is not confluent, we stay in the realm of C-reduced proof graphs.

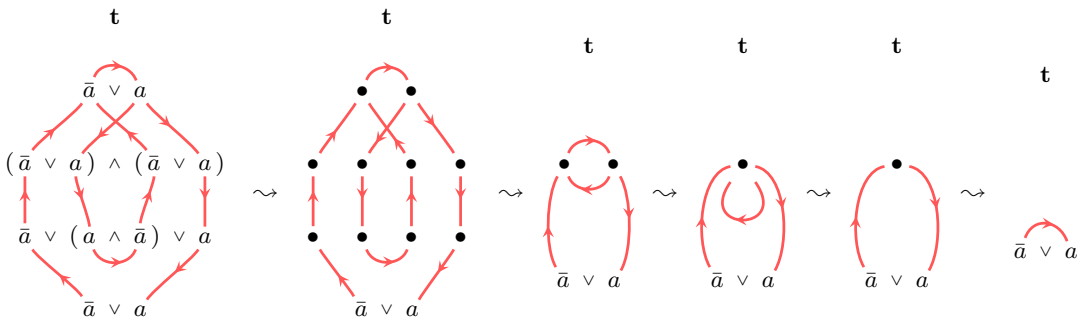
[16] In [DP04] and [LS05a] there is also a partial order structure on the Hom-sets, simply because the semigroup structure is idempotent. But this partial order structure has nothing to do with cut elimination, simply because everything is *a priory* cut-free.

to the left example in (3.20):

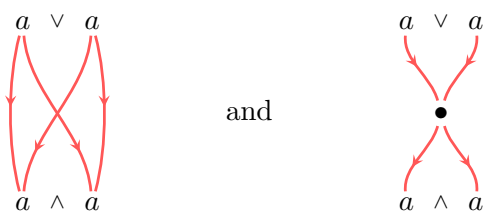


(3.21)

This allows us to use C-reduced prenets as "coherence graphs" for the categories studied in Chapter 2. As example, consider the contractibility axiom (applied to a single atom):



However, since we do not (yet) have a coherence theorem, we can use this notation only as help to guide the intuition. For example, in Figure 3.15 we use this notation to illustrate the idea behind the proof of Theorem 2.7.6. The middle equation in the second line is (2.39), i.e, the identity of the two nets below:



(3.22)

The left-most equation in the second line and the right-most equation in the first line in Figure 3.15 are both the contractibility equation (2.41). Everything else in Figure 3.15 is rather trivial from the viewpoint of proof nets.

3.8 Atomic Flows

In this section we present yet another variant of the basic idea underlying this chapter. We start from a countable set $\mathscr{A}$ of *atomic types*, equipped with an involutive bijection $\overline{(\cdot)} : \mathscr{A} \to \mathscr{A}$, such that for all $a \in \mathscr{A}$, we have $\bar{a} \neq a$ and $\bar{\bar{a}} = a$. A *(flow) type* is a finite list of atomic types, denoted by $p, q, r, \ldots$, and we write $p \,|\, q$ for the list concatenation of p and q, and we write 0 for the empty list. An *atomic flow* $\phi : p \to q$ is a two-dimensional

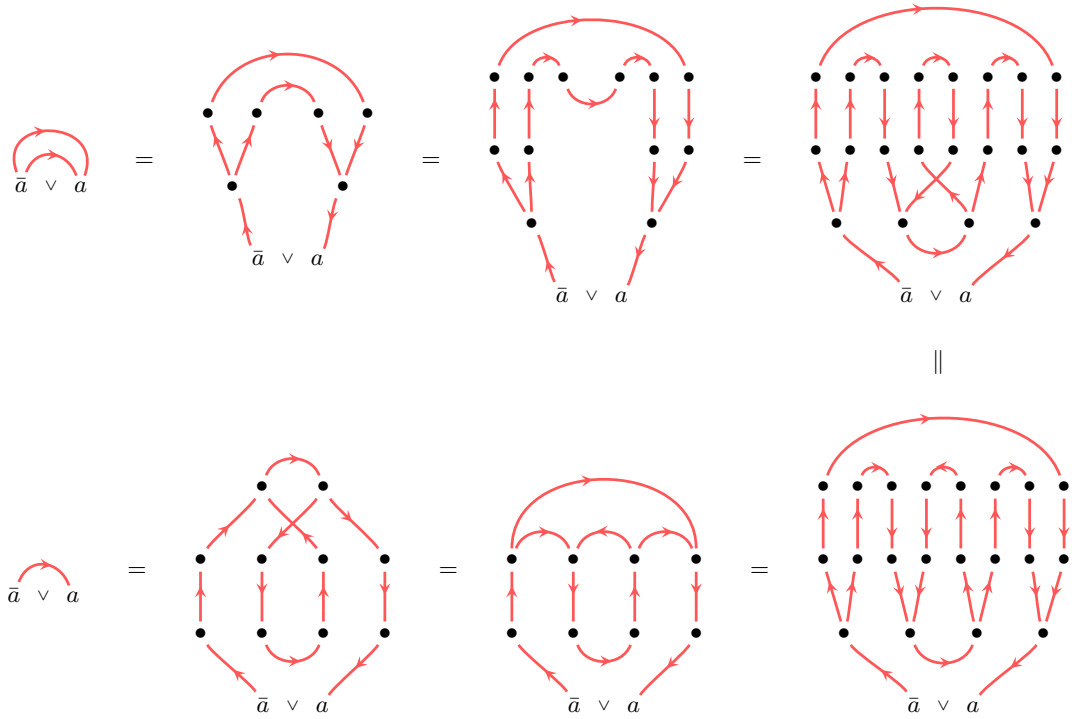
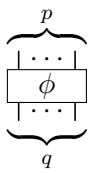


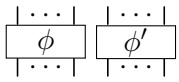
Figure 3.15: The idea of the proof of Theorem 2.7.6

diagram [Laf95a], written as

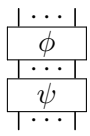


where p is the *input type* and q is the *output type*. The number of edges corresponds to the lengths of the lists, and each edge is labelled by the corresponding list element. For each type q, we have the *identity flow* id_q:

We can compose atomic flows horizontally: for $\phi\colon p \to q$ and $\phi'\colon p' \to q'$, we get $\phi|\phi'\colon p|p' \to q \mid q'$ of the shape



And we can compose atomic flows vertically: for $\phi\colon p \to q$ and $\psi\colon q \to r$, we get $\psi \circ \phi\colon p \to r$ of the shape



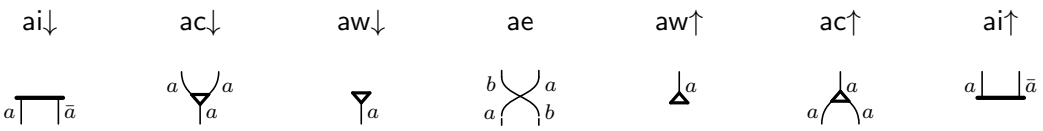


Figure 3.16: Generators for atomic flows

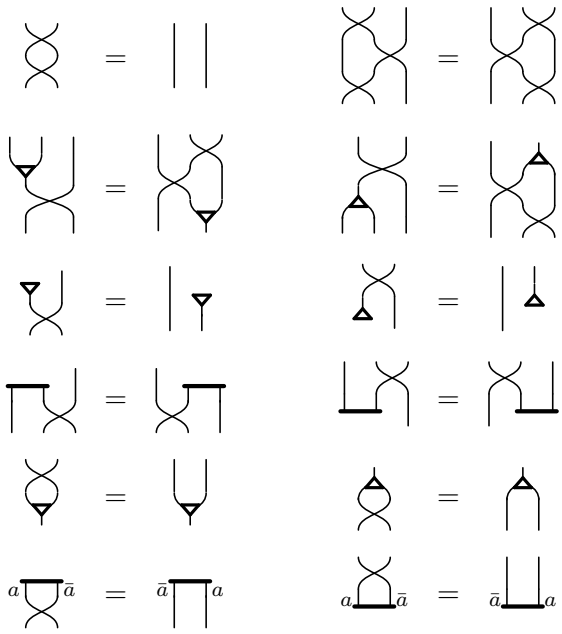
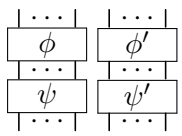


Figure 3.17: Relations for atomic flows

For $\phi\colon p \to q$ we have $\phi \circ \mathsf{id}_p = \phi = \mathsf{id}_q \circ \phi$ and $\phi \mid \mathsf{id}_0 = \phi = \mathsf{id}_0 \mid \phi$. We also have $(\psi \circ \phi) \mid (\psi' \circ \phi') = (\psi \mid \psi') \circ (\phi \mid \phi')$ which is pictured as



Finally, we have to give a set of generators and relations, which is done in Figures 3.16 and 3.17. It is easy to see that atomic flows form a (strict) monoidal category, that we denote by **AF**.

The generators in Figure 3.16 are called ai↓ (*atomic interaction down*), ac↓ (*atomic contraction down*), aw↓ (*atomic weakening down*), ae (*atomic exchange*), aw↑ (*atomic weakening up*), ac↑ (*atomic contraction up*), and ai↑ (*atomic interaction up*). Note that some of them have already been studied, e.g., [Bur91, Laf95a, ER06, MT10]. The typing information in Figure 3.16 says that

- for ai↓ (resp. ai↑) the two output edges (resp. input edges) carry opposite atomic types,

- for ac↓ (resp. ac↑) all input and output edges carry the same atomic type,
- for aw↓ (resp. aw↑) there are no typing restrictions, and
- for ae, the left input has to carry the same type as the right output, and vice versa.

When picturing an atomic flow we will omit the typing when this information is irrelevant or clear from context, as done in Figure 3.17. The typing is needed for two reasons: first, we need to exclude illegal flows like

.

Note, that for this it would suffice to have only two types, + and −, as done in [GG08]. The second reason for having the types here is the use of the flows as tool for proof transformations later in this chapter.

Definition 3.8.1. For a given atomic flow diagram ϕ, we define its *atomic flow graph* G_ϕ to be the directed acyclic graph whose vertices are the generators ai↓, ai↑, ac↓, ac↑, aw↓, aw↑ (i.e., all except ae) appearing in ϕ, whose incoming (resp. outcoming) edges are the incoming (resp. outcoming) edges of ϕ, and whose inner edges are downwards oriented as indicated by the flow diagram for ϕ. A *path* in ϕ is a path in G_ϕ.

Remark 3.8.2. If we label the edges in G_ϕ by the corresponding atomic type, then for every path in ϕ, all its edges carry the same label.

Remark 3.8.3. In [GG08], atomic flows have been defined as directed graphs, as done in Definition 3.8.1. Indeed, G_ϕ is the "canonical representative" of a class of flow diagrams wrt. to the equalities in Figure 3.17. However, with this definition the order of the input/output edges is lost, which makes the vertical composition and the mapping from formal derivations (done in the next section) more difficult to define.

Remark 3.8.4. We could add the relations

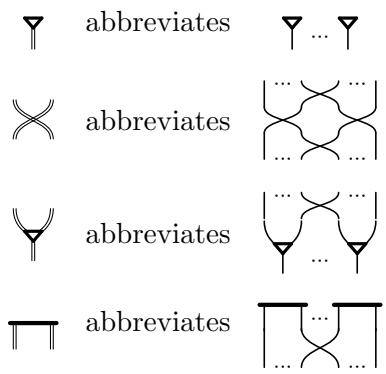
and (3.23)

and their duals to the ones given in Figure 3.17, and this would equip every object in **AF** with a monoid and a comonoid structure. We do not use the equations (3.23) here, in order to maintain the close relationship to the original definition of atomic flows in [GG08] and to derivations in SKS (see next section).

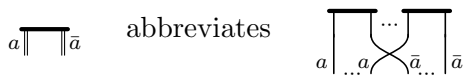
Notation 3.8.5. For making large atomic flows easier to read, we introduce the following notation:

‖ abbreviates | ⋯ |

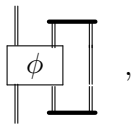
This can be extended to all generators:



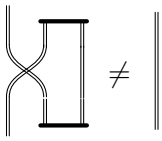
And similarly for aw↑, ac↑, and ai↑. In each case we allow the number of edges to be 0, which then yields the empty flow. Moreover, if we label an abbreviation with atomic type a, we mean that each edge being abbreviated has type a. For instance:



Remark 3.8.6. The category **AF** of atomic flows is strictly monoidal, but it is not compact closed (and also not star-autonomous): although we can for a given atomic flow $\phi\colon p|x \to q|x$ define the atomic flow $\mathrm{Tr}^x(\phi)\colon p \to q$ as

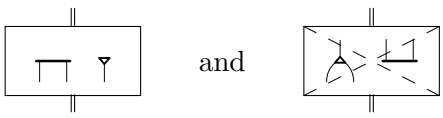


the category **AF** is not traced [JSV96], because it does not obey yanking:



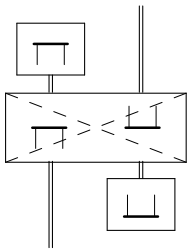
This makes the category **AF** very different from the category **PreC**.

Notation 3.8.7. A box containing some generators stands for an atomic flow generated only from these generators, and a box containing some generators crossed out stands for an atomic flow that does not contain any of these generators. For example, the two diagrams



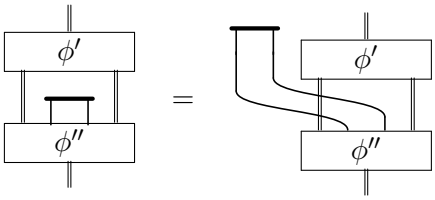
stand for a flow that contains only ai↓ and aw↓ generators and a flow that does not contain any ac↑ and ai↑ generators, respectively.

Proposition 3.8.8. *Every atomic flow ϕ can be written in the following form:*



(3.24)

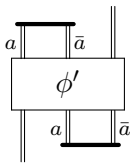
Proof. Let ϕ be given and pick an arbitrary occurrence of $\mathsf{ai}{\downarrow}$ inside ϕ. Then ϕ can be written as shown on the left below.



(3.25)

The equality follows by induction on the number of vertical edges to cross, For $\mathsf{ai}{\uparrow}$ we proceed dually. □

Proposition 3.8.9. *Let a be an atomic type. Then every atomic flow ϕ can be written as*



,

(3.26)

where ϕ' is ai*-free with respect to a.*

Proof. We apply the construction of the proof of Proposition 3.8.8 together with the relations in the last line of Figure 3.17. □

3.9 From Formal Deductions to Atomic Flows

We can assign to each formula, sequent, or list of sequents its flow type by forgetting the structural information of $\wedge$, $\vee$, t and f, and simply keeping the list of atomic types as they occur in the formulas. For a formula A, we denote this type by $\mathit{fl}(A)$.

Similarly, we will assign flows to inference rules. Rules like

$$\wedge\,\frac{\vdash \Gamma, A \qquad \vdash B, \Delta}{\vdash \Gamma, A \wedge B, \Delta} \qquad \text{and} \qquad \vee\,\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \vee B}$$

will be translated into the identity flows $\mathsf{id}_{\mathit{fl}(\Gamma)} | \mathsf{id}_{\mathit{fl}(A)} | \mathsf{id}_{\mathit{fl}(B)} | \mathsf{id}_{\mathit{fl}(\Delta)}$ and $\mathsf{id}_{\mathit{fl}(\Gamma)} | \mathsf{id}_{\mathit{fl}(A)} | \mathsf{id}_{\mathit{fl}(B)}$, respectively. And the rules

$$\mathsf{cont}\,\frac{\vdash \Gamma, A, A}{\vdash \Gamma, A} \qquad \text{and} \qquad \mathsf{weak}\,\frac{\vdash \Gamma}{\vdash \Gamma, A}$$

will be translated into flows of the shape

and .

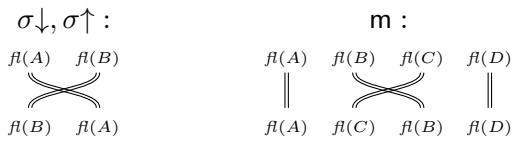
In this manner, we could translate whole sequent proofs into atomic flows. However, atomic flows carry more symmetries than present in the sequent calculus. In order to be able to mirror the richness of atomic flows inside a sound and complete deductive system for classical logic, we use here the deep inference system SKS (shown in Figure 3.7 on page 64). In the next section we will need the following property of SKS:

Proposition 3.9.1. *The inference rules*

$$\mathsf{c}{\downarrow}\,\frac{K\{A \vee A\}}{K\{A\}} \quad \textit{and} \quad \mathsf{c}{\uparrow}\,\frac{K\{A\}}{K\{A \wedge A\}}$$

are derivable in system SKS. [BT01]

I write **SKS** to denote the category whose objects are the formulas and whose arrows are the derivations of SKS. This system has the advantage that the rules for weakening, contraction, and identity and cut are already in atomic form. Thus, it is straightforward to translate SKS derivations into atomic flows. Formally, we assign to each context $K\{\cdot\}$ a left type and a right type denoted by $l(K\{\cdot\})$ and $r(K\{\cdot\})$, containing the lists of atomic types appearing in $K\{\cdot\}$ on the left, respectively on the right of the hole $\{\cdot\}$. For example, for $K\{\cdot\} = [a \vee (c \wedge [\{\cdot\} \vee \bar{a}])] \vee (b \wedge \bar{c})$ we have $l(K\{\cdot\}) = \langle a, c\rangle$ and $r(K\{\cdot\}) = \langle \bar{a}, b, \bar{c}\rangle$. Then, for each rule r of SKS we define the *rule flow* $\mathit{fl}(\mathsf{r})$ as follows: we map the rules $\mathsf{ai}{\downarrow}$, $\mathsf{ai}{\uparrow}$, $\mathsf{ac}{\downarrow}$, $\mathsf{ac}{\uparrow}$, $\mathsf{aw}{\downarrow}$, and $\mathsf{aw}{\uparrow}$ to the corresponding generator (with the appropriate typing), and we map the rules $\sigma{\downarrow}$, $\sigma{\uparrow}$, and m to the permutation flows shown below:



All remaining rules are mapped to the identity flow.

Then an inference step

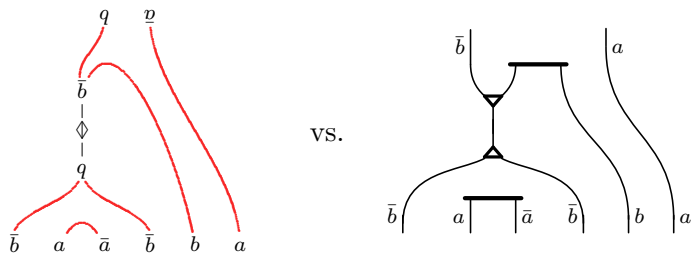
$$\mathsf{r}\,\frac{K\{A\}}{K\{B\}} \qquad \text{is mapped to} \qquad \mathsf{id}_{l(K\{\cdot\})} \mid \mathit{fl}(\mathsf{r}) \mid \mathsf{id}_{r(K\{\cdot\})}$$

A derivation Φ is mapped to the atomic flow $\phi = \mathit{fl}(\Phi)$, which is the vertical composition of the atomic flows obtained from the inference steps in Φ. This translation defines a forgetful functor $\mathit{fl}\colon \mathbf{SKS} \to \mathbf{AF}$. This functor has an interesting property: for every atomic flow there is a derivation that maps to it:

Theorem 3.9.2. *For every flow* $\phi\colon p \to q$ *there is a derivation* $\Phi\colon A \to B$ *with* $\mathit{fl}(A) = p$ *and* $\mathit{fl}(B) = q$ *and* $\mathit{fl}(\Phi) = \phi$. [GG08]

Theorem 3.9.2 only works because the flows forget the structural information about $\wedge$, $\vee$, t and f. In that respect, atomic flows are different from C-reduced prenets. However,

if we compare the results of translating SKS-derivations into C-prenets respectively atomic flows, then both are very similar, as the following example shows, which is the same as the one on the left in (3.20):



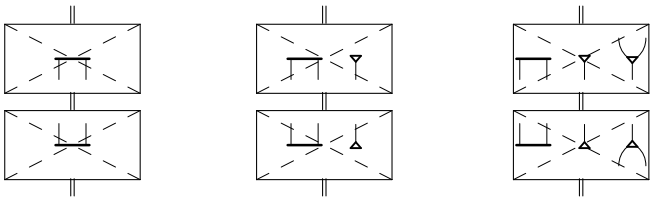
If we fix $\phi\colon p \to q$ together with A and B with $fl(A) = p$ and $fl(B) = q$, we can in general not provide a derivation $\Phi\colon A \to B$ with $fl(\Phi) = \phi$. We are thus interested in properties of atomic flows that can be lifted to derivations, in the following sense:

Definition 3.9.3. We say that a binary relation R on atomic flows *can be lifted to* SKS, if $R(\phi, \phi')$ implies that for every derivation $\Phi\colon A \to B$ with $fl(\Phi) = \phi$ there is a derivation $\Phi'\colon A \to B$ with $fl(\Phi') = \phi'$.

3.10 Normalizing Derivations via Atomic Flows

In this section I will show how atomic flows can be used to provide a normalization procedure for SKS-derivations. This is done by showing transformations on atomic flows that can be lifted to derivations in SKS.

Definition 3.10.1. An atomic flow is *weakly streamlined* (resp., *streamlined* and *strongly streamlined*) if it can be represented as the flow on the left (resp., in the middle and on the right):



Proposition 3.10.2. *An atomic flow ϕ is weakly streamlined if and only if in G_ϕ there is no path from an* ai↓*-vertex to an* ai↑*-vertex.*

Proof. Immediate from (3.25), read from right to left. □

Definition 3.10.3. An atomic flow ϕ is *weakly streamlined with respect to an atomic type a* if in G_ϕ there is no path from an ai↓-vertex to an ai↑-vertex, whose edges are labelled by a or $\bar{a}$.

Definition 3.10.4. A derivation $\Phi\colon A \to B$ is *weakly streamlined (resp. streamlined, resp. strongly streamlined)* if $fl(\Phi)$ is weakly streamlined (resp. streamlined, resp. strongly streamlined).

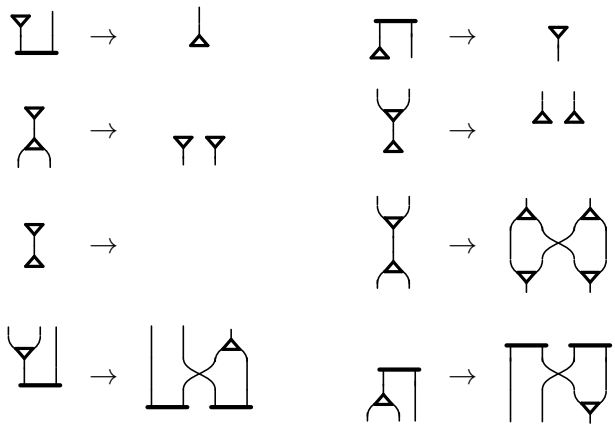


Figure 3.18: Local rewrite rules

The property *strongly streamlined* can indeed be seen as the up-down symmetric generalization being *cut-free*:

Proposition 3.10.5. *Every strongly streamlined proof in* SKS *does not contain any of the rules* ai↑*,* ac↑*,* aw↑*.*

Proof. If the premise of a derivation is t, then the upper box of its flow, as given in Definition 3.10.1, must be empty. □

In this section, we are going to show the following:

Theorem 3.10.6. *For every* SKS *derivation from A to B, there is a* SKS*-derivation from A to B that is strongly streamlined.*

From which we get immediately a cut elimination theorem for SKS:

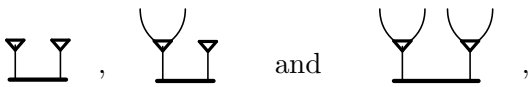
Corollary 3.10.7. *For every* SKS*-proof of A, there is a* SKS*-proof of A, not containing any of the rules* ai↑*,* ac↑*,* aw↑*.*

Proof. By Theorem 3.10.6 and Proposition 3.10.5. □

Consider now the rewrite relation on atomic flows generated by the rules shown in Figure 3.18, that is denoted by $\overset{\mathsf{cw}}{\to}$ (see [GG08]).

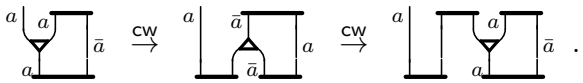
Proposition 3.10.8. *The rewrite relation $\overset{\mathsf{cw}}{\to}$ is locally confluent.*

Proof. The result follows from a case analysis on the critical peaks, which are:



and their duals. □

However, in general the reduction $\overset{\mathsf{cw}}{\to}$ is not terminating. This can easily be seen by the following example:



The reason is that there can be cycles composed of paths connecting instances of the $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$ generators. The purpose of the notion "weakly streamlined" (Definition 3.10.1) is precisely to avoid such a situation.

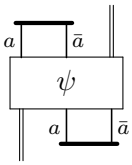
Theorem 3.10.9. *Every weakly streamlined atomic flow has a unique normal form with respect to* $\overset{\mathsf{cw}}{\to}$*, and this normal form is strongly streamlined.*

Proof. We do not show the proof of termination here since it can be found in [GG08]. We only note that the crucial point is Proposition 3.10.2. Then, by Proposition 3.10.8, we have uniqueness of the normal form. Since $\overset{\mathsf{cw}}{\to}$ preserves the property of being weakly streamlined, and in the normal form there are no more redexes for $\overset{\mathsf{cw}}{\to}$, there is no generator $\mathsf{ai}{\downarrow}$, $\mathsf{aw}{\downarrow}$, $\mathsf{ac}{\downarrow}$ above a generator $\mathsf{ai}{\uparrow}$, $\mathsf{aw}{\uparrow}$, $\mathsf{ac}{\uparrow}$. □

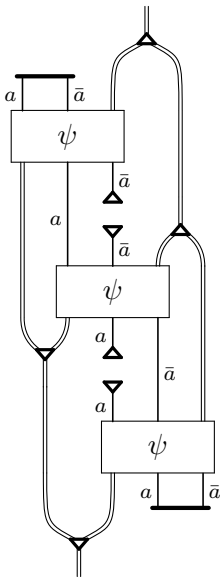
Theorem 3.10.10. *The relation* $\overset{\mathsf{cw}}{\to}$ *can be lifted to* SKS. [GG08]

Thus, it remains to find a method for transforming any atomic flow into a weakly streamlined one that can be lifted to SKS-derivations.

Definition 3.10.11. Let ϕ be an atomic flow of the shape

$\phi \quad =$  , (3.27)

where the edges of the selected $\mathsf{ai}{\downarrow}$ and $\mathsf{ai}{\uparrow}$ generators carry the same atomic types, as indicated in (3.27), and let ϕ' be the atomic flow

$\phi' \quad =$  . (3.28)

Then we call ϕ' a *path breaker of* ϕ *with respect to* a, and write $\phi \xrightarrow{\mathsf{pb}}_a \phi'$.

Lemma 3.10.12. *Let ϕ and ϕ' be given with $\phi \xrightarrow{\mathsf{pb}}_a \phi'$, and let b be any atomic type. If ϕ is weakly streamlined with respect to b, then so is ϕ'.*

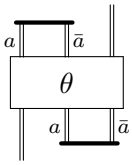
Proof. The only edges connecting an output of one copy of ψ to an input of another copy of ψ are typed by a and $\bar{a}$. Thus, the lemma is evident for $b \neq a$ and $b \neq \bar{a}$. Let us now assume $b = a$ and proceed by contradiction. Assume there is an $\mathsf{ai}{\downarrow}$ generator connected to an $\mathsf{ai}{\uparrow}$ generator via a path typed by a. If this is inside a copy of ψ, we have a contradiction; if it passes through the a-edge between the upper and the middle copy of ψ in (3.28), then this path connects to the $\mathsf{ai}{\downarrow}$ in (3.27), which also is a contradiction. Similarly for a path typed by $\bar{a}$. □

Lemma 3.10.13. *Let ϕ, ψ, and a be given as in* (3.27)*, and let $\phi \xrightarrow{\mathsf{pb}}_a \phi'$. If ψ is* ai*-free with respect to a, then ϕ' is weakly streamlined with respect to a.*

Proof. For not being weakly streamlined with respect to a, we would need a path connecting the upper $\mathsf{ai}{\downarrow}$ in (3.28) to the lower $\mathsf{ai}{\uparrow}$. However, such a path must pass through both the evidenced edge of type a and the evidenced edge of type $\bar{a}$, which is impossible (see Remark 3.8.2). □

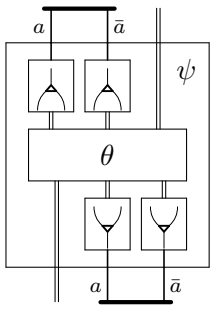
Lemmas 3.10.12 and 3.10.13 justify the name *path breaker* for the atomic flow in (3.28). It breaks all paths between the upper $\mathsf{ai}{\downarrow}$ and the lower $\mathsf{ai}{\uparrow}$ in (3.27), and it does not introduce any new paths. Furthermore, the interior of the flow ψ is not touched.

We now have to find a way to convert any atomic flow ϕ into one of shape (3.27) with ψ being ai-free with respect to a. For this, notice that by Proposition 3.8.9, we can write ϕ as

$\phi \quad =$ 

, (3.29)

where θ is ai-free with respect to a. This can be transformed into a flow ϕ':

$\phi' \quad =$ 

, (3.30)

which is of the desired shape and fulfills the condition of Lemma 3.10.13.

Definition 3.10.14. Let ϕ and ϕ' of shape (3.29) and (3.30) be given. If θ is ai-free with respect to a, then we call ϕ' a *taming of ϕ with respect to a*, and write $\phi \overset{\mathsf{tm}}{\to}_a \phi'$.

Lemma 3.10.15. *Let ϕ and ϕ' be given with $\phi \overset{\mathsf{tm}}{\to}_a \phi'$, and let b be any atomic type. If ϕ is weakly streamlined with respect to b, then so is ϕ'.*

Proof. Immediate from (3.29) and (3.30). □

Definition 3.10.16. On atomic flows, we define the *path breaking relation* $\overset{\mathsf{PB}}{\to}$ as follows. We have $\phi \overset{\mathsf{PB}}{\to} \phi'$ if and only if there is a flow ϕ'' and an atomic type a, such that $\phi \overset{\mathsf{tm}}{\to}_a \phi'' \overset{\mathsf{pb}}{\to}_a \phi'$ and ϕ is not weakly streamlined with respect to a.

Theorem 3.10.17. *The relation $\overset{\mathsf{PB}}{\to}$ is terminating, and its normal forms are weakly streamlined.*

Proof. Let ϕ be given. We proceed by induction on the number of atomic types occurring in ϕ, with respect to which ϕ is not weakly streamlined. Whenever we have $\phi \overset{\mathsf{PB}}{\to} \phi'$, this number is decreased by one for ϕ' (by Lemmas 3.10.12, 3.10.13 and 3.10.15). By the constructions in (3.28) and (3.30), there is always such a ϕ' if ϕ is not weakly streamlined. □

It remains to show that the path breaker can be lifted to SKS-derivations. For the remainder of this section, we use the following convention for saving space:

$$= \frac{A}{B} \quad \text{abbreviates} \quad \begin{array}{c} A \\ \Big\| \{\alpha\downarrow,\alpha\uparrow,\sigma\downarrow,\sigma\uparrow,\mathbf{t}\downarrow,\mathbf{f}\uparrow,\mathbf{f}\downarrow,\mathbf{t}\uparrow\} \\ B \end{array} .$$

Lemma 3.10.18. *Given a context $K\{\cdot\}$ and a formula A, there exist derivations*

$$\begin{array}{c} A \wedge K\{\mathsf{t}\} \\ \Big\| \{\mathsf{s},=\} \\ K\{A\} \end{array} \quad \textit{and} \quad \begin{array}{c} K\{A\} \\ \Big\| \{\mathsf{s},=\} \\ K\{\mathsf{f}\} \vee A \end{array} .$$

Proof. By structural induction on $K\{\cdot\}$. [Str03a, Brü03]. □

We use this lemma to show that the transformation in the proof of Proposition 3.8.8, which does nothing to the flows can "be lifted" to SKS in the following sense. Let Φ be a derivation. Then for every instance of the rule $\mathsf{ai}\downarrow$, we can do the transformation:

$$\mathsf{ai}\downarrow \frac{\begin{array}{c} A \\ \Phi' \Big\| \\ K\{\mathsf{t}\} \end{array}}{\begin{array}{c} K\{a \vee \bar{a}\} \\ \Phi'' \Big\| \\ B \end{array}} \quad \to \quad \begin{array}{c} \mathsf{ai}\downarrow \dfrac{\mathsf{t}\downarrow \dfrac{A}{\mathsf{t} \wedge A}}{[a \vee \bar{a}] \wedge A} \\ \Phi' \Big\| \\ [a \vee \bar{a}] \wedge K\{\mathsf{t}\} \\ \Big\| \{\mathsf{s},=\} \\ K\{a \vee \bar{a}\} \\ \Phi'' \Big\| \\ B \end{array} , \tag{3.31}$$

which does not change the atomic flow, and dually for $\mathsf{ai}{\uparrow}$.

Lemma 3.10.19. *The relation* $\overset{\mathsf{tm}}{\to}_a$ *can be lifted to* SKS.

Proof. Let $\Phi\colon A \to B$ with $fl(\Phi) = \phi$ and a be given. By repeatedly applying (3.31) we get the derivation

$$\begin{array}{c} A \\ \Vert\{\mathsf{ai}{\downarrow},=\} \\ [a \vee \bar{a}] \wedge \cdots \wedge [a \vee \bar{a}] \wedge A \\ \Theta\Vert \\ B \vee (a \wedge \bar{a}) \vee \cdots \vee (a \wedge \bar{a}) \\ \Vert\{\mathsf{ai}{\uparrow},=\} \\ B \end{array},$$

with $fl(\Theta) = \theta$, from which we can obtain a derivation

$$\begin{array}{c} \mathsf{ai}{\downarrow}, \mathbf{t}{\downarrow}\,\dfrac{\mathbf{t}{\downarrow}\,\dfrac{A}{\mathsf{t} \wedge A}}{[a \vee \bar{a}] \wedge A} \\ \Vert\{\mathsf{ac}{\uparrow},=\} \\ [(a \wedge \cdots \wedge a) \vee (\bar{a} \wedge \cdots \wedge \bar{a})] \wedge A \\ \Vert\{\mathsf{m},=\} \\ [a \vee \bar{a}] \wedge \cdots \wedge [a \vee \bar{a}] \wedge A \\ \Theta\Vert \\ B \vee (a \wedge \bar{a}) \vee \cdots \vee (a \wedge \bar{a}) \\ \Vert\{\mathsf{m},=\} \\ B \vee ([a \vee \cdots \vee a] \wedge [\bar{a} \vee \cdots \vee \bar{a}]) \\ \Vert\{\mathsf{ac}{\downarrow},=\} \\ \mathsf{ai}{\uparrow}\,\dfrac{B \vee (a \wedge \bar{a})}{\mathbf{t}{\uparrow}\,\dfrac{B \vee \mathsf{f}}{B}} \end{array},$$

whose flow is as shown in (3.30). □

Lemma 3.10.20. *The relation* $\overset{\mathsf{pb}}{\to}_a$ *can be lifted to* SKS.

Proof. Let $\Phi\colon A \to B$ with $fl(\Phi) = \phi$ and a be given. By applying (3.31) we have a derivation

$$\begin{array}{c} \mathsf{ai}{\downarrow}, \mathbf{t}{\downarrow}\,\dfrac{A}{[a \vee \bar{a}] \wedge A} \\ \Psi\Vert \\ \mathbf{t}{\uparrow}, \mathsf{ai}{\uparrow}\,\dfrac{B \vee (a \wedge \bar{a})}{B} \end{array},$$

with $fl(\Psi) = \psi$. We also have the derivations

$$\begin{array}{r} \mathsf{s}\,\dfrac{\mathsf{aw}\downarrow\,\dfrac{=\,\dfrac{\mathsf{aw}\uparrow\,\dfrac{[B \vee (a \wedge \bar{a})] \wedge A}{[B \vee (a \wedge \mathsf{t})] \wedge A}}{[B \vee [a \vee \mathsf{f}]] \wedge A}}{[B \vee [a \vee \bar{a}]] \wedge A}}{B \vee ([a \vee \bar{a}] \wedge A)} \end{array} \quad \text{and} \quad \begin{array}{r} \mathsf{s}\,\dfrac{\mathsf{aw}\downarrow\,\dfrac{=\,\dfrac{\mathsf{aw}\uparrow\,\dfrac{[B \vee (a \wedge \bar{a})] \wedge A}{[B \vee (\mathsf{t} \wedge \bar{a})] \wedge A}}{[B \vee [\mathsf{f} \vee \bar{a}]] \wedge A}}{[B \vee [a \vee \bar{a}]] \wedge A}}{B \vee ([a \vee \bar{a}] \wedge A)} \end{array}$$

that we call Φ_a and $\Phi_{\bar{a}}$, respectively. We can now build

$$\begin{array}{c} A \\ \Big\Vert\{\mathsf{c}\uparrow,\mathsf{ai}\downarrow,=\} \\ (([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\ (\Psi \wedge A) \wedge A \Big\Vert \\ ([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\ \Phi_a \wedge A \Big\Vert \\ [B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\ [B \vee \Psi] \wedge A \Big\Vert \\ B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\ B \vee \Phi_{\bar{a}} \Big\Vert \\ B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\ B \vee [B \vee \Psi] \Big\Vert \\ B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\ \Big\Vert\{\mathsf{c}\downarrow,\mathsf{ai}\uparrow,=\} \\ B \end{array} ,$$

whose atomic flow is as shown in (3.28). □

Theorem 3.10.21. *The relation* $\overset{\mathsf{PB}}{\to}$ *can be lifted to* SKS.

Proof. Immediate from Lemmas 3.10.19 and 3.10.20. □

Proof of Theorem 3.10.6. For every SKS-derivation $\Phi\colon A \to B$ there exists a weakly streamlined SKS-derivation $\Phi'\colon A \to B$ by Theorem 3.10.17 and Theorem 3.10.21; for every weakly streamlined SKS-derivation $\Phi'\colon A \to B$ there exists a strongly streamlined SKS-derivation $\Phi''\colon A \to B$ by Theorem 3.10.9 and Theorem 3.10.10. □

4

Towards a Combinatorial Characterization of Proofs in Classical Logic

In the previous chapter I have mentioned the problem that there is no correctness criterion for $\mathbb{N}$-nets, C-nets, nor atomic flows. Part of the problem is the lack of understanding of the "linear" rules s (switch) and m (medial) in system SKS (see Figure 3.7 on page 64), i.e., those rules which are represented by simple wires in the atomic flows. In this chapter I will investigate the combinatorial properties of rewriting paths formed with these rules (modulo associativity and commutativity of $\wedge$ and $\vee$). As an application, I will show a combinatorial proof of a decomposition theorem for classical logic. The results of this chapter have been published in [Str07a].

4.1 Rewriting with medial

For simplicity, we consider here only the formulas not containing the units $\mathbf{t}$ and $\mathbf{f}$, i.e., we consider the set $\mathscr{T}$ of formulas defined by the grammar

$$\mathscr{T} ::= \mathscr{A} \mid (\mathscr{T} \wedge \mathscr{T}) \mid [\mathscr{T} \vee \mathscr{T}]$$

where $\mathscr{A} = \{a, b, c, \ldots, \bar{a}, \bar{b}, \bar{c}, \ldots\}$ is the countable set of atoms (propositional variables and their duals). As before, to ease the readability, we use different types of parentheses: $(\ldots)$ for $\wedge$ and $[\ldots]$ for $\vee$. We use capital Latin letters to denote formulas. To ease readability, we will sometimes write $(A \wedge B \wedge C)$ for $((A \wedge B) \wedge C)$ and $[A \vee B \vee C]$ for $[[A \vee B] \vee C]$.

Let AC be the following set of equations on formulas, saying that $\wedge$ and $\vee$ are both associative and commutative:

$$\begin{array}{ll} (A \wedge B) \approx (B \wedge A) & ((A \wedge B) \wedge C) \approx (A \wedge (B \wedge C)) \\ [A \vee B] \approx [B \vee A] & [[A \vee B] \vee C] \approx [A \vee [B \vee C]] \end{array} \tag{4.1}$$

and let $\approx_{\mathsf{AC}}$ be the equational theory induced by AC, i.e., the smallest congruence relation

containing AC. Then we have $P \approx_{\mathsf{AC}} Q$ if and only if there is an SKS-derivation from P to Q using the inference rules $\sigma\!\downarrow, \sigma\!\uparrow, \alpha\!\downarrow, \alpha\!\uparrow$ (see Figure 3.7 on page 64).

Now let M be the rewriting system consisting only of the medial rule

$$[(A \wedge B) \vee (C \wedge D)] \to ([A \vee C] \wedge [B \vee D]) \quad , \tag{4.2}$$

We are now interested in the rewrite relation $\to_{\mathsf{M/AC}}$, i.e., rewriting via the medial rule modulo associativity and commutativity of the two binary operations. More formally: Let P and Q be formulas. Then $P \to_{\mathsf{M/AC}} Q$, if and only if there are formulas P' and Q' such that $P \approx_{\mathsf{AC}} P'$ and $P' \to_{\mathsf{M}} Q'$ and $Q' \approx_{\mathsf{AC}} Q$, where $P' \to_{\mathsf{M}} Q'$ means there is a single rewriting step from P' to Q' using the rule in (4.2). For more details on the formal definitions see, e.g, [BN98]. Since no ambiguity is possible here, we omit the index AC and simply write $P \approx Q$ instead of $P \approx_{\mathsf{AC}} Q$. Further, we write $P \xrightarrow[\mathsf{M}]{} Q$ instead of $P \to_{\mathsf{M/AC}} Q$, and we define $\xrightarrow[\mathsf{M}]{*}$ to be the reflexive transitive closure of $\xrightarrow[\mathsf{M}]{}$. Then $P \xrightarrow[\mathsf{M}]{*} Q$ iff there is a derivation in SKS from P to Q that uses only the rules $\mathsf{m}, \sigma\!\downarrow, \sigma\!\uparrow, \alpha\!\downarrow, \alpha\!\uparrow$.

We are interested in the question: *Under which conditions do we have* $P \xrightarrow[\mathsf{M}]{*} Q$ *?*

4.2 Relation webs

For simplifying the definitions, I will in the following assume that every atom appears at most once in a formula. This allows us to ignore the distinction between atoms and atom occurrences. What matters in this and the next section are the positions occupied by the atoms in the formulas.

For a given formula P, let $\mathscr{V}_P$ denote the set of atoms occurring in P. Let us now treat a formula as a binary tree whose inner nodes are labeled by either $\wedge$ or $\vee$, and whose leaves are the elements of $\mathscr{V}_P$. For $a, b \in \mathscr{V}_P$ we write $a \overset{\wedge}{\underset{P}{\frown}} b$ if their first common ancestor in P is an $\wedge$ and we write $a \overset{\vee}{\underset{P}{\frown}} b$ if it is an $\vee$. Furthermore, we define the sets

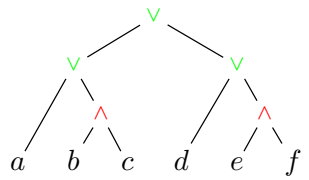
$$\begin{aligned} \mathscr{E}_P^{\wedge} &= \{(a,b) \in \mathscr{V}_P \times \mathscr{V}_P \mid a \overset{\wedge}{\underset{P}{\frown}} b\} \\ \mathscr{E}_P^{\vee} &= \{(a,b) \in \mathscr{V}_P \times \mathscr{V}_P \mid a \overset{\vee}{\underset{P}{\frown}} b\} \end{aligned}$$

Note that $\mathscr{E}_P^{\wedge}$ and $\mathscr{E}_P^{\vee}$ are symmetric, i.e., $(a,b) \in \mathscr{E}_P^{\wedge}$ iff $(b,a) \in \mathscr{E}_P^{\wedge}$, and similarly for $\mathscr{E}_P^{\vee}$. We also have:

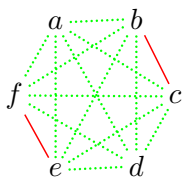
$$\mathscr{E}_P^{\wedge} \cap \mathscr{E}_P^{\vee} = \emptyset \qquad \text{and} \qquad \mathscr{E}_P^{\wedge} \cup \mathscr{E}_P^{\vee} = (\mathscr{V}_P \times \mathscr{V}_P) \setminus \{(a,a) \mid a \in \mathscr{V}_P\} \quad .$$

The triple $\boxast P = \langle \mathscr{V}_P; \mathscr{E}_P^{\wedge}, \mathscr{E}_P^{\vee} \rangle$ is called the *relation web of* P. We can think of it as a complete undirected graph with vertices $\mathscr{V}_P$ and edges $\mathscr{E}_P^{\wedge} \cup \mathscr{E}_P^{\vee}$ where we color the edges in $\mathscr{E}_P^{\wedge}$ red and the edges in $\mathscr{E}_P^{\vee}$ green.

Consider for example the formula $P = [[a \vee (b \wedge c)] \vee [d \vee (e \wedge f)]]$. Its syntax tree and its relation web are, respectively,



and



(4.3)

where the red lines are solid and green lines are drawn as dotted lines.[17]

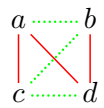
It is now easy to see that we have the following:

Proposition 4.2.1. *Let P and Q be formulas. Then $\boxast P = \boxast Q$ iff $P \approx Q$.*

More interesting, however, is the question, under which circumstances a triple $\langle \mathscr{V}; \mathscr{E}^\wedge, \mathscr{E}^\vee \rangle$ is indeed the relation web of a formula. Let us define a *preweb* to be a triple $\langle \mathscr{V}; \mathscr{E}^\wedge, \mathscr{E}^\vee \rangle$ where $\mathscr{E}^\wedge$ and $\mathscr{E}^\vee$ are symmetric subsets of $\mathscr{V} \times \mathscr{V}$ such that

$$\mathscr{E}^\wedge \cap \mathscr{E}^\vee = \emptyset \qquad \text{and} \qquad \mathscr{E}^\wedge \cup \mathscr{E}^\vee = (\mathscr{V} \times \mathscr{V}) \setminus \{(a,a) \mid a \in \mathscr{V}\} \quad . \tag{4.4}$$

Proposition 4.2.2. *Let $\mathscr{G} = \langle \mathscr{V}; \mathscr{E}^\wedge, \mathscr{E}^\vee \rangle$ be a preweb. Then $\mathscr{G} = \boxast P$ for some formula P if and only if we do not have any $a, b, c, d \in \mathscr{V}$ with*



(4.5)

Proof. See, e.g., [Möh89, Ret93, BdGR97, Gug07]. □

The term "relation web" first appears in [Gug07]. The basic idea, however, is much older. In graph theory, a graph $\langle \mathscr{V}; \mathscr{E}^\wedge \rangle$ not containing configuration (4.5) is called P_4-*free*. It is also called a *cograph* because its complement $\langle \mathscr{V}; \mathscr{E}^\vee \rangle$ has the same property. Cographs are used in [Ret96] to provide a correctness criterion for linear logic proof nets, where $\langle \wedge, \vee \rangle$ is $\langle \otimes, \parr \rangle$. One can also find the terms N-*free* or Z-*free* if configuration (4.5) is forbidden, depending on how the picture is drawn. A comprehensive survey is for example [Möh89]. If $\wedge$ is not commutative, but only associative, then $\mathscr{E}^\wedge$ becomes a partial order, more precisely, a *series-parallel order* (by Proposition 4.2.2 it can be obtained from the singletons via series- and parallel composition of orders). The inclusion relation for these orders has been characterized by a rewriting system in [BdGR97].

Remark 4.2.3. Proposition 4.2.2 also scales to the case with more than two binary operations. For example in [Ret93, BdGR97, Gug07] it is proved for the case of two commutative operations and one non-commutative operation. This is the reason why we use here the more general notion of relation web, instead of cographs.

Let P be a formula and let $\mathscr{W} \subseteq \mathscr{V}_P$. Then we can obtain from $\boxast P$ a new relation web $(\boxast P)|_{\mathscr{W}} = \langle \mathscr{W}; \mathscr{F}^\wedge, \mathscr{F}^\vee \rangle$ by simply removing all vertices not belonging to $\mathscr{W}$ and all edges adjacent to them. Similarly we can obtain from P a formula $P|_{\mathscr{W}}$ by removing in the formula tree all leaves not in $\mathscr{W}$ and then systematically removing all $\vee$- and $\wedge$-nodes that

[17] Note that formally, in $\mathscr{E}_P^\wedge$ and $\mathscr{E}_P^\vee$ every edge appears twice, namely as (a,b) and as (b,a), while in the graph we draw it only once. We chose, nonetheless, this way of presentation because it easily scales to cases where non-commutative operations are added.

became unary by this. More formally, we define inductively on the formula structure:

$$a|_{\mathscr{W}} = \begin{cases} a & \text{if } a \in \mathscr{W} \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$[A \vee B]|_{\mathscr{W}} = \begin{cases} [A|_{\mathscr{W}} \vee B|_{\mathscr{W}}] & \text{if } \mathscr{V}_A \cap \mathscr{W} \neq \emptyset \text{ and } \mathscr{V}_B \cap \mathscr{W} \neq \emptyset \\ A|_{\mathscr{W}} & \text{if } \mathscr{V}_A \cap \mathscr{W} \neq \emptyset \text{ and } \mathscr{V}_B \cap \mathscr{W} = \emptyset \\ B|_{\mathscr{W}} & \text{if } \mathscr{V}_A \cap \mathscr{W} = \emptyset \text{ and } \mathscr{V}_B \cap \mathscr{W} \neq \emptyset \\ \text{undefined} & \text{otherwise} \end{cases}$$

$$(A \wedge B)|_{\mathscr{W}} = \begin{cases} (A|_{\mathscr{W}} \wedge B|_{\mathscr{W}}) & \text{if } \mathscr{V}_A \cap \mathscr{W} \neq \emptyset \text{ and } \mathscr{V}_B \cap \mathscr{W} \neq \emptyset \\ A|_{\mathscr{W}} & \text{if } \mathscr{V}_A \cap \mathscr{W} \neq \emptyset \text{ and } \mathscr{V}_B \cap \mathscr{W} = \emptyset \\ B|_{\mathscr{W}} & \text{if } \mathscr{V}_A \cap \mathscr{W} = \emptyset \text{ and } \mathscr{V}_B \cap \mathscr{W} \neq \emptyset \\ \text{undefined} & \text{otherwise} \end{cases}$$

Clearly we then have $\boxast(P|_{\mathscr{W}}) = (\boxast P)|_{\mathscr{W}}$, but note that $P|_{\mathscr{W}}$ is not necessarily a subformula of P. For example, let $P = [(a \wedge b) \vee (c \wedge [(d \wedge e) \vee f])]$ and $\mathscr{W} = \{a, c, f\}$. Then $P|_{\mathscr{W}} = [a \vee (c \wedge f)]$. If we have another formula Q with $\mathscr{V}_P \cap \mathscr{V}_Q \neq \emptyset$ then we write $P|_Q$ to abbreviate $P|_{\mathscr{V}_P \cap \mathscr{V}_Q}$.

4.3 The Characterization of Medial

For two formulas P and Q, we write $P \blacktriangleleft\!\blacktriangleright Q$ if their relation webs obey the following three properties:

(i) $\mathscr{V}_P = \mathscr{V}_Q$,

(ii) $\mathscr{E}_P^{\wedge} \subseteq \mathscr{E}_Q^{\wedge}$ (or, equivalently, $\mathscr{E}_Q^{\vee} \subseteq \mathscr{E}_P^{\vee}$), and

(iii) for all $a, d \in \mathscr{V}_P$ $(= \mathscr{V}_Q)$ with $a \overset{\vee}{\underset{P}{\frown}} d$ and $a \overset{\wedge}{\underset{Q}{\frown}} d$, there are $b, c \in \mathscr{V}_P$ such that we have the following configurations

in $\boxast P$: in $\boxast Q$:

(4.6)

The motivation for this definition is the following theorem.

Theorem 4.3.1. *For any two formulas P and Q we have $P \xrightarrow[\mathsf{M}]{*} Q$ iff $P \blacktriangleleft\!\blacktriangleright Q$.*

In the proof of this theorem, we make crucial use of two lemmas.

Lemma 4.3.2. *Let P and Q be formulas with $P \xrightarrow[\mathsf{M}]{*} Q$. If P' is a subformula of P, then $P' \xrightarrow[\mathsf{M}]{*} Q|_{P'}$. And if Q_1 is a subformula of Q, then $P|_{Q_1} \xrightarrow[\mathsf{M}]{*} Q_1$.*

Proof. Since $P \xrightarrow[\mathsf{M}]{*} Q$, we have an $n \geq 0$ and formulas $R_0, \ldots, R_n$, such that $P \approx R_0 \xrightarrow[\mathsf{M}]{} R_1 \xrightarrow[\mathsf{M}]{} \cdots \xrightarrow[\mathsf{M}]{} R_n \approx Q$. We will say an R_i (for $0 \leq i \leq n$) is *nested* if there is a formula $R \approx R_i$ which has a subformula $[(A_1 \wedge B_1) \vee (A_2 \wedge B_2)]$ such that $\mathscr{V}_{A_1} \cap \mathscr{V}_{P'} \neq \emptyset$ and $\mathscr{V}_{A_2} \cap \mathscr{V}_{P'} \neq \emptyset$ and $\mathscr{V}_{B_1} \cap \mathscr{V}_{P'} = \emptyset$. We first show that none of the R_i can be nested. Clearly R_0 ($\approx P$) is not nested. Now we proceed by way of contradiction and pick the smallest i such that R_i is nested. Since R_i is obtained from R_{i-1} via a medial rewriting step, we can, without loss of generality, assume that $A_1 = [A \vee C]$ and $B_1 = [B \vee D]$ such that $\mathscr{V}_A \cap \mathscr{V}_{P'} \neq \emptyset$ and $\mathscr{V}_{[B \vee D]} \cap \mathscr{V}_{P'} = \emptyset$, and that R_{i-1} has $[(A \wedge B) \vee (C \wedge D) \vee (A_2 \wedge B_2)]$ as subformula. But then R_{i-1} is also nested. Contradiction. Now we define $R'_i = R_i|_{P'}$ for all $0 \leq i \leq n$. We are going to show that $R'_i \approx R'_{i+1}$ or $R'_i \xrightarrow[\mathsf{M}]{} R'_{i+1}$ for all $0 \leq i \leq n$. We have $R_i \xrightarrow[\mathsf{M}]{} R_{i+1}$. Hence, R_i has a subformula $[(A \wedge B) \vee (C \wedge D)]$ which is replaced by $([A \vee C] \wedge [B \vee D])$ in R_{i+1}. Now we proceed by way of contradiction: since $R'_i \not\approx R'_{i+1}$ we have (without loss of generality) that $\mathscr{V}_A \cap \mathscr{V}_{P'} \neq \emptyset$ and $\mathscr{V}_D \cap \mathscr{V}_{P'} \neq \emptyset$. Since additionally $R'_i \not\xrightarrow[\mathsf{M}]{} R'_{i+1}$, we must have $\mathscr{V}_A \cap \mathscr{V}_{P'} = \emptyset$ or $\mathscr{V}_C \cap \mathscr{V}_{P'} = \emptyset$. Hence R_i is nested, which is a contradiction. Now the first statement of the lemma follows by an induction on n. The second statement is shown analogously. □

Lemma 4.3.3. *Let P and Q be formulas with $P \blacktriangleleft\!\!\blacktriangleright Q$. If P' is a subformula of P, then $P' \blacktriangleleft\!\!\blacktriangleright Q|_{P'}$. And if Q_1 is a subformula of Q, then $P|_{Q_1} \blacktriangleleft\!\!\blacktriangleright Q_1$.*

Proof. For proving the first statement, let $Q' = Q|_{P'}$. We have $\mathscr{V}_{P'} = \mathscr{V}_{Q'}$ and $\mathscr{E}^{\wedge}_{P'} \subseteq \mathscr{E}^{\wedge}_{Q'}$. Now let $a, d \in \mathscr{V}_{P'}$ with $a \overset{\vee}{\underset{P'}{\frown}} d$ and $a \overset{\wedge}{\underset{Q'}{\frown}} d$. Then we also have $a \overset{\vee}{\underset{P}{\frown}} d$ and $a \overset{\wedge}{\underset{Q}{\frown}} d$, and therefore we have $b, c \in \mathscr{V}_P$ such that (4.6). In order to complete the proof of the lemma, we need to show that $b, c \in \mathscr{V}_{P'}$. By way of contradiction, assume that b occurs in the context of P'. Then b has the same first common ancestor with a and d in P. Hence, the edges (a, b) and (d, b) have the same color in $\boxtimes P$. Contradiction. The second statement is shown analogously. □

Remark 4.3.4. It is important to observe that it is crucial for both lemmas that P' is a subformula of P (or that Q_1 is a subformula of Q). If we just have $P \blacktriangleleft\!\!\blacktriangleright Q$ (resp. $P \xrightarrow[\mathsf{M}]{*} Q$) and a subset $\mathscr{W} \subseteq \mathscr{V}_P$, then in general we do *not* have that $P|_{\mathscr{W}} \blacktriangleleft\!\!\blacktriangleright Q|_{\mathscr{W}}$ (resp. $P|_{\mathscr{W}} \xrightarrow[\mathsf{M}]{*} Q|_{\mathscr{W}}$). A simple example is given by $P = [(a \wedge b) \vee (c \wedge d)]$ and $Q = ([a \vee c] \wedge [b \vee d])$ and $\mathscr{W} = \{a, b, d\}$. Then $P|_{\mathscr{W}} = [(a \wedge b) \vee d]$ and $Q|_{\mathscr{W}} = (a \wedge [b \vee d])$. We clearly have $P \blacktriangleleft\!\!\blacktriangleright Q$ (resp. $P \xrightarrow[\mathsf{M}]{*} Q$) but not $P|_{\mathscr{W}} \blacktriangleleft\!\!\blacktriangleright Q|_{\mathscr{W}}$ (resp. $P|_{\mathscr{W}} \xrightarrow[\mathsf{M}]{*} Q|_{\mathscr{W}}$).

Proof of Theorem 4.3.1. First, assume we have $P \xrightarrow[\mathsf{M}]{*} Q$. Then there is an $n \geq 0$ with $P \xrightarrow[\mathsf{M}]{n} Q$. Obviously, we have $\mathscr{V}_P = \mathscr{V}_Q$ because no rewriting step can change the set of atom occurrences, and $\mathscr{E}^{\wedge}_P \subseteq \mathscr{E}^{\wedge}_Q$ because every rewriting step transforms some green edges into red ones and never the other way around. Hence Conditions (i) and (ii) are satisfied. For proving Condition (iii), we proceed by induction on n. For $n = 0$ this is trivial. Now let $n \geq 1$, and assume we have a and d with $a \overset{\vee}{\underset{P}{\frown}} d$ and $a \overset{\wedge}{\underset{Q}{\frown}} d$. Then there are formulas R and T such that $P \xrightarrow[\mathsf{M}]{*} R \xrightarrow[\mathsf{M}]{} T \xrightarrow[\mathsf{M}]{*} Q$ and $a \overset{\vee}{\underset{R}{\frown}} d$ and $a \overset{\wedge}{\underset{T}{\frown}} d$. Because of Proposition 4.2.1, we can assume without loss of generality that that R has a subformula $[(A \wedge B) \vee (C \wedge D)]$,

which is in T replaced by $([A \vee C] \wedge [B \vee D])$. We can without loss of generality assume that $a \in \mathscr{V}_A$ and $d \in \mathscr{V}_D$. Then we have for all $b \in \mathscr{V}_B$ and $c \in \mathscr{V}_C$ the following configurations:

in $\boxast P$: in $\boxast R$: in $\boxast T$: in $\boxast Q$:

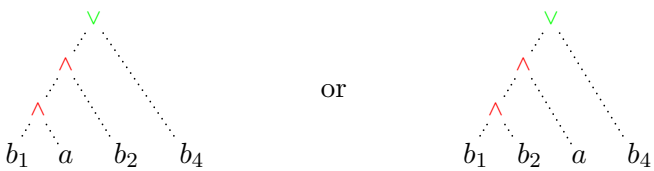
We will now show that there is a $b \in \mathscr{V}_B$ with $a \overset{\wedge}{\underset{P}{\frown}} b$ and $b \overset{\vee}{\underset{Q}{\frown}} d$. For this, we need an auxiliary definition. For a formula S and an atom $a \in \mathscr{V}_S$ we define a partial order $\overset{a}{\underset{S}{\prec}}$ on the set $\mathscr{V}_S$ as follows: $b_1 \overset{a}{\underset{S}{\prec}} b_2$ iff the first common ancestor of a and b_2 in the formula tree of S is also an ancestor of b_1, and the first common ancestor of b_1 and b_2 is also an ancestor of a. It is easy to see that $\overset{a}{\underset{S}{\prec}}$ is reflexive, transitive, and antisymmetric, and hence, a partial order. For example, in (4.3) we have $b \overset{c}{\underset{P}{\prec}} e$, and d, e are incomparable wrt. $\overset{c}{\underset{P}{\prec}}$. Now pick $b_1 \in \mathscr{V}_B$ which is minimal wrt. $\overset{a}{\underset{P}{\prec}}$. We claim that $a \overset{\wedge}{\underset{P}{\frown}} b_1$. By way of contradiction, assume $a \overset{\vee}{\underset{P}{\frown}} b_1$. Then we apply the induction hypothesis to $P \xrightarrow[\mathsf{M}]{*} R$, which gives us a' and b' with the following configurations:

in $\boxast P$: in $\boxast R$:

It follows (by the same argumentation as in the proof of Lemma 4.3.3) that $b' \in \mathscr{V}_B$ and that $b' \overset{a}{\underset{P}{\prec}} b_1$, contradicting the minimality of b_1. If $b_1 \overset{\vee}{\underset{Q}{\frown}} d$, then we have found our desired b. So, assume $b_1 \overset{\wedge}{\underset{Q}{\frown}} d$, and pick a $b_4 \in \mathscr{V}_B$ which is minimal wrt. $\overset{d}{\underset{Q}{\prec}}$. With a similar argument as above, we can show that $b_4 \overset{\vee}{\underset{Q}{\frown}} d$. If $a \overset{\wedge}{\underset{P}{\frown}} b_4$, then, as before, we have our b. So, let us assume that $a \overset{\vee}{\underset{P}{\frown}} b_4$. Since we also have that $b_1 \overset{a}{\underset{P}{\prec}} b_4$ and $b_4 \overset{d}{\underset{Q}{\prec}} b_1$, it follows that $b_1 \overset{\vee}{\underset{P}{\frown}} b_4$ and $b_1 \overset{\wedge}{\underset{Q}{\frown}} b_4$. By Lemma 4.3.2 we have $P|_B \xrightarrow[\mathsf{M}]{*} B \xrightarrow[\mathsf{M}]{*} Q|_B$. Now we can apply the induction hypothesis to $P|_B \xrightarrow[\mathsf{M}]{*} Q|_B$ and get $b_2, b_3 \in \mathscr{V}_B$ such that we have:

in $\boxast P|_B$: in $\boxast Q|_B$:

Note that $b_2, b_3 \in \mathscr{V}_B$ and that $b_2 \overset{b_1}{\underset{P}{\prec}} b_4$. Hence, in the formula tree for P, we have one of the following situations:



In both cases $a \overset{\wedge}{\underset{P}{\frown}} b_2$. Similarly, it follows that $b_2 \overset{\vee}{\underset{Q}{\frown}} d$. With a similar argumentation, we can find $c_2 \in \mathscr{V}_C$ with $c_2 \overset{\wedge}{\underset{P}{\frown}} d$ and $a \overset{\vee}{\underset{Q}{\frown}} c_2$. Hence, Condition (iii) is fulfilled, and we have $P \blacktriangleleft\!\!\blacktriangleright Q$.

Conversely, assume we have $P \blacktriangleleft\!\!\blacktriangleright Q$. We proceed by induction on the cardinality of $\mathscr{V}_P$, to show that $P \xrightarrow[\mathsf{M}]{*} Q$. The base case, where $\mathscr{V}_P$ is a singleton, is trivial. Now we make a case analysis on the formula structure of P and Q.

1. $P = [P' \vee P'']$ and $Q = [Q_1 \vee Q_2]$. We define the following four sets:

$$\mathscr{V}_1' = \mathscr{V}_{P'} \cap \mathscr{V}_{Q_1} \ , \ \mathscr{V}_2' = \mathscr{V}_{P'} \cap \mathscr{V}_{Q_2} \ , \ \mathscr{V}_1'' = \mathscr{V}_{P''} \cap \mathscr{V}_{Q_1} \ , \ \mathscr{V}_2'' = \mathscr{V}_{P''} \cap \mathscr{V}_{Q_2} \ .$$

First, note that we cannot have that one of $\mathscr{V}_1'$ and $\mathscr{V}_2''$ is empty, and at the same time that one of $\mathscr{V}_2'$ and $\mathscr{V}_1''$ is empty because then one of $\mathscr{V}_{P'}, \mathscr{V}_{P''}, \mathscr{V}_{Q_1}, \mathscr{V}_{Q_2}$ would be empty, which is impossible. The remaining two possibilities of two empty sets are:

- If $\mathscr{V}_2' = \emptyset$ and $\mathscr{V}_1'' = \emptyset$, then $\mathscr{V}_{P'} = \mathscr{V}_{Q_1}$ and $\mathscr{V}_{P''} = \mathscr{V}_{Q_2}$. Hence, by Lemma 4.3.3 we have $P' \blacktriangleleft\!\!\blacktriangleright Q_1$ and $P'' \blacktriangleleft\!\!\blacktriangleright Q_2$. By induction hypothesis we have therefore

$$P = [P' \vee P''] \xrightarrow[\mathsf{M}]{*} [Q_1 \vee P''] \xrightarrow[\mathsf{M}]{*} [Q_1 \vee Q_2] = Q$$

- If $\mathscr{V}_1' = \emptyset$ and $\mathscr{V}_2'' = \emptyset$, then $\mathscr{V}_{P'} = \mathscr{V}_{Q_2}$ and $\mathscr{V}_{P''} = \mathscr{V}_{Q_1}$, and we proceed similarly.

Let us now assume that one of the four sets is empty, say $\mathscr{V}_1' = \emptyset$. We let

$$P_2' = P'|_{Q_2} \quad , \qquad P_1'' = P''|_{Q_1} \quad , \qquad P_2'' = P''|_{Q_2} \quad .$$

Then $P_2' = P'$ and $P'' \approx [P_1'' \vee P_2'']$ because $\mathscr{E}_Q^{\vee} \subseteq \mathscr{E}_P^{\vee}$. By Lemma 4.3.3 we have $P_1'' \blacktriangleleft\!\!\blacktriangleright Q_1$ and $[P_2' \vee P_2''] \blacktriangleleft\!\!\blacktriangleright Q_2$. Hence, by induction hypothesis we have

$$P \approx [P_2' \vee [P_1'' \vee P_2'']] \approx [P_1'' \vee [P_2' \vee P_2'']] \xrightarrow[\mathsf{M}]{*} [Q_1 \vee [P_2' \vee P_2'']] \xrightarrow[\mathsf{M}]{*} [Q_1 \vee Q_2] = Q$$

If one of $\mathscr{V}_2', \mathscr{V}_1'', \mathscr{V}_2''$ is empty, we can proceed analogously. Let us now consider the case where none of $\mathscr{V}_1', \mathscr{V}_2', \mathscr{V}_1'', \mathscr{V}_2''$ is empty. Then we can define

$$P_1' = P'|_{Q_1} \quad , \quad P_2' = P'|_{Q_2} \quad , \quad P_1'' = P''|_{Q_1} \quad , \quad P_2'' = P''|_{Q_2} \quad .$$

We have $P' \approx [P_1' \vee P_2']$ and $P'' \approx [P_1'' \vee P_2'']$. By Lemma 4.3.3 we have $[P_1' \vee P_1''] \blacktriangleleft\!\!\blacktriangleright Q_1$ and $[P_2' \vee P_2''] \blacktriangleleft\!\!\blacktriangleright Q_2$. Hence, by induction hypothesis:

$$P \approx [[P_1' \vee P_2'] \vee [P_1'' \vee P_2'']] \approx [[P_1' \vee P_1''] \vee [P_2' \vee P_2'']] \xrightarrow[\mathsf{M}]{*} [Q_1 \vee Q_2] = Q$$

2. $P = (P' \wedge P'')$ and $Q = (Q_1 \wedge Q_2)$. This is analogous to the previous case.
3. $P = (P' \wedge P'')$ and $Q = [Q_1 \vee Q_2]$. As before, we let

$$\mathscr{V}_1' = \mathscr{V}_{P'} \cap \mathscr{V}_{Q_1} \ , \ \mathscr{V}_2' = \mathscr{V}_{P'} \cap \mathscr{V}_{Q_2} \ , \ \mathscr{V}_1'' = \mathscr{V}_{P''} \cap \mathscr{V}_{Q_1} \ , \ \mathscr{V}_2'' = \mathscr{V}_{P''} \cap \mathscr{V}_{Q_2} \ .$$

Note that if $\mathscr{V}_1' \neq \emptyset$ and $\mathscr{V}_2'' \neq \emptyset$ then we have immediately a contradiction to Condition (ii), and similarly if $\mathscr{V}_2' \neq \emptyset$ and $\mathscr{V}_1'' \neq \emptyset$. Hence, one of $\mathscr{V}_1'$ and $\mathscr{V}_2''$ must be empty, and one of $\mathscr{V}_2'$ and $\mathscr{V}_1''$ must be empty. But this is impossible as observed in Case 1 above.

4. $P = [P' \vee P'']$ and $Q = (Q_1 \wedge Q_2)$. This is the most interesting case. As before, we let

$$\mathscr{V}_1' = \mathscr{V}_{P'} \cap \mathscr{V}_{Q_1} \ , \ \mathscr{V}_2' = \mathscr{V}_{P'} \cap \mathscr{V}_{Q_2} \ , \ \mathscr{V}_1'' = \mathscr{V}_{P''} \cap \mathscr{V}_{Q_1} \ , \ \mathscr{V}_2'' = \mathscr{V}_{P''} \cap \mathscr{V}_{Q_2} \ .$$

We first show that none of the sets $\mathscr{V}_1', \mathscr{V}_1'', \mathscr{V}_2', \mathscr{V}_2''$ is empty. So, assume, by way of contradiction, that $\mathscr{V}_1'' = \emptyset$. By a similar argumentation as before it follows that $\mathscr{V}_1' \neq \emptyset$ and $\mathscr{V}_2'' \neq \emptyset$. So, pick $a \in \mathscr{V}_1'$ and $d \in \mathscr{V}_2''$. We have $a \overset{\vee}{\underset{P}{\frown}} d$ and $a \overset{\wedge}{\underset{Q}{\frown}} d$. Since $P \blacktriangleleft\!\!\blacktriangleright Q$, we have $b, c \in \mathscr{V}_P$ such that (4.6). Because $c \overset{\wedge}{\underset{P}{\frown}} d$ we must have that $c \in \mathscr{V}_{P''}$, and because of $a \overset{\vee}{\underset{Q}{\frown}} c$, we must have that $c \in \mathscr{V}_{Q_1}$. Hence $c \in \mathscr{V}_1''$. Contradiction. We can therefore define:

$$P_1' = P'|_{Q_1} \quad , \quad P_2' = P'|_{Q_2} \quad , \quad P_1'' = P''|_{Q_1} \quad , \quad P_2'' = P''|_{Q_2} \quad ,$$

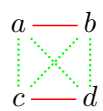
and

$$Q_1' = Q_1|_{P'} \quad , \quad Q_2' = Q_2|_{P'} \quad , \quad Q_1'' = Q_1|_{P''} \quad , \quad Q_2'' = Q_2|_{P''} \quad .$$

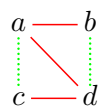
We now want to show that $P_1' \blacktriangleleft\!\!\blacktriangleright Q_1'$. But by Remark 4.3.4 we cannot apply Lemma 4.3.3. However, we have $\mathscr{V}_{P_1'} = \mathscr{V}_{Q_1'}$ and $\mathscr{E}^{\wedge}_{P_1'} \subseteq \mathscr{E}^{\wedge}_{Q_1'}$. Now let $a, d \in \mathscr{V}_{P_1'}$ with $a \overset{\vee}{\underset{P_1'}{\frown}} d$ and $a \overset{\wedge}{\underset{Q_1'}{\frown}} d$. Hence, we have $a \overset{\vee}{\underset{P}{\frown}} d$ and $a \overset{\wedge}{\underset{Q}{\frown}} d$. Since $P \blacktriangleleft\!\!\blacktriangleright Q$, we have $b, c \in \mathscr{V}_P$ such that (4.6). Note that because $a, d \in \mathscr{V}_{P'}$, we also have $b \in \mathscr{V}_{P'}$ (otherwise we would have $a \overset{\vee}{\underset{P}{\frown}} b$) and $c \in \mathscr{V}_{P'}$ (otherwise we would have $c \overset{\vee}{\underset{P}{\frown}} d$). Similarly, because $a, d \in \mathscr{V}_{Q_1}$, we also have $b, c \in \mathscr{V}_{Q_1}$ (otherwise we would have $a \overset{\wedge}{\underset{Q}{\frown}} c$ and $b \overset{\wedge}{\underset{Q}{\frown}} d$, respectively). Hence $b, c \in \mathscr{V}_{P_1'}$, and therefore $P_1' \blacktriangleleft\!\!\blacktriangleright Q_1'$. Similarly, we get $P_1'' \blacktriangleleft\!\!\blacktriangleright Q_1''$ and $P_2' \blacktriangleleft\!\!\blacktriangleright Q_2'$ and $P_2'' \blacktriangleleft\!\!\blacktriangleright Q_2''$. Hence, we have by induction hypothesis

$$P_1' \xrightarrow[\mathsf{M}]{*} Q_1' \quad , \quad P_1'' \xrightarrow[\mathsf{M}]{*} Q_1'' \quad , \quad P_2' \xrightarrow[\mathsf{M}]{*} Q_2' \quad , \quad P_2'' \xrightarrow[\mathsf{M}]{*} Q_2'' \quad . \tag{4.7}$$

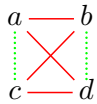
Now let $P_{12}' = (P_1' \wedge P_2')$. We clearly have $\mathscr{V}_{P'} = \mathscr{V}_{P_{12}'}$ and $\mathscr{E}^{\wedge}_{P'} \subseteq \mathscr{E}^{\wedge}_{P_{12}'}$. Now let us assume we have $a, d \in \mathscr{V}_{P'}$ with $a \overset{\vee}{\underset{P'}{\frown}} d$ and $a \overset{\wedge}{\underset{P_{12}'}{\frown}} d$. Then we must have $a \in \mathscr{V}_{P_1'}$ and $d \in \mathscr{V}_{P_2'}$, or vice versa (otherwise the two edges would have the same color in P' and P_{12}'). Hence, we have $a \overset{\vee}{\underset{P}{\frown}} d$ and $a \overset{\wedge}{\underset{Q}{\frown}} d$. Since $P \blacktriangleleft\!\!\blacktriangleright Q$, we have $b, c \in \mathscr{V}_Q$ such that (4.6). Note that because $a, d \in \mathscr{V}_{P'}$, we also have $b, c \in \mathscr{V}_{P'}$ (otherwise we would have $a \overset{\vee}{\underset{P}{\frown}} b$ and $d \overset{\vee}{\underset{P}{\frown}} c$). This means we have in $\boxtimes P'$ the configuration



Since we have $a \overset{\vee}{\underset{Q}{\frown}} c$ and $b \overset{\vee}{\underset{Q}{\frown}} d$, we must also have $a \overset{\vee}{\underset{P_{12}'}{\frown}} c$ and $b \overset{\vee}{\underset{P_{12}'}{\frown}} d$. And since we have $a \overset{\wedge}{\underset{P}{\frown}} b$ and $c \overset{\wedge}{\underset{P}{\frown}} d$, we also have $a \overset{\wedge}{\underset{P_{12}'}{\frown}} b$ and $c \overset{\wedge}{\underset{P_{12}'}{\frown}} d$. Furthermore, we have $a \overset{\wedge}{\underset{P_{12}'}{\frown}} d$ (because $a \in \mathscr{V}_{P_1'}$ and $d \in \mathscr{V}_{P_2'}$). Hence, we have in $\boxtimes P_{12}'$ the configuration



By Proposition 4.2.2, we must have



Hence, $P' \blacktriangleleft\!\!\blacktriangleright (P'_1 \wedge P'_2)$. By the same argumentation, we get $P'' \blacktriangleleft\!\!\blacktriangleright (P''_1 \wedge P''_2)$ and $[Q'_1 \vee Q''_1] \blacktriangleleft\!\!\blacktriangleright Q_1$ and $[Q'_2 \vee Q''_2] \blacktriangleleft\!\!\blacktriangleright Q_2$. By induction hypothesis we have therefore

$$\begin{aligned} P' &\xrightarrow[\mathsf{M}]{*} (P'_1 \wedge P'_2) & [Q'_1 \vee Q''_1] &\xrightarrow[\mathsf{M}]{*} Q_1 \\ P'' &\xrightarrow[\mathsf{M}]{*} (P''_1 \wedge P''_2) & [Q'_2 \vee Q''_2] &\xrightarrow[\mathsf{M}]{*} Q_2 \end{aligned} \tag{4.8}$$

Now we can combine (4.7) and (4.8) to get

$$\begin{aligned} [P' \vee P''] \xrightarrow[\mathsf{M}]{*} [(P'_1 \wedge P'_2) \vee (P''_1 \wedge P''_2)] &\xrightarrow[\mathsf{M}]{} ([P'_1 \vee P''_1] \wedge [P'_2 \vee P''_2]) \\ &\xrightarrow[\mathsf{M}]{*} ([Q'_1 \vee Q''_1] \wedge [Q'_2 \vee Q''_2]) \xrightarrow[\mathsf{M}]{*} (Q_1 \wedge Q_2) \end{aligned}$$

In other words: $P \xrightarrow[\mathsf{M}]{*} Q$. □

Corollary 4.3.5. *The relation* $\blacktriangleleft\!\!\blacktriangleright \subseteq \mathscr{T} \times \mathscr{T}$ *is transitive.*

4.4 The Characterization of Switch

Let us compare the result of the previous section to the one in [BdGR97], where one of the two binary operations was not commutative but only associative. Although this has some consequences for the characterization of relation webs (Proposition 4.2.2), the consequences for the main result (Theorem 4.3.1) are only cosmetic. For this reason let us recall here the commutative version of the results in [BdGR97]. Let P be the rewriting system

$$\begin{aligned} ([A \vee B] \wedge [C \vee D]) &\to [(A \wedge C) \vee (B \wedge D)] \\ (A \wedge [B \vee C]) &\to [(A \wedge B) \vee C] \\ (A \wedge B) &\to [A \vee B] \end{aligned} \tag{4.9}$$

Note that it is *not* a typo that the first rewrite rule is the inversion of medial. Analogous to $\xrightarrow[\mathsf{M}]{*}$, we define $\xrightarrow[\mathsf{P}]{*}$ to be the transitive closure of the rewriting relation via (4.9) modulo AC. The result of [BdGR97] can be stated as follows:

Theorem 4.4.1. *For any two formulas P, Q we have $P \xrightarrow[\mathsf{P}]{*} Q$ iff $\mathscr{V}_P = \mathscr{V}_Q$ and $\mathscr{E}^{\vee}_P \subseteq \mathscr{E}^{\vee}_Q$.*

In other words, the main difference to Theorem 4.3.1 is that the Condition (iii) is absent in [BdGR97]. Let us now look at the case where we remove the first rule from P. Let S be the rewrite system

$$\begin{aligned} (A \wedge [B \vee C]) &\to [(A \wedge B) \vee C] \\ (A \wedge B) &\to [A \vee B] \end{aligned} \tag{4.10}$$

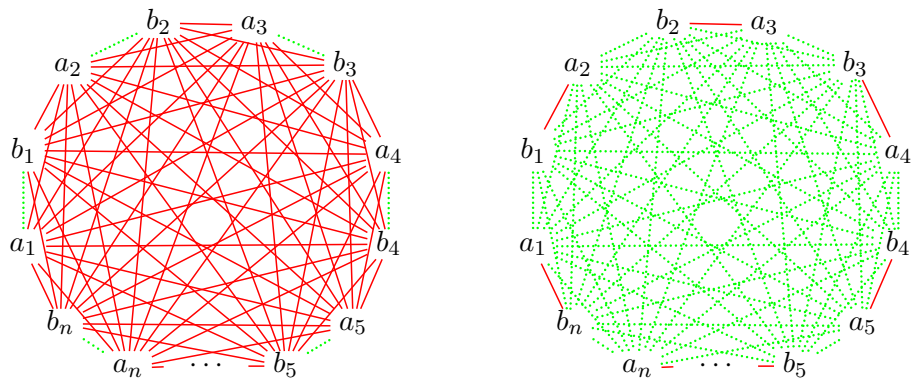
We define $\xrightarrow[\mathsf{S}]{*}$ as the transitive and reflexive closure of $\to_{\mathsf{S/AC}}$. This means that $P \xrightarrow[\mathsf{S}]{*} Q$ if and only if there is a derivation in (a variant of) SKS, using only the rules s, $\alpha\downarrow$, $\alpha\uparrow$, $\sigma\downarrow$, $\sigma\uparrow$, and mix, where mix is the deep inference rule obtained from the mix map, as discussed in Section 2.3. The characterization of this relation is the following:

Theorem 4.4.2. *We have* $P \xrightarrow[\mathsf{S}]{*} Q$ *if and only if* $\mathscr{V}_P = \mathscr{V}_Q$*, and for all* $n \geq 1$ *and all subsets* $\mathscr{W} = \{a_1, b_1, \ldots, a_n, b_n\} \subseteq \mathscr{V}_P$ *we do not have that*

$$P|_{\mathscr{W}} \approx ([a_1 \vee b_1] \wedge \cdots \wedge [a_n \vee b_n]) \quad \textit{and} \quad Q|_{\mathscr{W}} \approx [(b_1 \wedge a_2) \vee (b_2 \wedge a_3) \vee \cdots \vee (b_n \wedge a_1)]$$

In other words, we are not allowed to have the following configurations in the relation webs of P and Q:

in $\boxast P$: in $\boxast Q$:



Note that $\mathscr{E}_P^{\vee} \subseteq \mathscr{E}_Q^{\vee}$ follows by letting $n = 1$.

The characterization in Theorem 4.4.2 is simply an alternative formulation of the correctness criterion for proof nets for multiplicative linear logic with mix [Ret96]. It is equivalent to the acyclicity condition of [DR89].[18]

It is interesting to note the different nature of the three characterizations of the rewrite systems M, P, and S. This is the reason for the difficulty to give a characterization of the rewrite system MS, which combines M and S:

$$\begin{aligned} [(A \wedge B) \vee (C \wedge D)] &\to ([A \vee C] \wedge [B \vee D]) \\ (A \wedge [B \vee C]) &\to [(A \wedge B) \vee C] \\ (A \wedge B) &\to [A \vee B] \end{aligned} \tag{4.11}$$

Finding a characterization of the rewrite relation $\xrightarrow[\mathsf{MS}]{*}$ in terms of relation webs remains an open problem.

4.5 A Combinatorial Proof of a Decomposition Theorem

If a formula I is of the shape

$$([a_1 \vee \bar{a}_1] \wedge [a_2 \vee \bar{a}_2] \wedge \cdots \wedge [a_n \vee \bar{a}_n])$$

for some $n \geq 1$ and atoms $a_1, a_2, \ldots, a_n$, then we say I is an *initial formula*.

[18] If mix is absent, then an additional condition (connectedness) would be needed. For more details on the relation between S and linear logic, see, e.g., [DHPP99, Ret93, Gug07, Str03a], and for relating the condition in Theorem 4.4.2 to multiplicative proof nets, see, e.g., [Ret03]. For more information on mix, see [FR94], and for a direct proof of Theorem 4.4.2, see, e.g., [Str03b, Str03a].

It is well-known that classical logic is multiplicative linear logic plus contraction and weakening. Let us therefore introduce two more rewrite systems. Let W be the rewrite system containing only the rule

$$A \to [A \vee B] \tag{4.12}$$

and let C be the system containing only the rule

$$[A \vee A] \to A \tag{4.13}$$

Now let $\mathsf{K1} = \mathsf{S} \cup \mathsf{W} \cup \mathsf{C}$. Then we have the following theorem, which says that a proof in classical logic is a rewrite path in $\mathsf{K1}$.

Theorem 4.5.1. *A formula Q is a Boolean tautology if and only if there is an initial formula I with $I \xrightarrow[\mathsf{K1}]{*} Q$.* [BT01]

The main reason for introducing medial in [BT01] was that with medial we can reduce the contraction to atoms. Consequently, in SKS the contraction rule is restricted to an atomic version. Let C' be the rewrite system consisting of a rule

$$[a \vee a] \to a \tag{4.14}$$

for every atom symbol (including their duals). If we let $\mathsf{K2} = \mathsf{MS} \cup \mathsf{W} \cup \mathsf{C}'$, then we have

Theorem 4.5.2. *Let P and Q be formulas. Then $P \xrightarrow[\mathsf{K1}]{*} Q$ iff $P \xrightarrow[\mathsf{K2}]{*} Q$.* [BT01]

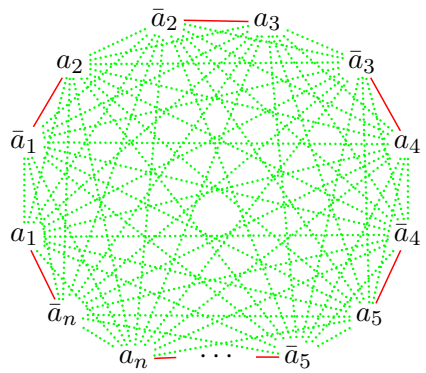
While [BT01] and related work (e.g., [GS01, Gug07, Brü03, Str03a]) are mainly concerned with the syntactic manipulation of formulas/formulas, Hughes proposes in [Hug06] the notion of *combinatorial proof*, which is based on a variant of Theorem 4.4.2 and the notion of *skew fibration*: Given two prewebs $\mathscr{G}_1 = \langle \mathscr{V}_1; \mathscr{E}_1^\wedge, \mathscr{E}_1^\vee \rangle$ and $\mathscr{G}_2 = \langle \mathscr{V}_2; \mathscr{E}_2^\wedge, \mathscr{E}_2^\vee \rangle$, then a skew fibration $\mathsf{h}\colon \mathscr{G}_1 \to \mathscr{G}_2$ is a mapping $\mathscr{V}_1 \to \mathscr{V}_2$ such that

(a) $(a, b) \in \mathscr{E}_1^\wedge$ implies $(\mathsf{h}(a), \mathsf{h}(b)) \in \mathscr{E}_2^\wedge$ (i.e., h is a graph homomorphism for the red edges), and

(b) for all $a \in \mathscr{V}_1$ and $d \in \mathscr{V}_2$, if $(\mathsf{h}(a), d) \in \mathscr{E}_2^\wedge$, then there is a $b \in \mathscr{V}_1$ with $(a, b) \in \mathscr{E}_1^\wedge$ and $(\mathsf{h}(b), d) \notin \mathscr{E}_2^\wedge$.

A *combinatorial proof* of a Boolean formula Q is a skew fibration $\mathsf{h}\colon \boxast P \to \boxast Q$ for a formula P such that

(c) $\boxast P$ does not contain a configuration



(4.15)

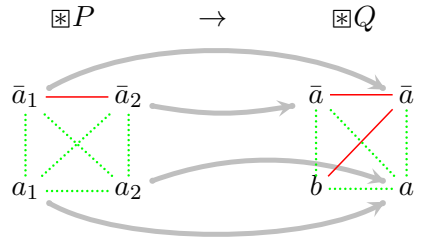
for any $n \geq 1$ and atoms $a_1, a_2, \ldots, a_n$, and

(d) h maps only non-negated atoms to non-negated atoms and negated atoms to negated ones.

Theorem 4.5.3. *A formula Q is a Boolean tautology, if and only if it has a combinatorial proof.* [Hug06]

Remark 4.5.4. Note that for Theorems 4.5.1 and 4.5.3 to make sense, we have to allow more than one occurrence of an atom in a formula. This means that in the relation web $\boxast P$ of a formula P, the set $\mathscr{V}_P$ is the set of atom occurrences. Then we can call a map $\mathsf{h}\colon \mathscr{V}_P \to \mathscr{V}_Q$ *label preserving* if the name of an atom is not changed by h.

To give an example, we show here the combinatorial proof of Pierce's law $Q = [([\bar{a} \vee b] \wedge \bar{a}) \vee a]$, taken from [Hug06]. We let $P = [(\bar{a}_1 \wedge \bar{a}_2) \vee a_1 \vee a_2]$. The skew fibration $\mathsf{h}\colon \boxast P \to \boxast Q$ is given as follows:



Theorem 4.5.5. *Let P and Q be formulas. Then $P \blacktriangleleft\!\blacktriangleright Q$ if and only if $\mathscr{V}_P = \mathscr{V}_Q$ and the identity function on $\mathscr{V}_P$ is a skew fibration $\boxast P \to \boxast Q$.*

Proof. First, assume $P \blacktriangleleft\!\blacktriangleright Q$. Since $\mathscr{E}_P^{\wedge} \subseteq \mathscr{E}_Q^{\wedge}$, Condition (a) above is fulfilled. Now let $a, d \in \mathscr{V}_P$ with $a \underset{Q}{\overset{\wedge}{\frown}} d$. If $a \underset{P}{\overset{\wedge}{\frown}} d$, then we let $b = d$ and we are done. If $a \underset{P}{\overset{\vee}{\frown}} d$, then we have $b, c \in \mathscr{V}_P$ with (4.6). Now b has the desired properties. Conversely, assume that $\mathscr{V}_P = \mathscr{V}_Q$ and the identity $\mathscr{V}_P \to \mathscr{V}_Q$ is a skew fibration. By (a) we have $\mathscr{E}_P^{\wedge} \subseteq \mathscr{E}_Q^{\wedge}$. Now let $a, d \in \mathscr{V}_P$ with $a \underset{P}{\overset{\vee}{\frown}} d$ and $a \underset{Q}{\overset{\wedge}{\frown}} d$. Then by (b) there is a $b \in \mathscr{V}_P$ with $a \underset{P}{\overset{\wedge}{\frown}} b$ and $b \underset{Q}{\overset{\vee}{\frown}} d$. Since $\mathscr{E}_P^{\wedge} \subseteq \mathscr{E}_Q^{\wedge}$, we also have $a \underset{Q}{\overset{\wedge}{\frown}} b$ and $b \underset{P}{\overset{\vee}{\frown}} d$. By exchanging the roles of a and d and applying (b) again, we get $c \in \mathscr{V}_P$ with $d \underset{P}{\overset{\wedge}{\frown}} c$ and $c \underset{Q}{\overset{\vee}{\frown}} a$. Since $\mathscr{E}_P^{\wedge} \subseteq \mathscr{E}_Q^{\wedge}$, it follows that $d \underset{Q}{\overset{\wedge}{\frown}} c$ and $c \underset{P}{\overset{\vee}{\frown}} a$. Hence $c \neq b$. By Proposition 4.2.2, we conclude that $b \underset{P}{\overset{\vee}{\frown}} c$ and $b \underset{Q}{\overset{\wedge}{\frown}} c$. $\square$

In the following, we establish a precise relation between the notion of proof as rewriting path (in a deep inference deductive system) and the notion of proof as a combinatorial object using relation webs and skew fibrations. For this, we first have to characterize the rewrite systems W and C'. Let P and Q be formulas. A map $\mathsf{w}\colon \boxast P \to \boxast Q$ is called a *weakening*, if

(e) w is an injective skew fibration, and

(f) for all $a, b \in \mathscr{V}_P$, we have $a \underset{P}{\overset{\wedge}{\frown}} b$ iff $\mathsf{w}(a) \underset{Q}{\overset{\wedge}{\frown}} \mathsf{w}(b)$.

A map $\mathsf{c}\colon \boxast P \to \boxast Q$ is called an *atomic contraction*, if

(g) c is surjective, and

(h) for all $a, b \in \mathscr{V}_P$, we have $a \underset{P}{\overset{\wedge}{\frown}} b$ iff $\mathsf{c}(a) \underset{Q}{\overset{\wedge}{\frown}} \mathsf{c}(b)$.

Note that it follows that c is a skew fibration. We have the following:

Proposition 4.5.6. *For all formulas P and Q,*

1. $P \xrightarrow[\mathsf{W}]{*} Q$ *iff there is a label preserving weakening* $\mathsf{w}\colon \boxast P \to \boxast Q$.
2. $P \xrightarrow[\mathsf{C}']{*} Q$ *iff there is a label preserving atomic contraction* $\mathsf{c}\colon \boxast P \to \boxast Q$.

Proof. The "only if" direction is trivial for both statements. The "if" direction for the first statement follows by observing that condition (b) implies that for all d not in the image of w there is in Q a subformula D containing only material (including d) not appearing in P, and a subformula B containing only material (including b) appearing in P, such that $[B \vee D]$ is also a subformula of Q. Injectivity and Condition (f) ensure that B is also a subformula of P. Hence, we can rewrite B into $[B \vee D]$. For the second statement it suffices to note that whenever two occurrences of an atom a in P are mapped onto the same occurrence in Q, then they must appear as subformula $[a \vee a]$ in P. □

Lemma 4.5.7. *A label preserving skew fibration* $\mathsf{h}\colon \mathscr{V}_P \to \mathscr{V}_Q$ *is surjective if and only if there is a formula R with $\mathscr{V}_R = \mathscr{V}_P$ such that $P \blacktriangleleft\!\!\blacktriangleright R$ and* h *is an atomic contraction when seen as map $\boxast R \to \boxast Q$.*

Proof. Let h be surjective. We construct R from Q by replacing each atom occurrence a by $[a \vee \cdots \vee a]$ where the number of a's is the cardinality of the preimage $\mathsf{h}^{-1}(a)$ in P. Then obviously the canonical map $\mathscr{V}_R \to \mathscr{V}_Q$ is an atomic contraction, and the identity map $\mathscr{V}_P \to \mathscr{V}_R$ inherits from h the property of being a skew fibration. Finally we apply Theorem 4.5.5. The converse follows from the fact that the composition of a skew fibration with an atomic contraction is again a skew fibration.[19] □

Now we can put everything together to give a combinatorial proof for the following theorem:

Theorem 4.5.8. *A formula Q is a Boolean tautology if and only if there is an initial formula I, such that*

$$I \xrightarrow[\mathsf{S}]{*} P \xrightarrow[\mathsf{M}]{*} R \xrightarrow[\mathsf{C}']{*} S \xrightarrow[\mathsf{W}]{*} Q$$

for some formulas P, R, and S.

Proof. The "if" direction follows immediately from Theorems 4.5.1 and 4.5.2. For the "only if" direction we start with the combinatorial proof for Q given by Theorem 4.5.3. We have a skew fibration $\mathsf{h}\colon \boxast P \to \boxast Q$. By Theorem 4.4.2 and Condition (c) we can obtain an initial formula I with $I \xrightarrow[\mathsf{S}]{*} P$. Now we let $\mathscr{V}_S \subseteq \mathscr{V}_Q$ be the image of $\mathsf{h}\colon \mathscr{V}_P \to \mathscr{V}_Q$, and let $S = Q|_{\mathscr{V}_S}$. This gives us a surjective skew fibration $\mathsf{h}'\colon \boxast P \to \boxast S$. We can rename in P (and in I) all appearing atoms such that h' becomes label preserving. Then we apply Lemma 4.5.7 to get R. By Theorem 4.3.1 we have $P \xrightarrow[\mathsf{M}]{*} R$, and by Proposition 4.5.6.2 we have $R \xrightarrow[\mathsf{C}']{*} S$. Finally, note that the embedding $\boxast S \to \boxast Q$ is a weakening. So, by Proposition 4.5.6.1 we get $S \xrightarrow[\mathsf{W}]{*} Q$. □

[19] An anonymous referee pointed out that it is in general not true that the composition of two skew fibrations is again a skew fibration because they are defined on prewebs.

Remark 4.5.9. The proof of Theorem 4.5.8, together with the rule permutation results in the calculus of structures [Brü03] can be used to show that skew fibrations are closed under composition when their definition is restricted to relation webs (cf. Footnote 19).

Remark 4.5.10. Theorem 4.5.8 can also be proved without using relation webs and skew fibrations by using the permutability of inference rules in the calculus of structures [Brü03, Str03a]. However, that proof is rather tedious, and certainly much longer.

5

Comparing Mechanisms of Compressing Proofs in Classical Logic

If we study the problem of the identity of proofs, we also have to address the size of proofs. In fact, a satisfactory notion of proof identity should take into account the proof size. However, there are proof normalization procedures that cause an exponential blow-up. The two most important ones are *cut elimination* and *extension elimination*. Thus, the use of cut and extension can both be seen as ways of compressing proofs in classical logic with a (potentially) exponential speed-up. Unfortunately, the two concepts are generally studied for different reasons by different communities.

The purpose of this chapter is to present a deductive system that allows to study cut and extension together as well as independently. This will be done by the use of deep inference. With the help of the resulting system, I will provide a new proof of the p-equivalence of Frege systems with extension and Frege systems with substitution.

5.1 Deep Inference and Frege Systems

The concept of extension is usually studied within *Frege systems* (also known as *Hilbert systems* or *Hilbert-Frege-systems* or *Hilbert-Ackermann-systems* [Hil22, HA28]), which consists of a set of axioms (more precisely, axiom schemes) and a set of inference rules, which in the case of classical propositional logic only contains *modus ponens*:

$$\mathsf{mp}\ \frac{A \qquad A \Rightarrow B}{B}$$

As before, I use $A \Rightarrow B$ as abbreviation for $\bar{A} \vee B$ and $A \Leftrightarrow B$ for $[\bar{A} \vee B] \wedge [\bar{B} \vee A]$. I assume the reader to be familiar with Frege systems, and I will not go into further details. The important facts are that the set of axioms in a Frege system has to be sound and complete, and that all Frege systems p-simulate each other. We also immediately have:

Proposition 5.1.1. *Every Frege system p-simulates* SKS.

Proof. Notice that $\bar{a} \vee a$ has a Frege proof, and for every rule

$$\mathsf{r}\,\frac{K\{A\}}{K\{B\}}$$

in SKS, we can show by induction on $K\{\ \}$ that there is a Frege proof of $\overline{K\{A\}} \vee K\{B\}$ whose size is polynomial in the size of $K\{B\}$. Then the application of an inference rule in SKS can be simulated by modus ponens. □

Let us use KS to denote the system obtained from SKS by removing the rules $\mathsf{ai}\uparrow$, $\mathsf{aw}\uparrow$, and $\mathsf{ac}\uparrow$. Then, KS is considered to be the cut-free version of SKS [BT01, Brü03] (see also Section 3.10).

Proposition 5.1.2. *The rules*

$$\mathsf{i}\!\downarrow\frac{K\{\mathbf{t}\}}{K\{\bar{A} \vee A\}} \qquad \mathsf{w}\!\downarrow\frac{K\{\mathbf{f}\}}{K\{A\}} \qquad \mathsf{c}\!\downarrow\frac{K\{A \vee A\}}{K\{A\}} \qquad \mathsf{d}\!\downarrow\frac{K\{[A \vee C] \wedge [B \vee C]\}}{K\{(A \wedge B) \vee C\}}$$

are derivable in KS. *More precisely,* KS *p-simulates* $\mathsf{KS} \cup \{\mathsf{i}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{c}\!\downarrow, \mathsf{d}\!\downarrow\}$. [BT01]

The rules $\mathsf{i}\!\downarrow$, $\mathsf{w}\!\downarrow$, and $\mathsf{c}\!\downarrow$ are the general (non-atomic) versions of $\mathsf{ai}\!\downarrow$, $\mathsf{aw}\!\downarrow$, and $\mathsf{ac}\!\downarrow$, respectively.

Proposition 5.1.3. *The system* KS *p-simulates cut-free sequent calculus.* [BT01]

The converse is not true, i.e., cut-free sequent calculus cannot p-simulate KS. A counter-example can be found in [BG09], where Bruscoli and Guglielmi show that the example used by Statman [Sta78] to prove an exponential lower bound for cut-free sequent calculus admits polynomial size proofs in KS. This situation changes when we add cut, i.e., go to SKS.

Proposition 5.1.4. *The rules*

$$\mathsf{i}\!\uparrow\frac{K\{A \wedge \bar{A}\}}{K\{\mathbf{f}\}} \qquad \mathsf{c}\!\uparrow\frac{K\{A\}}{K\{A \wedge A\}} \qquad \mathsf{w}\!\uparrow\frac{K\{A\}}{K\{\mathbf{t}\}} \tag{5.1}$$

are derivable in SKS. *More precisely,* SKS *p-simulates* $\mathsf{SKS} \cup \{\mathsf{i}\!\uparrow, \mathsf{c}\!\uparrow, \mathsf{w}\!\uparrow\}$. [BT01]

Proposition 5.1.5. SKS *is p-equivalent to every sequent system with cut.* [BT01]

As before, we use

$$= \frac{A}{B} \quad \text{to abbreviate} \quad \begin{array}{c} A \\ \Big\Vert_{\{\alpha\downarrow,\alpha\uparrow,\sigma\downarrow,\sigma\uparrow,\mathbf{t}\downarrow,\mathbf{f}\uparrow,\mathbf{f}\downarrow,\mathbf{t}\uparrow\}} \\ B \end{array} .$$

In this chapter, we also need the following inference rule, that we call *open*:

$$\mathsf{open}\!\downarrow\frac{A \qquad B}{A \wedge B} \tag{5.2}$$

It is a branching rule that says, if we have a proof of A and a proof of B, then we can get a proof of $A \wedge B$. Note that it cannot be applied deep inside a context. This rule is needed in

order give the substitution rule (to be discussed in Section 5.3) the same power as in Frege systems.[20]

We write SKSo to denote the system SKS extended by the rule open↓, and we write KSo to denote the system KS extended by the rule open↓. We can easily prove the following propositions:

Proposition 5.1.6. SKS *and* SKSo *are p-equivalent, and* KS *and* KSo *are p-equivalent.*

Proof. Note that SKSo trivially p-simulates SKS. For showing that SKS p-simulates SKSo, we replace

$$\mathsf{open}\downarrow\dfrac{\mathsf{ai}\downarrow\dfrac{\mathbf{t}}{\bar{a}\vee a}}{A}\quad\mathsf{ai}\downarrow\dfrac{\mathbf{t}}{\bar{b}\vee b}\,\Big\|\,B\,\Big\|\,A\wedge B \qquad \text{by} \qquad \mathsf{ai}\downarrow\dfrac{\mathsf{ai}\downarrow\dfrac{\mathbf{t}}{\bar{a}\vee a}}{A}\,\Big\|\,A\wedge[\bar{b}\vee b]\,\Big\|\,A\wedge B$$

everywhere in a given SKSo-proof to obtain an SKS-proof, whose size is quadratic in the size of the original proof. For KSo and KS the proof is the same. □

Proposition 5.1.7. SKSo *p-simulates any Frege system* F.

Proof. For every axiom B in F one can give a proof

$$\mathsf{SKS}\Big\|\,B$$

and modus ponens can be simulated as follows:

$$\mathsf{modus\ ponens}\ \dfrac{A\qquad \bar{A}\vee B}{B} \qquad\rightsquigarrow\qquad \mathsf{i}\uparrow\dfrac{\mathsf{s}\dfrac{\mathsf{open}\downarrow\dfrac{A\qquad\bar{A}\vee B}{A\wedge[\bar{A}\vee B]}}{(A\wedge\bar{A})\vee B}}{B}$$

which gives a p-simulation (by a constant factor). □

We have just shown:

Theorem 5.1.8. SKS *and* SKSo *are p-equivalent to each other and to every Frege-system.*

5.2 Extension

Let us now turn to the actual interest of this chapter, the extension rule (first formulated by Tseitin [Tse68]), which allows to use abbreviations in the proof. I.e., there is a finite set of fresh and mutually distinct propositional variables $a_1, \ldots, a_n$ which can abbreviate

[20] As observed by Bruscoli, substitution in plain SKS is weaker than substitution in Frege systems.

formulas $A_1, \ldots, A_n$, that obey the side condition that for all $1 \leq i \leq n$, the variable a_i does not appear in $A_1, \ldots, A_i$. Extension can easily be integrated in a Frege-system by simply adding the formulas $a_i \Leftrightarrow A_i$, for $1 \leq i \leq n$, to the set of axioms. In that case we speak of an *extended Frege-system* [CR79]. In the sequent calculus one could add these formulas as non-logical axioms, with the consequence that cut-elimination would not hold anymore. This very idea is used by Bruscoli and Guglielmi in [BG09] for adding extension to a system in the calculus of structures: instead of starting a proof from no premises, they use the conjunction

$$[\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \tag{5.3}$$

of all extension formulas as premise. Let us write xSKS to denote the system SKS with the extension incorporated this way, i.e., a proof of a formula B in xSKS is a derivation

$$\begin{array}{c} [\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \\ \mathsf{SKS}\Big\Vert \pi \\ B \end{array} \tag{5.4}$$

where

the propositional variables $a_1, \ldots, a_n$ are mutually distinct, and for all $1 \leq i \leq n$, the variable a_i does not appear in $A_1, \ldots, A_i$ nor in B. (5.5)

The system xSKSo is defined similarly, by additionally allowing the rule $\mathsf{open}{\downarrow}$.

Proposition 5.2.1. xSKS *and* xSKSo *are p-equivalent.*

Proof. As in the proof of Proposition 5.1.6: xSKSo trivially p-simulates xSKS, and for showing that xSKS p-simulates xSKSo, we replace

$$\mathsf{open}{\downarrow}\,\dfrac{\begin{array}{c} X \\ \Vert \\ A \end{array} \quad \begin{array}{c} X \\ \Vert \\ B \end{array}}{A \wedge B} \qquad \text{by} \qquad \begin{array}{c} \mathsf{c}{\uparrow}\,\dfrac{X}{X \wedge X} \\ \Vert \\ A \wedge X \\ \Vert \\ A \wedge B \end{array}$$

where X is the conjunction of extension formulas in (5.3). Again, the blow-up of the proof is only quadratic. □

Proposition 5.2.2. *Any Frege system with extension p-simulates* xSKS.

Proof. As in the proof of Proposition 5.1.1. Observe that the premise of (5.4) is provable in an extended Frege system. □

It should be clear that xSKS crucially relies on the presence of cut, in the same way as extended Frege-system rely on the presence of modus ponens: The premise of (5.4) contains the variables $a_1, \ldots, a_n$, which do not appear in the conclusion B. Thus, the derivation in (5.4) must contain cuts. This raises the question whether the virtues of extension can also

be used in a cut-free system. For this, let us for every extension axiom $a_i \Leftrightarrow A_i$ add the following two rules (we use the same name for both of them):

$$\mathsf{ext}{\downarrow}\,\frac{K\{a_i\}}{K\{A_i\}} \qquad \text{and} \qquad \mathsf{ext}{\downarrow}\,\frac{K\{\bar{a}_i\}}{K\{\bar{A}_i\}} \tag{5.6}$$

Note that the rule $\mathsf{ext}{\downarrow}$ is not sound. Consider for example the extension axiom $a \Leftrightarrow b \wedge c$ where a abbreviates $b \wedge c$. Applying it to $a \vee \bar{a}$ (which is a tautology) yields $(b \wedge c) \vee \bar{a}$ (which is *not* a tautology). Nonetheless, we allow to apply (5.6) in an arbitrary context $K\{\ \}$, provided that condition (5.5) is satisfied.

We write eKS to denote the system $\mathsf{KS} \cup \{\mathsf{ext}{\downarrow}\}$. A *proof in* eKS is a derivation in eKS that has premise $\mathbf{t}$ and that obeys condition (5.5). Similarly, we define eSKS as $\mathsf{SKS} \cup \{\mathsf{ext}{\downarrow}\}$, and eSKSo as $\mathsf{SKSo} \cup \{\mathsf{ext}{\downarrow}\}$, and eKSo as $\mathsf{KS} \cup \{\mathsf{ext}{\downarrow}, \mathsf{open}{\downarrow}\}$. Then we have the following:

Proposition 5.2.3. eKS *and* eSKS *and* eKSo *and* eSKSo *are all sound and complete for classical propositional logic.*

Proof. Completeness of all systems follows from completeness of KS, and soundness of eSKS follows from Theorem 5.2.8 below. This entails soundness of the other systems. □

Proposition 5.2.4. eSKS *and* eSKSo *are p-equivalent.*

Proof. As in the proof of Proposition 5.1.6. □

Proposition 5.2.5. eSKS *p-simulates* xSKS*, and* eSKSo *p-simulates* xSKSo*.*

Proof. Given a proof π of a formula B in xSKS, we can construct

$$\begin{array}{c}
\{\mathsf{ai}{\downarrow}\}\Big\|\pi_2 \\
[\bar{a}_1 \vee a_1] \wedge [\bar{a}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee a_n] \wedge [\bar{a}_n \vee a_n] \\
\{\mathsf{ext}{\downarrow}\}\Big\|\pi_1 \\
[\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \\
\mathsf{SKS}\Big\|\pi \\
B
\end{array}$$

where π_1 consists of $2n$ instances of $\mathsf{ext}{\downarrow}$ and π_2 of $2n$ instances of $\mathsf{ai}{\downarrow}$. □

Proposition 5.2.6. xSKS *p-simulates* eSKS*, and* xSKSo *p-simulates* eSKSo*.*

Proof. Assume we have an eSKS proof π of a formula B. We transform it as follows

$$\begin{array}{c}
\mathsf{eSKS}\Big\|\pi \\
B
\end{array}
\quad \rightsquigarrow \quad
\mathsf{w}{\uparrow}\,\dfrac{\begin{array}{c}
[\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \\
\mathsf{eSKS}\Big\|\pi' \\
[\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n] \wedge B
\end{array}}{B}$$

where π' is obtained from π by putting every line in conjunction with the formula (5.3). The instances of $\mathsf{ext}{\downarrow}$ in π' can now be removed as follows:

$$\mathsf{ext}{\downarrow}\,\frac{\cdots\wedge[\bar{a}_i\vee A_i]\wedge\cdots\wedge K\{a_i\}}{\cdots\wedge[\bar{a}_i\vee A_i]\wedge\cdots\wedge K\{A_i\}}\quad\rightsquigarrow\quad\begin{array}{c}\mathsf{c}{\uparrow}\,\dfrac{\cdots\wedge[\bar{a}_i\vee A_i]\wedge\cdots\wedge K\{a_i\}}{\cdots\wedge[\bar{a}_i\vee A_i]\wedge[\bar{a}_i\vee A_i]\wedge\cdots\wedge K\{a_i\}}\\ \{\mathsf{s}\}\Big\|\pi_{\mathsf{s}}\\ \mathsf{s}\,\dfrac{\cdots\wedge[\bar{a}_i\vee A_i]\wedge\cdots\wedge K\{a_i\wedge[\bar{a}_i\vee A_i]\}}{\cdots\wedge[\bar{a}_i\vee A_i]\wedge\cdots\wedge K\{(a_i\wedge\bar{a}_i)\vee A_i\}}\\ \mathsf{ai}{\uparrow}\,\dfrac{}{\cdots\wedge[\bar{a}_i\vee A_i]\wedge\cdots\wedge K\{A_i\}}\end{array}\tag{5.7}$$

where $K\{\ \}$ is an arbitrary (positive) context, and the existence of π_{s} (which contains only instances of the rule s) can be shown by an easy induction on $K\{\ \}$ (see Lemma 3.10.18 on page 82). The length of π_{s} is bound by the depth of $K\{\ \}$. Note the crucial use of the cut rule in (5.7). For eSKSo we proceed similarly. □

Proposition 5.2.7. eSKSo *p-simulates any Frege system with extension.*

Proof. As in the proof of Proposition 5.1.7. Observe that every extension axiom $a_i \Leftrightarrow A_i$, can be proved in eSKS with the rule $\mathsf{ext}{\downarrow}$. □

We can summarize the propositions of this section as follows:

Theorem 5.2.8. eSKS*,* xSKS*,* eSKSo*, and* xSKSo *are all p-equivalent to each other and to every extended Frege system.*

Proof. Immediate from Propositions 5.2.1, 5.2.2, 5.2.4, 5.2.5, 5.2.6, and 5.2.7. □

Systems eKS and eKSo give us a way of adding extension to a deductive system independently from cut. To show that extension without cut is potentially useful for giving short proofs for some of the standard benchmark tautologies, we give in Section 5.4 polynomial size proofs of the propositional pigeon hole principle in eKS.

5.3 Substitution

Let us next consider systems with substitution. A *substitution* is a function σ from the set $\mathscr{A}$ of propositional variables to the set $\mathscr{F}$ of formulas, such that $\sigma(a) = a$ for almost all $a \in \mathscr{A}$. We can define $\sigma(A)$ inductively for all formulas via $\sigma(A \wedge B) = \sigma(A) \wedge \sigma(B)$ and $\sigma(A \vee B) = \sigma(A) \vee \sigma(B)$ and $\sigma(\bar{A}) = \overline{\sigma(A)}$. Following the tradition, we write $A\sigma$ for $\sigma(A)$. For example, if $A = a \vee \bar{b} \vee b$ and $\sigma = \{a \mapsto a \wedge b, b \mapsto a \vee \bar{c}\}$ then $A\sigma = (a \wedge b) \vee (\bar{a} \wedge c) \vee a \vee \bar{c}$. We can define the inference rule for substitution

$$\mathsf{sub}{\downarrow}\,\frac{A}{A\sigma}\tag{5.8}$$

Note that the rule $\mathsf{sub}{\downarrow}$ cannot be applied inside a context $K\{\ \}$. Thus, it is exactly the same rule as in Frege systems and in strong contrast to all other rules in deep inference. Let us define $\mathsf{sSKS} = \mathsf{SKS} \cup \{\mathsf{sub}{\downarrow}\}$ and $\mathsf{sSKSo} = \mathsf{SKSo} \cup \{\mathsf{sub}{\downarrow}\}$. Contrary to the other systems we discussed so far, we cannot easily prove that sSKS p-simulates sSKSo, as in the proof of Proposition 5.1.6. However, we have the following two propositions:

Proposition 5.3.1. *Any Frege-system with substitution p-simulates* sSKS.

Proof. As in the proof of Proposition 5.1.1, because the substitution rule is the same in SKS and Frege-systems. □

Proposition 5.3.2. sSKSo *p-simulates any Frege-system with substitution.*

Proof. As in the proof of Proposition 5.1.7. □

Proposition 5.3.3. sSKS *p-simulates* xSKS.

Proof. This proof can already be found in [BG09]. For a given xSKS proof π of a formula B, we construct

$$
\begin{array}{c}
\mathsf{i}{\downarrow}\,\dfrac{\mathbf{t}}{(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee ([\bar{a}_1 \vee A_1] \wedge [\bar{A}_1 \vee a_1] \wedge \cdots \wedge [\bar{a}_n \vee A_n] \wedge [\bar{A}_n \vee a_n])} \\
\mathsf{SKS}\Big\|\,\pi' \\
= \dfrac{(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B}{(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee (\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B} \\
\mathsf{sub}{\downarrow}\,\dfrac{(\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B}{(\bar{A}_n \wedge A_n) \vee (\bar{A}_n \wedge A_n) \vee (\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B} \\
{}_{2*}\mathsf{i}{\uparrow}\,\dfrac{(\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B}{\mathsf{sub}{\downarrow}\,\dfrac{(\bar{a}_{n-1} \wedge A_{n-1}) \vee (\bar{A}_{n-1} \wedge a_{n-1}) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B}{\vdots}} \\
{}_{2*}\mathsf{i}{\uparrow}\,\dfrac{\dfrac{(\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1) \vee B}{\mathsf{sub}{\downarrow}\,\dfrac{(\bar{A}_1 \wedge A_1) \vee (\bar{A}_1 \wedge A_1) \vee B}{B}}}{}
\end{array}
\tag{5.9}
$$

where π' is obtained from π by putting every formula in disjunction with

$$(\bar{a}_n \wedge A_n) \vee (\bar{A}_n \wedge a_n) \vee \cdots \vee (\bar{a}_1 \wedge A_1) \vee (\bar{A}_1 \wedge a_1)$$

The derivation (5.9) is a valid derivation in sSKS because of condition (5.5). Note that we proceed backwards in eliminating the a_i in order to keep the size of the proof polynomial. □

Proposition 5.3.4. sSKSo *p-simulates* xSKSo.

Proof. By Theorem 5.2.1 we have that xSKS p-simulates xSKSo. The previous theorem tells us that sSKS p-simulates xSKS, and it is trivial that sSKSo p-simulates sSKS. □

For the other direction, the basic idea is to simulate the substitution inference step from A to $A\sigma$ by many extension inference steps, one for each occurrence of a variable a with

$\sigma(a) \neq a$ in A. Consider for example:

$$\mathsf{sub}\!\downarrow \dfrac{\begin{array}{c}\|\pi_2\\ K\{a \vee (b \wedge c) \vee \bar{a}\}\end{array}}{\begin{array}{c}K\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\}\\ \|\pi_1\\ B\end{array}} \quad \rightsquigarrow \quad \mathsf{ext}\!\downarrow \dfrac{\mathsf{ext}\!\downarrow \dfrac{\mathsf{ext}\!\downarrow \dfrac{\begin{array}{c}\|\pi_2\\ K\{a \vee (b \wedge c) \vee \bar{a}\}\end{array}}{K\{(a \wedge c) \vee (b \wedge c) \vee \bar{a}\}}}{K\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a}\}}}{\begin{array}{c}K\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\}\\ \|\pi_1\\ B\end{array}} \tag{5.10}$$

where the used substitution is $\{a \mapsto a \wedge c, c \mapsto a \vee c\}$ and the context $K\{\ \}$ does not contain any occurrences of a or c. The problem with this is that the result will, in general, not be a valid proof because both conditions in (5.5) might be violated. For this reason we first have to rename the variables a and c in π_2:

$$\mathsf{sub}\!\downarrow \dfrac{\begin{array}{c}\|\pi_2'\\ K\{a' \vee (b \wedge c') \vee \bar{a}'\}\end{array}}{\begin{array}{c}K\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\}\\ \|\pi_1\\ B\end{array}} \quad \rightsquigarrow \quad \mathsf{ext}\!\downarrow \dfrac{\mathsf{ext}\!\downarrow \dfrac{\mathsf{ext}\!\downarrow \dfrac{\begin{array}{c}\|\pi_2'\\ K\{a' \vee (b \wedge c') \vee \bar{a}'\}\end{array}}{K\{(a \wedge c) \vee (b \wedge c') \vee \bar{a}'\}}}{K\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a}'\}}}{\begin{array}{c}K\{(a \wedge c) \vee (b \wedge [a \vee c]) \vee \bar{a} \vee \bar{c}\}\\ \|\pi_1\\ B\end{array}} \tag{5.11}$$

Here a and c have been replaced everywhere in π_2 by fresh variables a' and c', respectively. The new substitution is $\{a' \mapsto a \wedge c, c' \mapsto a \vee c\}$, which can be replaced by instances of extension, without violating (5.5).

Theorem 5.3.5. eSKS *p-simulates* sSKS.

Proof. Let π be an sSKS proof of a formula B. Suppose π contains k instances of sub$\downarrow$, and let $\sigma_{1,1}, \ldots, \sigma_{k,1}$ be the k substitutions used in them. Then π is of the shape as shown in the left-most derivation in Figure 5.1. In the following, we use $\mathscr{A}_{i,j}$ to denote the set of variables a with $\sigma_{i,j}(a) \neq a$. As explained above, we can now iteratively rename the propositional variables in $\mathscr{A}_{1,1}, \ldots, \mathscr{A}_{k,1}$, starting from the bottommost instance of sub$\downarrow$, as indicated in Figure 5.1. The result of this renaming is shown in the rightmost derivation in Figure 5.1 and has the property that

$$\text{for all } i \text{ with } 1 \leq i \leq k, \text{ we have that no variable in } \mathscr{A}_{i,i+1} \text{ appears in any of } \pi_{1,1}, \pi_{2,2}, \ldots, \pi_{i,i}. \tag{5.12}$$

Let $\mathscr{A}_{i,i+1} = \{a_{i,1}, \ldots, a_{i,m_i}\}$, and let $A_{i,h} = \sigma_{i,i+1}(a_{i,h})$. We now have $n = m_1 + m_2 + \cdots + m_k$ extension variables, defined via

$$\mathsf{ext}\!\downarrow \dfrac{a_{i,h}}{A_{i,h}} \qquad \text{and} \qquad \mathsf{ext}\!\downarrow \dfrac{\bar{a}_{i,h}}{\bar{A}_{i,h}}$$

If we give the index pair (i, h) the lexicographic ordering, it immediately follows from (5.12) that condition (5.5) is fulfilled. Hence, we can trivially replace each instance of sub$\downarrow$ by a

$$
\begin{array}{c}
\mathsf{SKS}\Big\|\pi_{k+1,1} \\
\mathsf{sub}\downarrow\dfrac{B_{k,1}}{B_{k,1}\sigma_{k,1}} \\
\mathsf{SKS}\Big\|\pi_{k,1} \\
\vdots \\
\mathsf{SKS}\Big\|\pi_{3,1} \\
\mathsf{sub}\downarrow\dfrac{B_{2,1}}{B_{2,1}\sigma_{2,1}} \\
\mathsf{SKS}\Big\|\pi_{2,1} \\
\mathsf{sub}\downarrow\dfrac{B_{1,1}}{B_{1,1}\sigma_{1,1}} \\
\mathsf{SKS}\Big\|\pi_{1,1} \\
B
\end{array}
\rightsquigarrow
\begin{array}{c}
\mathsf{SKS}\Big\|\pi_{k+1,2} \\
\mathsf{sub}\downarrow\dfrac{B_{k,2}}{B_{k,2}\sigma_{k,2}} \\
\mathsf{SKS}\Big\|\pi_{k,2} \\
\vdots \\
\mathsf{SKS}\Big\|\pi_{3,2} \\
\mathsf{sub}\downarrow\dfrac{B_{2,2}}{B_{2,2}\sigma_{2,2}} \\
\mathsf{SKS}\Big\|\pi_{2,2} \\
\mathsf{sub}\downarrow\dfrac{B_{1,2}}{=\dfrac{B_{1,2}\sigma_{1,2}}{B_{1,1}\sigma_{1,1}}} \\
\mathsf{SKS}\Big\|\pi_{1,1} \\
B
\end{array}
\rightsquigarrow
\begin{array}{c}
\mathsf{SKS}\Big\|\pi_{k+1,3} \\
\mathsf{sub}\downarrow\dfrac{B_{k,3}}{B_{k,3}\sigma_{k,3}} \\
\mathsf{SKS}\Big\|\pi_{k,3} \\
\vdots \\
\mathsf{SKS}\Big\|\pi_{3,3} \\
\mathsf{sub}\downarrow\dfrac{B_{2,3}}{=\dfrac{B_{2,3}\sigma_{2,3}}{B_{2,2}\sigma_{2,2}}} \\
\mathsf{SKS}\Big\|\pi_{2,2} \\
\mathsf{sub}\downarrow\dfrac{B_{1,2}}{=\dfrac{B_{1,2}\sigma_{1,2}}{B_{1,1}\sigma_{1,1}}} \\
\mathsf{SKS}\Big\|\pi_{1,1} \\
B
\end{array}
\rightsquigarrow \cdots \rightsquigarrow
\begin{array}{c}
\mathsf{SKS}\Big\|\pi_{k+1,k+1} \\
\mathsf{sub}\downarrow\dfrac{B_{k,k+1}}{=\dfrac{B_{k,k+1}\sigma_{k,k+1}}{B_{k,k}\sigma_{k,k}}} \\
\mathsf{SKS}\Big\|\pi_{k,k} \\
\vdots \\
\mathsf{SKS}\Big\|\pi_{3,3} \\
\mathsf{sub}\downarrow\dfrac{B_{2,3}}{=\dfrac{B_{2,3}\sigma_{2,3}}{B_{2,2}\sigma_{2,2}}} \\
\mathsf{SKS}\Big\|\pi_{2,2} \\
\mathsf{sub}\downarrow\dfrac{B_{1,2}}{=\dfrac{B_{1,2}\sigma_{1,2}}{B_{1,1}\sigma_{1,1}}} \\
\mathsf{SKS}\Big\|\pi_{1,1} \\
B
\end{array}
$$

Figure 5.1: Renaming propositional variables in an sSKS proof

sequence of instances of ext↓, whose number is bound by the size of the $B_{i,i+1}$. Hence, the size of the resulting eSKS proof is at most quadratic in the size of π. □

Theorem 5.3.6. eSKSo *p-simulates* sSKSo.

Proof. The proof is almost literally the same as the previous one, which is not affected by the branching of the open↓-rule. □

I think that the proofs of Theorems 5.3.3 and 5.3.5 (and 5.2.8) are simpler than the ones in [CR79] and [KP89]. In fact, here the results look almost trivial, whereas the construction in [KP89] is rather involved. We have, in fact, more than enough to give alternative proofs of the results in [CR79] and [KP89]:

Theorem 5.3.7. *Frege systems with substitution p-simulate Frege systems with extension.*

Proof. By Theorem 5.2.7 we have that eSKSo p-simulates Frege systems with extension. By Theorem 5.2.8, we have that xSKS p-simulates eSKSo, by Theorem 5.3.3 we have that sSKS p-simulates xSKS, and finally, from Theorem 5.3.1, we get that Frege systems with substitution p-simulate sSKS. □

Theorem 5.3.8. *Frege systems with extension p-simulate Frege systems with substitution.*

Proof. By Theorem 5.3.2, we have that sSKSo p-simulates Frege systems with substitution. By Theorem 5.3.6 we have that eSKSo p-simulates sSKSo. By Theorem 5.2.8, we have that

xSKS p-simulates eSKSo, and by Theorem 5.2.2, we have that Frege systems with extension p-simulate xSKS. □

Furthermore, note that the transformation in the proof of Theorem 5.3.5 does not involve any cuts. Hence, we have also proved the following:

Theorem 5.3.9. eKS *p-simulates* sKS*, and* eKSo *p-simulates* sKSo*.*

5.4 Pigeonhole Principle and Balanced Tautologies

In this section I show two classes of tautologies which both admit polynomial-size proofs in eKS and sKS. The first one is the propositional pigeon-hole principle. The second one is a variation which has the property that every member of the class is a balanced tautology. A formula A is *balanced* if every propositional variable occurring in A occurs exactly twice, once positive and once negated. For example,

$$([a \vee b] \wedge [d \vee e]) \vee \left([\bar{a} \vee c] \wedge \left[\bar{d} \vee f\right]\right) \vee \left(\left[\bar{b} \vee \bar{c}\right] \wedge \left[\bar{e} \vee \bar{f}\right]\right)$$

is balanced (and a tautology), whereas

$$a \vee a \vee (\bar{a} \wedge \bar{a}) \qquad \text{and} \qquad a \wedge \bar{a} \wedge b$$

are not balanced. I use the notation $\bigwedge_{0 \leq i \leq n} F_i$ as abbreviation for $F_0 \wedge \cdots \wedge F_n$, and similarly for $\bigvee$. Furthermore, for a literal a, I abbreviate the formula $a \vee \cdots \vee a$ by a^n, if there are n copies of a. Consider now

$$\mathrm{PHP}_n = \bigwedge_{0 \leq i \leq n} \bigvee_{1 \leq j \leq n} p_{i,j} \Rightarrow \bigvee_{0 \leq i < m \leq n} \bigvee_{1 \leq j \leq n} (p_{i,j} \wedge p_{m,j}) \tag{5.13}$$

This formula is called the propositional pigeon hole principle because it expresses the fact that if there are $n+1$ pigeons and only n holes and every pigeon is in a hole then at least one hole contains two pigeons, provided one reads the propositional variable $p_{i,j}$ as "pigeon i sits in hole j".

The formulas (5.13) have been well investigated from the viewpoint of proof complexity. In [CR79] they where presented as a candidate for separating Frege systems and extended Frege systems (wrt. p-simulation). But Buss [Bus87] has shown that PHP_n admits a polynomial-size proof in a Frege system (and therefore in SKS) for every n.

I will here show that in eKS as well as in sKS we have cut-free polynomial-size proofs for (5.13). For this I use a new class of tautologies which also admit polynomial-size proofs in eKS, and which are defined as follows:

$$\mathrm{QHQ}_n = \bigvee_{0 \leq i \leq n} \bigwedge_{1 \leq j \leq n} \left[\bigvee_{1 \leq k \leq i} \bar{q}_{i,j,k} \vee \bigvee_{i < k \leq n} q_{k,j,i+1} \right] \tag{5.14}$$

Here are the first three examples:

$$
\begin{aligned}
\mathrm{QHQ}_1 &= q_{1,1,1} \vee \bar{q}_{1,1,1} \\
\mathrm{QHQ}_2 &= ([q_{1,1,1} \vee q_{2,1,1}] \wedge [q_{1,2,1} \vee q_{2,2,1}]) \vee ([\bar{q}_{1,1,1} \vee q_{2,1,2}] \wedge [\bar{q}_{1,2,1} \vee q_{2,2,2}]) \vee \\
&\quad ([\bar{q}_{2,1,1} \vee \bar{q}_{2,1,2}] \wedge [\bar{q}_{2,2,1} \vee \bar{q}_{2,2,2}]) \\
\mathrm{QHQ}_3 &= ([q_{1,1,1} \vee q_{2,1,1} \vee q_{3,1,1}] \wedge [q_{1,2,1} \vee q_{2,2,1} \vee q_{3,2,1}] \wedge [q_{1,3,1} \vee q_{2,3,1} \vee q_{3,3,1}]) \vee \\
&\quad ([\bar{q}_{1,1,1} \vee q_{2,1,2} \vee q_{3,1,2}] \wedge [\bar{q}_{1,2,1} \vee q_{2,2,2} \vee q_{3,2,2}] \wedge [\bar{q}_{1,3,1} \vee q_{2,3,2} \vee q_{3,3,2}]) \vee \\
&\quad ([\bar{q}_{2,1,1} \vee \bar{q}_{2,1,2} \vee q_{3,1,3}] \wedge [\bar{q}_{2,2,1} \vee \bar{q}_{2,2,2} \vee q_{3,2,3}] \wedge [\bar{q}_{2,3,1} \vee \bar{q}_{2,3,2} \vee q_{3,3,3}]) \vee \\
&\quad ([\bar{q}_{3,1,1} \vee \bar{q}_{3,1,2} \vee \bar{q}_{3,1,3}] \wedge [\bar{q}_{3,2,1} \vee \bar{q}_{3,2,2} \vee \bar{q}_{3,2,3}] \wedge [\bar{q}_{3,3,1} \vee \bar{q}_{3,3,2} \vee \bar{q}_{3,3,3}])
\end{aligned}
$$

The tautologies QHQ_n are balanced. This means that the size of a proof of such a tautology is directly related to the number of applications of $\mathsf{ac}{\downarrow}$. Furthermore, all proofs that we show here do not contain any weakening. This makes this class interesting for investigating the gap between linear logic and classical logic [Lam07, Str07b].

The formulas QHQ_1 and QHQ_2 are easily provable in $\mathsf{KS} \setminus \{\mathsf{ac}{\downarrow}\}$. One might be tempted to conjecture that $\mathsf{KS} \setminus \{\mathsf{ac}{\downarrow}\}$ or $\mathsf{eKS} \setminus \{\mathsf{ac}{\downarrow}\}$ is already complete for the class of balanced tautologies. But unfortunately, this is not the case. The smallest counterexample known to me is QHQ_3. Every possible application of $\mathsf{ai}{\downarrow}$, s, m, or $\mathsf{w}{\downarrow}$ leads to a non-tautologous formula. Thus also the extension rule is of no use. (The same is true for all formulas QHQ_n with $n \geq 3$.)

This is not surprising under the view of the following theorem, which says that balanced tautologies are not easier to prove than other tautologies.

Theorem 5.4.1. *The set of balanced tautologies is* **coNP***-complete.*

Proof. We can reduce provability of general tautologies to provability of balanced tautologies. For a formula B, we let B' be the formula obtained from B by doing the following replacement for every propositional variable a occurring in B: Let n be the number of occurrences of a in positive form in B, and let m be the number of occurrences of $\bar{a}$ in B. If $n \geq 1$ and $m \geq 1$, then introduce $n \cdot m$ fresh propositional variables $a_{i,j}$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Now replace for every $1 \leq i \leq n$ the ith occurrence of a by $a_{i,1} \vee \cdots \vee a_{i,m}$, and replace for every $1 \leq j \leq m$ the jth occurrence of $\bar{a}$ by $\bar{a}_{1,j} \vee \cdots \vee \bar{a}_{n,j}$. If $n = 0$, then introduce m fresh variables $a_1, \ldots a_m$ and replace the jth $\bar{a}$ by $\bar{a}_j \wedge a_j$. If $m = 0$, proceed similarly. Then B' is balanced, and its size is quadratic in the size of B. Furthermore, B' is a tautology if and only if B is a tautology. □

Let us now reduce PHP_n to QHQ_n. We first replace the implication by disjunction and negation, and then apply associativity and commutativity of $\vee$:

$$
\begin{aligned}
\mathrm{PHP}_n &= \bigvee_{0 \leq i \leq n} \bigwedge_{1 \leq j \leq n} \bar{p}_{i,j} \vee \bigvee_{0 \leq i < n} \bigvee_{i < m \leq n} \bigvee_{1 \leq j \leq n} (p_{i,j} \wedge p_{m,j}) && (5.15) \\
&= \bigvee_{0 \leq i \leq n} \bigwedge_{1 \leq j \leq n} \bar{p}_{i,j} \vee \bigvee_{0 \leq i \leq n} \bigvee_{1 \leq j \leq n} \bigvee_{i < m \leq n} (p_{i,j} \wedge p_{m,j}) && (5.16) \\
&= \bigvee_{0 \leq i \leq n} \left[\bigwedge_{1 \leq j \leq n} \bar{p}_{i,j} \vee \bigvee_{1 \leq j \leq n} \bigvee_{i < m \leq n} (p_{i,j} \wedge p_{m,j}) \right] && (5.17)
\end{aligned}
$$

Now consider the following class of formulas (where $\bar{p}^i_{i,j}$ abbreviates $\bar{p}_{i,j} \vee \cdots \vee \bar{p}_{i,j}$ with i copies of $\bar{p}_{i,j}$):

$$\mathrm{PHP}'_n = \bigvee\!\!\!\bigvee_{0\le i\le n} \bigwedge\!\!\!\bigwedge_{1\le j\le n} \left[\bar{p}^i_{i,j} \vee \bigvee\!\!\!\bigvee_{i<m\le n} p_{m,j}\right] \tag{5.18}$$

We have for each n a derivation from PHP'_n to PHP_n of length $O(n^3)$:

$$\begin{array}{rl}
 & = \dfrac{\mathrm{PHP}'_n}{\bigvee\!\!\!\bigvee_{0\le i\le n} \bigwedge\!\!\!\bigwedge_{1\le j\le n} [\bar{p}^i_{i,j} \vee \bigvee\!\!\!\bigvee_{i<m\le n} p_{m,j}]} \\
n(n+1)/2{*}\mathsf{ai}{\downarrow} & \dfrac{\bigvee\!\!\!\bigvee_{0\le i\le n} \bigwedge\!\!\!\bigwedge_{1\le j\le n} [\bar{p}^i_{i,j} \vee \bigvee\!\!\!\bigvee_{i<m\le n} ([\bar{p}_{i,j} \vee p_{i,j}] \wedge p_{m,j})]}{} \\
n(n+1)/2{*}\mathsf{s} & \dfrac{\bigvee\!\!\!\bigvee_{0\le i\le n} \bigwedge\!\!\!\bigwedge_{1\le j\le n} [\bar{p}^i_{i,j} \vee \bar{p}^{n-i}_{i,j} \vee \bigvee\!\!\!\bigvee_{i<m\le n} (p_{i,j} \wedge p_{m,j})]}{} \\
= & \dfrac{\bigvee\!\!\!\bigvee_{0\le i\le n} \bigwedge\!\!\!\bigwedge_{1\le j\le n} [\bar{p}^n_{i,j} \vee \bigvee\!\!\!\bigvee_{i<m\le n} (p_{i,j} \wedge p_{m,j})]}{} \\
n(n+1)(n-1){*}\mathsf{ac}{\downarrow} & \dfrac{\bigvee\!\!\!\bigvee_{0\le i\le n} \bigwedge\!\!\!\bigwedge_{1\le j\le n} [\bar{p}_{i,j} \vee \bigvee\!\!\!\bigvee_{i<m\le n} (p_{i,j} \wedge p_{m,j})]}{} \\
n(n+1){*}\mathsf{s} & \dfrac{\bigvee\!\!\!\bigvee_{0\le i\le n} [\bigwedge\!\!\!\bigwedge_{1\le j\le n} \bar{p}_{i,j} \vee \bigvee\!\!\!\bigvee_{1\le j\le n} \bigvee\!\!\!\bigvee_{i<m\le n} (p_{i,j} \wedge p_{m,j})]}{} \\
= & \dfrac{}{\mathrm{PHP}_n}
\end{array} \tag{5.19}$$

Remark 5.4.2. Since PHP'_n is just an instance of QHQ_n with $q_{i,j,k} = p_{i,j}$, every polynomial-size proof of QHQ_n yields also a polynomial-size proof of PHP_n. On the other hand, with the substitution (found by an anonymous referee)

$$p_{i,j} \mapsto \bigwedge\!\!\!\bigwedge_{1\le k\le i} q_{i,j,k} \wedge \bigwedge\!\!\!\bigwedge_{i<k\le n} \bar{q}_{k,j,i+1}$$

a polynomial-size proof of PHP_n can be transformed into a polynomial-size proof of QHQ_n. Thus the result by Buss [Bus87] can be used to give a polynomial-size proof of QHQ_n in SKS.

For a given number n, we define for all $0 \le i \le n$ and $1 \le j \le n$ the formula

$$\begin{aligned} Q_{i,j} &= \bigvee\!\!\!\bigvee_{1\le k\le i} \bar{q}_{i,j,k} \vee \bigvee\!\!\!\bigvee_{i<k\le n} q_{k,j,i+1} \\ &= \bar{q}_{i,j,1} \vee \bar{q}_{i,j,2} \vee \cdots \vee \bar{q}_{i,j,i} \vee q_{i+1,j,i+1} \vee q_{i+2,j,i+1} \vee \cdots \vee q_{n,j,i+1} \end{aligned} \tag{5.20}$$

Then $\mathrm{QHQ}_n = (Q_{0,1} \wedge \cdots \wedge Q_{0,n}) \vee (Q_{1,1} \wedge \cdots \wedge Q_{1,n}) \vee \cdots \vee (Q_{n,1} \wedge \cdots \wedge Q_{n,n})$. The formula $Q_{i,j}$ consists of n disjuncts. Let $Q^{\vee m}_{i,j}$ denote the formula obtained from $Q_{i,j}$ by removing the mth disjunct. Then for all $m \le i$ we have $Q_{i,j} = Q^{\vee m}_{i,j} \vee \bar{q}_{i,j,m}$ and for all $m > i$ we have $Q_{i,j} = Q^{\vee m}_{i,j} \vee q_{m,j,i+1}$. Figure 5.2 shows a derivation in sKS from QHQ_{n-1} to QHQ_n of length $O(n^3)$. In that figure, the number z abbreviates $n \cdot (n-1) \cdot (n-2)$. The used substitution is defined as follows:

$$q_{i,j,k} \mapsto [q_{i,j,k} \vee q_{n,j,k}] \wedge [\bar{q}_{n,j,i+1} \vee q_{i,n,k}] \quad .$$

Since the proof of QHQ_1 is trivial, we exhibited a cut-free polynomial-size proof of QHQ_n and PHP_n. We can transform the complete proof of QHQ_n into an eKS proof by renaming the variables $q_{i,j,k}$ at each stage (see proof of Theorem 5.3.5) and use the extension formulas[21]

$$q'_{i,j,k} \Leftrightarrow [q_{i,j,k} \vee q_{n,j,k}] \wedge [\bar{q}_{n,j,i+1} \vee q_{i,n,k}]$$

[21] To distinguish between the propositional variable occurrences in QHQ_n and the occurrences QHQ_{n-1}, we use q' for those in QHQ_{n-1}. This is more legible than adding yet another index to the q.

$$
\begin{array}{rl}
 & = \dfrac{\mathrm{QHQ}_{n-1}}{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} [\bigvee_{1\le k\le i} \bar{q}_{i,j,k} \vee \bigvee_{i<k<n} q_{k,j,i+1}]} \\[2ex]
\mathsf{sub}\downarrow & \dfrac{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} [\bigvee_{1\le k\le i} [(\bar{q}_{i,j,k} \wedge \bar{q}_{n,j,k}) \vee (q_{n,j,i+1} \wedge \bar{q}_{i,n,k})] \vee \bigvee_{i<k<n} ([q_{k,j,i+1} \vee q_{n,j,i+1}] \wedge [\bar{q}_{n,j,k+1} \vee q_{k,n,i+1}])]}{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} [\bigvee_{1\le k\le i} ([\bar{q}_{i,j,k} \vee q_{n,j,i+1}] \wedge [\bar{q}_{n,j,k} \vee \bar{q}_{i,n,k}]) \vee \bigvee_{i<k<n} ([q_{k,j,i+1} \vee q_{n,j,i+1}] \wedge [\bar{q}_{n,j,k+1} \vee q_{k,n,i+1}])]} \\[2ex]
z/2{*}\mathsf{m} & \dfrac{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} [\bigvee_{1\le k\le i} ([\bar{q}_{i,j,k} \vee q_{n,j,i+1}] \wedge [\bar{q}_{n,j,k} \vee \bar{q}_{i,n,k}]) \vee \bigvee_{i<k<n} ([q_{k,j,i+1} \vee q_{n,j,i+1}] \wedge [\bar{q}_{n,j,k+1} \vee q_{k,n,i+1}])]}{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} ([\bigvee_{1\le k\le i} [\bar{q}_{i,j,k} \vee q_{n,j,i+1}] \vee \bigvee_{i<k<n} [q_{k,j,i+1} \vee q_{n,j,i+1}]] \wedge} \\[2ex]
z\mathsf{m} & \dfrac{[\bigvee_{1\le k\le i} [\bar{q}_{n,j,k} \vee \bar{q}_{i,n,k}] \vee \bigvee_{i<k<n} [\bar{q}_{n,j,k+1} \vee q_{k,n,i+1}]]}{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} ([\bar{q}_{i,j,1} \vee \cdots \vee \bar{q}_{i,j,i} \vee q_{i+1,j,i+1} \vee \cdots \vee q_{n-1,j,i+1} \vee q^{n-1}_{n,j,i+1}] \wedge} \\[2ex]
 & = \dfrac{[\bar{q}_{n,j,1} \vee \cdots \vee \bar{q}_{n,j,i} \vee \bar{q}_{n,j,i+2} \vee \cdots \vee \bar{q}_{n,j,n} \vee \bar{q}_{i,n,1} \vee \cdots \vee \bar{q}_{i,n,i} \vee q_{i+1,n,i+1} \vee \cdots \vee q_{n-1,n,i+1}]}{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} ([Q^{\vee n}_{i,j} \vee q^{n-1}_{n,j,i+1}] \wedge [Q^{\vee i+1}_{n,j} \vee Q^{\vee n}_{i,n}])} \\[2ex]
z{*}\mathsf{ac}\downarrow & \dfrac{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} ([Q^{\vee n}_{i,j} \vee q_{n,j,i+1}] \wedge [Q^{\vee i+1}_{n,j} \vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n} \bigwedge_{1\le j<n} (Q_{i,j} \wedge [Q^{\vee i+1}_{n,j} \vee Q^{\vee n}_{i,n}])} \\[2ex]
 & = \dfrac{\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge [Q^{\vee i+1}_{n,1} \vee Q^{\vee n}_{i,n}] \wedge \cdots \wedge [Q^{\vee i+1}_{n,n-1} \vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge [(Q^{\vee i+1}_{n,1} \wedge \cdots \wedge Q^{\vee i+1}_{n,n-1}) \vee Q^{\vee n}_{i,n}])} \\[2ex]
n(n-2){*}\mathsf{d}\downarrow & \dfrac{\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge [(Q^{\vee i+1}_{n,1} \wedge \cdots \wedge Q^{\vee i+1}_{n,n-1} \wedge [\bar{q}_{n,n,i+1} \vee q_{n,n,i+1}]) \vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge [(Q^{\vee i+1}_{n,1} \wedge \cdots \wedge Q^{\vee i+1}_{n,n-1} \wedge \bar{q}_{n,n,i+1}) \vee q_{n,n,i+1} \vee Q^{\vee n}_{i,n}])} \\[2ex]
n{*}\mathsf{ai}\downarrow & \dfrac{\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge [(Q^{\vee i+1}_{n,1} \wedge \cdots \wedge Q^{\vee i+1}_{n,n-1} \wedge \bar{q}_{n,n,i+1}) \vee q_{n,n,i+1} \vee Q^{\vee n}_{i,n}])}{\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge [Q_{i,n} \vee (Q^{\vee i+1}_{n,1} \wedge \cdots \wedge Q^{\vee i+1}_{n,n-1} \wedge \bar{q}_{n,n,i+1})])} \\[2ex]
n{*}\mathsf{s} & \dfrac{\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge [Q_{i,n} \vee (Q^{\vee i+1}_{n,1} \wedge \cdots \wedge Q^{\vee i+1}_{n,n-1} \wedge \bar{q}_{n,n,i+1})])}{[\bigvee_{0\le i<n} (Q_{i,1} \wedge \cdots \wedge Q_{i,n-1} \wedge Q_{i,n})] \vee [\bigvee_{0\le i<n} (Q^{\vee i+1}_{n,1} \wedge \cdots \wedge Q^{\vee i+1}_{n,n-1} \wedge \bar{q}_{n,n,i+1})]} \\[2ex]
 & = \dfrac{[\bigvee_{0\le i<n} \bigwedge_{1\le j\le n} Q_{i,j}] \vee [(Q^{\vee 1}_{n,1} \wedge \cdots \wedge Q^{\vee 1}_{n,n-1} \wedge \bar{q}_{n,n,1}) \vee \cdots \vee (Q^{\vee n}_{n,1} \wedge \cdots \wedge Q^{\vee n}_{n,n-1} \wedge \bar{q}_{n,n,n})]}{[\bigvee_{0\le i<n} \bigwedge_{1\le j\le n} Q_{i,j}] \vee ([Q^{\vee 1}_{n,1} \vee \cdots \vee Q^{\vee n}_{n,1}] \wedge \cdots \wedge [Q^{\vee 1}_{n,n-1} \vee \cdots \vee Q^{\vee n}_{n,n-1}] \wedge [\bar{q}_{n,n,1} \vee \cdots \vee \bar{q}_{n,n,n}])} \\[2ex]
(n-1)(n-1){*}\mathsf{m} & \dfrac{[\bigvee_{0\le i<n} \bigwedge_{1\le j\le n} Q_{i,j}] \vee ([\bar{q}^{\,n-1}_{n,1,1} \vee \cdots \vee \bar{q}^{\,n-1}_{n,1,n}] \wedge \cdots \wedge [\bar{q}^{\,n-1}_{n,n-1,1} \vee \cdots \vee \bar{q}^{\,n-1}_{n,n-1,n}] \wedge [\bar{q}_{n,n,1} \vee \cdots \vee \bar{q}_{n,n,n}])}{[\bigvee_{0\le i<n} \bigwedge_{1\le j\le n} Q_{i,j}] \vee ([\bar{q}_{n,1,1} \vee \cdots \vee \bar{q}_{n,1,n}] \wedge \cdots \wedge [\bar{q}_{n,n-1,1} \vee \cdots \vee \bar{q}_{n,n-1,n}] \wedge [\bar{q}_{n,n,1} \vee \cdots \vee \bar{q}_{n,n,n}])} \\[2ex]
 & = \dfrac{[\bigvee_{0\le i<n} \bigwedge_{1\le j\le n} Q_{i,j}] \vee ([\bar{q}_{n,1,1} \vee \cdots \vee \bar{q}_{n,1,n}] \wedge \cdots \wedge [\bar{q}_{n,n-1,1} \vee \cdots \vee \bar{q}_{n,n-1,n}] \wedge [\bar{q}_{n,n,1} \vee \cdots \vee \bar{q}_{n,n,n}])}{[\bigvee_{0\le i<n} \bigwedge_{1\le j\le n} Q_{i,j}] \vee (Q_{n,1} \wedge \cdots \wedge Q_{n,n-1} \wedge Q_{n,n})} \\[2ex]
z{*}\mathsf{ac}\downarrow & \dfrac{\bigvee_{0\le i<n} \bigwedge_{1\le j\le n} Q_{i,j}}{\mathrm{QHQ}_n}
\end{array}
$$

Figure 5.2: Derivation from QHQ_{n-1} to QHQ_n

as extension axioms, i.e., the rules

$$
\mathsf{ext}\downarrow \frac{q'_{i,j,k}}{[q_{i,j,k} \vee q_{n,j,k}] \wedge [\bar{q}_{n,j,i+1} \vee q_{i,n,k}]} \qquad \mathsf{ext}\downarrow \frac{\bar{q}'_{i,j,k}}{(\bar{q}_{i,j,k} \wedge \bar{q}_{n,j,k}) \vee (q_{n,j,i+1} \wedge \bar{q}_{i,n,k})} \tag{5.21}
$$

In [Jap07], Japaridze provides another cut-free polynomial size proof of PHP_n. His system of *deep cirquents* uses a form of sharing instead of extension or substitution. But it is not known whether his method can also be used for QHQ_n.

6

Open Problems

In this final chapter I will mention some of the questions that have been left open by this thesis.

6.1 Full Coherence for Boolean Categories

One usually speaks of "coherence" [Mac71] if all diagrams of a certain kind commute. Very often a "coherence theorem" is based on so-called "coherence graphs" [KM71, DP04]. In certain cases (see, e.g., [Str05]) the notion of coherence graph is too restricted. For this reason, in [LS05a], the notion of "graphicality" is introduced.

Definition 6.1.1. Let $\mathscr{C}$ be a single-mixed $\mathsf{B1}$-category, and let $\mathscr{C}^{\between}$ be the category obtained from $\mathscr{C}$ by adding for each pair of objects A and B a map $\mathsf{mix}^{-1}_{A,B} \colon A \vee B \to A \wedge B$ which is inverse to $\mathsf{mix}_{A,B}$ (i.e., the two bifunctors $- \wedge -$ and $- \vee -$ are naturally isomorphic in $\mathscr{C}^{\between}$). We say that $\mathscr{C}$ is *graphical* if the canonical forgetful functor $F \colon \mathscr{C} \to \mathscr{C}^{\between}$ is faithful.

Note that $\mathscr{C}^{\between}$ is a star-autonomous category in which the two monoidal structures coincide, i.e., it is a compact closed category. The actual problem is usually to find a canonical way of making this collapse. But here, we can explore the fact that $\mathscr{C}$ is single-mixed and that the structure of a $\mathsf{B1}$-category does not induce any other natural maps $A \wedge B \to A \vee B$ or $A \vee B \to A \wedge B$. Although in general inverting arrows in a category can destroy the structure, it is harmless here since it only makes mix an iso, and hence $\mathscr{C}^{\between}$ compact closed. We do not go into further details of inverting arrows in categories. The whole point of Definition 6.1.1 is to provide the means of formulating the following open problem.

Open Problem 6.1.2. Let $\mathscr{E}$ be a set of equations and let $\mathscr{C}$ be the free $\mathsf{B1}$-category that is generated from a set $\mathscr{A}$ of generators (e.g., propositional variables) and that obeys all of $\mathscr{E}$. Is $\mathscr{C}$ graphical? This question is equivalent to asking for a general coherence result for Boolean categories. Chapter 2 of this work exhibits many equations that have to hold, but it gives no clue whether they are enough, or what could be missing.

For example the freely generated star-autonomous category without units [LS05a, HHS05, DP05] is graphical. This can be shown by using traditional proof nets for multiplicative

linear logic. However, the work of [LS06] can be used to show that the freely generated star-autonomous category with units is not graphical.

Clearly, in a graphical $\mathsf{B4}$-category the equations ($\mathsf{mix\text{-}m\text{-}\hat{t}}$), ($\mathsf{m\text{-}\check{t}\text{-}s}$), and ($\check{\mathsf{m}}^2\text{-}\mathsf{s}\text{-}\check{\mathsf{m}}^2$) all hold. However, at the current state of the art it is not known whether they hold in every $\mathsf{B4}$-category. I conjecture that this is not the case, but so far no counterexample could be constructed.

A related problem occurs in Section 3.6:

Open Problem 6.1.3. Is there a sound and complete deep inference proof system S for classical propositional logic such that its C-reduced prenets (obtained as shown in Sections 3.5 and 3.6) form a star-autonomous category, more precisely, a star-autonomous subcategory of **PreC**.

6.2 Correctness Criteria for Proof Nets for Classical Logic

The most important open problem left open by this work is the question of correctness criteria for $\mathbb{N}$-nets, C-nets, and atomic flows, as discussed in Chapter 3.

Open Problem 6.2.1. Given two formulas A and B and an atomic flow $\phi\colon \mathit{fl}(A) \to \mathit{fl}(B)$, is there an algorithm that decides in polynomial time whether there is an SKS-derivation from A to B whose atomic flow is ϕ?

Open Problem 6.2.2. Given a C-reduced prenet π with two conclusion $\bar{A}$ and B, is there an algorithm that decides in polynomial time whether there is an SKS-derivation from A to B which is translated to π (using the method in Sections 3.5 and 3.6)?

Open Problem 6.2.3. Given a cut-free $\mathbb{N}$-prenet $P \rhd A$, is there a polynomial algorithm deciding if there is a proof of A in KS or some sequent system translating into the given net.

All three open problems are closely related. The simplest one might be the last one, because it can be reduced to the problem mentioned at the end of Section 4.4:

Open Problem 6.2.4. Given two formulas A and B with the same atom occurrences, is there an algorithm that decides in polynomial time whether $A \xrightarrow[\mathsf{MS}]{*} B$.

6.3 The Relative Efficiency of Propositional Proof Systems

Chapter 5 of this thesis provides more new open problems than it provides answers. Figure 6.1 shows a refined version of Figure 1.2 on page 6 (see also [BG09]). A solid arrow $A \longrightarrow B$ means that A p-simulates B, the notation $A \not\longrightarrow B$ means that A does not p-simulate B, and a dotted arrow $A \dashrightarrow B$ means that it is not known whether A p-simulates B or not. The open problems indicated by these dotted arrows are surprisingly difficult:

(1) The question whether SKS p-simulates eSKS is equivalent to the question whether Frege systems p-simulate extended Frege systems. This question has already been asked in [CR79], and is one of the most important open problems in the area of proof complexity.

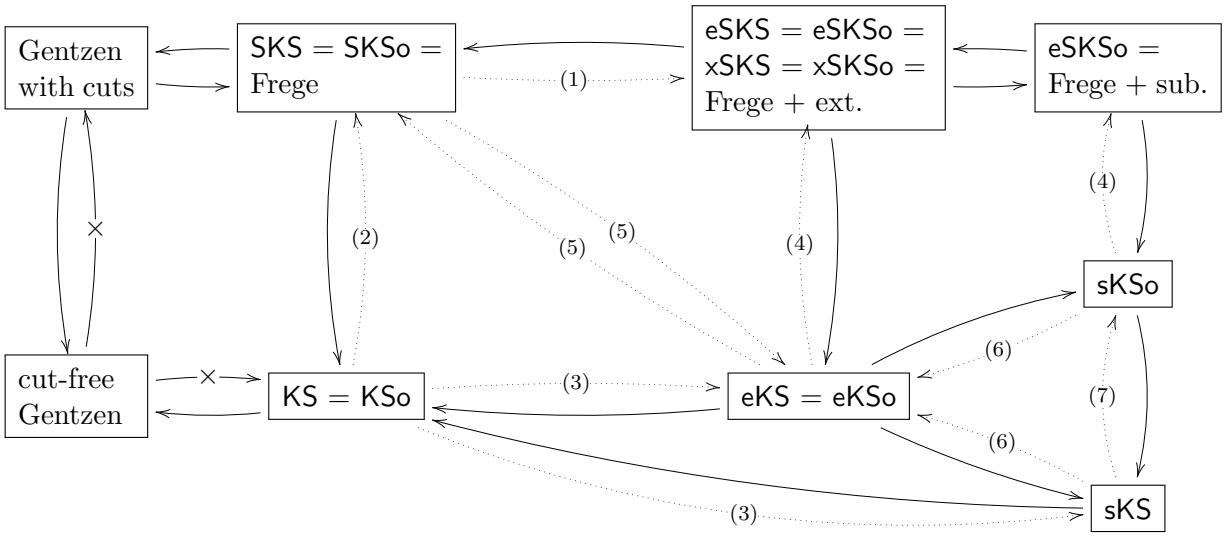


Figure 6.1: Classification of propositional proof systems

(2) We conjecture that KS does not p-simulate SKS (see also [BG09] and [BGGP10]),

(3) We also conjecture that KS does not p-simulate eKS. More precisely, it is conjectured that KS cannot provide polynomial size proofs of the formulas PHP_n (or QHQ_n), whereas this is possible in SKS (as shown in [Bus87]) as well as in eKS (as shown in Section 5.4). However, so far, no technique has been developed for showing that something *cannot be done* in KS.

(4) This is the question whether extension or substitution can simulate the behavior of the cut. It is one of the contributions of this paper that this question can now be asked. I conjecture that the answer is positive, but it is not clear how to prove it. Note that the naive cut elimination procedures fail in the presence of extension. Even if we manage to modify the technicalities such that we get a cut elimination procedure for eSKS, it is not clear how to avoid the exponential blow-up usually caused by cut elimination.

(5) The questions whether extension without cut is as powerful as the cut without extension, and vice-versa, can be seen as the little brothers of (1).

(6) It has already been shown in [CR79] that under the presence of cut substitution p-simulates extension, but without cut, this question is not trivial.

(7) We do not even know whether under the absence of cut the substitution rule alone is as powerful as the substitution rule together with the open↓-rule.

Remark 6.3.1. It has recently been shown [Jeř09, BGGP10] that the system KS + ac↑ quasipolynomially simulates SKS, and it is conjectured that this result can be improved to a polynomial simulation.

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