Chapter 58

The traveling salesman problem

The traveling salesman problem (TSP) asks for a shortest Hamiltonian circuit in a graph. It belongs to the most seductive problems in combinatorial optimization, thanks to a blend of complexity, applicability, and appeal to imagination.
The problem shows up in practice not only in routing but also in various other applications like machine scheduling (ordering jobs), clustering, computer wiring, and curve reconstruction.
The traveling salesman problem is an NP-complete problem, and no polynomial-time algorithm is known. As such, the problem would not fit in the scope of the present book. However, the TSP is closely related to several of the problem areas discussed before, like 2-matching, spanning tree, and cutting planes, which areas actually were stimulated by questions prompted by the TSP, and often provide subroutines in solving the TSP.
Being NP-complete, the TSP has served as prototype for the development and improvement of advanced computational methods, to a large extent utilizing polyhedral techniques. The basis of the solution techniques for the TSP is branch-and-bound, for which good bounding techniques are essential. Here 'good' is determined by two, often conflicting, criteria: the bound should be *tight* and *fast* to compute. Polyhedral bounds turn out to be good candidates for such bounds.

58.1. The traveling salesman problem

Given a graph $G = (V, E)$, a *Hamiltonian circuit* in G is a circuit C with $VC = V$. The *symmetric traveling salesman problem* (TSP) is: given a graph $G = (V, E)$ and a length function $l : E \to \mathbb{R}$, find a Hamiltonian circuit C of minimum length.

The directed version is as follows. Given a digraph $D = (V, A)$, a *directed Hamiltonian circuit*, or just a *Hamiltonian circuit*, in D is a directed circuit C with $VC = V$. The *asymmetric traveling salesman problem* (TSP or ATSP) is: given a digraph $D = (V, A)$ and a length function $l : A \to \mathbb{R}$, find a Hamiltonian circuit C of minimum length.

In the context of the traveling salesman problem, vertices are sometimes called *cities*, and a Hamiltonian circuit a *traveling salesman tour*. If the vertices are represented by points in the plane and each pair of vertices is connected by an edge of length equal to the Euclidean distance between the two points, one speaks of the *Euclidean traveling salesman problem*.

58.2. NP-completeness of the TSP

The problem of finding a Hamiltonian circuit and (hence) the traveling salesman problem are NP-complete. Indeed, in Theorem 8.11 and Corollary 8.11b we proved the NP-completeness of the directed and undirected Hamiltonian circuit problem. This implies the NP-completeness of the TSP, both in the undirected and the directed case:

Theorem 58.1. *The symmetric TSP and the asymmetric TSP are* NP-*complete.*

Proof. Given an undirected graph $G = (V, E)$, define $l(e) := 0$ for each edge e. Then G has a Hamiltonian circuit if and only if G has a Hamiltonian circuit of length ≤ 0. This reduces the undirected Hamiltonian circuit problem to the symmetric TSP.

One similarly shows the NP-completeness of the asymmetric TSP. ∎

This method also gives that the symmetric TSP remains NP-complete if the graph is complete and the length function satisfies the *triangle inequality*:

(58.1) $$l(uw) \leq l(uv) + l(vw) \text{ for all } u, v, w \in V.$$

Indeed, to test if a graph $G = (V, E)$ has a Hamiltonian circuit, define $l(uv) := 1$ if u and v are adjacent and $l(uv) := 2$ otherwise (for $u \neq v$). Then G has a Hamiltonian circuit if and only if there exists a traveling salesman tour of length $\leq |V|$.

Garey, Graham, and Johnson [1976] and Papadimitriou [1977a] showed that even the Euclidean traveling salesman problem is NP-complete. (Similarly for several other metrics, like l_1.) More on complexity can be found in Section 58.8b below.

58.3. Branch-and-bound techniques

The traveling salesman problem is NP-complete, and no polynomial-time algorithm is known. Most exact methods known are essentially enumerative, aiming at minimizing the enumeration. A general framework is that of *branch-and-bound*. The idea of branch-and-bound applied to the traveling salesman problem roots in papers of Tompkins [1956], Rossman and Twery [1958], and Eastman [1959]. The term 'branch and bound' was introduced by Little, Murty, Sweeney, and Karel [1963].

A rough, elementary description is as follows. Let $G = (V, E)$ be a graph and let $l : E \to \mathbb{R}$ be a length function. For any set $\mathcal{C}$ of Hamiltonian circuits, let $\mu(\mathcal{C})$ denote the minimum length of the Hamiltonian circuits in $\mathcal{C}$.

Keep a collection Γ of sets of Hamiltonian circuits and a function $\lambda : \Gamma \to \mathbb{R}$ satisfying:

(58.2) (i) $\bigcup \Gamma$ contains a shortest Hamiltonian circuit;
(ii) $\lambda(\mathcal{C}) \leq \mu(\mathcal{C})$ for each $\mathcal{C} \in \Gamma$.

A typical iteration is:

(58.3) Select a collection $\mathcal{C} \in \Gamma$ with $\lambda(\mathcal{C})$ minimal. Either find a circuit $C \in \mathcal{C}$ with $l(C) = \lambda(\mathcal{C})$ or replace $\mathcal{C}$ by (zero or more) smaller sets such that (58.2) is maintained.

Obviously, if we find $C \in \mathcal{C}$ with $l(C) = \lambda(\mathcal{C})$, then C is a shortest Hamiltonian circuit.

This method always terminates, but the method and its efficiency heavily depend on how the details in this framework are filled in: how to bound (that is, how to define and calculate $\lambda(\mathcal{C})$), how to branch (that is, which smaller sets replace $\mathcal{C}$), and how to find the circuit C.

As for branching, the classes $\mathcal{C}$ in Γ can be stored implicitly: for example, by prescribing sets B and F of edges such that $\mathcal{C}$ consists of all Hamiltonian circuits whose edge set contains B and is disjoint from F. Then we can split $\mathcal{C}$ by selecting an edge $e \in E \setminus (B \cup F)$ and replacing $\mathcal{C}$ by the classes determined by $B \cup \{e\}, F$ and by $B, F \cup \{e\}$ respectively.

As for bounding, one should choose $\lambda(\mathcal{C})$ that is fast to compute and close to $\mu(\mathcal{C})$. For this, polyhedral bounds seem good candidates, and in the coming sections we consider a number of them.

For finding the circuit $C \in \mathcal{C}$, a heuristic or exact method can be used. If it returns a circuit C with $l(C) > \lambda(\mathcal{C})$, we can delete all sets $\mathcal{C}'$ from Γ with $\lambda(\mathcal{C}') \geq l(C)$, thus saving computer space.

58.4. The symmetric traveling salesman polytope

The *(symmetric) traveling salesman polytope* of an undirected graph $G = (V, E)$ is the convex hull of the incidence vectors (in $\mathbb{R}^E$) of the Hamiltonian circuits. The TSP is equivalent to minimizing a function $l^\mathsf{T} x$ over the traveling salesman polytope. Hence this is NP-complete.

The NP-completeness of the TSP also implies that, unless NP=co-NP, no description in terms of inequalities of the traveling salesman polytope may be expected (Corollary 5.16a). In fact, as deciding if a Hamiltonian circuit exists is NP-complete, it is NP-complete to decide if the traveling salesman polytope is nonempty. Hence, if NP$\neq$co-NP, there exist no inequalities satisfied by the traveling salesman polytope such that their validity can be certified in polynomial time and such that they have no common solution.

58.5. The subtour elimination constraints

Polynomial-time computable lower bounds on the minimum length of a Hamiltonian circuit can be obtained by including the traveling salesman polytope in a larger polytope (a *relaxation*) over which $l^{\mathsf{T}}x$ can be minimized in polynomial time.

Dantzig, Fulkerson, and Johnson [1954a,1954b] proposed the following relaxation:

(58.4)
(i) $0 \leq x_e \leq 1$ for each edge e,
(ii) $x(\delta(v)) = 2$ for each vertex v,
(iii) $x(\delta(U)) \geq 2$ for each $U \subseteq V$ with $\emptyset \neq U \neq V$.

The integer solutions of (58.4) are precisely the incidence vectors of the Hamiltonian circuits. If (ii) holds, then (iii) is equivalent to:

(58.5) (iii') $x(E[U]) \leq |U| - 1$ for each $U \subseteq V$ with $\emptyset \neq U \neq V$.

These conditions are called the *subtour elimination constraints*.

It can be shown with the ellipsoid method that the minimum of $l^{\mathsf{T}}x$ over (58.4) can be found in strongly polynomial time (cf. Theorem 5.10). For this it suffices to show that the conditions (58.4) can be tested in polynomial time. This is easy for (i) and (ii). If (i) and (ii) are satisfied, we can test (iii) by taking x as capacity function, and test if there is a cut $\delta(U)$ of capacity less than 2, with $\emptyset \neq U \neq V$.

No combinatorial polynomial-time algorithm is known to minimize $l^{\mathsf{T}}x$ over (58.4). In practice, one can apply the simplex method to minimize $l^{\mathsf{T}}x$ over the constraints (i) and (ii), test if the solution satisfies (iii) by finding a cut $\delta(U)$ minimizing $x(\delta(U))$. If this cut has capacity at least 2, then x minimizes $l^{\mathsf{T}}x$ over (58.4). Otherwise, we can add the constraint $x(\delta(U)) \geq 2$ to the simplex tableau (a *cutting plane*), and iterate. (This method is implicit in Dantzig, Fulkerson, and Johnson [1954b].)

Branch-and-bound methods that incorporate such a cutting plane method to obtain bounds and that extend the cutting plane found to all other nodes of the branching tree to improve their bounds, are called *branch-and-cut*.

System (58.4) generally is not enough to determine the traveling salesman polytope: for the Petersen graph $G = (V, E)$, the vector x with $x_e = \frac{2}{3}$ for each $e \in E$ satisfies (58.4) but is not in the traveling salesman polytope of G (as it is empty).

Wolsey [1980] (also Shmoys and Williamson [1990]) showed that if G is complete and the length function l satisfies the triangle inequality, then the minimum of $l^{\mathsf{T}}x$ over (58.4) is at least $\frac{2}{3}$ times the minimum length of a Hamiltonian circuit. It is conjectured (cf. Carr and Vempala [2000]) that for any length function, a lower bound of $\frac{3}{4}$ holds (which is best possible). Related results are given by Papadimitriou and Vempala [2000] and Boyd and Labonté [2002] (who verified the conjecture for $n \leq 10$).

Maurras [1975] and Grötschel and Padberg [1979b] showed that, if G is the complete graph on V and $2 \leq |U| \leq |V|-2$, then the subtour elimination constraint (58.4)(iii) determines a facet of the traveling salesman polytope.

Chvátal [1989] showed the NP-completeness of recognizing if the bound given by the subtour elimination constraints is equal to the length of a shortest tour. He also showed that there is no nontrivial upper bound on the relative error of this bound.

58.6. 1-trees and Lagrangean relaxation

Held and Karp [1971] gave a method to find the minimum value of $l^{\mathsf{T}}x$ over (58.4), with the help of 1-*trees* and *Lagrangean relaxation*.

Let $G = (V, E)$ be a graph and fix a vertex, say 1, of G. A 1-*tree* is a subset F of E such that $|F \cap \delta(1)| = 2$ and such that $F \setminus \delta(1)$ forms a spanning tree on $V \setminus \{1\}$. So each Hamiltonian circuit is a 1-tree with all degrees equal to 2.

It is easy to find a shortest 1-tree F, as it consists of a shortest spanning tree of the graph $G - 1$, joined with the two shortest edges incident with vertex 1. Corollary 50.7c implies that the convex hull of the incidence vectors of 1-trees is given by:

(58.6)
$$
\begin{array}{rll}
\text{(i)} & 0 \leq x_e \leq 1 & \text{for each } e \in E, \\
\text{(ii)} & x(\delta(1)) = 2, & \\
\text{(iii)} & x(E[U]) \leq |U| - 1 & \text{for each nonempty } U \subseteq V \setminus \{1\}, \\
\text{(iv)} & x(E) = |V|. &
\end{array}
$$

Then (58.4) is equivalent to (58.6) added with (58.4)(ii).

The Lagrangean relaxation approach to find the minimum of $l^{\mathsf{T}}x$ over (58.4) is based on the following result. For any $y \in \mathbb{R}^V$ define

(58.7)
$$l_y(e) := l(e) - y_u - y_v$$

for $e = uv \in E$, and define

(58.8)
$$f(y) := 2y(V) + \min_F l_y(F),$$

where F ranges over all 1-trees. Christofides [1970] and Held and Karp [1970] observed that for each $y \in \mathbb{R}^V$:

(58.9)
$$f(y) \leq \text{the minimum length of a Hamiltonian circuit,}$$

since if C is a shortest Hamiltonian circuit, then $f(y) \leq 2y(V)+l_y(C) = l(C)$.

The function f is concave. Since a shortest 1-tree can be found fast, also $f(y)$ can be computed fast. Held and Karp [1970] showed:

Theorem 58.2. *The minimum value of* $l^{\mathsf{T}}x$ *over* (58.4) *is equal to the maximum value of* $f(y)$ *over* $y \in \mathbb{R}^V$.

Proof. This follows from general linear programming theory. Let $Ax = b$ be system (58.4)(ii) and let $Cx \geq d$ be system (58.6). As (58.4) is equivalent to $Ax = b$, $Cx \geq d$, we have, using LP-duality:

$$(58.10)\qquad \begin{aligned} \min_{\substack{Ax=b\\ Cx\geq d}} l^{\mathsf{T}}x &= \max_{\substack{y,z\\ z\geq \mathbf{0}\\ y^{\mathsf{T}}A+z^{\mathsf{T}}C=l^{\mathsf{T}}}} y^{\mathsf{T}}b + z^{\mathsf{T}}d \\ &= \max_y (y^{\mathsf{T}}b + \max_{\substack{z\geq \mathbf{0}\\ z^{\mathsf{T}}C=l^{\mathsf{T}}-y^{\mathsf{T}}A}} z^{\mathsf{T}}d) = \max_y \big(y^{\mathsf{T}}b + \min_{Cx\geq d}(l^{\mathsf{T}} - y^{\mathsf{T}}A)x\big) \\ &= \max_y f(y). \end{aligned}$$

The last inequality holds as $Cx \geq d$ determines the convex hull of the incidence vectors of 1-trees. ∎

This translates the problem of minimizing $l^{\mathsf{T}}x$ over (58.4) to finding the maximum of the concave function f. We can find this maximum with a subgradient method (cf. Chapter 24.3 of Schrijver [1986b]). The vector y (the *Lagrangean multipliers*) can be used as a correction mechanism to urge the 1-tree to have degree 2 at each vertex. That is, if we calculate $f(y)$, and see that the 1-tree F minimizing $l_y(F)$ has degree more than 2 at a vertex v, we can increase l_y on $\delta(v)$ by decreasing y_v. Similarly, if the degree is less than 2, we can increase y_v. This method was proposed by Held and Karp [1970,1971].

The advantage of this approach is that one need not implement a linear programming algorithm with a constraint generation technique, but that instead it suffices to apply the more elementary tools of finding a shortest 1-tree and updating y. More can be found in Jünger, Reinelt, and Rinaldi [1995].

58.7. The 2-factor constraints

A strengthening of relaxation (58.4) is obtained by using the facts that each Hamiltonian circuit is a 2-factor and that the convex hull of the incidence vectors of 2-factors is known (Corollary 30.8a) (this idea goes back to Robinson [1949] for the asymmetric TSP and Bellmore and Malone [1971] for the symmetric TSP, and was used for the symmetric TSP by Grötschel [1977a] and Pulleyblank [1979b]):

$$(58.11)\qquad \begin{array}{rll} \text{(i)} & 0 \leq x_e \leq 1 & \text{for each edge } e,\\ \text{(ii)} & x(\delta(v)) = 2 & \text{for each vertex } v,\\ \text{(iii)} & x(\delta(U)) \geq 2 & \text{for each } U \subseteq V \text{ with } \emptyset \neq U \neq V,\\ \text{(iv)} & x(\delta(U)\setminus F) - x(F) \geq 1 - |F| & \\ & & \text{for } U \subseteq V,\ F \subseteq \delta(U),\ F \text{ matching, } |F| \text{ odd.} \end{array}$$

Since a minimum-length 2-factor can be found in polynomial time, the inequalities (i), (ii), and (iv) can be tested in polynomial time (cf. Theorem 32.5). Hence the minimum of $l^{\mathsf{T}}x$ over (58.11) can be found in strongly polynomial time.

System (58.11) generally is not enough to determine the traveling salesman polytope, as can be seen, by taking the Petersen graph $G = (V, E)$ and $x_e := \frac{2}{3}$ for each edge e.

Grötschel and Padberg [1979b] showed that, for complete graphs, each of the inequalities (58.11)(iv) determines a facet of the traveling salesman polytope (if $|F| \geq 3$). Boyd and Pulleyblank [1991] studied optimization over (58.11).

58.8. The clique tree inequalities

Grötschel and Pulleyblank [1986] found a large class of facet-inducing inequalities, the 'clique tree inequalities', that generalize the 'comb inequalities' (see below), which generalize both the subtour elimination constraints (58.4)(iii) and the 2-factor constraints (58.11)(iv). However, no polynomial-time test of clique tree inequalities is known.

A *clique tree inequality* is given by:

(58.12) $$\sum_{i=1}^{r} x(\delta(H_i)) + \sum_{j=1}^{s} x(\delta(T_j)) \geq 2r + 3s - 1,$$

where $H_1, \ldots, H_r$ are pairwise disjoint subsets of V and $T_1, \ldots, T_s$ are pairwise disjoint proper subsets of V such that

(58.13) (i) no T_j is contained in $H_1 \cup \cdots \cup H_r$,
(ii) each H_i intersects an odd number of the T_j,
(iii) the intersection graph of $H_1, \ldots, H_r, T_1, \ldots, T_s$ is a tree.

(Here, the *intersection graph* is the graph with vertices $H_1, \ldots, H_r, T_1, \ldots, T_s$, two of them being adjacent if and only if they intersect. Each H_i is called a *handle* and each T_j a *tooth*.)

Theorem 58.3. *The clique tree inequality* (58.12) *is valid for the traveling salesman polytope.*

Proof. It suffices to show that each Hamiltonian circuit C satisfies:

(58.14) $$\sum_{i=1}^{r} d_C(H_i) + \sum_{j=1}^{s} d_C(T_j) \geq 2r + 3s - 1.$$

We apply induction on r, the case $r = 0$ being easy (as it implies $s = 1$). For each $i = 1, \ldots, r$, let β_i be the number of T_j intersecting H_i.

If there is an i with $d_C(H_i) \geq \beta_i$, say $i = 1$, then, by parity, $d_C(H_1) \geq \beta_1 + 1$. The sets $H_2, \ldots, H_r, T_1, \ldots, T_s$ fall apart into β_1 collections of type (58.13), to which we can apply induction. Adding up the inequalities obtained, we get:

$$\text{(58.15)} \qquad \sum_{i=2}^{r} d_C(H_i) + \sum_{j=1}^{s} d_C(T_j) \geq 2(r-1) + 3s - \beta_1.$$

Then (58.14) follows, as $d_C(H_1) \geq \beta_1 + 1$.

So we can assume that $d_C(H_i) \leq \beta_i - 1$ for each i. For all i, j, let $\alpha_{i,j} := 1$ if $T_j \cap H_i \neq \emptyset$ and C has no edge connecting $T_j \cap H_i$ and $T_j \setminus H_i$, and let $\alpha_{i,j} := 0$ otherwise. Then

$$\text{(58.16)} \qquad d_C(T_j) \geq 2 + 2\sum_{i=1}^{r} \alpha_{i,j},$$

since C restricted to T_j falls apart into at least $1 + \sum_{i=1}^{r} \alpha_{i,j}$ components (using (58.13)(i)).

Moreover, for each $i = 1, \ldots, r$, there exist at least $\beta_i - d_C(H_i)$ indices j with $\alpha_{i,j} = 1$. Hence

$$\begin{aligned} \text{(58.17)} \qquad \sum_{j=1}^{s} d_C(T_j) &\geq 2s + 2\sum_{i=1}^{r}\sum_{j=1}^{s} \alpha_{i,j} \geq 2s + 2\sum_{i=1}^{r}(\beta_i - d_C(H_i)) \\ &\geq 2s + r + \sum_{i=1}^{r}(\beta_i - d_C(H_i)) = 2r + 3s - 1 - \sum_{i=1}^{r} d_C(H_i), \end{aligned}$$

since $\sum_{i=1}^{r} \beta_i = r + s - 1$, as the intersection graph of the H_i and the T_j is a tree with $r + s$ vertices, and hence with $r + s - 1$ edges.

(58.17) implies (58.14). ∎

Notes. Grötschel and Pulleyblank [1986] also showed that, if G is a complete graph, then any clique tree inequality determines a facet if and only if each H_i intersects at least three of the T_j.

The clique tree inequalities are not enough to determine the traveling salesman polytope, as is shown again by taking the Petersen graph $G = (V, E)$ and $x_e := \frac{2}{3}$ for all $e \in E$.

The special case $r = 1$ of the clique tree inequality is called a *comb inequality*, and was introduced by Grötschel and Padberg [1979a] and proved to be facet-inducing (if G is complete and $s \geq 3$) by Grötschel and Padberg [1979b].

The special case of the comb inequality with $|H_1 \cap T_j| = 1$ for all $j = 1, \ldots, s$ is called a *Chvátal comb inequality*, introduced by Chvátal [1973b]. The special case of the Chvátal comb inequalities with $|T_j| = 2$ for each $j = 1, \ldots, s$ gives the 2-factor constraints (58.11)(iv) (since $2x(F) + \sum_{f \in F} x(\delta(f)) = 4|F|$).

No polynomial-time algorithm is know to test the clique tree inequalities, or the comb inequalities, or the Chvátal comb inequalities. Carr [1995,1997] showed that for each constant K, there is a polynomial-time algorithm to test the clique tree inequalities with at most K teeth and handles. (This can be done by first fixing intersection points of the $H_i \cap T_j$ (if nonempty) and points in $T_j \setminus (H_1 \cup \cdots \cup H_r)$, and next finding minimum-capacity cuts separating the appropriate sets of these points (taking x as capacity function). We can make them disjoint where necessary by the usual uncrossing techniques. As K is fixed, the number of vertices to be chosen is also bounded by a polynomial in $|V|$.)

Letchford [2000] gave a polynomial-time algorithm for testing a superclass of the comb inequalities in planar graphs. Related results are given in Carr [1996], Fleischer and Tardos [1996,1999], Letchford and Lodi [2002], and Naddef and Thienel [2002a,2002b].

58.8a. Christofides' heuristic for the TSP

Christofides [1976] designed the following algorithm to find a short Hamiltonian circuit in a complete graph $G = (V, E)$ (generally not the shortest however). It assumes a nonnegative length function l satisfying the following *triangle inequality*:

(58.18) $$l(uw) \leq l(uv) + l(vw)$$

for all $u, v, w \in V$.

First determine a shortest spanning tree T (with the greedy algorithm). Next, let U be the set of vertices that have odd degree in T. Find a shortest perfect matching M on U. Now $ET \cup M$ forms a set of edges such that each vertex has even degree. (If an edge occurs both in ET and in M, we take it as two parallel edges.) So we can make a closed path C such that each edge in $ET \cup M$ is traversed exactly once. Then C traverses each vertex at least once. By shortcutting we obtain a Hamiltonian circuit C' with $l(C') \leq l(C)$.

How far away is the length of C' from the minimum length μ of a Hamiltonian circuit?

Theorem 58.4. $l(C') \leq \frac{3}{2}\mu$.

Proof. Let C'' be a shortest Hamiltonian circuit. Then $l(T) \leq l(C'') = \mu$, since C'' contains a spanning tree. Also, $l(M) \leq \frac{1}{2}l(C'') = \frac{1}{2}\mu$, since we can split C'' into two collections of paths, each having U as set of end vertices. They give two perfect matchings on U, of total length at most $l(C'')$ (by the triangle inequality (58.18)). Hence one of these matchings has length at most $\frac{1}{2}l(C'')$. So $l(M) \leq \frac{1}{2}l(C'') = \frac{1}{2}\mu$.

Combining the two inequalities, we obtain

(58.19) $$l(C') \leq l(C) = l(T) + l(M) \leq \tfrac{3}{2}\mu,$$

which proves the theorem. ∎

The factor $\frac{3}{2}$ seems quite large, but it is the smallest factor for which a polynomial-time method is known. Don't forget moreover that it is a *worst-case* bound, and that in practice (or on average) the algorithm might have a much better performance.

Wolsey [1980] showed more strongly that (if l satisfies the triangle inequality) the length of the tour found by Christofides' algorithm, is at most $\frac{3}{2}$ times the lower bound based on the subtour elimination constraints (58.4). If all distances are 1 or 2, Papadimitriou and Yannakakis [1993] gave a polynomial-time algorithm with worst-case factor $\frac{7}{6}$. Hoogeveen [1991] analyzed the behaviour of Christofides' heuristic when applied to finding shortest Hamiltonian paths.