Private Announcements on Topological Spaces

Hans van Ditmarsch[1], Sophia Knight[2] and Aybüke Özgün[1,3]

[1]LORIA, CNRS - Université de Lorraine, France
[2]Uppsala University, Sweden
[3]ILLC, University of Amsterdam, the Netherlands

Abstract

In this work, we present a multi-agent logic of knowledge and change of knowledge interpreted on topological structures. Our dynamics are of the so-called semi-private character where a group G of agents is informed of some piece of information φ, while all the other agents observe that group G is informed, but are uncertain whether the information provided is φ or $\neg\varphi$. This article follows up on our prior work [30] where the dynamics were public events. We provide a complete axiomatization of our logic, and provide three detailed examples of situations with agents learning information through semi-private announcements.

1 Introduction

This work follows the tradition of modeling (dynamic) epistemic logics on spatial, rather than relational, structures, such as neighbourhood frames [36, 23], subset spaces [11, 12, 3, 35, 18, 19, 34], and more importantly topological spaces [13, 30, 29, 4, 6]. Unlike the rather standard approaches to modelling knowledge and information dynamics using relational semantics (see, e.g., [31, 28, 10] for a survey on this topic), our work acknowledges the observation based nature of these notions that demands richer, neighbourhood-like structures. In this work, we propose a multi-agent topological semantics for knowledge and semi-private announcements in the style of subset space semantics equipped with neighbourhood functions. While knowledge is entailed by the agents' current observation sets (roughly speaking, represented by some opens of the given topological model), the precondition of an announcement is captured by means of the topological interior operator that refers to the existence of a possible observation set entailing the announcement (we formalize these notions in Section 3). Therefore, we do not take the precondition of an announcement only that the announced formula be true, but that it be *'observable'*, as in [13]. It is this observational aspect of our work that makes it different than most approaches to knowledge and semi-private announcements based on standard relational semantics.

An initial investigation by Moss et al. [24] presented a single-agent subset space logic (SSL) for the notions of *knowledge* and *effort*. One of the crucial aspects of their proposal was that it was not only concerned with the representation of knowledge, but also aimed to give an account of information gain or knowledge increase in terms of *observational effort* ([24] is partly inspired by Vickers' work [32] on reconstruction of topology via a logic of finite observation). It is this latter feature of their work that makes the use of subset spaces significant. While the knowledge modality $K\varphi$ has the standard reading "the agent knows φ (is true)," in their setting, the effort modality $\Box\varphi$ captures a notion of effort as any action that results in an increase in knowledge and is read as "after (any) effort φ is true". Effort can be in the form of measurement, computation, approximation or even announcement, depending on the context and the information source. However, one obvious component of it is considered to be *observation* [24]. This point becomes clearer when we have a closer look at the semantic aspect of their setting. They evaluate the formulas in their bimodal language on *subset spaces* $(X, \mathcal{O})$, where X is a non-empty domain and $\mathcal{O}$ is a set of subsets of X. A subset space is not necessarily a topological space, but topological spaces constitute a particular case of subset spaces. The elements of $\mathcal{O}$ are taken to be *possible observations* or *possible observation sets*, and the formulas are interpreted not only with respect to the actual state, but also with respect to a (truthful) observation set. The unit of evaluation is a pair (x, U), where the point x represents the true state of affairs, and the set U represents all the points the agent considers possible, i.e., her epistemic range. The pairs of the form (x, U) are well-defined only if $x \in U \in \mathcal{O}$, meaning that U is a *truthful* observation set at x. According to subset space semantics, given a pair (x, U), the knowledge modality K quantifies over the elements of U ($K\varphi$ is true in (x, U), if φ is true in (y, U) for all $y \in U$), whereas the effort modality $\Box$ quantifies over all open subsets of U that include x ($\Box\varphi$ is true in (x, U), if φ is true in (x, V) for all $V \in \mathcal{O}$ with $x \in V \subseteq U$). More precisely, $\Box\varphi$ being true in (x, U) means that φ is true with respect to the actual state x and any further refinements of the current observation set U.

The epistemic motivation of subset space semantics and the dynamic nature of the effort modality clearly suggests a link with dynamic epistemic logic (DEL). The question of understanding the relation between some of the well-known dynamic modalities studied in the DEL literature, such as the public and arbitrary public announcement modalities, has recently received a considerable amount of attention. In spite of the obvious intuitive connections between SSL and the aforementioned informational attitudes studied in DEL, connecting SSL to dynamic epistemic logic is not entirely straightforward, even for a relatively simple case like public announcements. Connections between single-agent public announcement logic and SSL were made in [11, 12, 3, 35, 13]. Wang and Ågotnes [35] were the first to propose semantics for public announcements on subset spaces in terms of open set refinement rather than model restriction. Bjorndahl [13] then proposed a revision of the semantics in [35], based on topological spaces, and with an interior modality $int(\varphi)$ capturing the precondition of an announcement, which is associated with the interior operator of topological spaces. In previous work [29], we further extended the proposal in [13] with the arbitrary announcement modality capturing information change caused by *any announcement* (rather than any effort) and studied a particular type of 'effort' that is the shape of public announcements. While the standard requirement for a truthful public announcement in DEL literature is only that it be true, the precondition $int(\varphi)$ is stronger than φ simply being true (see also [13]) and it states that "φ is supported by truthful observation". In a framework where knowledge is based on truthful observations the agent possesses (such as the subset space setting), this precondition for the announcements seems to be the right notion to consider and we thus find this reading to be a good fit with the intuition behind the subset space/topological semantics and the observation-based dynamics we study in this paper: for an announcement to be successfully implemented, it is not sufficient that the announced formula be true, but there has to be a truthful observation set available to the agent, i.e., an open neighbourhood of the actual state, that entails the proposition in question. This has been explained in great detail in [13] with several examples and we adopt one of these examples to motivate and explain our semantics in later sections. Although using topological spaces rather than subset spaces restricts the class of models we work with, topological spaces come with more structure, especially on the observation sets. Topological spaces and some natural topological operators, such as the interior operator, come with intrinsic value that helps us model information change based on observation.

An independent issue, in a way a next step, was the generalization of this framework to a multi-agent setting. (It was a partially independent development, not necessarily related to dynamics.) Multi-agent subset space logics and topological logics were presented in [18, 19, 11, 34, 30]. The multi-agent case requires solving the complication of 'jumping out of the epistemic range'. We generalize our knowledge operator, so we now have formulas $K_i\varphi$, for 'agent i knows φ". For example, consider two agents i and j, each having an associated observation set such that the semantic primitive becomes a triple (x, U_i, U_j) instead of a pair (x, U). Now consider a formula like $K_i\hat{K}_jK_ip$, for 'agent i knows that agent j considers possible that agent i knows p'. If this is true for a triple (x, U_i, U_j), then $\hat{K}_jK_ip$ must be true for any $y \in U_i$; but y may not be in U_j, in which case (y, U_i, U_j) is not well-defined. It seems unclear how to interpret $\hat{K}_jK_ip$. This dilemma can be solved in different ways. In [34] it is solved with subsets containing partitions of the entire space, and in [30] by considering neighbourhoods that are not only relative to each agent, as usual in multi-agent subset space logics, but that are also relative to each state. When shifting the viewpoint from x to $y \in U_i$, in (x, U_i, U_j), this amounts to simultaneously shifting the neighbourhood (and not merely the point in the actual neighbourhood) for the other agent. So we then go from (x, U_i, U_j) to (y, U_i, V_j), where V_j may be different from U_j. Such a logic, including arbitrary announcement modalities that capture a particular type of effort, was axiomatized in [30]. The axiomatization is infinitary and completeness non-trivial, involving a somewhat tricky induction-cum-subinduction and a carefully customized complexity measure on formulas (motivated by [1, 2]). Although the setting was multi-agent, the dynamics was that of publicly observable events.

A next step in this story is non-public dynamics: actions such as announcements that are observed by some agents but not by other agents, or that are only partially observed by other agents. In dynamic epistemic logics, the first step away from public announcement is private announcement [17]. In this work, we present a multi-agent logic of knowledge and change of knowledge (with precondition semantically defined as the interior operator with respect to the information announced), interpreted on topological structures, where for the dynamic part we not only consider public events but also private events. As this is the topic of our contribution, we will discuss the issue of private announcements in somewhat greater detail with some discrete examples in this section.

Consider two agents 1 and 2 who are uncertain about the value of a proposition p, and such that this ignorance is known to them. Therefore, at this initial stage both agents are ignorant about p: $\neg K_1p$ and $\neg K_2p$ hold. After public announcement of p both agents know that p: we now have that K_1p and K_2p. Whereas the result of a *completely* private announcement to agent 1 that p should be that K_1p and $\neg K_2p$. The question now is what 1 and 2 know about each other at this stage. A main issue is what 2 notices about 1 being informed. In case that is nothing at all, then we can no longer describe the higher-order epistemic features of the agents in terms of knowledge. For example, initially 2 knows that 1 does not know p. But after 1 has become secretly informed, this 'knowledge' has become incorrect belief. In a system where the modality just means any epistemic stance, there is nothing wrong with this, but in a topological setting for knowledge, this is problematic. We want to preserve agents' *knowledge* of propositions and not merely their *belief* of them. This problem is not restricted to topological reasoning but was already considered before in the literature [17, 31]. A standard solution is to model the events that constitute private announcements as so-called *semi-public* or *semi-private announcements*. If 1 is semi-privately informed of p, we assume that 2 is aware that some action is occurring, but is merely uncertain which action exactly. As long as there is an infinity of actions available, this does not facilitate modelling the observational action greatly. But in many scenarios the uncertainty about what the privately informed agent is observing is restricted to the choice between a discrete number of well-described alternatives: the envelope agent 1 is opening contains p or contains $\neg p$, while agent 2 is seeing that 1 opens the envelope but cannot read the letter; the card agent 1 is picking up can be Hearts, Clubs, Spades, or Diamonds, but agent 2, another player in this card game, cannot see which card it is (and can only see the back of the card). After the semi-private announcement to 1 that p, where 2 is uncertain between p and $\neg p$, it holds that: 1 knows that p, and 2 knows that 1 knows *whether* p: $K_1 p$, and $K_2(K_1 p \vee K_1 \neg p)$. For the reader who is not familiar with these notions, we refer to Appendix A for examples on Kripke models, and [17, 27] for a more detailed discussion.

Rather than a logic of semi-private announcements to *individual* agents we present a more general logic of semi-private announcements to *groups* of agents, since public announcements and semi-private announcements to individuals are both particular cases of this logic. There are also a number of technical features of interest in our setup. One is related to the previous: instead of a *model restriction* semantics of announcements we implement a *neighbourhood refinement* semantics of announcements. Further, as the composition of two semi-private announcements to different agents is not a semi-private announcement (namely to whom?), we cannot employ a 'standard public announcement style' axiomatization with a composition axiom for announcements. Instead, we employ an alternative (equivalent) axiomatization, mentioned in [33], that has a derivation rule of substitution of equivalents. Another novelty is that the semi-private instead of public announcements necessitate a slightly different complexity measure in the completeness proof than in other works employing that style of proof [30, 2].

We proceed with an overview of the paper. In Section 2, we introduce the topological notions used throughout the paper. Section 3 starts with two examples motivating our framework and then provides the syntax and semantics for our multi-agent logic $\mathbf{sPAL}_{int}$ of knowledge and semi-private announcements. At the end of this section, we go back to the initial examples and illustrate the interpretation of the modal operators of our syntax. We finalize Section 3 with a rather abstract example based on real and irrational numbers. Section 4 includes the technical results of the paper where we give the axiomatization of the logic $\mathbf{sPAL}_{int}$ and present its soundness and completeness with respect to our proposed semantics. We then mention some further results in Section 5 and conclude.

2 Background on Topology

In this section, we introduce the topological concepts that will be used throughout this paper. All the topological notions presented in this section and a more thorough introduction to topology can be found in [14, 15].

Definition 1 (Topological Space). *A* topological space (X, τ) *is a pair consisting of a non-empty set X and a family τ of subsets of X satisfying $\emptyset \in \tau$ and $X \in \tau$, and closed under finite intersections and arbitrary unions.*

The set X is called the *space*. The subsets of X belonging to τ are called *open sets* (or *opens*) in the space; the family τ of open subsets of X is also called a *topology* on X. If for some $x \in X$ and an open $U \subseteq X$ we have $x \in U$, we say that U is an *open neighborhood* of x. Complements of opens are called *closed sets*.

A point x is called an *interior point* of a set $A \subseteq X$ if there is an open neighborhood U of x such that $U \subseteq A$. The set of all interior points of A is called the *interior* of A and denoted by $Int(A)$. We can then easily observe that for any $A \subseteq X$, $Int(A)$ is an open set and is indeed the largest open subset of A. Dually, $Cl(A)$ denotes the *closure* of A and it is the smallest closed set containing A. More precisely, $x \in Cl(A)$ iff for every open neighbourhood U of x, $U \cap A \neq \emptyset$.

Finally, for any subset $A \subseteq X$, we define the *boundary* of A, denoted by $Bd(A)$, to be the set $Bd(A) = Cl(A) \cap Cl(X \setminus A)$.

Proposition 2. *For any topological space (X, τ) and $A \subseteq X$, the point x belongs to $Bd(A)$ if and only if for every open neighbourhood U of x we have $U \cap A \neq \emptyset \neq U \setminus A$ (or, equivalently, $U \not\subseteq A$ and $U \not\subseteq X \setminus A$).*

Proof. See [15, Proposition 1.3.3]. □

Given a topological space (X, τ) and a non-empty set $Y \subseteq X$, a space (Y, τ_Y) is called a *subspace* of (X, τ) (induced by Y) where $\tau_Y = \{U \cap Y : U \in \tau\}$.

Definition 3 (Base)**.** *A family $\mathcal{B} \subseteq \tau$ is called a* base *for a topological space (X, τ) if every non-empty open subset of X can be written as a union of elements of $\mathcal{B}$.*

We can also give an equivalent definition of an interior point by referring only to a base $\mathcal{B}$ for a topological space (X, τ): for any $A \subseteq X$, $x \in Int(A)$ if and only if there is an open set $U \in \mathcal{B}$ such that $x \in U$ and $U \subseteq A$.

Given any family $\Sigma = \{A_i \mid i \in I\}$ of subsets of X, there exists a unique, smallest topology $\tau(\Sigma)$ with $\Sigma \subseteq \tau(\Sigma)$ [14, Theorem 3.1]. The family $\tau(\Sigma)$ consists of $\emptyset$, X, all finite intersections of the A_i, and all arbitrary unions of these finite intersections. Σ is called a *subbase* for $\tau(\Sigma)$, and $\tau(\Sigma)$ is said to be *generated* by Σ. The set of finite intersections of members of Σ forms a base for $\tau(\Sigma)$.

3 The Topological Logic of Semi-Private Announcements

3.1 Motivation

Dynamics of information change and higher-order knowledge becomes much more interesting when more than one agent is involved. In this section, we motivate our setting by two examples demonstrating different situations. We start with a rather simple example of a discrete nature that is adopted from [13] and modified to the multi-agent setting in [30]. The second example (also inspired by an example in [13]) concerns pairs of infinite binary strings (to represent a real number pair in the unit square $[0, 1] \times [0, 1]$). A suitable topology can be defined on the set of such pairs, and we investigate how different agents that are uncertain about one or both strings can be informed about individual digits in these binary strings; i.e., *semi-privately* informed, without the other agent getting the information but with the other agent knowing that the opponent is informed.

Example 4 (The Jewel and the Tomb-Revisited with two agents)**.** *Indiana Jones (i) and Emile Belloq (e) are both scouring for a priceless jewel placed in a tomb. The tomb could either contain a jewel (J) or not ($\neg J$), the tomb could have been rediscovered in modern times (D) or not ($\neg D$), and the tomb could be in the Valley of Tombs in Egypt (T) or not ($\neg T$). The propositional variables corresponding to these propositions are, respectively, J, D, and T. We represent a valuation of these variables by a triple xyz, where $x, y, z \in \{0, 1\}$. This scenario with the given relevant alternatives can be represented in a 8-state model with the domain $X = \{xyz \mid x, y, z \in \{0, 1\}\}$ and the topology τ that we consider is generated by the base consisting of the subsets $\{000, 100, 001, 101\}$, $\{010\}$, $\{110\}$, $\{011\}$, $\{111\}$. The idea is that one can only conceivably know (or learn) about the jewel or the location, on the condition that the tomb has been discovered (as in [13]). Therefore, $\{000, 100, 001, 101\}$ has no strict subsets besides empty set: if the tomb has not yet been discovered, no one can observe the jewel or the location. Moreover, in this example,* *we stipulate that the actual state is* 111 *stating that the tomb contains a jewel, the tomb has been rediscovered in modern times and it is in the Valley of Tombs in Egypt. We are then interested in designing a topological framework that could answer the following questions: (1) if both Indiana and Emile are initially ignorant about the jewel and the tomb, what facts would they come to know if Emile receives some further information? (2) Would they end up in a similar epistemic state? (3) What would they come to know about each other?*

The next example concerns the transmission of partial information about a probe's location, and different agents' perspective on this information.

Example 5 (The probe in the unit square)**.** *A group of scientists wants to launch a probe on a certain field to collect evidence for an experiment. They target a designated point in the field, but previous measurements show that the probe launches within a small square-shaped error range due to external reasons such as the weather conditions, possible mechanical problems etc. (see Figure 1).*

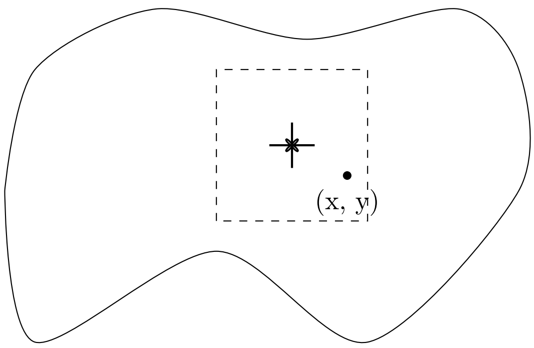

Figure 1: The arrows are pointing to the target point. Dashed square represents the error range and (x, y) denotes the real location of the probe.

They thus design a feedback mechanism from the probe to the source by encoding the coordinates as infinite binary strings. The probe can measure its location in a step by step manner and encodes it as pairs of binary strings. It transmits this information to two sources, a_x *and* a_y*, receiving the coding of coordinate-x and the coding of coordinate-y only, respectively. Since the precise location is described by a pair of infinite binary strings, observing the exact location of the probe corresponds to observing the entire infinite pair. Since the agents are finite beings, they can do this only for a finite amount of time. Moreover, for the same reason, the probe can measure at most a string of finite length, so they can only approximate the location of the probe: the exact location is not observable.*

This situation can be modelled on a topological space based on the set $\{0,1\}^\infty \times \{0,1\}^\infty$ *of the ordered pairs of infinite binary strings. Since the agents can learn the infinite binary sequence encoding the location of the probe up to a finite length* n*, the topology (representing what agents can in principle observe) will be generated by subsets of* X *whose every element has the same prefix up to length* n*, for all* $n \in \mathbb{N}$*.*

These two example will be reconsidered in Section 3.2 to their full (technical) extent and used to illustrate our proposed topological semantics for knowledge and semi-private announcements.

3.2 Syntax and Semantics

In this section, we introduce the syntax and semantics for the multi-agent logic of knowledge and semi-private announcements. This logic is a generalization of the public announcement logic introduced in [30] in the sense that it formalizes not only the information change within a group of agents when the new information is synchronously received by all the members of the group, but it also captures the information change when the new information is accessible to only a subset of the group. Here we do not consider completely private announcements and rather model semi-private announcements on topological spaces. Throughout the rest of this paper, we use the phrases semi-private announcements and private announcements interchangeably.

We let *Prop* denote a countable set of propositional variables and $\mathcal{A}$ a finite non-empty set of agents.

Definition 6 (Language)**.** *The language $\mathcal{L}$ is defined by*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \; int(\varphi) \mid K_i\varphi \mid [\varphi]_G\varphi$$

where $p \in Prop$, $i \in \mathcal{A}$ and $G \subseteq \mathcal{A}$. Abbreviations for the connectives $\vee$, $\rightarrow$ and $\leftrightarrow$ are standard, and $\bot$ is defined as an abbreviation of $p \wedge \neg p$. We employ $\hat{K}_i$ for $\neg K_i \neg\varphi$, and $\langle\varphi\rangle_G\psi$ for $\neg[\varphi]_G\neg\psi$. We denote the non-modal part of $\mathcal{L}$ (without the modalities K_i, int, $[\varphi]_G$) by $\mathcal{L}_{Pl}$, the part without $[\varphi]_G$ by $\mathcal{L}_{EL_{int}}$. We denote $[\varphi]_{\{i\}}\psi$ by $[\varphi]_i\psi$, and $[\varphi]_{\mathcal{A}}\psi$ by $[\varphi]\psi$ and the latter corresponds to the public announcement operator. We denote the extension of $\mathcal{L}_{EL_{int}}$ only with the public announcement modality $[\varphi]\psi$ by $\mathcal{L}_{PAL_{int}}$.

While the knowledge and semi-private announcement modalities K_i and $[\varphi]_G\psi$ are standard, the modality *int* intends to capture the precondition for an announcement in our setting. We read $int(\varphi)$ as "*φ is announceable*", where announceable is interpreted as being supported/entailed by a truthful observation set. This modality, as suggested by its notation, is interpreted as the topological interior operator and plays a crucial role in the formalization of the observation-based information dynamics we study in this paper. It is important to note that $\mathcal{L}_{EL_{int}}$ is the multi-agent extension of the single-agent epistemic language with the interior modality introduced in [13] and $\mathcal{L}_{PAL_{int}}$ is its extension with the public announcement modalities. The topological semantics for the corresponding single-agent languages has been investigated in [13], which was later extended to a multi-agent setting in [30] where only public events were considered.

We interpret the above language $\mathcal{L}$ of knowledge and semi-private announcements on topological spaces endowed with (partial) neighbourhood functions that assign an open neighbourhood for each agent $i \in \mathcal{A}$ at a given state x. The semantics introduced in this paper is similar to the topological semantics for the logic of public announcements proposed in [30]. In this paper, however, the neighbourhood functions are general enough to interpret semi-private announcements. We therefore generalize the approach in [30].

Definition 7 (Neighbourhood Function)**.** *Given a topological space (X, τ), a* neighbourhood function set *Φ on (X, τ) is a set of (partial)* neighbourhood functions *$\theta : X \rightharpoonup \mathcal{A} \to \tau$ such that for all $x, y \in \mathcal{D}(\theta)$, for all $i \in \mathcal{A}$, $G \subseteq \mathcal{A}$ and $Y \subseteq \mathcal{D}(\theta)$:*

1. $x \in \theta(x)(i)$,
2. $\theta(x)(i) \subseteq \mathcal{D}(\theta)$,
3. *if* $y \in \theta(x)(i)$ *then* $\theta(x)(i) = \theta(y)(i)$,
4. $\theta^Y_G \in \Phi$,

where $\mathcal{D}(\theta)$ is the domain of θ, θ_G^Y is the restricted neighbourhood function with $\mathcal{D}(\theta_G^Y) = Int(Y) \cup Int(\mathcal{D}(\theta) \setminus Y)$ and

$$\theta_G^Y(x)(j) = \begin{cases} \theta(x)(j) \cap \mathcal{D}(\theta_G^Y) & \text{for } \mathcal{D}(\theta_G^Y) \text{ and } j \notin G \\ \theta(x)(j) \cap Int(Y) & \text{for } x \in Int(Y) \text{ and } j \in G \\ \theta(x)(j) \cap Int(\mathcal{D}(\theta) \setminus Y) & \text{for } x \in Int(\mathcal{D}(\theta) \setminus Y) \text{ and } j \in G \end{cases}$$

The main role for the neighbour functions θ is to assign a *truthful* observation set to a given state for each agent. It simply defines the current observation set of each agent at the state in question. Each condition given in Definition 7 guarantees certain requirements that render the semantics well-defined and meaningful for the language $\mathcal{L}$. In particular, by the help of the neighbourhood functions, we also solve the problem of 'jumping out of the epistemic range' explained in the Introduction. We will provide a more detailed explanation regarding the definition of the neighbourhood functions together with our proposed semantics given in Definition 9.

Definition 8 (Topological Model)**.** *A* multi-agent topological model *(*topo-model*) is a tuple $\mathcal{M} = (X, \tau, \Phi, V)$, where (X, τ) is a topological space, Φ a neighbourhood function set, and $V : Prop \to X$ a valuation function. The part $\mathcal{X} = (X, \tau, \Phi)$ is a* multi-agent topological frame *(*topo-frame*).*

Given a topo-model $\mathcal{M} = (X, \tau, \Phi, V)$ (or a topo-frame $\mathcal{X} = (X, \tau, \Phi)$), τ is considered to be the set of observation sets that are 'potentially' available for all the agents and, following the intuition behind the subset space semantics [24], we refer to the opens as *possible observation sets.* A pair (x, θ) is a *neighbourhood situation* if $x \in \mathcal{D}(\theta)$, and $\theta(x)(i)$ is an *epistemic neighbourhood at x of agent i*. An epistemic neighbourhood $\theta(x)(i)$ represents the *actual, current* observation set of the agent i at x and it is her only source of knowledge at state x with respect to the neighbourhood situation (x, θ) (see Definition 9 below). If (x, θ) is a neighbourhood situation in $\mathcal{M}$ we write $(x, \theta) \in \mathcal{M}$. Similarly, if (x, θ) is a neighbourhood situation in $\mathcal{X}$ we write $(x, \theta) \in \mathcal{X}$. For any $(x, \theta) \in \mathcal{M}$, we call $\mathcal{M}, (x, \theta)$ a *pointed model.*

Definition 9 (Semantics for $\mathcal{L}$)**.** *Given a topo-model $\mathcal{M} = (X, \tau, \Phi, V)$ and a neighbourhood situation $(x, \theta) \in \mathcal{M}$, the semantics for the language $\mathcal{L}$ is defined recursively as:*

$$\begin{array}{lll} \mathcal{M}, (x, \theta) \models p & \text{iff} & x \in V(p) \\ \mathcal{M}, (x, \theta) \models \neg\varphi & \text{iff} & \text{not } \mathcal{M}, (x, \theta) \models \varphi \\ \mathcal{M}, (x, \theta) \models \varphi \wedge \psi & \text{iff} & \mathcal{M}, (x, \theta) \models \varphi \text{ and } \mathcal{M}, (x, \theta) \models \psi \\ \mathcal{M}, (x, \theta) \models K_i\varphi & \text{iff} & (\forall y \in \theta(x)(i))(\mathcal{M}, (y, \theta) \models \varphi) \\ \mathcal{M}, (x, \theta) \models int(\varphi) & \text{iff} & x \in Int([\![\varphi]\!]^\theta) \\ \mathcal{M}, (x, \theta) \models [\varphi]_G\psi & \text{iff} & \mathcal{M}, (x, \theta) \models int(\varphi) \text{ implies } \mathcal{M}, (x, \theta_G^\varphi) \models \psi \end{array}$$

where $p \in Prop$, $[\![\varphi]\!]^\theta = \{y \in \mathcal{D}(\theta) \mid \mathcal{M}, (y, \theta) \models \varphi\}$ and $\theta_G^\varphi = \theta_G^{[\![\varphi]\!]^\theta}$.

More precisely, $\theta_G^\varphi : X \rightharpoonup \mathcal{A} \to \tau$ is defined such that

$$\mathcal{D}(\theta_G^\varphi) = Int([\![\varphi]\!]^\theta) \cup Int(\mathcal{D}(\theta) \setminus [\![\varphi]\!]^\theta)$$

and

$$\theta_G^\varphi(x)(j) = \begin{cases} \theta(x)(j) \cap \mathcal{D}(\theta_G^\varphi) & \text{for } x \in \mathcal{D}(\theta_G^\varphi) \text{ and } j \notin G \\ \theta(x)(j) \cap Int([\![\varphi]\!]^\theta) & \text{for } x \in Int([\![\varphi]\!]^\theta) \text{ and } j \in G \\ \theta(x)(j) \cap Int(\mathcal{D}(\theta) \setminus [\![\varphi]\!]^\theta) & \text{for } x \in Int(\mathcal{D}(\theta) \setminus [\![\varphi]\!]^\theta) \text{ and } j \in G \end{cases}$$

It is not hard to see that the domain of an updated function θ^{φ}_G does not depend on G, it depends only on the initial function θ and the proposition φ. We thus write $\mathcal{D}(\theta^{\varphi})$ for $\mathcal{D}(\theta^{\varphi}_G)$ when confusion is unlikely to occur.

A formula $\varphi \in \mathcal{L}$ is *valid in a topo-model* $\mathcal{M}$, denoted $\mathcal{M} \models \varphi$, iff $\mathcal{M}, (x, \theta) \models \varphi$ for all $(x, \theta) \in \mathcal{M}$; φ is *valid*, denoted $\models \varphi$, iff for all topo-models $\mathcal{M}$ we have $\mathcal{M} \models \varphi$. Soundness and completeness with respect to topo-models are defined as usual.

For any topo-model $\mathcal{M} = (X, \tau, \Phi, V)$, the agents' current observation sets, i.e., the epistemic neighbourhood of each agent at a given state x, is defined by (partial) functions $\theta \in \Phi$, where $\theta : X \rightharpoonup \mathcal{A} \to \tau$. As briefly stated in Section 1, one important feature of the subset space semantics is the local interpretation of the propositions: once the evaluation pair of a state and an observation set (x, U) has been determined, the rest of the model does not have any effect on the truth of the proposition in question. Similarly in our setting, by choosing a neighbourhood situation (x, θ), we localize the interpretation to an open subdomain of the whole space, namely to $\mathcal{D}(\theta)$, that embeds an observation set at every state in $\mathcal{D}(\theta)$ for each agent $i \in \mathcal{A}$. For every $\theta \in \Phi$ and $x \in \mathcal{D}(\theta)$, the function $\theta(x) : \mathcal{A} \to \tau$ is defined to be a *total* function. Therefore, given a neighbourhood situation (x, θ), the neighbourhood function θ assigns a neighbourhood at x to *each* agent. Moreover, the conditions of neighbourhood functions given in Definition 7 make the semantics works for the multi-agent setting. To be more precise, Condition 1 guarantees that θ always gives a truthful observation set at the state in question for each agent. In particular, it also implies that the agents cannot have inconsistent, i.e., empty, observation sets. Since the neighbourhoods given by the neighbourhood functions depend not only on the agent but also on the current state of the agent, and since $x \in \theta(x)(i) \subseteq \mathcal{D}(\theta)$ for every $x \in \mathcal{D}(\theta)$ and every $i \in \mathcal{A}$ (due to conditions 1 and 2), our semantics do not face the problem of ending up with ill-defined evaluation pairs in the interpretation of iterated epistemic formulas such as $\hat{K}_j K_i p$ (see, e.g., [30, Section 2.5] for an example). Moreover, conditions 1 and 3 of Definition 7 make the axioms of the system **S5** for knowledge sound. We will give the weaker conditions for **S4**, **S4.2** and **S4.3** in Section 5. Finally, Condition 4 defines the refined neighbourhoods resulted by a semi-private announcement. By the nature of the semi-private announcements, the effect of the semi-private announcement of a proposition φ to the group $G \subseteq \mathcal{A}$ is different on the agents in G than it is on the ones in $\mathcal{A} \setminus G$. We therefore define the restricted neighbourhood function θ^{φ} in such a way that it captures this difference and assigns open sets to the agents accordingly. We continue analysing Definition 7 together with a discussion on the semantics for the modalities in $\mathcal{L}$.

The semantic clauses for the propositional variables and Booleans are standard and, as usual in the subset space setting, their truth value depends purely on the evaluation state:

Proposition 10. *Given a topo-model* $\mathcal{M} = (X, \tau, \Phi, V)$*, neighbourhood situations* $(x, \theta_1), (x, \theta_2) \in \mathcal{M}$*, and a formula* $\varphi \in \mathcal{L}_{Pl}$*,* $(x, \theta_1) \models \varphi$ *iff* $(x, \theta_2) \models \varphi$*.*

The neighbourhood functions, and thus, the observation sets become important in the evaluation of the modalities. Recall that for any neighbourhood situation (x, θ), the epistemic neighbourhood $\theta(x)(i)$ is the particular truthful observation set that the agent i curently has at the state x. For the semantic clause of the knowledge modalities, we can simply write

$$\mathcal{M}, (x, \theta) \models K_i \varphi \text{ iff } \theta(x)(i) \subseteq [\![\varphi]\!]^{\theta},$$

meaning that "*agent i knows φ iff her current observation set entails φ (with respect to the neigbourhood function θ)*". Therefore, as in the original subset space setting, K_i quantifies over the observation set of agent i.

Let us now focus on the semantics of the dynamic part, starting with the interpretation of the modality $int(\varphi)$ intending to capture the precondition of the announcement of φ. To be more precise, we note that

$$\mathcal{M},(x,\theta) \models int(\varphi) \text{ iff } (\exists U \in \tau)(x \in U \text{ and } U \subseteq [\![\varphi]\!]^\theta).$$

Given the observation-based interpretation of the open sets, we can read the above semantic clause as "*the precondition of the announcement of φ is satisfied at the state x (with respect to the neighbourhood function θ) iff there exists a truthful observation set that entails/supports φ (with respect to θ)*". Therefore, in our setting, the precondition of an announcement is not only that the announced formula be true but also that it be entailed by a possible observation set, as in [13]. Moreover, since $Int([\![\varphi]\!]^\theta)$ is the largest open neighbourhood contained in $[\![\varphi]\!]^\theta$, it is the largest, and consequently, the weakest observation entailing the proposition φ with respect to the neighbourhood function θ. Following [32], we say φ *can be verified* (via some true observation) at the neighbourhood situation (x,θ) if $x \in Int([\![\varphi]\!]^\theta)$. Similarly, φ *can be refuted* at the neighbourhood situation (x,θ) if $x \in Int([\![\neg\varphi]\!]^\theta)$ Moreover, the existence of a possible, truthful observation set supporting the new information φ, i.e., the verifiability of φ, in no way depends on the agents but only on the model in question and so is objectively determined. The definition of the restricted neighbourhood functions θ^φ (see Definition 7.4) is mainly based on this intuition behind the use of the interior operator as a precondition for announcements.

In our setting, i.e., in the setting of semi-private events, while a group of agents $G \subseteq \mathcal{A}$ is announced a proposition φ, the agents in $\mathcal{A} \setminus G$ are not totally blind to the new information. They get to learn that either φ or $\neg\varphi$ is announced, however, unlike the members of G, they do not receive any further information as to which one was announced. Given that the initial neigbourhood situation is (x,θ), the domain of the updated function θ^φ becomes $\mathcal{D}(\theta^\varphi) = Int([\![\varphi]\!]^\theta) \cup Int(\mathcal{D}(\theta)\backslash[\![\varphi]\!]^\theta)$ and this domain restriction does not depend on the group G. This represents the fact that the announcement of φ conveys the information to *all* the agents that either φ or $\neg\varphi$ can be verified. Therefore, from the opposite perspective, the refinement of $\mathcal{D}(\theta)$ to $\mathcal{D}(\theta^\varphi)$ simply leaves out the states in which neither φ nor $\neg\varphi$ is observable with respect to (x,θ). In fact, the set of states in which neither φ nor $\neg\varphi$ is observable with respect to a neighbourhood situation (x,θ) (or equivalently, the states in which φ is neither verifiable nor refutable) corresponds to another topological concept, namely to the set boundary points of $[\![\varphi]\!]^\theta$ in $\mathcal{D}(\theta)$:

Proposition 11. *For any topo-model $\mathcal{M} = (X,\tau,\Phi,V)$, any $(x,\theta) \in \mathcal{M}$ and any $\varphi \in \mathcal{L}$, we have*

$$\mathcal{D}(\theta) \setminus \mathcal{D}(\theta^\varphi) = Bd([\![\varphi]\!]^\theta) \cap \mathcal{D}(\theta).$$

Proof. Let $(\mathcal{D}(\theta),\tau_\theta)$ be the subspace of (X,τ) generated by $\mathcal{D}(\theta)$ and Int_θ, Cl_θ and Bd_θ be the interior, closure and boundary point operators of $(\mathcal{D}(\theta),\tau_\theta)$. Since $\mathcal{D}(\theta) \in \tau$, we have $Int(A)_\theta = Int(A)$ for any $A \subseteq \mathcal{D}(\theta)$. And, as usual, $Cl_\theta(A) = Cl(A) \cap \mathcal{D}(\theta)$ and $Bd_\theta(A) = Bd(A) \cap \mathcal{D}(\theta)$ for any $A \subseteq \mathcal{D}(\theta)$. We can therefore write

$$\begin{aligned}
\mathcal{D}(\theta) \setminus \mathcal{D}(\theta^\varphi) &= (Int_\theta([\![\varphi]\!]^\theta) \cup Bd_\theta([\![\varphi]\!]^\theta) \cup Int_\theta(\mathcal{D}(\theta) \setminus [\![\varphi]\!]^\theta)) \setminus (Int([\![\varphi]\!]^\theta) \cup Int(\mathcal{D}(\theta)\backslash[\![\varphi]\!]^\theta))\\
&= (Int_\theta([\![\varphi]\!]^\theta) \cup Bd_\theta([\![\varphi]\!]^\theta) \cup Int_\theta(\mathcal{D}(\theta) \setminus [\![\varphi]\!]^\theta)) \setminus (Int_\theta([\![\varphi]\!]^\theta) \cup Int_\theta(\mathcal{D}(\theta)\backslash[\![\varphi]\!]^\theta))\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\text{since } Int_\theta([\![\varphi]\!]^\theta) = Int([\![\varphi]\!]^\theta))\\
&= Bd_\theta([\![\varphi]\!]^\theta) \qquad (\text{since } Int_\theta([\![\varphi]\!]^\theta),\ Bd_\theta([\![\varphi]\!]^\theta) \text{ and } Int_\theta(\mathcal{D}(\theta) \setminus [\![\varphi]\!]^\theta) \text{ are disjoint.})\\
&= Bd([\![\varphi]\!]^\theta) \cap \mathcal{D}(\theta)
\end{aligned}$$

□

Therefore, topologically speaking, the domain restriction induced by the announcement of φ boils down to disregarding the boundary points of the truth set of φ under the domain of the initial neighbourhood function. Note that if the actual state x is an element of $Bd([\![\varphi]\!]^\theta) \cap \mathcal{D}(\theta)$, the update is not applicable. Therefore, a first step of a successful implementation of the announcement of φ neglects the states in which it is not possible to find a truthful observation set entailing either φ or $\neg\varphi$ and thus refines the domain of the initial neighbourhood functions. This naturally leads to a refinement of the current observation sets of all the agents (see Definition 7.2 and 7.4). While the observation sets of the members of $\mathcal{A} \setminus G$ who do not receive any further information are only restricted by the domain of the updated function θ^φ, the members of G can further strengthen their epistemic state depending on the content of the information received: without loss of generality, if φ can be verified at x (i.e., $x \in Int([\![\varphi]\!]^\theta)$), then the refined observation set becomes $\theta(x)(i) \cap Int([\![\varphi]\!]^\theta)$ for each agent in G. We can see how the semantics works on the Examples 4 and 5.

Example 12 (The Jewel and The Tomb-continued)**.** *Consider the topo-model $\mathcal{M} = (X, \tau, \Phi, V)$ based on the topological space (X, τ) described in Example 4 where Φ is the set of all neighbourhood functions that partition the domain X and is closed under the condition Definition 7.4. For instance, the neighbourhood function $\theta \in \Phi$ defined as $\theta(x)(i) = \theta(x)(e) = X$ for all $x \in X$ describes total ignorance of both agents. Consider this is the initial state and Emile is semi-privately announced that the tomb has jewel in it (J). This means that Indiana received the information that either J or $\neg J$ is observable but he does not learn which one. Modelling this situation on $\mathcal{M}$ amounts to calculating the following. $[\![J]\!]^\theta = \{100, 110, 101, 111\}$ and $Int([\![J]\!]^\theta) = \{110, 111\}$. The fact that $100 \notin Int([\![J]\!]^\theta)$ and $101 \notin Int([\![J]\!]^\theta)$ captures the intuition that one can only conceivably learn about the jewel: it is not possible to observe the jewel in tomb without the tomb being rediscovered in modern times. We moreover calculate that after the semi-private announcement of J to Emile, he does not only come to know J but he also comes to know D:*

$$(111, \theta) \models [J]_e(K_e J \wedge K_e D).$$

This amounts calculating $\theta_e^J(e)(111) = \theta(e)(111) \cap Int([\![J]\!]^\theta) = \{110, 111\}$. Observe that every state in $\theta_e^J(e)(111)$ makes both J and D true.

On the other hand, Indiana is still ignorant about whether the tomb has jewel or not, however, he comes to know that Emile knows whether the tomb has jewel and he also comes to know that the tomb has been discovered

$$(111, \theta) \models [J]_e(\neg K_i J \wedge \neg K_i \neg J \wedge K_i D \wedge K_i(K_e J \vee K_e \neg J)),$$

since $\theta_e^J(i)(111) = \theta(i)(111) \cap (Int([\![J]\!]^\theta) \cup Int(\mathcal{D}(\theta) \setminus [\![J]\!]^\theta)) = \{110, 111, 010, 011\}$. More precisely, $\theta_e^J(i)(111)$ includes some states falsifying J (namely, 010 and 011), some states falsifying $\neg J$ (namely, 110 and 111) and every state in $\theta_e^J(i)(111)$ makes D true. For the last conjunct $K_i(K_e J \vee K_e \neg J)$, we check whether every state in $\theta_e^J(i)(111)$ makes $K_e J \vee K_e \neg J$ true with respect to the updated neighbourhood function θ_e^J. An intermediate step to reach this result is calculating $\theta_e^J(e)(x)$, for all $x \in \theta_e^J(i)(111)$. We then obtain (1) $\theta_e^J(e)(110) = \theta_e^J(e)(111) = \theta(e)(111) \cap Int([\![J]\!]^\theta) = \{110, 111\}$ and (2) $\theta_e^J(e)(010) = \theta_e^J(e)(011) = \theta(e)(111) \cap Int([\![\neg J]\!]^\theta) = \{010, 011\}$. We then have that Emile knows J (i.e., $K_e J$ holds) at 110 and 111 and he knows $\neg J$ (i.e., $K_e \neg J$ holds) at 010 and 011 (with respect to θ_e^J). Therefore, every state in the observation set $\theta_e^J(i)(111)$ of Indiana satisfies $K_e J \vee K_e \neg J$, hence, Indiana knows Emile knows whether the tomb has jewel after the announcement of J to Emile.

Example 13 (The probe in the unit square-continued)**.** *Recall that the probe is located in a square region and it transmits information about the x- and y-coordinates of its location to two different agents, a_x and a_y, in the form of binary bits giving increasingly precise information* *about each coordinate. We model this situation and the corresponding information dynamics described in Example 5 on a topological space based on the domain* $\{0,1\}^\infty \times \{0,1\}^\infty$ *of the ordered pairs of infinite binary strings.*

In order to define our model more formally, we need to introduce some notation. If $s \in \{0,1\}^\infty$, *for* $n \in \mathbb{N}^+$, *we let* $s|_n$ *be the first* n *bits of* s, *and we let* $s[n]$ *be the* n*th bit of* s. *As usual, we let* $\{0,1\}^*$ *be the set of finite strings over* $\{0,1\}$ *and for* $d \in \{0,1\}^*$, $|d|$ *denotes the length of the finite string* d. *For* $d \in \{0,1\}^*$ *we define* $S_d = \{x \in \{0,1\}^\infty \mid x|_{|d|} = d\}$, *in other words,* S_d *is the set of all infinite binary strings that have* d *as a prefix. Note that* S_ϵ *is* $\{0,1\}^\infty$, *since* ϵ *is the empty string. Note also that when we consider the elements of* $\{0,1\}^\infty$ *as points on the unit interval, we can think of* S_d *as a certain subinterval of the unit interval. More precisely, each* S_d *is the interval bounded by* $\frac{d}{2^{|d|}}$ *and* $\frac{d+1}{2^{|d|}}$ *when* d *is viewed as the binary representation of a natural number. We cannot, however, go in the opposite direction and consider all such intervals to be sets of the form* S_d, *since there are multiple possible representations of some of the points in* $[0,1]$ *as binary strings.*

Now consider the topology τ *generated by the set* $\mathcal{B} = \{S_d \mid d \in \{0,1\}^*\}$. *It is not hard to see that* $\mathcal{B}$ *indeed constitutes a base over the domain* $\{0,1\}^\infty$*:*

1. *Since* $S_\epsilon \in \mathcal{B}$, *we have* $\bigcup \mathcal{B} = \{0,1\}^\infty$
2. *For any* $U_1, U_2 \in \mathcal{B}$, *we have either* $U_1 \cap U_2 = \emptyset$, $U_1 \cap U_2 = U_1$ *or* $U_1 \cap U_2 = U_2$. *Therefore,* $\mathcal{B}$ *is closed under finite intersections.*

For our example, we use the product space $(\{0,1\}^\infty \times \{0,1\}^\infty, \tau \times \tau)$ *and we have two agents* a_x *and* a_y. *This scenario concerns the following propositional variables:*

$$Prop = \{x_i \mid i \in \mathbb{N}^+\} \cup \{y_i \mid i \in \mathbb{N}^+\}$$

where

$$V(x_i) = \{(x,y) \in \{0,1\}^\infty \times \{0,1\}^\infty \mid x[i] = 1\};$$
$$V(y_i) = \{(x,y) \in \{0,1\}^\infty \times \{0,1\}^\infty \mid y[i] = 1\}.$$

We read x_i *as* "the ith bit of x is 1" *and* y_i *as* "the ith bit of y is 1".

By using this model, we will model the agents observing the bits of the binary sequences and what they come to know about the location of the probe and about what the other agent knows. Moreover, we will further extend this model and have a look at a situation in which the agent receives some information that is not observable, i.e., whose truth set does not correspond to any open set in $\tau \times \tau$.

We start by describing the situation in which both agents are totally ignorant about the exact location of the probe within the unit square. This is described by the neighbourhood function θ *such that* $\theta((x,y))(a_x) = \theta((x,y))(a_y) = \{0,1\}^\infty \times \{0,1\}^\infty$. *In order to obtain a well-defined neighbourhood function set* Φ *on the topology* $(\{0,1\}^\infty \times \{0,1\}^\infty, \tau \times \tau)$, *we must close the singleton set* $\{\theta\}$ *under open domain restrictions described in Definition 7.4, so we let* $\Phi = \{\theta' : \{0,1\}^\infty \times \{0,1\}^\infty \rightharpoonup \{a_x, a_y\} \to \tau \mid Y \subseteq \{0,1\}^\infty \times \{0,1\}^\infty$ *and* $G \subseteq \{a_x, a_y\}$ *such that* $\theta' = \theta_G^Y\}$. *It is easy to see that* Φ *satisfies the properties of a neighbourhood function set given in Definition 7.*

We now evaluate some formulas on the topo-model $\mathcal{M} = (\{0,1\}^\infty \times \{0,1\}^\infty, \tau \times \tau, \Phi, V)$ *at the neighbourhood situation* $((x,y),\theta) = ((11000..., 01000...), \theta)$. *In other words, in the initial situation where the both agents are totally ignorant about the binary sequences, i.e., the location of the probe within the unit square. The ordered pair* $(11000..., 01000...)$ *represents the actual state, i.e., the exact location of the probe in the unit square.*

We can model the agents learning bits of x *and* y*:* a_x *learning bits of* x *and* a_y *learning bits of* y *in the following way:*

a_x learning that the first bit of x is 1, induces the function $\theta^{x_1}_{a_x}$. We note that

$$[\![x_1]\!]^\theta = S_1 \times \{0,1\}^\infty = Int([\![x_1]\!]^\theta),$$

and similarly $(\{0,1\}^\infty \times \{0,1\}^\infty)\backslash[\![x_1]\!]^\theta = S_0 \times \{0,1\}^\infty = Int((\{0,1\}^\infty \times \{0,1\}^\infty)\backslash[\![x_1]\!]^\theta)$. *Therefore,*

$$\theta^{x_1}_{a_x}((x,y))(a) = \begin{cases} \{0,1\}^\infty \times \{0,1\}^\infty & \text{for } a = a_y \\ S_0 \times \{0,1\}^\infty & \text{for } a = a_x \text{ and } (x,y) \in S_0 \times \{0,1\}^\infty \\ S_1 \times \{0,1\}^\infty & \text{for } a = a_x \text{ and } (x,y) \in S_1 \times \{0,1\}^\infty \end{cases}$$

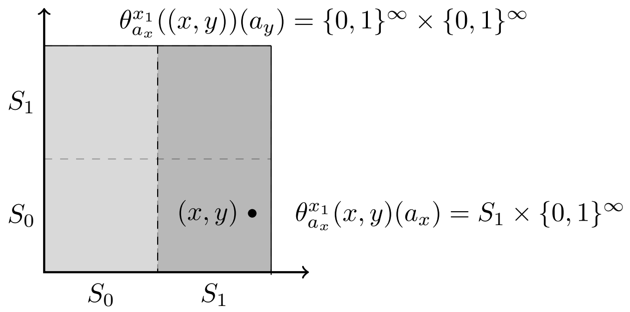

Figure 2: Updated situation where a_x knows the first bit of x is 1 and a_y is ignorant about the first bit of x and she does not know that a_x knows the first bit of x is 1. On the other hand, a_y know that a_x is informed of the first bit of x. We have $\theta^{x_1}_{a_x}((x,y)(a_x) = S_1 \times \{0,1\}^\infty$ and $\theta^{x_1}_{a_x}((x,y)(a_y) = X$.

Other functions for updating with propositional variables work similarly. We now see for example that

$$((x,y),\theta) \models [x_1]_{a_x}(K_{a_x}x_1 \wedge \neg K_{a_y}K_{a_x}x_1)$$

after a_x learns that the first bit of x is 1, a_x knows this, but a_y does not know that a_x knows this. On the other hand, as a result of the semi-private nature of the announcement, a_y knows that a_x knows the value of the first bit of x (see Figure 13):

$$((x,y),\theta) \models [x_1]_{a_x}K_{a_y}((x_1 \to K_{a_x}x_1) \wedge (\neg x_1 \to K_{a_x}\neg x_1)).$$

Observe that this was not the case before the announcement:

$$((x,y),\theta) \not\models K_{a_y}((x_1 \to K_{a_x}x_1) \wedge (\neg x_1 \to K_{a_x}\neg x_1)).$$

In case of iterative private announcements to different agents, for example, we have

$$((x,y),\theta) \models [x_1]_{a_x}[\neg y_1]_{a_y}(K_{a_x}x_1 \wedge K_{a_y}y_1 \wedge \neg K_{a_x}y_1 \wedge \neg K_{a_y}x_1).$$

meaning that after a_x was announced that the first bit of x is 1 *and then a_y was announced that the first bit of y is* 0*, they come to know the first bit of their own sequences, however, neither of the two knows the other's first bit. A public announcement of $x_1 \wedge \neg y_1$, on the other hand, results in the situation*

$$((x,y),\theta) \models [x_1 \wedge \neg y_1](K_{a_x}(x_1 \wedge \neg y_1) \wedge K_{a_y}(x_1 \wedge \neg y_1) \wedge K_{a_x}K_{a_y}(x_1 \wedge \neg y_1) \wedge K_{a_y}K_{a_x}(x_1 \wedge \neg y_1)).$$

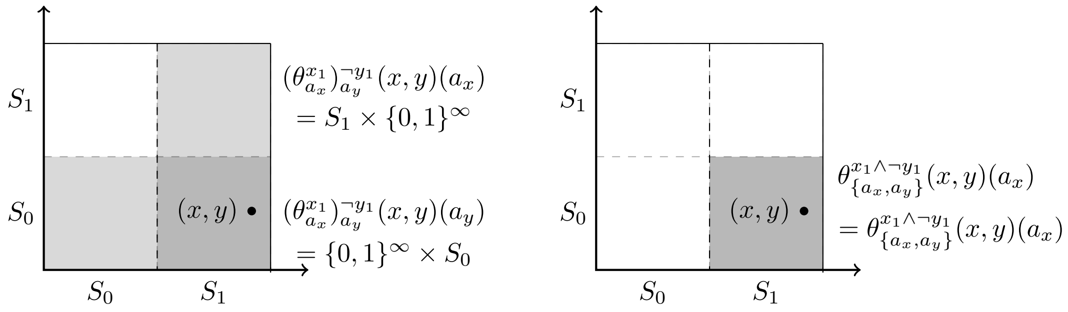

Figure 3: The left figure depicts the iterative private announcements of x_1 and $\neg y_1$ to agent a_x and agent a_y, respectively, whereas the right one represents the public announcement of $x_1 \wedge \neg y_1$.

where each agent knows the first bit of both sequences and they also know that everyone knows the first bit of both sequences (see Figure 13). The public announcement of propositional variables and their boolean combinations indeed lead to common knowledge among the informed agents.

The above case captures a scenario in which the agents receive only observable information from the probe that corresponds to opens in the given topology. We can further extend this example and talk about a situation where the agents receive intercepted information from the probe that is known to be errant in the sense that the shape of the signal does not match the one that could be sent by the probe in terms of the first nth bit of the binary sequences. In this case, the agents only consider the observable part of the new information by the help of the interior modality.

In order to capture such a scenario, we extend the set of propositional variables by the set

$$Prop' = \{z_{(.i,.j)} \mid i,j \in \mathbb{N}\}$$

with the valuation

$$V(z_{(.i,.j)}) = \{(x,y) \in \{0,1\}^\infty \times \{0,1\}^\infty \mid (x,y) \sim (.i,.j)\},$$

where $\sim$ is defined as

$$(x,y) \sim (.i,.j) \textit{ iff } (x,y) \textit{ corresponds to the pair } (.i,.j) \textit{ on } \mathbb{R}^2.$$

For example, $V(z_{(.5,.5)}) = \{(0111...,0111...),(1000...,1000...),(0111...,1000...),(1000...,0111...)\}$, *and the proposition $z_{(.5,.5)}$ states that "the probe is at one of the coordinates in $V(z_{(.5,.5)})$". The propositions of type $z_{(.i,.j)}$ clearly form discrete options as to where the probe stands and require measurement of infinitely many bits of both sequences. Since the probe, by design, cannot calculate as accurately, even if the agents receive such information, they simply focus on the observable part of the new information that corresponds to what could actually be measured and sent by the probe. This phenomena in our setting is captured by the help of the interior modality int.*

We again consider the initial neighbourhood situation $((x,y),\theta)$ and suppose the agent a_x receives the information $z_{(.75,.25)}$. Observe that

$$V(z_{(.75,.25)}) = \{(11000...,01000...),(11000...,00111....),(10111...,01000...),(10111...,00111....)\}.$$

Even though $z_{(.75,.25)}$ is true (since its truth set at θ contains the actual state $(11000...,01000...)$*), it is not observable, or equivalently for this example, it cannot be sent by the probe. Therefore,* *it is not an announceable formula in our setting. This is formalized by the interior modality:* $Int([\![z_{(.75,.25)}]\!]^\theta) = \emptyset$. *We therefore obtain*

$$((x,y),\theta) \not\models \langle z_{(.75,.25)}\rangle_{a_x}\varphi$$

for any $\varphi \in \mathcal{L}$. *More interestingly, consider the case the agent* a_y *receives the information* $y_2 \vee z_{(.75,.25)}$. *Note that*

$$[\![y_2 \vee z_{(.75,.25)}]\!]^\theta = \{0,1\}^\infty \times (S_{01} \cup S_{11}) \cup \{(11000..., 00111....), (10111..., 00111....)\},$$

and

$$Int([\![y_2 \vee z_{(.75,.25)}]\!]^\theta) = \{0,1\}^\infty \times (S_{01} \cup S_{11}),$$

corresponding to the weakest observation set that entails $y_2 \vee z_{(.75,.25)}$. *As the agent knows that the probe cannot calculate infinite sequences such as the discrete signals*

$$\{(11000..., 00111....), (10111..., 00111....)\},$$

he only considers the observation set entailing the announcement while updating his information state. We therefore obtain, for example, $((x,y),\theta) \models [y_2 \vee z_{(.75,.25)}]_{a_y} K_{a_y} y_2$, *although some states in* $[\![y_2 \vee z_{(.75,.25)}]\!]^\theta$, *namely* $((10111..., 00111....),\theta)$, *falsifies* y_2: $((10111..., 00111....),\theta) \models \neg y_2$.

Our last example is of a more abstract character and it is based on the natural topology of open intervals on the set of irrational numbers.

Example 14 (Irrational Intervals)**.** *Let* $(\mathbb{R},\tau^*)$ *be the standard topology on the reals*[1] *and let* $\mathbb{I}$ *denote the set of irrational numbers. Let* τ *be the subspace of* τ^* *induced by* $\mathbb{I}$. *We use* $(\mathbb{I},\tau)$ *as our space. We suppose there are two agents, whom we call* a_1 *and* a_2. *We let*

$$Prop = \{<t \mid t \in \mathbb{R}\} \cup \{>t \mid t \in \mathbb{R}\} \cup \{=t \mid t \in \mathbb{R}\}.$$

At this stage, it should be noted that we have an uncountable set of propositional variables, which goes beyond our usual logical settings (for example, as used to demonstrate completeness of the axiomatization by way of a canonical model technique). However, to demonstrate the use of our logic it does not matter; although some topological models have uncountable features, in any formula that we wish to evaluate in these models only a countable (and even finite) number of propositional variables occur.

We let $V(<t) = \{x \in \mathbb{I} \mid x < t\}$, $V(>t) = \{x \in \mathbb{I} \mid t < x\}$, *and* $V(=t) = \{x \in \mathbb{I} \mid x = t\}$. *In other words,* $<t$ *is true only at those points in* $\mathbb{I}$ *whose values are below* t, *similarly for* $>t$, *and* $=t$ *is only true at the point* t.

We define the neighbourhood function θ *as* $\theta(y)(i) = \mathbb{I}$ *for all* $y \in \mathbb{I}$ *and* $i \in \{a_1,a_2\}$ *and the neighbourhood function set is defined to be the smallest set* S *such that* $\theta \in S$ *and whenever* $\theta' \in S$, $Y \subseteq \mathbb{I}$, *and* $G \subseteq \{a_1,a_2\}$, $(\theta')^Y_G \in S$.

We can use these propositions to model a situation where a_1 *and* a_2 *are learning about the value of an irrational point,* x. *Agent* a_1 *receives sequential announcements that* x *is less than certain rationals, and agent* a_2 *receives sequential announcements that* x *is greater than certain rationals. We suppose* $x = \pi$ *and examine the effects of different private announcements to each agent. We suppose that in the initial situation neither agent knows anything about the value of* x, *so we model this state of affairs with the initial neighbourhood function* θ *where* $\theta(y)(i) = \mathbb{I}$ *for all* $y \in \mathbb{I}$ *and* $i \in \{a_1,a_2\}$.

[1]The natural topology on the reals is the topology with base $\{(a,b) \mid a,b \in \mathbb{R} \text{ and } a < b\}$.

Suppose that first a_2 receives the announcement >3. We verify that $(\pi,\theta) \models [>3]_2(K_2 > 3 \wedge \neg K_1 > 3)$, in other words, after the announcement a_2 knows >3 but a_1 still does not. $(\pi,\theta) \models [>3]_2(K_2 > 3 \wedge \neg K_1 > 3)$ iff $(\pi,\theta) \models int(>3)$ implies $(\pi,\theta_2^{>3}) \models K_2 > 3 \wedge \neg K_1 > 3$. $\pi \in Int([\![>3]\!])$ so we should verify that $(\pi,\theta_2^{>3}) \models K_2 > 3 \wedge \neg K_1 > 3$. Note that $Int([\![>3]\!]^\theta) \cup Int(\mathbb{I} \backslash [\![>3]\!]^\theta) = \mathbb{I}$ so $\mathcal{D}(\theta_2^{>3}) = \mathbb{I}$. So

$$\theta_2^{>3}(y)(a_i) = \begin{cases} \mathbb{I} & \text{if } i = 1 \\ \{t \in \mathbb{I} \mid t > 3\} & \text{for } y > 3 \text{ and } i = 2 \\ \{t \in \mathbb{I} \mid t < 3\} & \text{for } y < 3 \text{ and } i = 2 \end{cases}$$

This makes it clear that $(\pi,\theta_2^{>3}) \models K_2 > 3 \wedge \neg K_1 > 3$, and thus $(\pi,\theta) \models [>3]_2(K_2 > 3 \wedge \neg K_1 > 3)$.

Similarly, we can verify other properties like $(\pi,\theta) \models [<4]_1(K_1 <4 \wedge \neg K_2 <4)$. In fact, it is easy to see that for $q \in \mathbb{Q}$, if we let $\{i,j\} = \{1,2\}$ and $i \neq j$, then $\theta_i^{>q}(y)(a_j) = \mathbb{I}$ for any $y \in \mathbb{I}$, and similarly $\theta_i^{<q}(y)(a_j) = \mathbb{I}$ for any y. However, for $z \in \mathbb{I}$, $\theta_i^{>z}(y)(a_j) = \theta_i^{<z}(y)(a_j) = \mathbb{I} \backslash \{z\}$ for any $y \in \mathbb{I}$. From these facts it follows that for any $x \in \mathbb{I}$, and for any $q \in \mathbb{Q}$,

$$(x,\theta) \models K_j\varphi \leftrightarrow [>q]_i K_j\varphi \quad \text{and} \quad (x,\theta) \models K_j\varphi \leftrightarrow [<q]_i K_j\varphi$$

But, this would not be the case if $q \in \mathbb{I}$. Moreoever, for all $z \in \mathbb{I}$, we have $(x,\theta) \models [>z]_i K_j \neg = z$.

Note that $(\pi,\theta) \models \neg\langle = \pi\rangle_1 K_1 = \pi$. This is because $[\![=\pi]\!] = \{\pi\}$ and $Int(\{\pi\}) = \emptyset$, so π is not announceable anywhere. Finally, we also can see that there is no announcement α such that $(x,\theta) \models [\alpha]_G K_1(=\pi)$. Furthermore, there is no (finite) sequence of announcements $(\alpha_1, \ldots, \alpha_n)$ such that $(\pi,\theta) \models \langle\alpha_1\rangle_i \ldots \langle\alpha_n\rangle_i K_i(=\pi)$, for either i because the singleton $\{\pi\}$ is not an open set of the space $(\mathbb{I},\tau)$, and thus any open neighbourhood of π includes some point y such that $y \neq \pi$. However, we cannot express this nonexistence property in our logic, since we do not have expressions about arbitrary announcements. A logic rich enough to express such properties is of interest but we leave such extensions to future work.

4 Axiomatization, Soundness and Completeness

In this section, we define an axiomatization for the multi-agent (topological) semi-private announcement logic with the interior modality $\mathbf{sPAL}_{int}$, comment on alternative and equivalent axiomatizations (following [33]), and prove soundness and completeness results for the given axiomatization.

The logic $\mathbf{sPAL}_{int}$ is the smallest subset of $\mathcal{L}$ containining the following axioms and closed under the derivation rules. An element of $\mathbf{sPAL}_{int}$ is called a *theorem* (of $\mathbf{sPAL}_{int}$), notation $\vdash \varphi$, or equivalently $\varphi \in \mathbf{sPAL}_{int}$.

(P) all instantiations of propositional tautologies

(K-K) $K_i(\varphi \to \psi) \to (K_i\varphi \to K_i\psi)$

(K-T) $K_i\varphi \to \varphi$

(K-4) $K_i\varphi \to K_iK_i\varphi$

(K-5) $\neg K_i\varphi \to K_i\neg K_i\neg\varphi$

(int-K) $int(\varphi \to \psi) \to (int(\varphi) \to int(\psi))$

(int-T) $int(\varphi) \to \varphi$

(int-4) $int(\varphi) \to int(int(\varphi))$

(K_{int}) $K_i\varphi \to int(\varphi)$

(Rp1) $[\varphi]_G p \leftrightarrow (int(\varphi) \to p)$

(Rp2) $[\varphi]_G \neg\psi \leftrightarrow (int(\varphi) \to \neg[\varphi]_G\psi)$

(Rp3) $[\varphi]_G(\psi \wedge \chi) \leftrightarrow [\varphi]_G\psi \wedge [\varphi]_G\chi$

(Rp4) $[\varphi]_G int(\psi) \leftrightarrow (int(\varphi) \to int([\varphi]_G\psi))$

(Rp5-i) $[\varphi]_G K_i\psi \leftrightarrow (int(\varphi) \to K_i[\varphi]_G\psi)$ where $i \in G$

(Rp5-$\bar{\text{i}}$) $[\varphi]_G K_i\psi \leftrightarrow (int(\varphi) \to K_i[\varphi]_G\psi \wedge K_i[\neg\varphi]_G\psi)$ where $i \notin G$

(DRp1) From φ and $\varphi \to \psi$, infer ψ

(DRp2) From φ, infer $K_i\varphi$

(DRp3) From φ, infer $int(\varphi)$

(DRp4) From $\varphi \leftrightarrow \psi$, infer $\chi \leftrightarrow \chi[\varphi/\psi]$

Table 1: The axiomatization of $\mathbf{sPAL}_{int}$. Without the (Rp-) named axioms and (DRp4) we get the axiomatization of $\mathbf{EL}_{int}$. The (non-standard) meaning of the substitution $\chi[\varphi/\psi]$ is given in the accompanying text.

Let us elaborate on the meaning of some axioms and rules. While the (K-) named axioms express the **S5** character of the knowledge modality K_i, the first three axioms for the modality int reflects the **S4** nature of the topological interior operator. Moreover, the occurances of $int(\varphi)$ on the right-hand-side of the reduction axioms capture that this modality is used as a precondition for the announcements. Moreover, the axioms (Rp5-i) $[\varphi]_G K_i\psi \leftrightarrow (int(\varphi) \to K_i[\varphi]_G\psi)$, where $i \in G$, and (Rp5-$\bar{\text{i}}$) $[\varphi]_G K_i\psi \leftrightarrow (int(\varphi) \to K_i[\varphi]_G\psi \wedge K_i[\neg\varphi]_G\psi)$, where $i \notin G$, are the obvious instantiations of the [8]-style reduction of knowledge after action model execution. In the former case, agent $i \in G$ completely observes the announcement. Therefore, after the announcement she knows something (say, $[\varphi]_G K_i\psi$ is the case), iff, on the condition that the announcement can be executed (the precondition $int(\varphi)$), she knows that after the announcement it is true (i.e., $K_i[\varphi]_G\psi$ holds). In the latter case, agent $i \notin G$ only partially observes the announcement. She cannot distinguish the announcement of φ from the announcement of $\neg\varphi$. Therefore, after the announcement she only knows something (i.e., $[\varphi]_G K_i \psi$), iff, again on the condition $int(\varphi)$, she knows that after the announcement it is true (i.e., $K_i[\varphi]_G\psi$), but she also knows that after the announcement of $\neg\varphi$ it is true (i.e., $K_i[\neg\varphi]_G\psi$). This is because the announcements φ and $\neg\varphi$ are indistinguishable for her. To illustrate, consider agent a successfully being informed of the truth of p. Then after this announcement, a comes to know that p. Whereas agent b, who is partially observing this semi-private announcement, only learns that a knows whether p is true. So we have that $[p]_a K_a p$, but not that $[p]_a K_b p$. In order for a formula ψ to be known by b after $[p]_a$, it also has to be true after $[\neg p]_a$. Such a formula ψ for which this holds is $K_a p \vee K_a \neg p$. Indeed, we have that $[p]_a K_b(K_a p \vee K_a \neg p)$.

The derivation rule (DRp4) is one of *replacement of equivalents*, where $\chi[\varphi/\psi]$ denotes any formula obtained by replacing one or more non-dynamic occurrences of φ in χ with ψ. Non-dynamic occurrences of φ are the occurrences of φ which are not inside any $[\ ]_G$ [33]. As usual, we could also give alternative equivalent axiomatizations for the logic $\mathbf{sPAL}_{int}$. One possible choice would be replacing (DRp4) by the Necessitation rule for the dynamic modality $[\ ]_G$ (i.e., from φ, infer $[\psi]_G\varphi$) and adding the K-axiom for $[\psi]_G$ ((RpK) $[\psi]_G(\varphi \to \chi) \to ([\psi]_G \to [\psi]_G\chi)$). This would slightly change the completeness proof and we elaborate on this issue after Theorem 20.

We now continue with the soundness and completeness proofs with respect to the class of all topo-models for the system $\mathbf{sPAL}_{int}$. While the soundness proof consists of a standard validity check, the completeness proof will be given via *reduction*, a method commonly used in the DEL literature (see, e.g., [31] for a more detailed discussion of DEL). In order to be able to apply this method, we need the static fragment $\mathbf{EL}_{int}$ of $\mathbf{sPAL}_{int}$ to be complete with respect to the topo-models:

Theorem 15 ([30]). $\mathbf{EL}_{int}$ *is sound and complete with respect to the class of all topo-models.*

Proof. See [30, Section 4.1].

□

Theorem 16. $\mathbf{sPAL}_{int}$ *is sound with respect to the class of all topo-models.*

Proof. The proof strategy is standard and the details for (Rp4), (Rp5-i), (Rp5-$\bar{\text{i}}$) and (DRp4) are presented in Appendix B. □

We prove the completeness of $\mathbf{sPAL}_{int}$ via reduction. More precisely, we will define an inductive translation from the language $\mathcal{L}$ to $\mathcal{L}_{EL_{int}}$ that provides us an algorithm converting each formula of the language $\mathcal{L}$ to a *semantically and provably equivalent* formula in $\mathcal{L}_{EL_{int}}$. This method is commonly used in the DEL literature to prove completeness of public announcement logics and, more generally, of action model logics. For a more detailed discussion on this proof method, we refer the reader to [31].

Definition 17 (Translation). *The translation $t : \mathcal{L} \to \mathcal{L}_{EL_{int}}$ is defined recursively as follows:*

$$\begin{array}{lcl}
t(p) & = & p \\
t(\neg\varphi) & = & \neg t(\varphi) \\
t(\varphi \wedge \psi) & = & t(\varphi) \wedge t(\psi) \\
t(int(\varphi)) & = & int(t(\varphi)) \\
t(K_i\varphi) & = & K_i t(\varphi) \\
t([\varphi]_G p) & = & t(int(\varphi) \to p) \\
t([\varphi]_G \neg\psi) & = & t(int(\varphi) \to \neg[\varphi]_G\psi) \\
t([\varphi]_G(\psi \wedge \chi)) & = & t([\varphi]_G\psi) \wedge t([\varphi]_G\chi) \\
t([\varphi]_G int(\psi)) & = & t(int(\varphi) \to int([\varphi]_G\psi)) \\
t([\varphi]_G K_i(\psi)) & = & t(int(\varphi) \to K_i[\varphi]_i\psi), \ \textit{when } i \in G \\
t([\varphi]_G K_i(\psi)) & = & t(int(\varphi) \to K_i[\varphi]_G\psi \wedge K_i[\neg\varphi]_G\psi), \ \textit{when } i \notin G \\
t([\varphi]_G[\psi]_{G'}\chi) & = & t([\varphi]_G t([\psi]_{G'}\chi))
\end{array}$$

If we compare the left-hand side of each equation in the translation to its right-hand side, we can observe that the main logical connective bound by t on the left is out of the scope of t on the right. For example, in $t(\varphi \wedge \psi)$, on the left, the translation function t binds the conjunction, but not in $t(\varphi) \wedge t(\psi)$, on the right. However, when the main connective is an announcement modality, t operates on the main logical connective of the formula bound by the announcement. For example, with $t([\varphi]_G\neg\psi)$ on the left (t binds announcement, which binds negation), we get with some further rewriting $t(int(\varphi)) \to \neg t([\varphi]_G\psi)$ on the right (negation binds t, which binds announcement). More importantly, whereas on the left the announcement binds the negation, on the right the negation binds the announcement. We can see this as pushing the announcement operator deeper into the formula: on the right, it has been 'pushed' beyond the negation. All the cases for announcement, except the last one, have the main operator that is bound by the announcement on the left, bind the announcement on the right. If the announcement binds a propositional variable on the left, then on the right the announcement has disappeared: $t([\varphi]_G p)$ versus $t(int(\varphi) \to p)$.

In order to show that for each formula, this translation produces an equivalent formula without announcements, we need to address two concerns: (i) the formula without announcements thus produced is equivalent to the original formula, and (ii) such a formula is always produced, i.e., the rewrite procedure terminates. Concerning equivalence it is sufficient to show that each step in the translation is truth preserving. For example, $t([\varphi]_G p) = t(int(\varphi) \to p)$ is a proper translation step, because $[\varphi]_G p \leftrightarrow (int(\varphi) \to p)$ is an axiom of the *sound* system $\mathbf{sPAL}_{int}$.

But we also have to show that the process of iteratively translating formulas terminates. To prove that, we will define a complexity measure on formulas and show (A) that the right-hand side is always less complex than the left-hand side, and (B) that eventually all the announcement operators disappear from the formula (the proof is by induction on the number of announcements in a formula). Even so, this is tricky: comparing $t([\varphi]_G(\psi \wedge \chi))$ (left) to $t([\varphi]_G\psi) \wedge t([\varphi]_G\chi)$ (right), we even have more announcement operators on the right! Will they eventually disappear? Two different ways to produce formulas with fewer announcements by rewriting are the *outside-in reduction strategy* and the *inside-out reduction strategy*. In the first case, take an *outermost* announcement modality (an announcement modality that is not in the scope of another announcement modality), and apply one or more of the translation rules above. Eventually, you will get a logically equivalent formula with at least one less announcement modality. In the second case, take an *innermost* announcement modality (an announcement modality with no announcement modality in its scope). Then do the same. Either way, we will have to apply the derivation rule (DRp4), 'replacement of equivalents' (from $\varphi \leftrightarrow \psi$, infer $\chi \leftrightarrow \chi[\varphi/\psi]$).

In our logic, the outside-in strategy fails since we cannot always express two or more consecutive announcements by means of a single announcement. One way to do that is to have a *composition axiom for announcements* that makes a single announcement out of the two announcements of the form $[\varphi]_G[\psi]_{G'}$. We do not have that: there is no $[\chi]_{G''}$ that has the same effect as $[\varphi]_G[\psi]_{G'}$. Namely, as already mentioned, what should then be the group G'' learning this χ? But the inside-out reduction works in our case: comparing $t([\varphi]_G[\psi]_{G'}\chi)$ on the left to $t([\varphi]_G t([\psi]_{G'}\chi))$ on the right, one can envisage first getting rid of $[\psi]_{G'}$ in $[\psi]_{G'}\chi$, producing an equivalent ξ, and then continue by reducing $[\varphi]_G\xi$. To illustrate the inside-out style reduction, we can consider the simple dynamic proposition $[p][q]K_1r$. We then obtain the formula corresponding to $t([p][q]K_1r)$ recursively by starting the reduction with the innermost dynamic modality and following the rules given in Definition 17:

$$t([p][q]K_1r) = t([p]t([q]K_1r)) = t([p](int(q) \to K_1r)) = int(p) \to (int(q) \to K_1r).$$

For a more detailed discussion on alternative reduction rules, see e.g., [28, Chapter 6] and [33].

The soundness of $\mathbf{sPAL}_{int}$ shows that the translation given in Definition 17 preserves the truth of a formula. We now define a complexity measure on the formulas of the language $\mathcal{L}$ which will help us to obtain the desired completeness result.

Definition 18 (Complexity)**.** *The complexity measure $c : \mathcal{L} \to \mathbb{N}$ defined recursively as follows:*

$$\begin{array}{lcl} c(p) & = & 1 \\ c(\neg\varphi) & = & 1 + c(\varphi) \\ c(\varphi \wedge \psi) & = & 1 + max(c(\varphi), c(\psi)) \\ c(int(\varphi)) & = & 1 + c(\varphi) \\ c(K_i\varphi) & = & 1 + c(\varphi) \\ c([\varphi]_G\psi) & = & c(\varphi) + 6c(\psi) \end{array}$$

Lemma 19. *For any $\varphi, \psi, \chi \in \mathcal{L}$ and $i \in \mathcal{A}$*

1. $c(\varphi) \geq c(\psi)$, *if* $\psi \in Sub(\varphi)$;
2. $c([\varphi]_G p) > c(int(\varphi) \to p)$;
3. $c([\varphi]_G \neg\psi) > c(int(\varphi) \to \neg[\varphi]_G\psi)$;
4. $c([\varphi]_i(\psi \wedge \chi)) > c([\varphi]_G\psi \wedge [\varphi]_G\chi)$;
5. $c([\varphi]_i int(\psi)) > c(int(\varphi) \to int([\varphi]_G\psi))$;
6. $c([\varphi]_G K_i\psi) > c(int(\varphi) \to K_i[\varphi]_G\psi)$, *when* $i \in G$;
7. $c([\varphi]_G K_i(\psi)) > c(int(\varphi) \to K_i[\varphi]_G\psi \wedge K_i[\neg\varphi]_G\psi)$, *when* $i \notin G$.

Therefore,

- $c(\varphi) > c(t(\varphi))$, *for any* $\varphi \in \mathcal{L} \setminus \mathcal{L}_{Pl}$ *and*
- $c(\varphi) = c(t(\varphi))$, *for any* $\varphi \in \mathcal{L}_{Pl}$.

Proof. The proof is elementary and follows from routine complexity calculations. □

Theorem 20. *For any $\varphi \in \mathcal{L}$, $\vdash \varphi \leftrightarrow t(\varphi)$.*

Proof. The proof follows by induction on the complexity of φ. We only prove the case $\varphi = [\psi]_G[\chi]_{G'}\eta$. For the other cases, see [21, p. 188].

IH: For all $\psi \in \mathcal{L}$ with $c(\psi) \leq c(\varphi)$, $\vdash \psi \leftrightarrow t(\psi)$.
Case: $\varphi = [\psi]_G[\chi]_{G'}\eta$.

1.	$\vdash [\chi]_{G'}\eta \leftrightarrow t([\chi]_{G'}\eta)$	by IH
2.	$\vdash [\psi]_G[\chi]_{G'}\eta \leftrightarrow [\psi]_G t([\chi]_{G'}\eta)$	(DRp4)
3.	$\vdash [\psi]_G t([\chi]_{G'}\eta) \leftrightarrow t([\psi]_G t([\chi]_{G'}\eta))$	by IH*
4.	$\vdash [\psi]_G[\chi]_{G'}\eta \leftrightarrow t([\psi]_G t([\chi]_{G'}\eta))$	Propositional taut., (DRp1), 2, 3

*: $c([\psi]_G t([\chi]_{G'}\eta)) < c([\psi]_G[\chi]_{G'}\eta)$ by Lemma 19.

Therefore, since $t([\psi]_G t([\chi]_{G'}\eta)) = t([\psi]_G[\chi]_{G'}\eta)$ (see Definition 17), we have $\vdash [\psi]_G[\chi]_{G'}\eta \leftrightarrow t([\psi]_G[\chi]_{G'}\eta)$. □

Theorem 20 shows that we can do inside-out reduction in the proof system of $\mathbf{sPAL}_{int}$. In case we were to axiomatize $\mathbf{sPAL}_{int}$ by the Necessitation Rule and the K-axiom for the dynamic modality for $[]_G$ instead of (DRp4), the derivation presented for the case $[\psi]_G[\chi]_{G'}\eta$ would be slightly different in the following way:

1.	$\vdash [\chi]_{G'}\eta \leftrightarrow t([\chi]_{G'}\eta)$	by IH
2.	$\vdash [\psi]_G([\chi]_{G'}\eta \leftrightarrow t([\chi]_{G'}\eta))$	(Nec for $[\ ]_G$)
3.	$\vdash [\psi]_G[\chi]_{G'}\eta \leftrightarrow [\psi]_G t([\chi]_{G'}\eta))$	(K-Axiom for $[\ ]_G$, DRp1, 1, 2)
4.	$\vdash [\psi]_G t([\chi]_{G'}\eta) \leftrightarrow t([\psi]_G t([\chi]_{G'}\eta))$	by IH*
5.	$\vdash [\psi]_G[\chi]_{G'}\eta \leftrightarrow t([\psi]_G t([\chi]_{G'}\eta))$	Propositional taut., (DRp1), 3, 4

Theorem 21. $\mathbf{sPAL}_{int}$ *is complete with respect to the class of all topo-models.*

Proof. Let $\varphi \in \mathcal{L}$ such that $\varphi \notin \mathbf{sPAL}_{int}$. Then, by Theorem 20, we obtain $t(\varphi) \notin \mathbf{sPAL}_{int}$. Since $\mathbf{EL}_{int} \subseteq \mathbf{sPAL}_{int}$, we have $t(\varphi) \notin \mathbf{EL}_{int}$ (note that $t(\varphi) \in \mathcal{L}_{EL_{int}}$). Then, by Theorem 15, there exists a topo-model $\mathcal{M} = (X, \tau, \Phi, V)$ and a neighbourhood situation such that $\mathcal{M}, (x, \theta) \not\models t(\varphi)$. Then, by the soundness of $\mathbf{sPAL}_{int}$, we have $\mathcal{M}, (x, \theta) \not\models \varphi$. □

5 Conclusion, Further Results, and Future Work

We presented a multi-agent logic of knowledge and change of knowledge interpreted on topological structures. The dynamic part consisted of semi-private announcements to subgroups. We then modeled public announcements as a special case. We provided a complete axiomatization of our logic. We presented three detailed examples. While the first example consists in an 8-state model, the other examples are about infinite binary strings and about rational and irrational numbers.

Our results generalize to weaker kinds of knowledge than **S5**: our setting also accounts for the **S4**, **S4.2** and **S4.3** types of knowledge. These logics have also been defended as true characterizations of knowledge: see (e.g.) [20] for **S4**, [22, 25] for **S4.2** and [26, 9] for **S4.3**. Such logics have also been studied on topological spaces as epistemic logics for agents with different reasoning powers, and also with proper dynamic extensions [5, 6, 4, 23]. One can adapt the notion of neighbourhood function (Definition 7) such that these weaker notions of knowledge can be combined with the interior modality. To get an **S4**-type topo-model, replace Condition 3 of Definition 7 by

3. *if* $y \in \theta(x)(i)$ *then* $\theta(y)(i) \subseteq \theta(x)(i)$*,*

and remove Condition 4. Similarly, to get an **S4.2**-type topo-model, we add the following condition to the **S4**-type topo-model

3′. *if* $y, z \in \theta(x)(i)$ *then* $\theta(y)(i) \cap \theta(z)(i) \neq \emptyset$*,*

and for **S4.3** we add

3′. *if* $y, z \in \theta(x)(i)$*, then either* $\theta(y)(i) \subseteq \theta(z)(i)$ *or* $\theta(z)(i) \subseteq \theta(y)(i)$*.*

We can then also work with the weakenings of $\mathbf{sPAL}_{int}$ based on the epistemic systems **S4** and **S4.3** for the modalities K_i by simply closing the neighbourhood function sets under Condition 4 of Definition 7, however we cannot obtain such a dynamic extension for **S4.2**: the characterizing axiom $\hat{K}_i K_i \varphi \to K_i \hat{K}_i \varphi$ (for 'confluence', 'Church-Rosser') may no longer hold after update, as the intersection of updated open neighbourhoods $\theta^\varphi(y)(i) \cap \theta^\varphi(z)(i)$ may have become empty after the refinement. The details of our various results for these **S4** extensions are not presented in this paper.

For further research we wish to investigate whether $\mathbf{sPAL}_{int}$ is expressive enough to model all non-public forms of dynamics (that are **S5** preserving), and not merely semi-private announcements. A first step in such a project could be to study completely private annoucenements because in combination with obvious program manipulations they are already sufficiently expressive to describe all dynamics, e.g., all action models, see [17, 16]. Such results may possibly carry over to a topological setting.

In this work, the observation component of knowledge and information dynamics is represented mostly in the semantics by means of open sets. We can further extend our syntax by observation modalities and belief and study the connection between knowledge, belief and observation together in one framework (as in [7]). An extension with the original effort modality of [24] is also of great interest.

References

[1] Balbiani, P., Baltag, A., van Ditmarsch, H., Herzig, A., Hoshi, T., and de Lima, T. (2008) 'Knowable' as 'known after an announcement'. *The Review of Symbolic Logic*, **1**, 305–334.

[2] Balbiani, P. and van Ditmarsch, H. (2015) A simple proof of the completeness of APAL. *Studies in Logic*, **8(1)**, 65–78.

[3] Balbiani, P., van Ditmarsch, H., and Kudinov, A. (2013) Subset space logic with arbitrary announcements. *Proc. of the 5th ICLA*, pp. 233–244, Springer.

[4] Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2013) The topology of belief, belief revision and defeasible knowledge. *Proc. of the 4th LORI*, pp. 27–40.

[5] Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2014) The topology of full and weak belief, ILLC report PP-2014-03.

[6] Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2015) The topological theory of belief, ILLC report PP-2015-18.

[7] Baltag, A., Bezhanishvili, N., Özgün, A., and Smets, S. (2016) Justified belief and the topology of evidence. *Proc. of WOLLIC 2016*, pp. 83–103.

[8] Baltag, A., Moss, L., and Solecki, S. (1998) The logic of public announcements, common knowledge, and private suspicions. *Proc. of 7th TARK*, pp. 43–56, Morgan Kaufmann.

[9] Baltag, A. and Smets, S. (2008) A qualitative theory of dynamic interactive belief revision. *Logic and the Foundations of Game and Decision Theory (LOFT7)*, pp. 11–58.

[10] Baltag, A., van Ditmarsch, H., and Moss, L. S. (2008) Epistemic logic and information update. *Handbook of Phil. of Sciences*, vol. 8, pp. 361–455, Elsevier Science Publishers.

[11] Baskent, C. (2007) *Topics in Subset Space Logic*. Master's thesis, University of Amsterdam, University of Amsterdam.

[12] Baskent, C. (2012) Public announcement logic in geometric frameworks. *Fundam. Inform.*, **118**, 207–223.

[13] Bjorndahl, A. (2013) Subset space public announcement logic revisited. *CoRR*, **abs/1302.4009**.

[14] Dugundji, J. (1965) *Topology*. Allyn and Bacon Series in Advanced Mathematics, Prentice Hall.

[15] Engelking, R. (1989) *General Topology*. Sigma series in Pure Mathematics, Berlin: Heldermann, revised and completed edn.

[16] French, T., Hales, J., and Tay, E. (2014) A composable language for action models. Goré, R., Kooi, B., and Kurucz, A. (eds.), *Advances in Modal Logic 10*, pp. 197–216, College Publications.

[17] Gerbrandy, J. (1999) *Bisimulations on Planet Kripke*. Ph.D. thesis, University of Amsterdam, ILLC Dissertation Series DS-1999-01.

[18] Heinemann, B. (2008) Topology and knowledge of multiple agents. *Proc. of the 11th IBERAMIA*, pp. 1–10, Springer.

[19] Heinemann, B. (2010) Logics for multi-subset spaces. *Journal of Applied Non-Classical Logics*, **20**, 219–240.

[20] Hintikka, J. (1962) *Knowledge and Belief*. Ithaca, N.Y.,Cornell University Press.

[21] Kooi, B. (2011) Dynamic epistemic logic. van Benthem, J. and ter Meulen, A. (eds.), *Handbook of Logic and Language*, pp. 671–690, Elsevier, second edition.

[22] Lenzen, W. (1978) *Recent Work in Epistemic Logic*, vol. 30. North Holland: Acta Philosophica Fennica.

[23] Ma, M. and Sano, K. (2015) How to update neighbourhood models. *Journal of Logic and Computation*, published online `doi:10.1093/logcom/exv026`.

[24] Moss, L. S. and Parikh, R. (1992) Topological reasoning and the logic of knowledge. *Proc. of the 4th TARK*, pp. 95–105, Morgan Kaufmann.

[25] Stalnaker, R. (2006) On logics of knowledge and belief. *Philosophical Studies*, **128**, 169–199.

[26] van der Hoek, W. (1991) Systems for knowledge and beliefs. van Eijck, J. (ed.), *Logics in AI*, vol. 478 of *LNCS*, pp. 267–281, Springer Berlin Heidelberg.

[27] van Ditmarsch, H. (2000) *Knowledge games*. Ph.D. thesis, University of Groningen, ILLC Dissertation Series DS-2000-06.

[28] van Ditmarsch, H., Halpern, J., van der Hoek, W., and Kooi, B. (2015) *Handbook of Epistemic Logic*. College Publications.

[29] van Ditmarsch, H., Knight, S., and Özgün, A. (2014) Arbitrary announcements on topological subset spaces. *Proc. of the 12th EUMAS*, pp. 252–266, Springer.

[30] van Ditmarsch, H., Knight, S., and Özgün, A. (2015) Announcement as effort on topological spaces. *Proc. of the 15th TARK*, pp. 95–102.

[31] van Ditmarsch, H., van der Hoek, W., and Kooi, B. (2007) *Dynamic Epistemic Logic*, vol. 337 of *Synthese Library*. Springer.

[32] Vickers, S. (1989) *Topology via logic*. Cambridge Tracts in Theoretical Computer Science, University Press.

[33] Wang, Y. and Cao, Q. (2013) On axiomatizations of public announcement logic. *Synthese*, **190**, 103–134.

[34] Wang, Y. N. and Ågotnes, T. (2013) Multi-agent subset space logic. *Proc. of the 23rd IJCAI*, pp. 1155–1161, IJCAI/AAAI.

[35] Wáng, Y. N. and Ågotnes, T. (2013) Subset space public announcement logic. *Proc. of 5th ICLA*, pp. 245–257, Springer.

[36] Zvesper, J. (2010) *Playing with Information*. Ph.D. thesis, University of Amsterdam, ILLC, PhD Thesis.

Appendices

A From public to private announcement in Kripke models

Consider two agents 1 and 2 who are uncertain about the value of a proposition p, and such that this ignorance is known to them. Model *i* in Figure 4 encodes this situation. The result of a public announcement of p is model *ii*, where both 1 and 2 have learned p. The $\neg p$ world has been eliminated by this announcement. At this stage, we observe that an equally good way to describe the result of this public announcement is the model *iii*, where the $\neg p$ world is not eliminated but where the accessibility links between the p and the $\neg p$ world are cut. *Refinement* is an alternative semantics for public announcement logic (rather than *restriction*). From the perspective of the agents there is indeed no difference between the two models. Given that p is true, both in *ii* and *iii* the agents 1 and 2 have common knowledge of p. The $\neg p$ world is inaccessible.

The result of a private announcement to agent 1 that p is the model *iv*. We note that in the top right p world agent 1 knows that p, whereas agent 2 believes that both agents are still ignorant about p. In other words, 2 believes that nothing happened. This belief is incorrect. It is not knowledge. Just like for public announcement, for private announcement it does not hurt to also keep the original state where p is false, as in model *v*. From that perspective it represents 1 being privately informed that $\neg p$. The beliefs of both 1 and 2 in the top right world remain the same in *iv* and in *v*, as the top left $\neg p$ world is inaccessible.

The difference between models v and vi is that now 2 is no longer unaware of 1 being informed of p, but that 2 considers it possible that 1 is informed about the value of p. However, 2 still keeps his options open; he also considers it possible that 1 was not informed at all. We note that 2 has universal access in this model. Unlike v, in model vi the agents' epistemic stances are again described as *knowledge*: their accessibility relations are equivalence relations.

Now that we have vi, the step to vii is fairly small: vii represents the value of p being privately announced to 1 (or, from the perspective of the actual world where p is true: p is privately announced to 1), whereas 2 learns that 1 learns the value of p; and where both agents are aware of this (so that the action has some public character as well, it is not truly private). The model vii consists of the top row of model vi: the alternative where 1 did not learn anything is now ruled out. Again, this is a model where both agents have knowledge: their accessibility relations are equivalence relations. The transition from model i to model vii goes under the name of *semi-private* (or *semi-public*) announcement.

Instead of private announcements to individual agents we can consider private announcements to subgroups of agents: among the members of the addressed subgroup, it functions as a public announcement, whereas the remaining agents think nothing happens. In that sense the public announcement of p is a private announcement of p to agents 1 and 2. When we have private announcement to subgroups, there are other connections between the depicted models. A private announcement of φ to agents in group G can be called the announcement where the agents in G *learn* that φ. A semi-public announcement of p to 1 can alternatively be described as the action where 1 and 2 learn that (1 learns p or 1 learns $\neg p$) (and where the first is really the case). In other words, a semi-private announcement can be described in terms of private announcements.

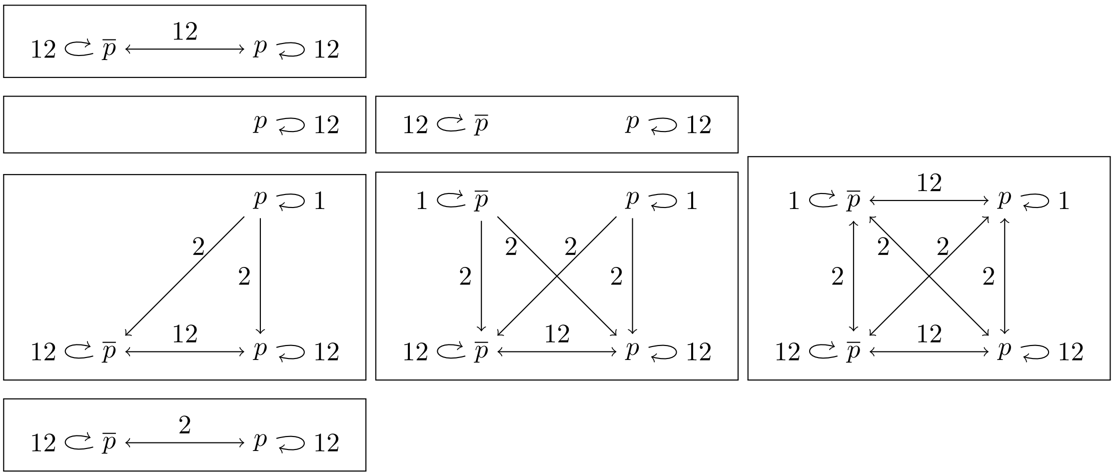

Figure 4: Different ways to announce to agent 1 that p is true. Figures are numbered i to vii from top to bottom and from left to right. In worlds denoted $\overline{p}$ atom p is false. In all cases we assume that the (top) right world is the actual world.

B Proof of Theorem 16

Lemma 22. *For any $\mathcal{M} = (X, \tau, \Phi, V)$, $\theta \in \Phi$ and $\varphi, \psi \in \mathcal{L}$;*

1. $[\![int(\varphi)]\!]^\theta = Int([\![\varphi]\!]^\theta)$
2. $[\![int(\varphi) \wedge [\varphi]_G\psi]\!]^\theta = [\![\langle\varphi\rangle_G\psi]\!]^\theta$
3. $[\![\langle\varphi\rangle_G\psi]\!]^\theta \subseteq [\![\psi]\!]^{\theta^\varphi_G} \subseteq [\![[\varphi]_G\psi]\!]^\theta$

Proof. See [30, Proposition 15] for (1) and [30, Proposition 16.2] for (2). For 3, we have:

$$[\![\langle\varphi\rangle_G\psi]\!]^\theta = Int([\![\varphi]\!]^\theta) \cap [\![\psi]\!]^{\theta^\varphi_G} \subseteq [\![\psi]\!]^{\theta^\varphi_G} \subseteq (\mathcal{D}(\theta) \setminus Int([\![\varphi]\!]^\theta)) \cup [\![\psi]\!]^{\theta^\varphi_G} = [\![[\varphi]_G\psi]\!]^\theta.$$

□

Theorem 16. $\mathbf{sPAL}_{int}$ *is sound with respect to the class of all topo-models.*

Here we only show that (Rp4), (Rp5-i) and (Rp5-ī) are valid and (DRp4) preserves validity. The rest is straightforward and the soundness of the static part follows from the soundness of EL_{int} (see [30]).

Let $\mathcal{M} = (X, \tau, \Phi, V)$ be a topo-model and $(x, \theta) \in \mathcal{M}$.
(Rp4):

($\Rightarrow$) Let $(x, \theta) \models [\varphi]_G int(\psi)$.

$$\begin{array}{lll} (x,\theta) \models [\varphi]_G int(\psi) & \text{iff} & x \in Int([\![\varphi]\!]^\theta) \text{ implies } (x, \theta^\varphi_G) \models int(\psi) \\ & \text{iff} & x \in Int([\![\varphi]\!]^\theta) \text{ implies } x \in Int([\![\psi]\!]^{\theta^\varphi_G}) \end{array}$$

Now suppose $(x, \theta) \models int(\varphi)$, i.e., $x \in Int([\![\varphi]\!]^\theta)$. Then, by assumption, $x \in Int([\![\psi]\!]^{\theta^\varphi_G})$. Therefore, by Lemma 22.3, we obtain $x \in Int([\![[\varphi]_G\psi]\!]^\theta)$. Then, by Lemma 22.1, we have $x \in [\![int([\varphi]_G\psi)]\!]^\theta$, i.e., $(x, \theta) \models int([\varphi]_G\psi)$.

($\Leftarrow$) Suppose $(x, \theta) \models int(\varphi) \to int([\varphi]_G\psi)$ and suppose $(x, \theta) \models int(\varphi)$ (i.e., $x \in Int([\![\varphi]\!]^\theta)$). Then, we have $(x, \theta) \models int([\varphi]_G\psi)$, i.e., $x \in Int([\![[\varphi]_G\psi)]\!]^\theta)$. Therefore $x \in Int([\![\varphi]\!]^\theta) \cap Int([\![[\varphi]_G\psi)]\!]^\theta) = Int([\![int(\varphi)^\theta]\!]) \cap Int([\![[\varphi]_G\psi)]\!]^\theta) = Int([\![int(\varphi)^\theta]\!] \cap [\![[\varphi]_G\psi]\!]^\theta) = Int([\![int(\varphi) \wedge [\varphi]_G\psi]\!]^\theta)$. Then, by Lemma 22.2 and 22.3, we have $x \in Int([\![\psi]\!]^{\theta^\varphi_G})$, i.e, $(x, \theta^\varphi_G) \models int(\psi)$.

(Rp5-i): $j \in G$

($\Rightarrow$) Let $(x, \theta) \models [\varphi]_G K_j\psi$.

$$\begin{array}{lll} (x,\theta) \models [\varphi]_G K_j\psi & \text{iff} & x \in Int([\![\varphi]\!]^\theta) \text{ implies } (x, \theta^\varphi_G) \models K_j\psi \\ & \text{iff} & x \in Int([\![\varphi]\!]^\theta) \text{ implies } \forall y \in \theta^\varphi_G(x)(j), (y, \theta^\varphi_G) \models \psi \end{array}$$

Let $z \in \theta(x)(j)$ and suppose $(z, \theta) \models int(\varphi)$, i.e., $z \in Int([\![\varphi]\!]^\theta)$. By Definition 9, we have $\theta^\varphi_G(x)(j) = \theta(x)(j) \cap Int([\![\varphi]\!]^\theta)$. Therefore, $z \in \theta^\varphi_G(x)(j)$. Then, by assumption, we obtain $(z, \theta^\varphi_G) \models \psi$. Thus, $(z, \theta^\varphi_G) \models [\varphi]_G\psi$. Since z has been chosen arbitrarily from $\theta(x)(j)$, we have $(x, \theta) \models K_j[\varphi]_G\psi$.

(⇐) Let $(x,\theta) \models int(\varphi) \to K_j[\varphi]_G\psi$. Suppose moreover that $(x,\theta) \models int(\varphi)$ and let $z \in \theta^{\varphi}_G(x)(j)$. Note that $\theta^{\varphi}_G(x)(j)$ is non-empty since $x \in \theta^{\varphi}_G(x)(j)$ and $\theta^{\varphi}_G(x)(j) = \theta(x)(j) \cap Int(\llbracket\varphi\rrbracket^{\theta})$, by Definition 9. By assumption, we have $(x,\theta) \models K_j[\varphi]_G\psi$. Then, as $z \in \theta(x)(j) \cap Int(\llbracket\varphi\rrbracket^{\theta})$, we obviously obtain $(z,\theta^{\varphi}_G) \models \psi$. Since z has been chosen arbitrarily from $\theta^{\varphi}_G(x)(j)$, we have $(x,\theta^{\varphi}_G) \models K_j\psi$.

(Rp5-ī): $j \notin G$

(⇒) Let $(x,\theta) \models [\varphi]_G K_j\psi$.

$$\begin{array}{lll}(x,\theta) \models [\varphi]_G K_j\psi & \text{iff} & x \in Int(\llbracket\varphi\rrbracket^{\theta}) \text{ implies } (x,\theta^{\varphi}_G) \models K_j\psi \\ & \text{iff} & x \in Int(\llbracket\varphi\rrbracket^{\theta}) \text{ implies } \forall y \in \theta^{\varphi}_G(x)(j), (y,\theta^{\varphi}_G) \models \psi\end{array}$$

Now suppose $(x,\theta) \models int(\varphi)$ and $z \in \theta(x)(j)$. We want to show that $(z,\theta) \models [\varphi]_G\psi$ and $(z,\theta) \models [\neg\varphi]_G\psi$, i.e., we want to show:

1. $z \in Int(\llbracket\varphi\rrbracket^{\theta})$ implies $(z,\theta^{\varphi}_G) \models \psi$, and
2. $z \in Int(\llbracket\neg\varphi\rrbracket^{\theta})$ implies $(z,\theta^{\varphi}_G) \models \psi$.

1. Suppose $z \in Int(\llbracket\varphi\rrbracket^{\theta})$. Then, $z \in \theta(x)(j) \cap Int(\llbracket\varphi\rrbracket^{\theta})$. Therefore, by definition of θ^{φ}_G and since $j \notin G$, we obtain $z \in \theta^{\varphi}_G(x)(j)$. Then, since by assumption $(x,\theta) \models [\varphi]_G K_j\psi$, as shown above it follows that, $(z,\theta^{\varphi}_G) \models \psi$.

2. Suppose $z \in Int(\llbracket\neg\varphi\rrbracket^{\theta})$. Then, $z \in \theta(x)(j) \cap Int(\mathcal{D}(\theta) \setminus \llbracket\varphi\rrbracket^{\theta})$. The rest follows similarly.

Therefore, $(x,\theta) \models K_j[\varphi]_G\psi$ and $(x,\theta) \models K_j[\neg\varphi]_G\psi$.

(⇐) Let $(x,\theta) \models int(\varphi) \to K_j[\varphi]_G\psi \land K_j[\neg\varphi]_G\psi$. We want to show $(x,\theta) \models [\varphi]_G K_j\psi$. Suppose $(x,\theta) \models int(\varphi)$. Then, by assumption, $(x,\theta) \models K_j[\varphi]_G\psi \land K_j[\neg\varphi]_G\psi$. This means, for all $y \in \theta(x)(j)$:

1. if $y \in Int(\llbracket\varphi\rrbracket^{\theta})$, then $(y,\theta^{\varphi}_G) \models \psi$, and
2. if $y \in Int(\llbracket\neg\varphi\rrbracket^{\theta})$, then $(y,\theta^{\neg\varphi}_G) \models \psi$

Observe that $\theta^{\varphi}_G = \theta^{\neg\varphi}_G$. Therefore, (2) means that if $y \in Int(\llbracket\neg\varphi\rrbracket^{\theta})$, then $(y,\theta^{\varphi}_G) \models \psi$. Since $\mathcal{D}(\theta^{\varphi}_G) = Int(\llbracket\varphi\rrbracket^{\theta}) \cup Int(\llbracket\neg\varphi\rrbracket^{\theta})$, we obtain, by (1) and (2) that for all $y \in \theta^{\varphi}_G(x)(j)$, $(y,\theta^{\varphi}_G) \models \psi$, i.e., $(x,\theta^{\varphi}_G) \models K_j\psi$. We therefore have $(x,\theta) \models [\varphi]_G K_j\psi$.

(DRp4): Let $\varphi,\psi,\chi \in \mathcal{L}$ and suppose $\models \varphi \leftrightarrow \psi$. We want to show that $\models \chi \leftrightarrow \chi[\varphi/\psi]$, where $\chi[\varphi/\psi]$ denotes any formula obtained by replacing one or more non-dynamic occurrences of φ in χ with ψ. The proof follows by induction on χ in case $\varphi \in Sub(\chi)$. Observe that if $\varphi \notin Sub(\chi)$, we have $\chi := \chi[\varphi/\psi]$, therefore $\models \chi \leftrightarrow \chi[\varphi/\psi]$ is vacuously true. Now suppose $\varphi \in Sub(\chi)$.

Base Case: $\chi = \varphi$

Then, $\chi[\varphi/\psi] = \psi$. Therefore, $\models \chi \leftrightarrow \chi[\varphi/\psi]$ can be written as $\models \varphi \leftrightarrow \psi$ and this is the case by assumption.

IH: For all $\eta \in \mathcal{L}$ with $c(\eta) < c(\chi)$, $\models \eta \leftrightarrow \eta[\varphi/\psi]$.

Case: $\chi = \neg\eta$

Note that $(\neg\eta)[\varphi/\psi] = \neg(\eta[\varphi/\psi])$. Therefore,

$$\begin{array}{rll} (x,\theta) \models \neg\eta & \text{iff} & (x,\theta) \not\models \eta \\ & \text{iff} & (x,\theta) \not\models \eta[\varphi/\psi] \qquad \text{by (IH)} \\ & \text{iff} & (x,\theta) \models \neg(\eta[\varphi/\psi]) \\ & \text{iff} & (x,\theta) \models (\neg\eta)[\varphi/\psi] \end{array}$$

Case: $\chi = \eta \wedge \zeta$
Note that $(\eta \wedge \zeta)[\varphi/\psi] = \eta[\varphi/\psi] \wedge \zeta[\varphi/\psi]$. Therefore,

$$\begin{array}{rll} (x,\theta) \models (\eta \wedge \zeta)[\varphi/\psi] & \text{iff} & (x,\theta) \models \eta[\varphi/\psi] \wedge \zeta[\varphi/\psi] \\ & \text{iff} & (x,\theta) \models \eta \wedge \zeta \qquad \text{by (IH)} \end{array}$$

Case: $\chi = int(\eta)$
Note that $(int(\eta))[\varphi/\psi] = int(\eta[\varphi/\psi])$. Therefore,

$$\begin{array}{rll} (x,\theta) \models (int(\eta))[\varphi/\psi] & \text{iff} & (x,\theta) \models int(\eta[\varphi/\psi]) \\ & \text{iff} & x \in Int([\![\eta[\varphi/\psi]]\!]^\theta) \\ & \text{iff} & x \in Int([\![\eta]\!]^\theta) \qquad \text{by (IH)} \\ & \text{iff} & (x,\theta) \models int(\eta) \end{array}$$

Case: $\chi = K_i(\eta)$
Note that $(K_i\eta)[\varphi/\psi] = K_i\eta[\varphi/\psi]$. Therefore,

$$\begin{array}{rll} (x,\theta) \models (K_i\eta)[\varphi/\psi] & \text{iff} & (x,\theta) \models K_i\eta[\varphi/\psi] \\ & \text{iff} & \forall y \in \theta(x)(i), (y,\theta) \models \eta[\varphi/\psi] \\ & \text{iff} & \forall y \in \theta(x)(i), (y,\theta) \models \eta \qquad \text{by (IH)} \\ & \text{iff} & (x,\theta) \models K_i\eta \end{array}$$

Case: $\chi = [\eta]_G\zeta$
Note that $([\eta]_G\zeta)[\varphi/\psi] = [\eta]_G\zeta[\varphi/\psi]$. Recall that we replace non-dynamic occurrences of φ in $[\eta]_G\zeta$, i.e., the occurrences outside of any $[\ ]_G$. Therefore, in this particular case, there is no replacing in η. Then, we have

$$\begin{array}{rll} (x,\theta) \models ([\eta]_G\zeta)[\varphi/\psi] & \text{iff} & (x,\theta) \models [\eta]_G\zeta[\varphi/\psi] \\ & \text{iff} & (x,\theta) \models int(\eta) \text{ implies } (x,\theta_G^\varphi) \models \zeta[\varphi/\psi] \\ & \text{iff} & (x,\theta) \models int(\eta) \text{ implies } (x,\theta_G^\varphi) \models \zeta \qquad \text{by (IH)} \\ & \text{iff} & (x,\theta) \models [\eta]_i\zeta \end{array}$$