Master Thesis

on

Properties of a Logical System in the Calculus of Structures

by

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Properties of a Logical System in the Calculus of Structures

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Abstract *The calculus of structures is a new framework for presenting logical systems. It is a generalisation of a traditional framework, the one-sided sequent calculus. One of the main features of the calculus of structures is that the inference rules are deep: they can be applied anywhere inside logical expressions. Rules in the sequent calculus are, in contrast, shallow. A certain logical system in the calculus of structures, called System* BV*, is studied here. We see that the deep-nesting of rules is a real distinguishing feature between the two frameworks. To this purpose a notion of shallow systems is introduced, such that sequent systems are particular instances. A counterexample, sort of a fractal structure, is then presented to show that there is no shallow system for the logic behind* BV*, and hence no sequent system for* BV*. This result contributes to justifying the claim that the calculus of structures is a better logical framework than sequent calculus, for certain logics.*

Keywords *Proof theory, sequent calculus, calculus of structures, non-commutativity.*

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1 Introduction

The sequent calculus [2] was developed by Gentzen as a framework for specifying formal logical systems. The presentation of logical systems in the sequent calculus has many interesting properties, especially from the viewpoint of proof search. Logical systems in sequent calculus enjoy the so-called subformula property, which enables locally finite search in proof building.

However, sequent calculus is unnecessarily rigid from a certain point of view. In the sequent presentation of linear logic [3] for example, observing certain logical relations might be impossible. Furthermore, it seems very difficult, if not impossible, to introduce a self-dual non-commutative operator in the sequent presentation of linear logic. These were partly the reasons behind the development of the calculus of structures by Guglielmi in [4]. The calculus of structures is more general than sequent calculus, but still conserves certain desired notions like cut-elimination and the subformula property. Moreover, any sequent system that admits a one-sided presentation can be easily ported to the calculus of structures.

In this paper, a logical system in the calculus of structures, called System BV [6], is studied. It is an extension of multiplicative linear logic plus mix with a self-dual non-commutative operator. The purpose of this study is to see whether the calculus of structures is necessary by showing that BV admits no sequent systems of a reasonable kind that will be made precise. One of the difficulties is that there is currently no single accepted precise definition of sequent calculus. The definition of sequent systems used in the following sections comes from a simple observation on most sequent systems: the logical rules in these systems are *shallow*, in a sense that is going to be made precise.

Logical rules in sequent calculus, if viewed in bottom-up fashion, decompose a formula based on its main connective, i.e., on the outermost connective appearing in the formula. For example, consider the following *tensor* rule of linear logic:

$$\otimes\frac{\vdash A,\Phi \quad \vdash B,\Psi}{\vdash A\otimes B,\Phi,\Psi} \quad .$$

The formula $A\otimes B$ is the main formula and it has $\otimes$ as its main connective. The symbols Φ and Ψ denote the contexts, i.e., multisets of formulae. Another way to see the rules in sequent calculus is to consider them as formula trees transformations. The tensor rule above, for example, transforms the formula-tree of $A\otimes B$ into two formula-trees A and B, which correspond to the two immediate subformulae of $A\otimes B$. We can generalise this further, taking into account the contexts of the rules, to allow finitely branching trees transformation. Moreover, the trees produced in the premise do not need to correspond to the subtrees of the tree in the conclusion. A rule is called *shallow* if the depth of the trees (i.e., the maximum length of the branches) involved in the rule is bounded. A shallow system is then a system in which all the logical rules are shallow. Note that here we are referring to the *scheme*, not the instances, of the rule. In this view, sequent systems are instances of shallow systems.

Let us now look at the calculus of structures and what generalisations it offers. The basic building blocks in the calculus of structures are *structures*, which can be thought of as generalised sequents. There is no distinction made between formulae and sequents. Instead, all atoms in a sequent are considered connected by certain structural relations. In BV, there are three structural relations: the par, the copar (or times) and the seq (sequential). The first two correspond to the connectives $\parr$ and $\otimes$ in linear logic. The structure $[S_1,\dots,S_n]$ is called a par structure, where all the structures in it are considered connected by par. Similarly, the structure $(S_1,\dots,S_n)$ is a copar structure where all the structures in it are connected by copar, and the structure $\langle S_1;\dots;S_n\rangle$ is a seq structure where all the structures in it are connected by seq. The par and copar structures are subject to commutativity and associativity, while the seq structures are subject to associativity. In addition, there is a unit $\circ$ common to all three structural relations. Formulae can be translated into structures by treating the $\parr$ connective as par and $\otimes$ as copar. For example, the formula $(a\parr b)\parr(c\otimes d)$ translates to the structure $[a,b,(c,d)]$ in BV. This translation extends also to sequents by treating the commas in sequents as the $\parr$ connectives.

The main novelty of the calculus of structures is that the branches of the derivation trees in sequent calculus are treated as connected by a certain connective. In the case of BV, it is the $\otimes$ connective. This has the consequence that every inference rule in the calculus of structures has at most one premise. The context of the rules is also generalised to allow the rules to be applied anywhere inside a structure. Typical rules have the form:

$$\rho\frac{S\{T\}}{S\{R\}}$$

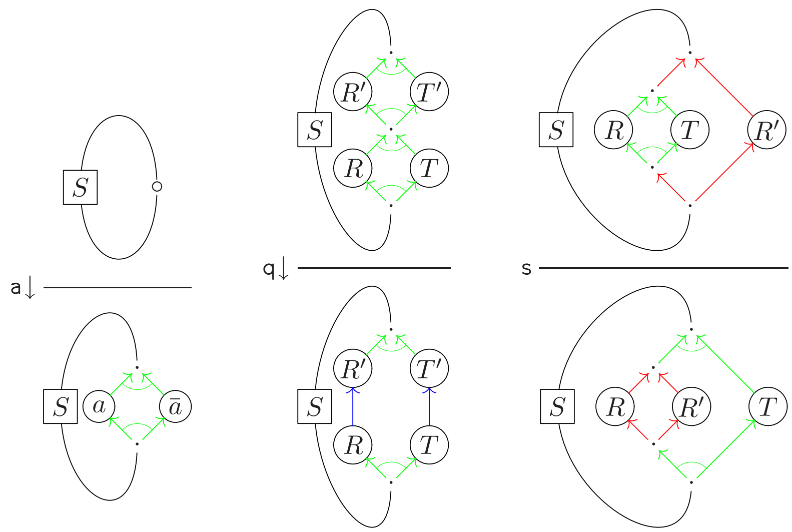


Fig. 1 *Graphical representation of the rules in* BV

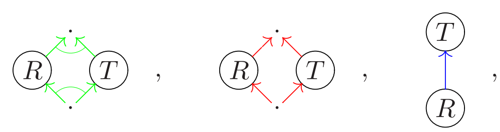
where ρ is the name of the rule, R and T are structures and $S\{\ \}$ is a *structure context*, i.e., a structure with a hole. The structure R is called *redex*. A derivation in the calculus of structures is a sequence of instances of rules. It is a proof if it ends at the top with a unit. Common to all systems already studied in the calculus of structures are the interaction rules:

$$\mathsf{i}{\downarrow}\,\frac{S\{\circ\}}{S[R,\overline{R}]} \quad \text{and} \quad \mathsf{i}{\uparrow}\,\frac{S(R,\overline{R})}{S\{\circ\}}$$

which correspond to the identity rule and the cut rule in sequent calculus, respectively.

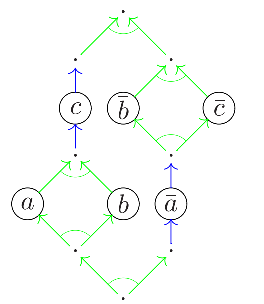
It is obvious that the rules in the calculus of structures are not shallow. The question is whether this deep-nesting of rules is really necessary in order to get all the provable structures, i.e., whether there exists a shallow version of BV that proves the same structures as BV. The main part of this paper is about establishing a counterexample to show that there cannot exist any shallow systems that is equivalent to BV, hence no sequent system corresponding to BV. The counterexample consists of a class of provable structures with a certain property: the bottom most instance of rule of all possible proofs must be applied to certain innermost redexes inside the structures. In other words, those structures impose an innermost reduction strategy on its proofs.

To better appreciate the idea behind the counterexample, let us consider another, more intuitive, representation of structures [4], i.e., as graphs composed using the following building blocks:

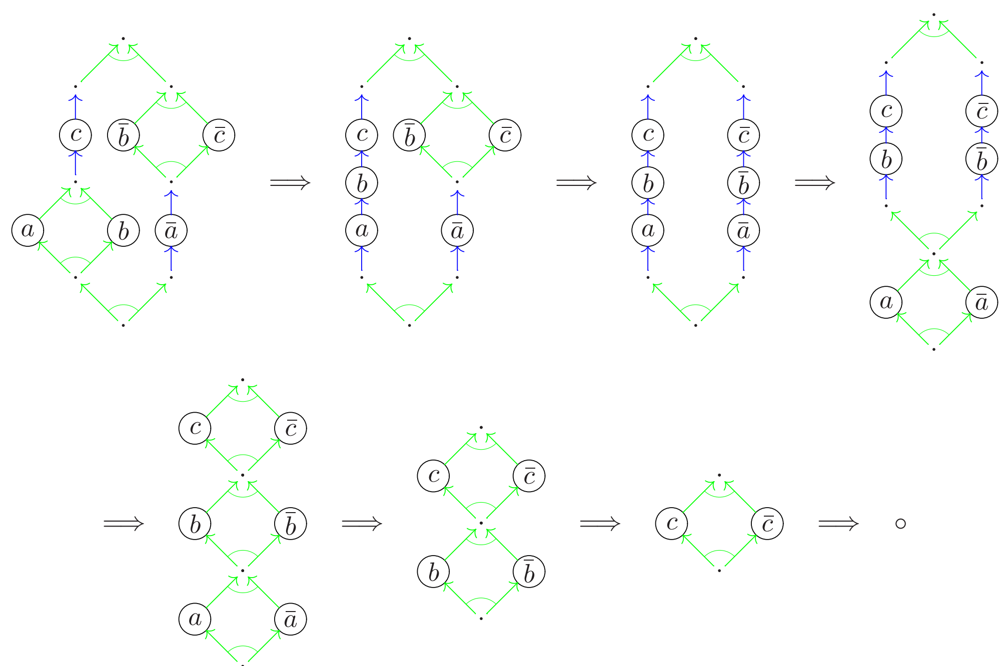


which correspond to the par structure $[R,T]$, copar structure (R,T) and seq structure $\langle R;T\rangle$. The nodes in the graph denote points in time and the arrows connecting the nodes represent temporal relations between those nodes. Two atoms that are in the same time span (i.e., they share the starting and ending time points) can be "wired" (represented as an arc connecting the arrows) and thus can communicate. Two dual atoms that are in communication can annihilate each other. A proof in BV can be understood as a process of changing temporal relations between atoms to a more and more ordered state, until all dual atoms are in communication and annihilate each other. The inference rules in BV in graphical forms are shown in Figure 1.

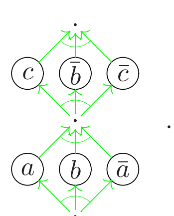
The counterexample is based on the structure $[\langle [a,b];c\rangle,\langle \bar{a};[\bar{b},\bar{c}]\rangle]$, let us call it S_0, which can be presented graphically as follows:



All the dual atoms in this structure can be made into communication by identifying two time points between two structures connected by a par. We can "prove" the structure in the following way:



The proof above has a certain interesting property. The substructures $[a, b]$ and $[\bar{b}, \bar{c}]$ in S_0 above must decide their temporal relations first, before the proof can proceed. Let us see what happens if we identify the time points outside those two substructures:



The structure now is no longer provable because now b comes before $\bar{b}$ and can no more communicate. There are several other possibilities not shown here, but none of them leads to a proof. Thus the structure $[\langle [a, b]; c\rangle, \langle \bar{a}; [\bar{b}, \bar{c}]\rangle]$ is provable if we change either $[a, b]$ to $\langle a; b\rangle$ or $[\bar{b}, \bar{c}]$ to $\langle \bar{b}; \bar{c}\rangle$ first. Now, the crucial part of establishing the counterexample is that we can *delay* this changing of temporal relation between

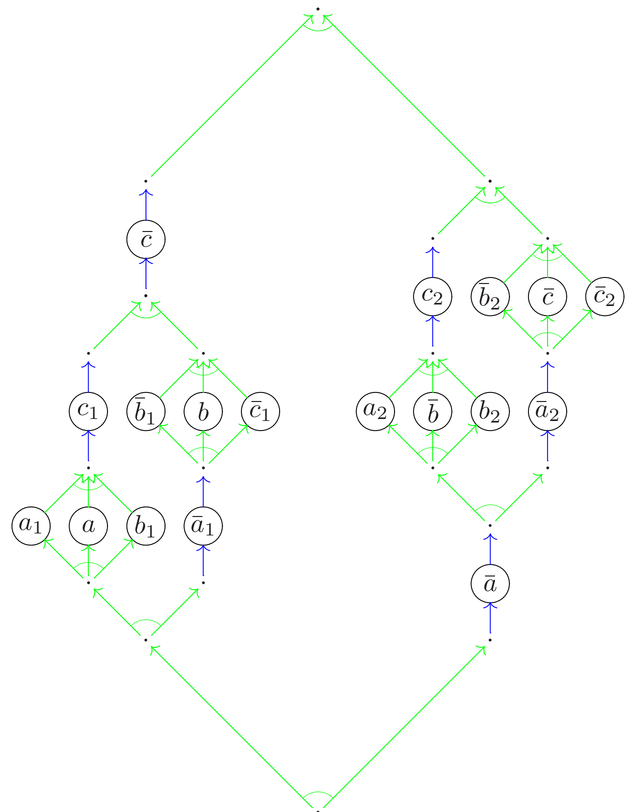
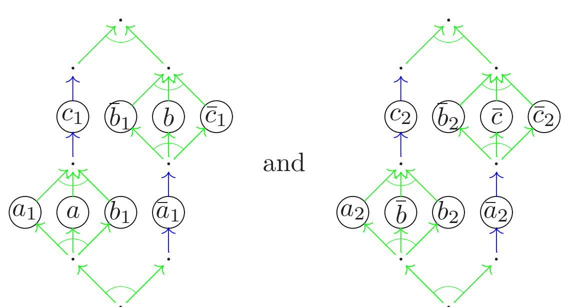


Fig. 2 *Graphical representation of the structure S_1*

a and b (or $\bar{b}$ and $\bar{c}$) by nesting them inside other S_0 structures:



The one step nesting of S_0 produces the structure S_1 shown in Figure 2. We are now forced to change the relation $[a_1, b_1, a]$ to $\langle a_1; b_1; a\rangle$ before we can change the temporal relation between a and b. This process of nesting of substructures of a certain S_0 structure inside other S_0 structures can be repeated to generate larger and larger provable structures with the same property: their proofs must start by changing the innermost redexes. Given a particular shallow system with a certain depth, we are then able to produce a structure such that its innermost redexes are beyond the depth of the shallow rules in the system, and thus establish the proof that no shallow system can be equivalent to BV. Of course, this is a rather simplified explanation. The formal proof in Section 5 will use a different representation, but the principle is still the same.

This paper is organised as follows. Section 2 covers some basic definitions concerning structures. In Section 3, a representation of structures, called trace, is introduced. It was originally developed to give semantics to System BV. In fact, System BV was discovered through this semantics [4]. Section 4 introduces System BV along with a partial characterisation of its provable structures. The formal proof of the deep-nesting property, based on traces, is given in Section 5. This paper ends with some conclusions and suggestions for further developments.

Associativity

$$[\vec{R},[\vec{T}]]=[\vec{R},\vec{T}]$$
$$(\vec{R},(\vec{T}))=(\vec{R},\vec{T})$$
$$\langle\vec{R};\langle\vec{T}\rangle;\vec{U}\rangle=\langle\vec{R};\vec{T};\vec{U}\rangle$$

Unit

$$[\circ,\vec{R}]=[\vec{R}]$$
$$(\circ,\vec{R})=(\vec{R})$$
$$\langle\circ;\vec{R}\rangle=\langle\vec{R};\circ\rangle=\langle\vec{R}\rangle$$

Singleton

$$[R]=(R)=\langle R\rangle=R$$

Commutativity

$$[\vec{R},\vec{T}]=[\vec{T},\vec{R}]$$
$$(\vec{R},\vec{T})=(\vec{T},\vec{R})$$

Negation

$$\bar{\circ}=\circ$$
$$\overline{[R_1,\dots,R_h]}=(\bar{R}_1,\dots,\bar{R}_h)$$
$$\overline{(R_1,\dots,R_h)}=[\bar{R}_1,\dots,\bar{R}_h]$$
$$\overline{\langle R_1;\dots;R_h\rangle}=\langle\bar{R}_1;\dots;\bar{R}_h\rangle$$
$$\bar{\bar{R}}=R$$

Contextual Closure

$$\text{if } R=T \text{ then } S\{R\}=S\{T\}$$

Fig. 3 *Syntactic equivalence =*

2 Structures

2.1 Definition There are infinitely many *positive literals* and *negative literals*. Literals, positive or negative, are denoted by a, b, ... *Structures* are denoted by S, P, Q, R, T, U and V. The structures of the language BV are generated by

$$S ::= a \mid \circ \mid [\underbrace{S,\dots,S}_{>0}] \mid (\underbrace{S,\dots,S}_{>0}) \mid \langle\underbrace{S;\dots;S}_{>0}\rangle \mid \bar{S} \quad ,$$

where $\circ$, the *unit*, is not a literal; $[S_1,\dots,S_h]$ is a *par structure*, $(S_1,\dots,S_h)$ is a *copar structure* and $\langle S_1;\dots;S_h\rangle$ is a *seq structure*; $\bar{S}$ is the *negation* of the structure S. Structures with a hole not occuring in the scope of a negation are denoted by $S\{\ \}$. The structure R is a *substructure* of $S\{R\}$, and $S\{\ \}$ is its *context*. We simplify the indication of context in cases where structural parentheses fill the hole exactly: for example, $S[R,T]$ stands for $S\{[R,T]\}$.

2.2 Definition The structures of the language BV are equivalent modulo the relation $=$, defined in Figure 3. There, $\vec{R}$, $\vec{T}$ and $\vec{U}$ stand for finite, non-empty sequences of structures (sequences may contain ',' or ';' separators as appropriate in the context).

2.3 Definition Structures are said to be *in normal form* when the only negated structures appearing in them are atoms. A structure S is *in canonical form* when $S=\circ$ or S is in normal form and no unit $\circ$ appears in it. Similarly, a structure context $S\{\ \}$ is in normal form when the only negated structures appearing in it are atoms. A structure context $S\{\ \}$ is in canonical form when it is the empty context $\{\ \}$ or $S\{\ \}$ is in normal form and no unit $\circ$ appears in it.

2.4 Definition Given a structure S, we talk about the atom occurrences of S when we consider all the atoms appearing in S as distinct. Therefore, in the structure $\langle a;a\rangle$ there are two different occurrences of the atom a. The set of all the atom occurrences in S is denoted with $\mathsf{occ}\,S$.

We will need to measure the depth of a structure context in order to prove certain properties concerning the nesting of rules in BV. In applying the rules, structures are always considered modulo associativity and commutativity (for par and copar). Therefore, the depth of a structure context is also considered modulo these equalities. Intuitively, we can view structures as finitely branching trees. A structure context is then a particular tree with a hole $\{\ \}$ as a leaf. The depth is measured as the length of the branch ending with $\{\ \}$. The formal definition is as follows.

2.5 Definition The *depth* of a canonical structure context $S\{\ \}$ is defined as follows:

$$\text{depth}\{\ \}=0,$$

$$\text{depth}[S_1,S_2\{\ \}]=\begin{cases}\text{depth}\,S_2\{\ \}, & \text{if } S_2\{\ \}=[S_2',S_2''\{\ \}],\\ \text{depth}\,S_2\{\ \}+1, & \text{otherwise,}\end{cases}$$

$$\text{depth}(S_1,S_2\{\ \})=\begin{cases}\text{depth}\,S_2\{\ \}, & \text{if } S_2\{\ \}=(S_2',S_2''\{\ \}),\\ \text{depth}\,S_2\{\ \}+1, & \text{otherwise,}\end{cases}$$

$$\text{depth}\langle S_1;S_2\{\ \}\rangle=\begin{cases}\text{depth}\,S_2\{\ \}, & \text{if } S_2\{\ \}=\langle S_2';S_2''\{\ \}\rangle\\ & \text{or } S_2\{\ \}=\langle S_2'\{\ \};S_2''\rangle,\\ \text{depth}\,S_2\{\ \}+1, & \text{otherwise,}\end{cases}$$

$$\operatorname{depth}\langle S_1\{\ \}; S_2\rangle = \begin{cases} \operatorname{depth} S_1\{\ \}, & \text{if } S_1\{\ \} = \langle S_1'; S_1''\{\ \}\rangle \\ & \text{or } S_1\{\ \} = \langle S_1'\{\ \}; S_1''\rangle, \\ \operatorname{depth} S_1\{\ \} + 1, & \text{otherwise.} \end{cases}$$

All the structures and structure contexts above are assumed to be in canonical forms.

For example, the structure context $[a, b, \{\ \}]$ has depth 1 and $[\langle\{\ \}; c\rangle, \langle b; c\rangle]$ has depth 2. The depth of a substructure R in $S\{R\}$ is the depth of its context.

2.6 Definition A substructure R in $S\{R\}$ is *n-deep* if $\operatorname{depth} S\{\ \} = n$.

3 Characterisation of Structures

In this section we consider a different representation for structures, a special kind of graph called *trace* [4]. It is a generalisation of series-parallel orders [7]. This representation makes it possible to characterise a structure by certain forbidden configurations in its trace. This characterisation is useful when we want to study general properties of inference rules. The definitions and the main theorems in this section are taken from [4].

3.1 Let S be a structure in normal form. The four *structural relations* $\triangleleft_S$ (*seq*), $\triangleright_S$ (*aseq*), $\downarrow_S$ (*par*) and $\uparrow_S$ (*anti-par*) are defined as the minimal sets such that $\triangleleft_S, \triangleright_S, \downarrow_S, \uparrow_S \subset (\mathsf{occ}\, S)^2$ and, for every $S'\{\ \}$, U and V and for every a in U and b in V, they hold:

1 if $S = S'\langle U; V\rangle$ then $a \triangleleft_S b$ and $b \triangleright_S a$;

2 if $S = S'[U, V]$ then $a \downarrow_S b$;

3 if $S = S'(U, V)$ then $a \uparrow_S b$.

To a structure that is not in normal form we associate the structural relations obtained from any of its normal forms. The quadruple $(\mathsf{occ}\, S, \triangleleft_S, \downarrow_S, \uparrow_S)$ is called the *trace* of S, written $\mathsf{tr}\, S$. The subscripts in $\triangleleft_S$, $\triangleright_S$, $\downarrow_S$ and $\uparrow_S$ are omitted when they are not necessary. Given two sets of atom occurrences μ and ν, we write $\mu \triangleleft \nu$, $\mu \triangleright \nu$, $\mu \downarrow \nu$ and $\mu \uparrow \nu$ to indicate situations where, for every a in μ and for every b in ν, they hold, respectively, $a \triangleleft b$, $a \triangleright b$, $a \downarrow b$ and $a \uparrow b$.

For example, in $(\overline{\langle \bar{a}, b\rangle}, \overline{(c, \bar{d})}) = (\langle a, \bar{b}\rangle, [\bar{c}, d])$ these relations are determined: $a \triangleleft \bar{b}$, $a \uparrow \bar{c}$, $a \uparrow d$, $\bar{b} \triangleright a$, $\bar{b} \uparrow \bar{c}$, $\bar{b} \uparrow d$, $\bar{c} \uparrow a$, $\bar{c} \uparrow \bar{b}$, $\bar{c} \downarrow d$, $d \uparrow a$, $d \uparrow \bar{b}$, $d \downarrow \bar{c}$.

All the atoms in a sub-structure are in the same structural relation with respect to each of the atoms surrounding them:

3.2 Proposition *Given a structure $S\{R\}$ and two atom occurrences a in $S\{\ \}$ and b in R, if $a \triangleleft b$ (respectively, $a \triangleright b$, $a \downarrow b$, $a \uparrow b$) then $a \triangleleft c$ (respectively, $a \triangleright c$, $a \downarrow c$, $a \uparrow c$) for all the atom occurrences c in R.*

3.3 Theorem *Given S and its associated structural relations $\triangleleft$, $\triangleright$, $\downarrow$ and $\uparrow$, the following properties hold, where a, b, c and d are distinct atom occurrences in S:*

s_1 *None of $\triangleleft$, $\triangleright$, $\downarrow$ and $\uparrow$ is reflexive*: $\neg(a \triangleleft a)$, $\neg(a \triangleright a)$, $\neg(a \downarrow a)$, $\neg(a \uparrow a)$.

s_2 *One and only one among $a \triangleleft b$, $a \triangleright b$, $a \downarrow b$ and $a \uparrow b$ holds.*

s_3 *The relations $\triangleleft$ and $\triangleright$ are mutually inverse*: $a \triangleleft b \Leftrightarrow b \triangleright a$.

s_4 *The relations $\triangleleft$ and $\triangleright$ are transitive*: $(a \triangleleft b) \wedge (b \triangleleft c) \Rightarrow a \triangleleft c$ *and* $(a \triangleright b) \wedge (b \triangleright c) \Rightarrow a \triangleright c$.

s_5 *The relations $\downarrow$ and $\uparrow$ are symmetric*: $a \downarrow b \Leftrightarrow b \downarrow a$ *and* $a \uparrow b \Leftrightarrow b \uparrow a$.

s_6 Triangular property: *for $\sigma_1, \sigma_2, \sigma_3 \in \{\triangleleft \cup \triangleright, \downarrow, \uparrow\}$, it holds*

$$(a\ \sigma_1\ b) \wedge (b\ \sigma_2\ c) \wedge (c\ \sigma_3\ a) \Rightarrow (\sigma_1 = \sigma_2) \vee (\sigma_2 = \sigma_3) \vee (\sigma_3 = \sigma_1) \quad .$$

s_7 Square property:

$\mathsf{s}_7^{\triangleleft}$ $$(a \triangleleft b) \wedge (a \triangleleft d) \wedge (c \triangleleft d) \Rightarrow (a \triangleleft c) \vee (b \triangleleft c) \vee (b \triangleleft d) \vee (c \triangleleft a) \vee (c \triangleleft b) \vee (d \triangleleft b) \quad ,$$

$\mathsf{s}_7^{\downarrow}$ $$(a \downarrow b) \wedge (a \downarrow d) \wedge (c \downarrow d) \Rightarrow (a \downarrow c) \vee (b \downarrow c) \vee (b \downarrow d) \quad ,$$

$\mathsf{s}_7^{\uparrow}$ $$(a \uparrow b) \wedge (a \uparrow d) \wedge (c \uparrow d) \Rightarrow (a \uparrow c) \vee (b \uparrow c) \vee (b \uparrow d) \quad .$$

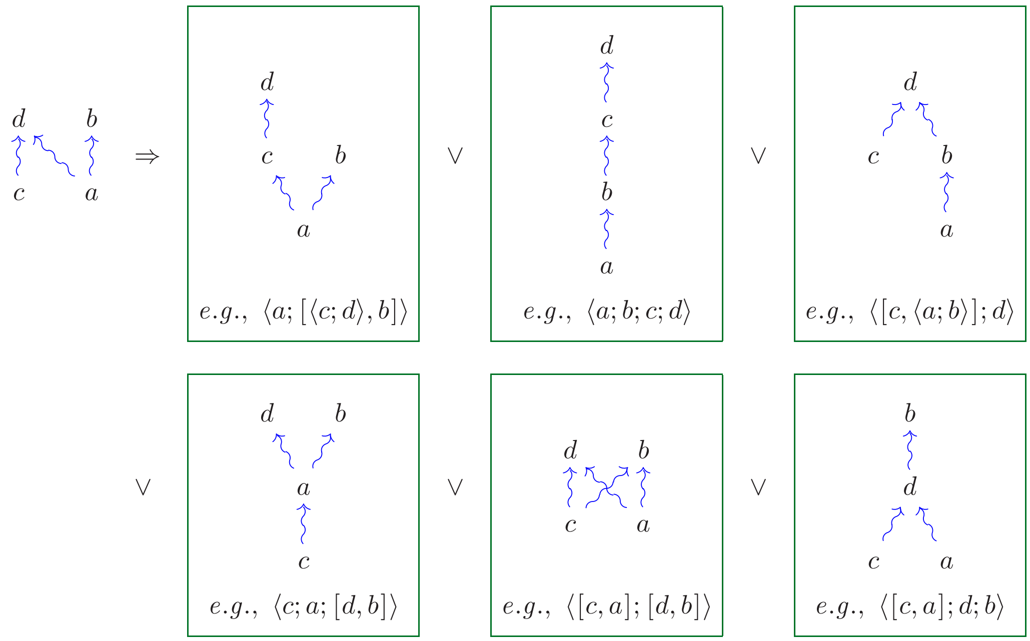


Fig. 4 *Square property for* $\triangleleft$

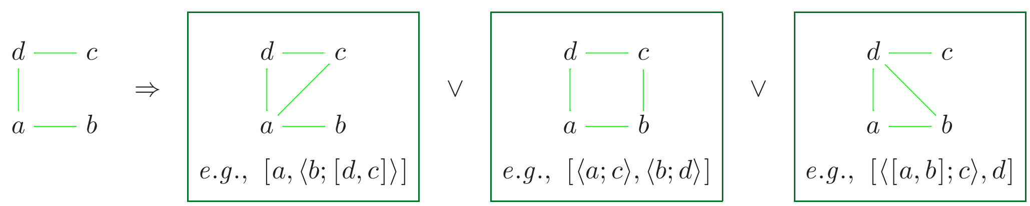


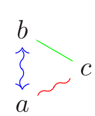
Fig. 5 *Square property for* $\downarrow$

3.4 Structural relations between occurrences of atoms are represented by drawing

$$a \rightsquigarrow b \quad , \; a \leftrightsquigarrow b \quad , \; a \;—\; b \quad \text{and} \quad a \sim b$$

when $a \triangleleft b$ (and $b \triangleright a$), $a \triangleleft b$ or $a \triangleright b$, $a \downarrow b$ and $a \uparrow b$, respectively. Dashed arrows represent negations of structural relations.

The triangular property says that there are no structures such that the following configuration may be found in them:



in every triangle at least two sides must represent the same structural relation.

The square property for $\triangleleft$ can be represented as in Figure 4. Figure 5 shows the square property for $\downarrow$. The square property for $\uparrow$ is similar to the one for $\downarrow$.

3.5 Theorem *Two structures are equivalent if and only if they have the same trace.*

4 System BV

4.1 Definition An *(inference) rule* is any scheme $\rho \frac{T}{R}$, where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*. Rule names are denoted by ρ and π. A *(formal) system*, denoted by $\mathscr{S}$, is a set of rules. A *derivation* in a system $\mathscr{S}$ is a finite chain of instances of rules of $\mathscr{S}$, and is denoted by Δ. A derivation can consist of just one structure. The topmost structure in a derivation is called its *premise*; the lowest structure is called *conclusion*. A derivation Δ whose premise is T, conclusion is R,

$$\mathsf{a}\!\downarrow \frac{S\{\ \}}{S[a,\bar{a}]} \qquad \mathsf{a}\!\uparrow \frac{S(a,\bar{a})}{S\{\ \}}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R,T];[R',T']\rangle}{S[\langle R;R'\rangle,\langle T;T'\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R;T\rangle,\langle R';T'\rangle)}{S\langle (R,R');(T,T')\rangle}$$

$$\mathsf{s}\frac{S([R,T],R')}{S[(R,R'),T]}$$

Fig. 6 *System* SBV

$$\circ\!\downarrow \frac{}{\circ} \qquad \mathsf{a}\!\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \mathsf{s}\frac{S([R,T],R')}{S[(R,R'),T]} \qquad \mathsf{q}\!\downarrow \frac{S\langle [R,T];[R',T']\rangle}{S[\langle R;R'\rangle,\langle T;T'\rangle]}$$

Fig. 7 *System* BV

and whose rules are in $\mathscr{S}$ is denoted by $\begin{array}{c}T\\ \Delta\| \mathscr{S}\\ R\end{array}$.

4.2 Definition An instance of a rule $\rho\frac{T}{R}$ is called *non-trivial* if $R \neq T$, otherwise it is *trivial*.

4.3 Definition System SBV [6] is shown in Fig. 6. The rules in SBV are of the form $\rho\frac{S\{T\}}{S\{R\}}$. The structure R is called *redex* and the structure T is called *contractum*.

The rules $\mathsf{a}\!\downarrow$ and $\mathsf{a}\!\uparrow$ are called the *interaction* rules. They are the atomic versions of the more general interaction rules $\mathsf{i}\!\downarrow\frac{S\{\circ\}}{S[R,\overline{R}]}$ and $\mathsf{i}\!\uparrow\frac{S(R,\overline{R})}{S\{\circ\}}$. The non-atomic interaction rules can be reduced to the atomic ones, as shown in [6]. System SBV consists of a down fragment $\{\mathsf{a}\!\downarrow,\mathsf{s},\mathsf{q}\!\downarrow\}$ and an up fragment $\{\mathsf{a}\!\uparrow,\mathsf{s},\mathsf{q}\!\uparrow\}$. The rules $\mathsf{a}\!\downarrow$ and $\mathsf{q}\!\uparrow$ in the up fragment are admissible (for proofs) as a consequence of the cut elimination theorem for this system [4, 5]. Therefore, it is enough to study the down fragment for its proof search properties. We need an extra rule, a logical axiom rule, to define proofs in SBV.

4.4 Definition The down fragment of SBV, together with the rule $\circ\!\downarrow\frac{}{\circ}$, is called System BV (Figure 7).

4.5 Definition A *proof*, denoted by Π, is a derivation whose top is an instance of $\circ\!\downarrow$. A system $\mathscr{S}$ *proves* R if there is in $\mathscr{S}$ a proof Π whose conclusion is R, written $\begin{array}{c}\Pi\|\mathscr{S}\\ R\end{array}$.

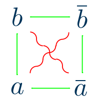
4.6 Definition Two systems are *equivalent* if they prove the same structures.

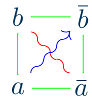
The following proposition states a simple property of derivations in BV that will be used in proving several lemmas.

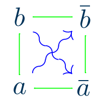
4.7 Proposition *If there is a derivation* $\begin{array}{c}\mathsf{a}\!\downarrow\frac{T\{\circ\}}{T[a,\bar{a}]}\\ \Delta\|\mathsf{BV}\\ R\end{array}$ *then then there is a derivation* $\begin{array}{c}T\{\circ\}\\ \Delta'\|\mathsf{BV}\\ R'\end{array}$ *where* R' *is obtained from* R *by replacing the atom occurrences* a *and* $\bar{a}$ *with units, and* Δ' *is obtained from* Δ *by replacing all the occurrences of* a *and* $\bar{a}$ *in* Δ *with units.*

Given a provable structure in BV, we can remove all the dual instances of atoms, except maybe for some pairs, and get another provable structure. In this way, we can see whether there are certain local properties obeyed by all provable structures. There are, indeed, such properties as shown in the following lemma.

4.8 Lemma *Let* S *be a provable structure in* BV *that consists of pairwise distinct atoms. The trace* *of S does not contain any of the following configurations*:



,


and


.

Proof Suppose that S is provable in BV but $\mathsf{tr}\,S$ contains such configurations, i.e., there exists atoms $a, \bar{a}, b, \bar{b} \in \mathsf{occ}\,S$ that are interrelated as in one of the configurations above. Take any proof $\begin{array}{c}\Pi \Vert \mathsf{BV} \\ S\end{array}$. By repeated applications of Proposition 4.7 we can remove all the atom occurrences from Π, except for the atoms $\{a, b, \bar{a}, \bar{b}\}$, and obtain the proof $\begin{array}{c}\Pi' \Vert \mathsf{BV} \\ S'\end{array}$. The trace of S' must be in one of the configurations above, which correspond to the structures $[(a, \bar{b}), (\bar{a}, b)]$, $[\langle a; \bar{b}\rangle, (\bar{a}, b)]$ and $[\langle a; \bar{b}\rangle, \langle b; \bar{a}\rangle]$, respectively. As can be easily checked, none of these structures is provable in BV, which contradicts the fact that S' is provable. □

The characterisation in the lemma above is only partial. Consider for example the structure $[(a, b), (c, \bar{b}), (\bar{a}, \bar{c})]$. The forbidden configurations are absent, but the structure is clearly not provable in BV. A complete characterisation will probably have to be global, i.e., it would involve all the atom occurrences in the structure.

5 Shallow Systems

We have seen the intuitive explanation of the proof of deep-nesting of BV in the introduction. The essential idea is that a certain structure, called S_0, requires an innermost reduction strategy in its proofs, that is, its proofs must start with the innermost redex in the structure. The proof of deep-nesting also relies on the fact that shallow rules have limited "vision" on the redexes in a structure. Therefore, we achieve the deep-nesting effect by deepening the redexes by merging several S_0-structures together, producing the structure S_n. Given a shallow system, we are then able to construct a certain structure S_n that is provable in BV but deep enough not to be provable in that shallow system.

The informal definition of shallow rules in the introduction, i.e., as tree transformations with bounded depth, does not capture fully the intended meaning of shallowness mentioned above. There are rules that duplicate structures, like the contraction rule $\mathsf{c}{\downarrow}\,\dfrac{[R, R]}{R}$ for example, which cannot really be considered shallow, since to copy a structure it must look into the entire structure. In general, it is still arguable whether this rule should be excluded, since the main target of this study is to show that there are logics which cannot be captured by any sequent systems, and the contraction rule is quite common in sequent systems. In the case of BV, contraction and weakening are disallowed because they would lead to a system which proves more structures than BV.

Apart from the problem in defining shallowness, there is another issue of what kind of rules we should consider inference rules. Without any restriction, we can have a rule where the premise has no logical relation with the conclusion, that is, the premise does not logically imply the conclusion. For example, the addition of a rule like $\rho\,\dfrac{\langle R; T\rangle}{(R, T)}$ to BV does not make any difference in terms of provability, since R and T cannot communicate, either in the copar relation or seq relation. Since De Morgan laws hold for BV, logical implication $P \Rightarrow Q$ is the same as the structure $[\overline{P}, Q]$. To see whether the rule ρ above corresponds to logical implication in BV, it is enough to check whether $[(R, T), \overline{\langle R; T\rangle}]$ is provable, which is clearly not the case. The definition of shallow rules adopted here is restricted to rules that correspond to logical implications.

Before we proceed to the definition of shallow rules and shallow systems, we need to extend the language of structures with structure variables. The intention is to capture the scheme of the rule at the object level. A scheme of a rule is just a rule where the structures in both premise and conclusion are made of structure variables.

5.1 Definition We enrich the language of structures defined in 2.1 with an infinite set of structure variables. Structure variables are denoted with A, B and C. The extended language of structures is defined as follows:

$$S ::= a \mid A \mid \circ \mid [\underbrace{S, \dots, S}_{>0}] \mid (\underbrace{S, \dots, S}_{>0}) \mid \langle \underbrace{S; \dots; S}_{>0} \rangle \mid \bar{S} \quad ,$$

where a is an atom and A is a structure variable. Structure variables representing atomic structures will be denoted with x and y. The definitions that apply to structures without variables apply also to structures with variables. In particular, $\mathsf{occ}\,S$ now denotes the set of occurances of atoms and variables in the structure S. The set of structure variables in S is denoted by $\mathrm{vars}\,S$. A *ground* structure is a structure that does not contain any variable. A structure with variables can be *instantiated* by replacing the variables with other structures.

In the following definition of shallow systems, only the logical rules are required to be shallow. The logical axiom and the interaction rules are still the same as in BV. Moreover, since the systems that are of interest will be those with cut-elimination property, we will always refer to cut-free systems when we will be talking about shallow systems.

5.2 Definition The *depth* of a structure S is defined as follows:

$$\mathrm{depth}\,S = \max\{\mathrm{depth}\,S'\{\ \} \mid S'\{R\} = S \text{ and } R \in \mathsf{occ}\,S\}$$

5.3 Definition A rule $\rho\dfrac{T}{R}$ is *shallow* if R and T are structures such that $\mathsf{occ}\,R = \mathrm{vars}\,R = \mathrm{vars}\,T = \mathsf{occ}\,T \neq \emptyset$. The depth of ρ is defined as $\max(\mathrm{depth}\,R, \mathrm{depth}\,T)$.

Consider for example the rule

$$\rho\frac{[A,([B,C],C')]}{[A,B,(C,C')]}.$$

It is a shallow rule with depth 3, which is the depth of the structure in the premise.

5.4 Definition A shallow system $\mathscr{S}$ consists of the rules $\{\mathsf{i}{\downarrow}, \circ{\downarrow}\}$ and a set of shallow rules such that

a. for every shallow rule $\rho\dfrac{T}{R}$ all the ground instances of $[R,\bar{T}]$ are provable in $\mathscr{S}$, and

b. $\exists n \in \mathrm{N}$ such that $\mathrm{depth}\,\rho \leq n$ for every shallow rule ρ.

5.5 Definition A shallow system is called n-deep if the maximum depth of its shallow rules is n.

In proving properties of general shallow rules, we need a uniform representation of the rules. We cannot list all the possible shallow rules simply because there is an infinite number of them. A more general strategy is adopted here. Instead of considering particular shapes of shallow rules, we look into their operations in the traces of the structures involved in the application of the rules. Any non-interaction rule works by modifying structural relations between atom occurrences in a structure. The square and triangular properties can later be used to infer the effects on the overall structural relations in the structure. In addition to the square and triangular properties, there are two additional properties induced by the definitions of shallow rules and shallow systems. The first is that there are some structural relations that cannot be modified by a shallow rule, if they happen to belong to a substructure that is nested deeper than the depth of the shallow rule. The second is that the changing of structural relations must obey the ordering $\downarrow \sqsubseteq \triangleleft \sqsubseteq \uparrow$. Both properties are shown in the following lemmas.

5.6 Definition Let $\rho\dfrac{T}{R}$ be an instance of ρ. The structural relation $a\ \sigma_R\ b$, where $a, b \in \mathsf{occ}\,R$ and $\sigma \in \{\downarrow, \triangleleft, \uparrow\}$, is *preserved* by ρ if $a, b \in \mathsf{occ}\,T$ and $a\ \sigma_T\ b$.

5.7 Lemma *Let $\rho\dfrac{T}{R}$ be an instance of a shallow rule ρ of depth n and P be an m-deep substructure in R. If $m > n$ then all the structural relations in P are preserved by ρ.*

5.8 Lemma *Let $\rho\dfrac{T}{R}$ be an instance of a shallow rule in a shallow system $\mathscr{S}$. Let $a, b \in \mathsf{occ}\,R$. If $\mathscr{S}$ is equivalent to BV then:*

a. if $\neg(a \downarrow_R b)$ then $\neg(a \downarrow_T b)$, and

b. if $a \triangleleft_R b$, then $a \triangleleft_T b$ or $a \uparrow_T b$.

Proof

a. Suppose that $\neg(a \downarrow_R b)$, i.e., $a \triangleleft_R b$ or $a \uparrow_R b$, but $a \downarrow_T b$. Then we can replace all the atoms in R and T, except a and b, with units. We have the following instances of ρ:

$$\rho\frac{[a,b]}{\langle a;b\rangle} \quad \text{and} \quad \rho\frac{[a,b]}{(a,b)}.$$

By the definition of shallow systems, $[\langle a;b\rangle,(\bar{a},\bar{b})]$ and $[(a,b),(\bar{a},\bar{b})]$ must be provable in $\mathscr{S}$. But these structures are not provable in BV, contrary to the assumption that $\mathscr{S}$ is equivalent to BV.

b. Suppose that $a \triangleleft_R b$, but $\neg(a \triangleleft_T b)$ and $\neg(a \uparrow_T b)$. As shown in case a, it cannot be that $a \downarrow_T b$, so suppose that it is $b \triangleleft_T a$. This means that $\rho \frac{\langle b; a\rangle}{\langle a; b\rangle}$ is an instance of ρ and that $[\langle a; b\rangle, \langle \bar{b}; \bar{a}\rangle]$ must be provable in $\mathscr{S}$. Again, since $[\langle a; b\rangle, \langle b; a\rangle]$ is not provable in BV, we get a contradiction to the assumption that $\mathscr{S}$ is equivalent to BV.

□

6 Deep-Nesting Property of System BV

The main part of the proof of deep-nesting is showing the class of structures generated from the structure $S_0 = [\langle [a, b]; c\rangle, \langle \bar{a}; [\bar{b}, \bar{c}]\rangle]$. This is done by merging variants of S_0 with the innermost par-substructures of the original structure. For example, the structure S_1 is obtained by merging $[\langle [a_0, b_0]; c_0\rangle, \langle \bar{a}_0; [\bar{b}_0, \bar{c}_0]\rangle]$ with the substructure $[a, b]$ and $[\langle [a_1, b_1]; c_1\rangle, \langle \bar{a}_1; [\bar{b}_1, \bar{c}_1]\rangle]$ with the substructure $[\bar{b}, \bar{c}]$. The resulting structure is

$$[\langle [\langle [a_0, b_0, a]; c_0\rangle, \langle \bar{a}_0; [\bar{b}_0, \bar{c}_0, b]\rangle]; c\rangle, \langle \bar{a}; [\langle [a_1, b_1, \bar{b}]; c_1\rangle, \langle \bar{a}_1; [\bar{b}_1, \bar{c}_1, \bar{c}]\rangle]\rangle].$$

The formal definition of this merging process is as follows.

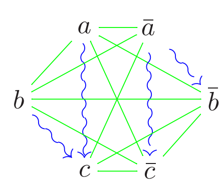
6.1 Definition A non-unit structure R is called *flat* if all the structural relations between atom occurrences in R are of the same type. The structure $[a_1, \ldots, a_n]$ is a *flat par structure*. Likewise, $(a_1, \ldots, a_n)$ and $\langle a_1; \ldots; a_n\rangle$ are a *flat copar structure* and a *flat seq structure*, respectively. An atomic structure can be seen as a flat par structure, flat copar structure or flat seq structure.

6.2 Definition Let C be an infinite set of positive atoms and let N^* be the set of finite sequences of natural numbers. Elements of N^* are denoted by u, v, w and z. Concatenation of two sequences of natural number u and v is denoted by $u.v$. The set of indexed atoms is defined as $\mathrm{C}_{\mathrm{N}^*} = \{a_u \mid a \in \mathrm{C},\ u \in \mathrm{N}^*\}$. Two atoms with different names or indexes are considered different. The following functions generate a class of structures:

$$\begin{aligned} \alpha_0(u, R, T) &= [\langle [a_u, b_u, R]; c_u\rangle, \langle \bar{a}_u; [\bar{b}_u, \bar{c}_u, T]\rangle] \\ \alpha_n(u, R, T) &= [\langle \alpha_{n-1}(u.0, a_u, [b_u, R]); c_u\rangle, \langle \bar{a}_u; \alpha_{n-1}(u.1, \bar{b}_u, [\bar{c}_u, T])\rangle], \text{ for } n > 0. \end{aligned}$$

where $u \in \mathrm{N}^*$, $a \neq b \neq c \neq a$ and R, T are either units or flat par-structures. A structure of the form $\alpha_n(u, R, T)$ is called an α_n-structure. We denote with S_n the structure $\alpha_n(0, \circ, \circ)$.

The formal proof of the deep-nesting property makes use of a representation of structures in traces. It is technically easier to understand the behaviour of shallow rules in this setting, by using the square and triangular properties and the results from Lemma 4.8, Lemma 5.7 and Lemma 5.8. To see how this works, consider the trace of S_0 (the indexes are omitted):



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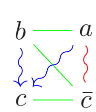
If, for example, there is a proof of S_0 that starts with a rule ρ of depth 1, then by Lemma 5.7, the structural relations $a \downarrow b$ and $\bar{b} \downarrow \bar{c}$ must be preserved by ρ since they correspond to substructures at depth 2. Assuming the application of ρ is non-trivial, some other structural relations must be changed. The changing, however, must preserve provability, which means the par relations between dual atoms must also be preserved. Given these constraints, any other changes to the structural relations will either result in a violation of a certain structural property or destroy the provability. For example, if we change $a \downarrow \bar{c}$ to $a \uparrow \bar{c}$, the triangular and square properties will be violated:



.

This idea can be generalised to the case

$$S_n = [\langle [P\{a\}, Q\{b\}]; c\rangle, \langle \bar{a}; [P'\{\bar{b}\}, Q'\{\bar{c}\}]\rangle].$$

If we preserve all the structural relations inside $[P\{a\}, Q\{b\}]$ and $[P'\{\bar{b}\}, Q'\{\bar{c}\}]$, then all other structural relations cannot be changed without changing the provability of S_n. Both ideas are explored in Lemma 6.7 and Lemma 6.8.

6.3 Proposition *For every flat par structures R and T, and for every $u \in \mathrm{N}^*$ there exists a derivation* $\begin{array}{c}[R,T]\\ \| \mathsf{BV}\\ \alpha_n(u,R,T)\end{array}$ *in* BV.

Proof

Base Case

$$\alpha_0(u,R,T) = \mathsf{q}\!\downarrow\frac{\mathsf{q}\!\downarrow\frac{\mathsf{a}\!\downarrow\frac{\mathsf{q}\!\downarrow\frac{\mathsf{q}\!\downarrow\frac{\mathsf{a}\!\downarrow\frac{\mathsf{a}\!\downarrow\frac{[R,T]}{[R,T,c_u,\bar{c}_u]}}{\langle[b_u,\bar{b}_u,R,T];[c_u,\bar{c}_u]\rangle}}{[\langle[b_u,\bar{b}_u,T,T];c_u\rangle,\bar{c}_u]}}{[\langle[a_u,R];c_u\rangle,\bar{b}_u,\bar{c}_u,T]}}{\langle[a_u,\bar{a}_u];[\langle[b_u,R];c_u\rangle,\bar{b}_u,\bar{c}_u,T]\rangle}}{[\langle a_u;[b_u,R];c_u\rangle,\langle\bar{a}_u;[\bar{b}_u,\bar{c}_u,T]\rangle]}}{[\langle[a_u,b_u,R];c_u\rangle,\langle\bar{a}_u;[\bar{b}_u,\bar{c}_u,T]\rangle]} \quad .$$

Inductive Case

Assume that $\begin{array}{c}[U,V]\\ \| \mathsf{BV}\\ \alpha_i(v,U,V)\end{array}$ for all $i < n$, for all $v \in \mathrm{N}^*$ and for all flat par-structures U, V. Then

$$\begin{array}{c}[R,T]\\ \|\\ \alpha_0(u,R,T) = [\langle[a_u,b_u,R];c_u\rangle,\langle\bar{a}_u;[\bar{b}_u,\bar{c}_u,T]\rangle]\\ \|\\ [\langle[a_u,b_u,R];c_u\rangle,\langle\bar{a}_u;\alpha_{n-1}(u.1,\bar{b}_u,[\bar{c}_u,T])\rangle]\\ \|\\ \alpha_n(u,R,T) = [\langle\alpha_{n-1}(u.0,a_u,[b_u,R]);c_u\rangle,\langle\bar{a}_u;\alpha_{n-1}(u.1,\bar{b}_u,[\bar{c}_u,T])\rangle]\end{array}$$

□

6.4 Theorem *For every n, S_n is provable in* BV.

Proof Since $S_n = \alpha_n(0,\circ,\circ)$, by Proposition 6.3: $\begin{array}{c}\circ\!\downarrow\frac{}{\circ}\\ \| \mathsf{BV}\\ \alpha_n(0,\circ,\circ)\end{array}$. □

6.5 Proposition *There are no substructures of the form $[R,\overline{R}]$ in S_n.*

Proof

Base Case

$S_0 = \alpha_0(0,\circ,\circ) = [\langle[a_0,b_0];c_0\rangle,\langle\bar{a}_0;[\bar{b}_0,\bar{c}_0]\rangle]$, obvious.

Inductive Case

Assume that $S_{n-1} = \alpha_{n-1}(0,\circ,\circ)$ has no substructures of the form $[R,\bar{R}]$. The change of index from $\alpha_{n-1}(0,\circ,\circ)$ to $\alpha_{n-1}(u,\circ,\circ)$ does not affect the form of the structure, since all the indexed atoms are distinct by definition. The addition of new atoms to $\alpha_{n-1}(u,\circ,\circ)$, i.e., $\alpha_{n-1}(u,U,V)$, does not introduce any new dual substructures, provided that U and V have no dual atoms and consist of different atoms than $\alpha_{n-1}(u,\circ,\circ)$. Therefore the structure

$$S_n = \alpha_n(0,\circ,\circ) = [\langle\alpha_{n-1}(0.0,a_0,b_0);c_0\rangle,\langle\bar{a}_0;\alpha_{n-1}(0.1,\bar{b}_0,\bar{c}_0)\rangle]$$

does not contain any substructures of the form $[R,\bar{R}]$ either. □

6.6 Proposition *Every α_0-substructure in $\alpha_n(u,R,T)$ is $2n$-deep.*

Proof The base case is obvious. For the inductive case, assume that every α_0-substructure in $\alpha_{n-1}(w,P,Q)$ is $2(n-1)$-deep. Since

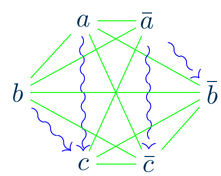
$$\alpha_n(u,R,T) = [\langle\alpha_{n-1}(u.0,a_u,[b_u,R]);c_u\rangle,\langle\bar{a}_u;\alpha_{n-1}(u.1,\bar{b}_u,[\bar{c}_u,T])\rangle],$$

the depth of α_0-substructure in $\alpha_n(u,R,T)$ is $2(n-1)+2 = 2n$. □

6.7 Lemma *Let* $\rho\,\dfrac{\overset{\Vert \mathscr{S}}{T}}{R}$ *be a proof in a shallow system* $\mathscr{S}$. *If* $\mathscr{S}$ *is equivalent to* BV *and the following conditions hold*:

a. R *consists of pairwise distinct atoms*,

b. $\mathtt{tr}\,R$ *contains the following configuration*:

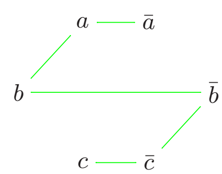


(i)

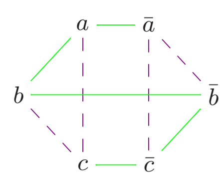
c. $a \downarrow_T b$ *and* $\bar{b} \downarrow_T \bar{c}$,

then the configuration (i) is preserved by ρ.

Proof Since T must be provable, the par links between dual atoms must be preserved. This, together with conditions a-c, give us the following configuration in $\mathtt{tr}\,T$:



We assume above that $\mathscr{S}$ is equivalent to BV. Therefore, Lemma 5.8 applies: the structural relations $\triangleleft$ and $\uparrow$ cannot be changed to $\downarrow$. So it must hold in $\mathtt{tr}\,T$:

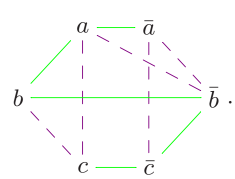


(ii)

We first show that all par relations $x \downarrow_R y$ in (i) must be preserved, and then, based on this result we prove that any seq relations $x \triangleleft_R y$ in (i) must also be preserved.

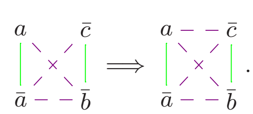
1. $a \downarrow_T \bar{a}$, $b \downarrow_T \bar{b}$, $c \downarrow_T \bar{c}$, $a \downarrow_T b$ and $\bar{b} \downarrow_T \bar{c}$, see diagram (ii).

2. $a \downarrow_T \bar{b}$. Suppose that $\neg(a \downarrow_T \bar{b})$. Adding this information to (ii) we have in $\mathtt{tr}\,T$:

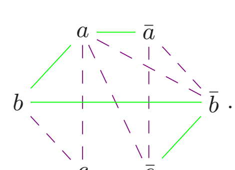


(ii.1)

Take the atoms a, $\bar{a}$, $\bar{b}$ and $\bar{c}$. By the square property, we can infer that $\neg(a \downarrow_T \bar{c})$, as shown in the following figure:

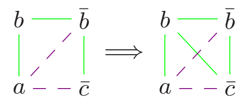


Now we combine this new information with (ii.1):

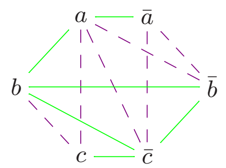


(ii.2)

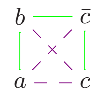
Take the atoms b, $\bar{b}$, a and c. Again, by the square property, $b \downarrow_T \bar{c}$ as shown below:



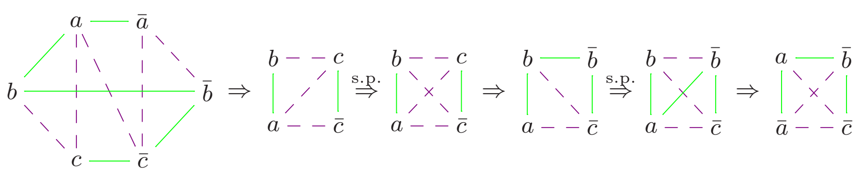
This together with (ii.2) yield:



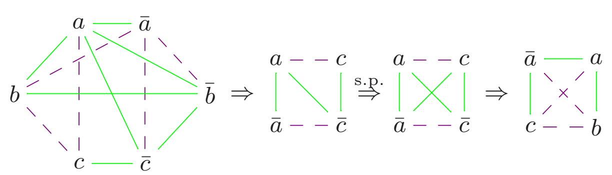
(ii.3)

But if this is the case, T cannot be a valid structure, because  violates the square property. Therefore $a \downarrow \bar{b}$ must hold in T.

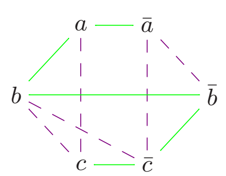
3. $a \downarrow \bar{c}$ in T. Suppose that $\neg(a \downarrow \bar{c})$, following a similar reasoning as in Case 2, we have (s.p. stands for square property):



4. $b \downarrow_T \bar{a}$. Suppose $\neg(b \downarrow_T \bar{a})$. It follows from the result established in cases 1-3 that in $\mathsf{tr}\, T$:

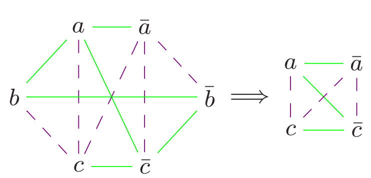


5. $b \downarrow_T \bar{c}$. This case is analogous to case 2. That is if we suppose that $\neg(b \downarrow_T \bar{c})$, then the trace:

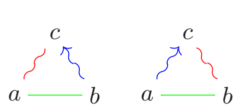
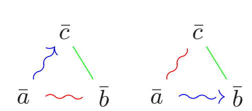


is isomorphic to (ii.1) in case 2.

6. $\bar{a} \downarrow_T c$. Suppose $\neg(\bar{a} \downarrow_T c)$. Since $a \downarrow_T c$, as shown in Case 3, we have the following forbidden configuration:

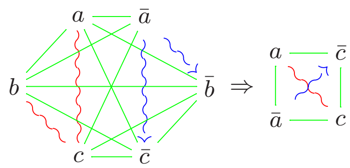


We are left with the relations $a \triangleleft_R c$, $b \triangleleft_R c$, $\bar{a} \triangleleft_R \bar{b}$ and $\bar{a} \triangleleft_R \bar{c}$. Since the structural relation $\triangleleft$ can change only to $\uparrow$, there are 16 possible configurations to consider. By applying the triangular property, it is easy to see that $a \uparrow_T c$ iff $b \uparrow_T c$, and that $\bar{a} \uparrow_T \bar{b}$ iff $\bar{a} \uparrow_T \bar{b}$. This is the case because otherwise we would have the following forbidden configurations:

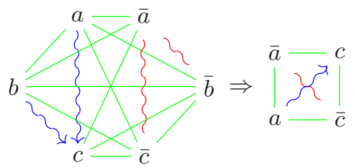
 and 

Therefore, we need only to consider the following three cases.

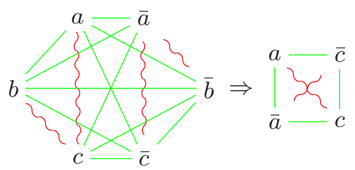
7. $a \uparrow_T c$, $b \uparrow_T c$, $\bar{a} \lhd_T \bar{b}$ and $\bar{a} \lhd_T \bar{c}$.



8. $a \lhd_T c$, $b \lhd_T c$, $\bar{a} \uparrow_T \bar{b}$ and $\bar{a} \uparrow_T \bar{c}$.



9. $a \uparrow_T c$, $b \uparrow_T c$, $\bar{a} \uparrow_T \bar{b}$ and $\bar{a} \uparrow_T \bar{c}$.



The configurations in cases 7, 8 and 9 are forbidden by Lemma 4.8, which means that the relations $a \lhd_R c$, $b \lhd_R c$, $\bar{a} \lhd_R \bar{b}$ and $\bar{a} \lhd_R \bar{c}$ must be preserved by ρ. Therefore, the configuration (i) must hold in T. □

6.8 Lemma *Let* $\rho\,\dfrac{\overset{\Pi\mathscr{S}}{U}}{S\{P\}}$ *be a proof in a shallow system* $\mathscr{S}$ *where* $\rho \neq \mathsf{i}{\downarrow}$, P *is an* α_n*-structure and* $S\{P\}$ *consists of pairwise distinct atoms. If* $\mathscr{S}$ *is equivalent to* BV *and all the structural relations in* α_{n-1}*-substructures of* P *are preserved by* ρ *then all the structural relations in* P *are also preserved by* ρ.

Proof Let us recall the definition of α_n-structure:

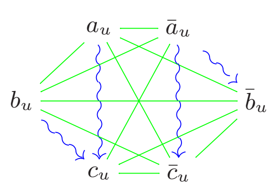
$$P = \alpha_n(u, R, T) = [\langle \alpha_{n-1}(u.0, a_u, [b_u, R]); c_u \rangle, \langle \bar{a}_u; \alpha_{n-1}(u.1, \bar{b}_u, [\bar{c}_u, T]) \rangle].$$

All the structural relations in $\alpha_{n-1}(u.0, a_u, [b_u, R])$ (likewise, $\alpha_{n-1}(u.1, \bar{b}_u, [\bar{c}_u, T])$) are preserved by ρ.
Let the variables x, y stand for atoms in $\mathsf{occ}\, P$. The following case analyses show that the structural relation between x and y, for all possible values of x and y, must be preserved by ρ as a consequence of the structural properties of the underlying traces of $S\{P\}$ and U.

1. $x, y \in \mathsf{occ}\, \alpha_{n-1}(u.0, a_u, [b_u, R])$, or $x, y \in \mathsf{occ}\, \alpha_{n-1}(u.1, \bar{b}_u, [\bar{c}_u, T])$, follows immediately from the condition of this lemma.

2. $x \in \mathsf{occ}\, \alpha_{n-1}(u.0, a_u, [b_u, R])$, $y = c_u$. It holds in P: $x \lhd y$.

   a. $x \in \{a_u, b_u\}$. The structural relations between atoms in $\{a_u, b_u, c_u, \bar{a}_u, \bar{b}_u, \bar{c}_u\}$ is

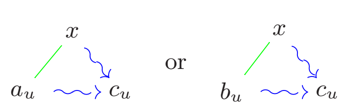


   for some structure contexts $Q_1\{\ \}$ and $Q_2\{\ \}$. By Lemma 6.7, this configuration is preserved by ρ.

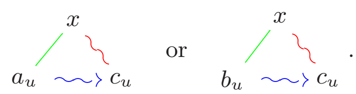
   b. $x \notin \{a_u, b_u\}$. Then $x \downarrow_P a_u$ or $x \downarrow_P b_u$, because

$$\alpha_{n-1}(u.0, a_u, [b_u, R]) = [Q_1\{a_u\}, Q_2\{b_u\}].$$

   Depending on whether $x \downarrow_P b_u$ or $x \downarrow_P c_u$, one of the following configurations must hold in P:



The relations between x and $\{a_u, b_u\}$ are preserved by ρ, so it holds in U: $x \downarrow_U a_u$ or $x \downarrow_U b_u$. The structural relations $a_u \lhd_U c_u$ and $b_u \lhd_U c_u$ must hold as shown in Case 2.a. The structural relation $x \lhd_U c_u$ must hold in U, because otherwise $x \uparrow_U c_u$ by Lemma 5.8 and one of the following configuration would occur in U, depending on whether $x \downarrow_U a_u$ or $x \downarrow_U b_u$.

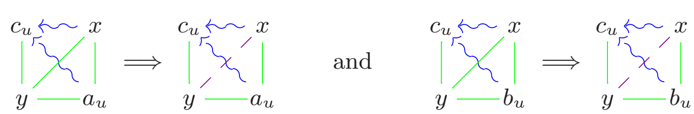


3. $x \in \mathsf{occ}\,\alpha_{n-1}(u.1, \bar{b}_u, [\bar{c}_u, T])$, $y = \bar{a}_u$. It holds in P: $y \lhd_P x$.
   a. $x \in \{\bar{b}_u, \bar{c}_u\}$, see Case 2.a.
   b. $x \notin \{\bar{b}_u, \bar{c}_u\}$, then $x \downarrow_U \bar{b}_u$ or $x \downarrow_U \bar{c}_u$. Using a similar argument as in Case 2.b, we establish that the structural relation $\bar{a}_u \lhd_U x$ must hold, because otherwise the following forbidden configurations would occur in U:

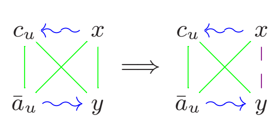
   which correspond to the cases $x \downarrow_U \bar{b}_u$ and $x \downarrow_U \bar{c}_u$, respectively.

4. $x \in \mathsf{occ}\langle \mathsf{occ}\,\alpha_{n-1}(u.0, a_u, [b_u, R]); c_u\rangle$, $y \in \mathsf{occ}\langle \bar{a}_u; \mathsf{occ}\,\alpha_{n-1}(u.1, \bar{b}_u, [\bar{c}_u, T])\rangle$.
   a. $x \in \{a_u, b_u, c_u\}$, $y \in \{\bar{a}_u, \bar{b}_u, \bar{c}_u\}$, see Case 2.a.
   b. $x \in \{a_u, b_u, c_u\}$, $y \notin \{\bar{a}_u, \bar{b}_u, \bar{c}_u\}$. Then it holds in U: $y \downarrow_U \bar{b}_u$ or $y \downarrow_U \bar{c}_u$. The strutural relations between atoms in $\{x, \bar{a}_u, \bar{b}_u, \bar{c}_u\}$ are preserved by ρ, as shown in Case 2.a, and the structural relations between atoms in $\{y, \bar{a}_u, \bar{b}_u, \bar{c}_u\}$ are also preserved by ρ, as shown in Case 1 and Case 3. Therefore, if we change the relation $x \downarrow_P y$ to $\neg(x \downarrow_U y)$, the square property will be violated, as shown in the following diagrams:

   which correspond to the cases $y \downarrow_U \bar{b}_u$ and $y \downarrow_U \bar{c}_u$, respectively.
   c. $x \notin \{a_u, b_u, c_u\}$, $y \in \{\bar{a}_u, \bar{b}_u, \bar{c}_u\}$. Then it holds in U: $x \downarrow_U a_u$ or $x \downarrow_U b_u$. The structural relations between atoms in $\{x, a_u, b_u, c_u\}$ must be preserved by ρ, as shown in Case 1 and 2, and the structural relations between atoms in $\{y, a_u, b_u, c_u\}$ must also be preserved as shown in Case 2.a. The changing of $x \downarrow_P y$ to $\neg(x \downarrow_U y)$ leads to the violation of square property as shown in the following diagrams:



   which correspond to the cases $x \downarrow_U a_u$ and $x \downarrow_U b_u$, respectively.
   d. $x \notin \{a_u, b_u, c_u\}$, $y \notin \{\bar{a}_u, \bar{b}_u, \bar{c}_u\}$. The structural relations $\bar{a}_u \downarrow_P c_u$, $x \lhd_P c_u$, $\bar{a}_u \lhd_P y$, $c_u \downarrow_P y$ and $\bar{a}_u \downarrow_P x$ are preserved by ρ as shown in Case 3, 4.a, 4.b and 4.c. Therefore, if the relation $x \downarrow_P y$ is changed to $\neg(x \downarrow_U y)$, the square property would be violated.



□

6.9 Lemma *Let* $\rho \dfrac{\overset{\Vert \mathscr{S}}{U}}{S\{P\}}$ *be a proof in a shallow system* $\mathscr{S}$*, where* P *is an* α_n*-struture and* $\rho \neq \mathsf{i}{\downarrow}$*. If* $\mathscr{S}$ *is equivalent to* BV *and all the structural relations in every* α_0*-substructure of* P *are preserved by* ρ*, then all the structural relations in* P *are also preserved by* ρ*.*

Proof The case where $P = \alpha_0(u, R, T)$ is obvious. So suppose that

$$P = \alpha_n(u, R, T) = [\langle \alpha_{n-1}(u.0, a_u, [b_u, R]); c_u\rangle, \langle \bar{a}_u; \alpha_{n-1}(u.1, \bar{b}_u, [\bar{c}_u, T])\rangle]$$

By inductive hypothesis, the structural relations in α_{n-1} substructures of P are preserved by ρ. This means, by Lemma 6.8, that all the structural relations in P must also be preserved by ρ. □

6.10 Theorem *No shallow system can be equivalent to* BV.

Proof Suppose $\mathscr{S}$ is a n-deep system equivalent to BV. Consider the structure S_{n+1}. By Theorem 6.4 S_{n+1} is provable in BV. Since $\mathscr{S}$ is equivalent to BV, there must exist also a proof

$$\rho\,\frac{\begin{array}{c}\Vert\,\mathscr{S}\\ T\end{array}}{S_{n+1}}\ .$$

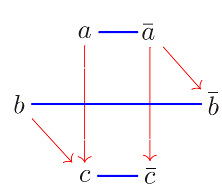
We can assume without loss of generality that the instance of ρ above is non-trivial, i.e., $T \neq S_{n+1}$.
The rule ρ cannot be an interaction rule since by Proposition 6.5 there are no substructures of the form $[R, \overline{R}]$ in S_{n+1}. Therefore it must be a shallow rule a depth of at most n. From Proposition 6.6 all the α_0-substructures of S_{n+1} are $2(n+1)$-deep, so all the structural relations in any α_0-substructures of S_{n+1} must be preserved by ρ. But by Lemma 6.9, this implies that all the structural relations in S_{n+1} must also be preserved by ρ, that is, $T = S_{n+1}$, contrary to the assumption that the application of ρ is non-trivial. Therefore, S_{n+1} is not provable in $\mathscr{S}$ and consequently, $\mathscr{S}$ cannot be equivalent to BV. □

7 Conclusions and Open Problems

The proof of the deep-nesting property of System BV we have just seen essentially makes use of the property of the self-dual sequential operator. The proof shows that finding a sequent system for BV is impossible, under the definition of sequent systems as shallow systems. The readers may question certain restrictions put in the definition of shallow systems in Section 5, for example the exclusion of contraction and weakening. In fact, the proof of deep-nesting scales also to the case where contraction and weakening are allowed, but still under control. Take for example the one-sided sequent presentation of MELL, where the contraction and weakening are only applicable in the presence of the ?-connective. If we want to extend MELL with a self-dual non-commutative operator, we certainly want to do it without destroying cut-elimination, and hence the separation property. Contraction and weakening are thus not aplicable to the structures in the counterexample shown in Section 6.

The trace representation is highly combinatoric in nature. The proof of deep-nesting based on this representation involves a lot of case analyses. This complication is probably unnecessary if we are able to show that SBV is *implicationally complete*. That is, given a provable structure $[R, \overline{T}]$ which corresponds to the implication $T \Rightarrow R$, there is a derivation $\begin{array}{c}T\\ \Vert\,\mathsf{SBV}-\{\mathsf{a}\uparrow\}\\ R\end{array}$. If this is true, any shallow rule would be strongly admissible for $\{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\}$. Then it is enough to prove the deep nesting property for $\{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow\}$, which is considerably easier compared to the proof using the trace.

System BV is not the only logical system that incorporates both commutativity and non-commutativity. Retoré's Pomset logic [8] achieves the same thing. However, Pomset logic is based on proof-nets, and its sequentialisation is still an open question. It will be interesting to see the connection between Pomset logic and BV. There are reasons to believe that they are indeed the same. At the language level, for example, there is a striking correspondence between the trace representation of structures and Retoré's proof-nets. He uses a new presentation of proof-nets, called R&B-directed cographs. They are basically traces, with additional axiom links between dual atoms. The structure S_0 defined in Section 6, for example, corresponds to the proof-net:



where the par links are removed, except for the ones connecting dual atoms. One possible way to establish the connection would be to apply the correctness criteria used in proof-nets to characterise provable structures in BV. If this is true, then it would be impossible to find a sequent system for Pomset Logic.

This result on the deep-nesting property of BV goes along with two other ongoing researches on presenting known logical systems in the calculus of structures. In [1], it is shown that the classical logic enjoys a local presentation in the calculus of structures, which seems impossible to achieve in sequent calculus. And in [6], the global promotion rule in the sequent presentation of MELL becomes local in its presentation in the calculus of structures. These provide a strong indication that the calculus of structures is a richer proof-theoretical framework compared to the sequent calculus, for logics with De Morgan rules.

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Statement of Academic Honesty

I hereby declare that I have not used any auxiliary means for my thesis work other than that what has been cited in my thesis.

Dresden, 20 July 2001

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