Chapter 25

The matching polytope

As a by-product of his weighted matching algorithm (to be discussed in Chapter 26), Edmonds obtained a characterization of the matching polytope in terms of defining inequalities. It forms the first class of polytopes whose characterization does not simply follow just from total unimodularity, and its description was a breakthrough in polyhedral combinatorics.

25.1. The perfect matching polytope

The *perfect matching polytope* of a graph $G = (V, E)$ is the convex hull of the incidence vectors of the perfect matchings in G. It is denoted by $P_{\text{perfect matching}}(G)$:

(25.1) $$P_{\text{perfect matching}}(G) = \text{conv.hull}\{\chi^M \mid M \text{ perfect matching in } G\}.$$

So $P_{\text{perfect matching}}(G)$ is a polytope in $\mathbb{R}^E$.

Consider the following set of linear inequalities for $x \in \mathbb{R}^E$:

(25.2)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) = 1 & \text{for each } v \in V, \\ \text{(iii)} & x(\delta(U)) \geq 1 & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

In Section 18.1 we saw that if G is bipartite, the perfect matching polytope is fully determined by the inequalities (25.2)(i) and (ii). These inequalities are not enough for, say, K_3: taking $x_e := \frac{1}{2}$ for each edge e of K_3 gives a vector x satisfying (25.2)(i) and (ii) but not belonging to the perfect matching polytope of K_3 (as it is empty).

Edmonds [1965b] showed that for general graphs, adding (25.2)(iii) is enough. It is clear that for any perfect matching M in G, the incidence vector χ^M satisfies (25.2). So $P_{\text{perfect matching}}(G)$ is contained in the polytope determined by (25.2). The essence of Edmonds' theorem is that one needs no more inequalities.

Theorem 25.1 (Edmonds' perfect matching polytope theorem)**.** *The perfect matching polytope of any graph $G = (V, E)$ is determined by* (25.2).

Proof. Clearly, the perfect matching polytope is contained in the polytope Q determined by (25.2). Suppose that the converse inclusion does not hold. So we can choose a vertex x of Q that is not in the perfect matching polytope.

We may assume that we have chosen this counterexample such that $|V| + |E|$ is as small as possible. Hence $0 < x_e < 1$ for all $e \in E$ (otherwise, if $x_e = 0$, we can delete e, and if $x_e = 1$, we can delete e and its ends). So each degree of G is at least 2, and hence $|E| \geq |V|$. If $|E| = |V|$, each degree is 2, in which case the theorem is trivially true. So $|E| > |V|$. Note also that $|V|$ is even, since otherwise $Q = \emptyset$ (consider $U := V$ in (25.2)(iii)).

As x is a vertex, there are $|E|$ linearly independent constraints among (25.2) satisfied with equality. Since $|E| > |V|$, there is an odd subset U of V with $3 \leq |U| \leq |V| - 3$ and $x(\delta(U)) = 1$.

Consider the projections x' and x'' of x to the edge sets of the graphs $G/\overline{U}$ and G/U, respectively (where $\overline{U} := V \setminus U$). Here we keep parallel edges.

Then x' and x'' satisfy (25.2) for $G/\overline{U}$ and G/U, respectively, and hence belong to the perfect matching polytopes of $G/\overline{U}$ and G/U, by the minimality of $|V| + |E|$.

So $G/\overline{U}$ has perfect matchings $M'_1, \ldots, M'_k$ and G/U has perfect matchings $M''_1, \ldots, M''_k$ with

$$(25.3) \qquad x' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M'_i} \text{ and } x'' = \frac{1}{k}\sum_{i=1}^{k} \chi^{M''_i}.$$

(Note that x is rational as it is a vertex of Q.)

Now for each $e \in \delta(U)$, the number of i with $e \in M'_i$ is equal to $kx'(e) = kx(e) = kx''(e)$, which is equal to the number of i with $e \in M''_i$. Hence we can assume that, for each $i = 1, \ldots, k$, M'_i and M''_i have an edge in $\delta(U)$ in common. So $M_i := M'_i \cup M''_i$ is a perfect matching of G. Then

$$(25.4) \qquad x = \frac{1}{k}\sum_{i=1}^{k} \chi^{M_i}.$$

Hence x belongs to the perfect matching polytope of G. ∎

Notes. This proof was given by Aráoz, Cunningham, Edmonds, and Green-Krótki [1983] and Schrijver [1983c], with ideas of Seymour [1979a]. For other proofs, see Balinski [1972], Hoffman and Oppenheim [1978], and Lovász [1979b]. A proof can also be derived from Edmonds' weighted matching algorithm (Chapter 26).

25.2. The matching polytope

The characterization of the perfect matching polytope implies Edmonds' matching polytope theorem. It characterizes the *matching polytope* of a graph $G = (V, E)$, denoted by $P_{\text{matching}}(G)$, which is the convex hull of the incidence vectors of the matchings in G:

(25.5) $$P_{\text{matching}}(G) = \text{conv.hull}\{\chi^M \mid M \text{ matching in } G\}.$$

Again, $P_{\text{matching}}(G)$ is a polytope in $\mathbb{R}^E$.

Corollary 25.1a (Edmonds' matching polytope theorem)**.** *For any graph $G = (V, E)$, the matching polytope is determined by:*

(25.6)
$$\begin{array}{rll} \text{(i)} & x_e \geq 0 & \text{for each } e \in E, \\ \text{(ii)} & x(\delta(v)) \leq 1 & \text{for each } v \in V, \\ \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

Proof. Clearly, each vector x in the matching polytope satisfies (25.6). To see that the inequalities (25.6) are enough, let x satisfy (25.6). Make a copy $G' = (V', E')$ of G, and add edges vv' for each vertex $v \in V$, where v' is the copy of v in V'. This makes the graph $\widetilde{G} = (\widetilde{V}, \widetilde{E})$.

Define $\tilde{x}_e := \tilde{x}_{e'} := x_e$ for each $e \in E$, where e' is the copy of e in E', and $\tilde{x}(vv') := 1 - x(\delta(v))$ for each $v \in V$. Then by Theorem 25.1, $\tilde{x}$ belongs to the perfect matching polytope of $\widetilde{G}$, since $\tilde{x}$ satisfies (25.2) with respect to $\widetilde{G}$.

Indeed, for each $v \in V$ one has $\tilde{x}(\tilde{\delta}(v)) = \tilde{x}(\tilde{\delta}(v')) = 1$ (where $\tilde{\delta} := \delta_{\widetilde{G}}$). Moreover, consider any odd subset U of $\widetilde{V} = V \cup V'$, say $U = W \cup X'$ with $W, X \subseteq V$. Then $\tilde{x}(\tilde{\delta}(U)) \geq \tilde{x}(\tilde{\delta}(W \setminus X)) + \tilde{x}(\tilde{\delta}(X' \setminus W'))$. So we may assume that $W \cap X = \emptyset$, and by symmetry we may assume that W is odd, and hence that $X = \emptyset$. So it suffices to show that for any odd $U \subseteq V$ one has $\tilde{x}(\tilde{\delta}(U)) \geq 1$. Now

(25.7) $$\tilde{x}(\tilde{\delta}(U)) + 2\tilde{x}(\widetilde{E}[U]) = \sum_{v \in U} \tilde{x}(\tilde{\delta}(v)) = |U|,$$

and hence

(25.8) $$\tilde{x}(\tilde{\delta}(U)) = |U| - 2\tilde{x}(\widetilde{E}[U]) \geq |U| - 2\lfloor \tfrac{1}{2}|U| \rfloor = 1.$$

So by Theorem 25.1, $\tilde{x}$ belongs to the perfect matching polytope of $\widetilde{G}$, and hence x belongs to the matching polytope of G. ∎

25.3. Total dual integrality: the Cunningham-Marsh formula

With linear programming duality one can derive from Corollary 25.1a a min-max relation for the maximum weight of a matching:

Corollary 25.1b. *Let $G = (V, E)$ be a graph and let $w \in \mathbb{R}^E_+$ be a weight function. Then the maximum weight of a matching is equal to the minimum value of*

(25.9) $$\sum_{v\in V} y_v + \sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor,$$

where $y \in \mathbb{R}_+^V$ *and* $z \in \mathbb{R}_+^{\mathcal{P}_{\text{odd}}(V)}$ *satisfy*

(25.10) $$\sum_{v\in V} y_v \chi^{\delta(v)} + \sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U]} \geq w.$$

Proof. Directly with LP-duality from Corollary 25.1a. ∎

The constraints (25.6) determining the matching polytope in fact are totally dual integral, as was shown by Cunningham and Marsh [1978]. This implies that a stronger min-max relation holds than obtained by linear programming duality from the matching polytope inequalities: if w is integer-valued, then in Corollary 25.1b we can restrict y and z to integer vectors:

Theorem 25.2 (Cunningham-Marsh formula)**.** *In Corollary* 25.1b*, if w is integer, we can take y and z integer. We can take z moreover such that the collection* $\{U \in \mathcal{P}_{\text{odd}}(V) \mid z_U > 0\}$ *is laminar.*[9]

Proof. We prove the theorem by induction on $|E| + w(E)$. If $w(e) = 0$ for some $e \in E$, we can delete e and apply induction. So we may assume that $w(e) \geq 1$ for each $e \in E$.

First assume that there exists a vertex u of G covered by every maximum-weight matching. Let $w' := w - \chi^{\delta(u)}$. By induction, there exist integer y'_v, z'_U that are optimum with respect to w'. Now increasing y'_u by 1, gives y_v, z_U as required for w, since the maximum of $w'(M)$ over all matchings M is strictly less than the maximum of $w(M)$ over all matchings M, as each maximum-weight matching M contains an edge e incident with u.

So we may assume that for each vertex v there exists a maximum-weight matching missing v. Hence if $y \in \mathbb{R}_+^V$ and $z \in \mathbb{R}_+^{\mathcal{P}_{\text{odd}}(V)}$ satisfying (25.10) attain the minimum of (25.9), then $y = \mathbf{0}$. (If $y_u > 0$, then each maximum-weight matching covers u, by complementary slackness.)

Now choose z attaining the minimum (with $y = \mathbf{0}$) such that

(25.11) $$\sum_{U\in\mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor^2$$

is as large as possible. Let $\mathcal{F} := \{U \in \mathcal{P}_{\text{odd}}(V) \mid z_U > 0\}$. Then $\mathcal{F}$ is laminar. For suppose not. Let $U, W \in \mathcal{F}$ with $U \cap W \neq \emptyset$ and $U \not\subseteq W \not\subseteq U$. Then $|U \cap W|$ is odd. To see this, choose $v \in U \cap W$. Then there is a maximum-weight matching M missing v. Since $z_U > 0$, $E[U]$ contains $\lfloor \frac{1}{2}|U| \rfloor$ edges in M, and hence each vertex in $U \setminus \{v\}$ is covered by an edge in M contained in U. Similarly, each vertex in $W \setminus \{v\}$ is covered by an edge in M contained in

[9] A collection $\mathcal{F}$ of sets is called *laminar* if $U \cap W = \emptyset$ or $U \subseteq W$ or $W \subseteq U$ for all $U, W \in \mathcal{F}$.

W. Hence each vertex in $(U \cap W) \setminus \{v\}$ is covered by an edge in M contained in $U \cap W$. So $|(U \cap W) \setminus \{v\}|$ is even, and hence $|U \cap W|$ is odd.

Now let $\alpha := \min\{z_U, z_W\}$, and decrease z_U and z_W by α and increase $z_{U \cap W}$ and $z_{U \cup W}$ by α. This resetting maintains (25.10), does not change (25.9), but increases (25.11), contradicting our assumption.

This shows that $\mathcal{F}$ is laminar. Now suppose that z is not integer-valued, and let U be an inclusionwise maximal set in $\mathcal{F}$ with $z_U \notin \mathbb{Z}$. Let $U_1, \ldots, U_k$ be the inclusionwise maximal sets in $\mathcal{F}$ properly contained in U (possibly $k = 0$). As $\mathcal{F}$ is laminar, the U_i are disjoint. Let $\alpha := z_U - \lfloor z_U \rfloor$. Then decreasing z_U by α and increasing each z_{U_i} by α would maintain (25.10) (by the integrality of w), but would strictly decrease (25.9) (since $\sum_{i=1}^{k} \lfloor \frac{1}{2}|U_i| \rfloor < \lfloor \frac{1}{2}|U| \rfloor$). This contradicts the minimality of (25.9). ∎

(This proof follows the method given by Schrijver and Seymour [1977]. Other proofs were given by Hoffman and Oppenheim [1978], Schrijver [1983a,1983c], and Cook [1985].)

Note that the Cunningham-Marsh formula has the Tutte-Berge formula (Corollary 24.1) as special case. The previous theorem is equivalent to:

Corollary 25.2a. *System* (25.6) *is totally dual integral.*

Proof. This follows from Theorem 25.2. ∎

25.3a. Direct proof of the Cunningham-Marsh formula

We give a direct proof of the Cunningham-Marsh formula, as given in Schrijver [1983a] (generalizing the proof of Lovász [1979b] of Edmonds' matching polytope theorem). It does not use Edmonds' matching polytope theorem, which rather follows as a consequence.

Let $G = (V, E)$ be a graph. For each weight function $w \in \mathbb{Z}_+^E$, let ν_w denote the maximum weight of a matching. We must show that for each $w \in \mathbb{Z}_+^E$ there exist $y \in \mathbb{Z}_+^V$ and $z \in \mathbb{Z}_+^{\mathcal{P}_{\text{odd}}(V)}$ such that

$$\text{(25.12)} \qquad \sum_{v \in V} y_v + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \lfloor \tfrac{1}{2}|U| \rfloor \leq \nu_w$$

and

$$\text{(25.13)} \qquad \sum_{v \in V} y_v \chi^{\delta(v)} + \sum_{U \in \mathcal{P}_{\text{odd}}(V)} z_U \chi^{E[U]} \geq w.$$

Suppose that G and w contradict this, with $|V|+|E|+w(E)$ as small as possible. Then G is connected (otherwise one of the components of G will form a smaller counterexample) and $w(e) \geq 1$ for each edge e (otherwise we can delete e). Now there are two cases.

Case 1: There is a vertex u covered by every maximum-weight matching. In this case, let $w' := w - \chi^{\delta(u)}$. Then $\nu_{w'} = \nu_w - 1$. Since $w'(E) < w(E)$, there are y' and z' satisfying (25.12) and (25.13) with respect to w'. Increasing y'_u by 1 gives y and z satisfying (25.12) and (25.13) with respect to w.

Case 2: No vertex is covered by every maximum-weight matching. Now let w' arise from w by decreasing all weights by 1. Let M be a matching with $w'(M) = \nu_{w'}$ and with $|M|$ as large as possible.

Then M does not cover all vertices, as, otherwise, for any matching N of maximum w-weight not covering all vertices:

$$w'(N) = w(N) - |N| > w(N) - |M| \geq w(M) - |M| = w'(M) = \nu_{w'}, \tag{25.14}$$

contradicting the definition of $\nu_{w'}$.

Suppose that M covers all but one vertex (in particular, $|V|$ is odd). Then

$$\nu_w \geq w(M) = w'(M) + |M| = \nu_{w'} + \lfloor \tfrac{1}{2}|V| \rfloor. \tag{25.15}$$

Since $w'(E) < w(E)$, there are y' and z' satisfying (25.12) and (25.13) with respect to w'. Increasing z'_V by 1 gives y and z satisfying (25.12) and (25.13) with respect to w (by (25.15)), a contradiction.

So we know that M leaves at least two vertices in V uncovered. Let u and v be not covered by M. We can assume that we have chosen M, u, v under the additional condition that the distance $d(u, v)$ of u and v in G is as small as possible. Then $d(u, v) > 1$, since otherwise we could augment M by edge $\{u, v\}$, thereby increasing $|M|$ while not decreasing $w'(M)$. Let t be an internal vertex of a shortest $u - v$ path. Let N be a matching not covering t, with $w(N) = \nu_w$.

Let P be the component of $M \cup N$ containing t. Then P forms a path starting at t and not covering both u and v (as t is not covered by N and u and v are not covered by M). We can assume that P does not cover u. Now the symmetric differences $M' := M \triangle P$ and $N' := N \triangle P$ are matchings again, and $|M'| \leq |M|$ (as M covers t), implying

$$\begin{aligned} w'(M') - w'(M) &= w(M') - |M'| - w(M) + |M| \geq w(M') - w(M) \\ &= w(N) - w(N') = \nu_w - w(N') \geq 0. \end{aligned} \tag{25.16}$$

So $w'(M') \geq w'(M) = \nu_{w'}$ and hence we have equality throughout. So $w(M') = w(M)$, $w'(M') = w'(M)$, and $|M'| = |M|$. However, M' does not cover t and u while $d(u, t) < d(u, v)$, contradicting our choice of M, u, v.

25.4. On the total dual integrality of the perfect matching constraints

System (25.2) determining the perfect matching polytope is generally not totally dual integral. Indeed, consider the complete graph $G = K_4$ on four vertices, with $w(e) := 1$ for each edge e; then the maximum weight of a perfect matching is 2, while the dual of optimizing $w^{\mathsf{T}} x$ subject to (25.2) is attained only by taking $y(\{v\}) = \frac{1}{2}$ for each vertex v.

However, consider the following system, again determining the perfect matching polytope (by Corollary 25.1a):

$$\begin{array}{lrll} (25.17) & \text{(i)} & x_e \geq 0 & \text{for each } e \in E; \\ & \text{(ii)} & x(\delta(v)) = 1 & \text{for each } v \in V; \\ & \text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for each } U \subseteq V \text{ with } |U| \text{ odd.} \end{array}$$

Corollary 25.2b. *System* (25.17) *is totally dual integral.*

Proof. Directly from Corollary 25.2a, with Theorem 5.25. ∎

This implies a result stated by Edmonds and Johnson [1970]:

Corollary 25.2c. *The perfect matching inequalities* (25.2) *form a totally dual half-integral system.*

Proof. Let $w \in \mathbb{Z}^E$, and minimize $w^\mathsf{T}x$ subject to (25.2). As it is the same as minimizing $w^\mathsf{T}x$ subject to (25.17), by Corollary 25.2b there is an optimum dual solution $y \in \mathbb{Z}^V$, $z \in \mathbb{Z}_+^{\mathcal{P}_{\mathrm{odd}}(V)}$. Since $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is half of the sum of the inequalities $x(\delta(v)) = 1$ ($v \in U$) and $-x(\delta(U)) \leq -1$, we obtain the total dual half-integrality of (25.2). ∎

This can be strengthened to (Barahona and Cunningham [1989]):

Corollary 25.2d. *If $w \in \mathbb{Z}^E$ and $w(C)$ is even for each circuit C, then the problem of minimizing $w^\mathsf{T}x$ subject to* (25.2) *has an integer optimum dual solution.*

Proof. If $w(C)$ is even for each circuit, there is a subset T of V with $\{e \in E \mid w(e) \text{ is odd}\} = \delta(T)$. Now replace w by $\tilde{w} := w + \sum_{v \in T} \chi^{\delta(v)}$. Then $\tilde{w}(e)$ is an even integer for each edge e. Hence by Corollary 25.2c there is an optimum dual solution $\tilde{y} \in \mathbb{Z}^V$, $z \in \mathbb{Z}_+^{\mathcal{P}_{\mathrm{odd}}(V)}$ for the problem of minimizing $\tilde{w}^\mathsf{T}x$ subject to (25.2). Now setting $y_v := \tilde{y}_v - 1$ if $v \in T$ and $y_v := \tilde{y}_v$ if $v \notin T$ gives an integer optimum dual solution for w. ∎

25.5. Further results and notes

25.5a. Adjacency and diameter of the matching polytope

Balinski and Russakoff [1974] and Chvátal [1975a] characterized adjacency on the matching polytope:

Theorem 25.3. *Let M and N be distinct matchings in a graph $G = (V, E)$. Then χ^M and χ^N are adjacent vertices of the matching polytope if and only if $M \triangle N$ is a path or circuit.*

Proof. To see necessity, let χ^M and χ^N be adjacent. Let P be any nontrivial component of $M \triangle N$ and let $M' := M \triangle P$ and $N' := N \triangle P$. So M' and N' are matchings again. Then

$$\frac{1}{2}(\chi^M + \chi^N) = \frac{1}{2}(\chi^{M'} + \chi^{N'}). \tag{25.18}$$

As χ^M and χ^N are adjacent, it follows that $\{M', N'\} = \{M, N\}$. So $M' = N$ and $N' = M$, and therefore $M \triangle N = P$.

To see sufficiency, let $P := M \triangle N$ be a path or circuit. Suppose that χ^M and χ^N are not adjacent. Then there exists a matching $L \neq M, N$ that belongs to the smallest face of the matching polytope containing $x := \frac{1}{2}(\chi^M + \chi^N)$. As $x_e = 0$ for each edge $e \notin M \cup N$ and $x_e = 1$ for each edge $e \in M \cap N$, we know that $M \cap N \subseteq L \subseteq M \cup N$. Moreover, $x(\delta(v)) = 1$ for each vertex v covered both by M and by N. Hence each vertex v covered both by M and by N is covered by L. As P is a path or a circuit, it follows that $L = M$ or $L = N$, a contradiction. ∎

This has as consequence for the diameter:

Corollary 25.3a. *The diameter of the matching polytope of any graph $G = (V, E)$ is equal to the maximum size $\nu(G)$ of the matchings.*

Proof. First, by Theorem 25.3, for any two matchings M and N, the distance of χ^M and χ^N is at most the number of nontrivial components of $M \triangle N$. Since each such component contains at least one edge and since these edges are pairwise disjoint, this number is at most $\nu(G)$. So the diameter is at most $\nu(G)$.

Equality follows from the fact that $\emptyset$ and any matching M have distance $|M|$. This follows from the fact that if M and N are adjacent, then $\big||M| - |N|\big| \leq 1$ by Theorem 25.3. ∎

Another direct consequence concerns adjacency on the *perfect* matching polytope:

Corollary 25.3b. *Let M and N be perfect matchings in a graph $G = (V, E)$. Then χ^M and χ^N are adjacent vertices of the perfect matching polytope if and only if $M \triangle N$ is a circuit.*

Proof. Directly from Theorem 25.3. ∎

This in turn implies for the diameter of the perfect matching polytope:

Corollary 25.3c. *The perfect matching polytope of a graph $G = (V, E)$ has diameter at most $\frac{1}{2}|V|$ ($\frac{1}{4}|V|$ if G is simple).*

Proof. For any two perfect matching M, N, the symmetric difference has at most $\frac{1}{2}|V|$ components (each being a circuit). Hence Corollary 25.3b implies that χ^M and χ^N have distance at most $\frac{1}{2}|V|$.

If G is simple the bounds can be sharpened to $\frac{1}{4}|V|$, as each even circuit has at least four vertices. ∎

Padberg and Rao [1974] showed that if G is a complete graph with an even number $2n$ of vertices, then $P_{\text{perfect matching}}(G)$ has diameter at most 2. (This can be derived from Theorem 18.5, since any two perfect matchings belong to some $K_{n,n}$-subgraph of G, which subgraph gives a face of $P_{\text{perfect matching}}(G)$.)

25.5b. Facets of the matching polytope

Pulleyblank and Edmonds [1974] (cf. Pulleyblank [1973]) characterized which of the inequalities (25.6) give a facet of the matching polytope:

Let $G = (V, E)$ be a graph. Define

(25.19) $I := \{v \in V \mid \deg_G(v) \geq 3$, or $\deg_G(v) = 2$ and v is contained in no triangle, or $\deg_G(v) = 1$ and the neighbour of v also has degree $1\}$,
$\mathcal{T} := \{U \subseteq V \bigm| |U| \geq 3$, $G[U]$ is factor-critical and 2-vertex-connected$\}$.

(Recall that graph G is *factor-critical* if, for each vertex v of G, $G - v$ has a perfect matching.)

Consider the system

$$
\text{(25.20)}\qquad
\begin{array}{rll}
\text{(i)} & x_e \geq 0 & \text{for } e \in E,\\
\text{(ii)} & x(\delta(v)) \leq 1 & \text{for } v \in I,\\
\text{(iii)} & x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor & \text{for } U \in \mathcal{T}.
\end{array}
$$

We first show:

Theorem 25.4. *Each inequality in* (25.6) *is a nonnegative integer combination of inequalities* (25.20).

Proof. First consider a vertex $v \notin I$. If $\deg_G(v) = 1$, let u be the neighbour of v. Then $u \in I$ and

$$
\text{(25.21)}\qquad x(\delta(v)) = x(\delta(u)) - \sum_{e \in \delta(u) - \delta(v)} x_e.
$$

If $\deg_G(v) = 2$ and v is contained in a triangle $G[U]$, then $x(\delta(v)) = x(E[U]) - x_e$, where e is the edge in $E[U]$ not incident with v.

Next consider a subset U of V with $|U|$ odd and $|U| \geq 3$. We show that $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of constraints (25.20), by induction on $|U|$. If $U \in \mathcal{T}$ we are done. So assume that $U \notin \mathcal{T}$. Let $H := G[U]$. If H is not factor-critical, there is a vertex v such that $H - v$ has no perfect matching. Let $U' = U \setminus \{v\}$. Then $x(E[U']) \leq \lfloor \frac{1}{2}|U| \rfloor - 1$ for the incidence vector x of any matching, and hence also for each vector x in the matching polytope. By the total dual integrality of the matching constraints (Corollary 25.2a), this constraint is a sum of constraints (25.6), and hence, by induction, of constraints (25.20). So $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of constraints (25.20), as $E[U] \subseteq E[U'] \cup \delta(v)$.

If H is factor-critical, it has a cut vertex v. Let $K_1, \ldots, K_t$ be the components of $H - v$ and let $U_i := K_i \cup \{v\}$ for each i. As H is factor-critical, each $|U_i|$ is odd. Hence $x(E[U]) \leq \lfloor \frac{1}{2}|U| \rfloor$ is a sum of the constraints $x(E[U_i]) \leq \lfloor \frac{1}{2}|U_i| \rfloor$. ∎

This implies that (25.20) is sufficient:

Corollary 25.4a. (25.20) *determines the matching polytope.*

Proof. Directly from Corollary 25.1a and Theorem 25.4. ∎

Another consequence is the result of Cunningham and Marsh [1978] that the irredundant system still is totally dual integral: