Arbitrary Announcements on Topological Subset Spaces

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Abstract. Subset space semantics for public announcement logic in the spirit of the effort modality have been proposed by Wang and Ågotnes [17] and by Bjorndahl [6]. They propose to model the public announcement modality by shrinking the epistemic range with respect to which a postcondition of the announcement is evaluated, instead of by restricting the model to the set of worlds satisfying the announcement. Thus we get an "elegant, model-internal mechanism for interpreting public announcements" [6, p.12]. In this work, we extend Bjorndahl's logic PAL_{int} of public announcement, which is modelled on topological spaces using subset space semantics and adding the interior operator, with an arbitrary announcement modality, and we provide topological subset space semantics for the corresponding arbitrary announcement logic $APAL_{int}$, and demonstrate completeness of the logic by proving that it is equal in expressivity to the logic without arbitrary announcements, employing techniques from [2, 13].

1 Introduction

In [7], Dabrowski et al. introduce a bimodal modal logic called *subset space logic* (SSL) in order to capture the notions of knowledge and effort (to obtain knowledge). It has a knowledge modality K and an effort modality $\Box$. The authors proposed a 'topological semantics' called subset space semantics for this logic. This semantics is not necessarily based on topological spaces, however, topological reasoning provides the intuition behind the semantics and constitutes an important instance; [1] treats the more purely topological case. In the setting of [7], unlike the standard evaluation of K on Kripke models, both modal operators K and $\Box$ are evaluated not only with respect to a state but also with respect to a *neighbourhood* of a given possible world, i.e., with respect to pairs of the form (x, U), where the evaluation state x represents *the real/actual world* and the neighbourhood U serves as a *truthful observation*: we can think of the neighbourhood U as what an agent can observe from where she stands, that is, a set of states that the agent thinks the actual world may belong to. Hence, by following the idea of 'obtaining knowledge by means of an observation,' they propose to evaluate K 'locally' in a given neighbourhood of a subset space. Moreover, the effort is interpreted as *open-set-shrinking* on subset spaces where more effort corresponds to a smaller neigbourhood, thus, to a better approximation of where the real world is [1].
More formally, the language used by Dabrowski et al. [7] is

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid \Box\varphi.$$

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A *subset space* is a pair consisting of a non-empty set called the *domain* and a certain collection of subsets of the domain. These subsets are called *open* sets and a *neighbourhood of a state x* is any open set including x. The crucial effort operator $\Box$ is interpreted as

> *Pair (x, U) satisfies $\Box\varphi$ iff for all V containing x and contained in U, (x, V) satisfies φ.*

where U and V are neighbourhoods of x. On the other hand, the knowledge formula $K\varphi$ is interpreted 'globally' within the corresponding neighbourhood U in a standard way as truth of φ at all points in U (this is why knowledge in SSL is of **S5**-character). However, restriction to a particular neighbourhood makes the evaluation of $K\varphi$ 'local' within the model in the sense that only the states in a given neighbourhood U need to be checked for the truth of φ. More precisely,

Pair (x, U) satisfies $K\varphi$ iff for all y in U, (y, U) satisfies φ.

A typical formula schema of this logic appearing in the SSL-literature (see, e.g., [1, 6]) is $\varphi \to \Diamond K\varphi$, which says that if φ is true, then after some effort the agent comes to know that it is true. This formula is of particular importance since it links SSL to the notion of 'knowability/learnability' (more details below). Besides its epistemic importance, if we evaluate this formula on a topological space and if φ is not a modal formula, the schema is true on the topological model iff the truth set of φ is an open set [1, 7]. Hence, SSL can be and is used to reason about elementary topology.

In [2], Balbiani et al. introduce a logic to quantify over announcements in the setting of epistemic logic. This *arbitrary public announcement logic* has (in the single agent version) the language

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid [\varphi]\varphi \mid \blacksquare\varphi$$

The construct $[\varphi]\psi$ stands for 'after public announcement of φ, ψ (is true)'. Throughout this work, we assume that announcements convey truthful, hard information. In a given model, the effect of the announcement of a formula in general is a model restriction to the subset satisfying the formula: $[\varphi]\psi$ is true iff after restriction of the model to the states satisfying φ, ψ is true in the restricted model. In this case the modality $\blacksquare$ quantifies over announcements and $\blacksquare\varphi$ means 'after any announcement, φ (is true)'. Its semantics is therefore

> *State x satisfies $\blacksquare\varphi$ iff for all announcements ψ true at x the model restriction to ψ satisfies φ at x,*

where the announcement ψ above does not contain $\blacksquare$.

A typical formula schema in this logic is again $\varphi \to \blacklozenge K\varphi$, which says that if φ is true, then there is an announcement after which the agent comes to know that it is true. This can be seen as an interpretation of 'knowability' à la Fitch [9, 14] where 'knowable' is interpreted as 'known after an announcement' [2, 14]. Clearly, the modality 'restriction to any submodel' ($\blacksquare$) is very much related in motivation to the modality 'restriction to any smaller neighbourhood' ($\Box$)' and this has indeed become the topic of subsequent works.

The effort modality has a dynamic nature as does the arbitrary announcement modality. As mentioned, it is evaluated on subset spaces by shrinking the initial open neighbourhood where *open-set-shrinking* represents receiving new information by means of *any effort* such as measurement, observation, computation, approximation etc. [7, 1, 4, 15]. More importantly for this work, the information intake represented by the effort modality is not necessarily via public announcements, however, it implicitly captures any kind of information gain including public announcements. Therefore, given such a dynamic operator on subset spaces, and extensive research on public announcement logics and the intuitive connection between the two, it is natural to investigate how to model public announcements on subset spaces and how to link the two in a formal setting. Proposals for the interpretation of public announcement on subset spaces as 'model restriction' include [4, 5, 3]. They propose to model public announcements on subset spaces by deleting the states and/or the neighbourhoods falsifying the announcement. However, this method is obviously not in the spirit of the effort modality in the sense that efforts do not lead to a global model change but lead to a 'local' neighbourhood shrinking. Hence, it is natural to search for an 'open-set-shrinking-like' interpretation of public announcements on subset spaces. To the best of our knowledge, Wang and Ågotnes [17] were the first to propose semantics for public announcements on subset spaces in the spirit of the effort modality, although this is not necessarily on topological spaces. Bjorndahl [6] then proposed a revised version of the [17] semantics. Bjorndahl's models are based on topological spaces and his topological usage of operators such as the interior operator $int(\varphi)$ we find quite natural and intuitive. This operation $int(\varphi)$ means 'φ is true and can be announced' (this will become clear below) and is therefore definable as $\langle\varphi\rangle\top$. Subject to this identity, Bjorndahl's language becomes

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid [\varphi]\varphi$$

where he mentions the arbitrary announcement as a future opportunity for research.

Our contribution to this emerging corpus of work is that we have extended Bjorndahl's proposal with such an arbitrary announcement modality so that we obtain (the language of [2])

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid [\varphi]\varphi \mid \blacksquare\varphi$$

and provide semantics for this language based on subset spaces rather than relational models, where we think that we have come close to the original [7] motivation for the effort modality. We then show completeness for this logic, by way of extending Bjorndahl's axiomatization with axioms and rules, and where the axioms are equivalences. The expressivity of the resulting logic is the same as that of the logic without the $\blacksquare$.

In Section 2 we review Bjorndahl's (topological) subset space logic with public announcements. This is the logical basis for our work. Section 3 contains our own contributions: we extend this logic with the effort-like arbitrary announcement modality and prove some of its properties, such as the **S4** character of this modality, and we demonstrate that this logic is complete and is not more expressive than the logic without the arbitrary announcement modality. Section 4 contains the conclusions and suggestions for further research.

2 Bjorndahl's subset space logic with public announcements

In this section, we start by introducing the basic topological concepts that will be used throughout this paper. For a more detailed discussion of general topology we refer the reader to [8]. We then present Bjorndahl's epistemic and public announcement logics [6], denoted by EL_{int} and PAL_{int} respectively, and the corresponding topological-based subset space semantics.

Definition 1 (Structures) *A* topological space *is a pair* (X, τ)*, where* X *is a non-empty set and* τ *is a family of subsets of* X *containing* X *and* $\emptyset$ *and closed under finite intersections and arbitrary unions. The set* X *is called the* space*. The subsets of* X *belonging to* τ *are called* open sets *(or* opens*) in the space; the family* τ *of open subsets of* X *is called a* topology *on* X*. We denote the opens of a topological space by capital letters such as* U, V, W *etc. Complements of opens are called* closed sets*. An open set containing* $x \in X$ *is called an* (open) neighbourhood *of* x*. The* interior $Int(A)$ *of a set* $A \subseteq X$ *is the largest open set contained in* A*.* [1] *A* topological model *(or* topo-model*)* $\mathcal{X} = (X, \tau, \nu)$ *is a topological space endowed with a valuation map* $\nu : \text{Prop} \to \mathcal{P}(X)$*.*

We denote an *epistemic scenario of a topological space* by (x, U) where $x \in U \in \tau$ and let $EL(\mathcal{X}) = \{(x, U) \mid x \in U \in \tau\}$, the set of epistemic scenarios on $\mathcal{X}$. It is important to emphasize that, in [6], Bjorndahl works with an extension of subset space semantics first introduced in [7] and summarized in Section 1, however, he restricts his models to topological spaces rather than all subset spaces.

Syntax. In [6], Bjorndahl considers the language $\mathcal{L}_{PAL_{int}}$ defined by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid int(\varphi) \mid [\varphi]\varphi$$

where $p \in$ Prop. Without the $[\varphi]$ operator, we get the language $\mathcal{L}_{EL_{int}}$. We employ the usual abbreviations for propositional operators and dual modalities, where in particular $\langle\varphi\rangle\psi$ is defined as $\neg[\varphi]\neg\psi$.

Definition 2 (Semantics for $\mathcal{L}_{PAL_{int}}$) *Given a topo-model* $\mathcal{X} = (X, \tau, \nu)$*, the semantics for the language* $\mathcal{L}_{PAL_{int}}$ *is defined recursively as:*

$$\begin{array}{lll}
\mathcal{X}, (x, U) \models p & \textit{iff} & x \in \nu(p) \\
\mathcal{X}, (x, U) \models \neg\varphi & \textit{iff} & \mathcal{X}, (x, U) \not\models \varphi \\
\mathcal{X}, (x, U) \models \varphi \wedge \psi & \textit{iff} & \mathcal{X}, (x, U) \models \varphi \textit{ and } \mathcal{X}, (x, U) \models \psi \\
\mathcal{X}, (x, U) \models K\varphi & \textit{iff} & (\forall y \in U)(\mathcal{X}, (y, U) \models \varphi) \\
\mathcal{X}, (x, U) \models int(\varphi) & \textit{iff} & x \in Int([\![\varphi]\!]^U) \\
\mathcal{X}, (x, U) \models [\varphi]\psi & \textit{iff} & \mathcal{X}, (x, U) \models int(\varphi) \textit{ implies } \mathcal{X}, (x, Int([\![\varphi]\!]^U)) \models \psi
\end{array}$$

where $p \in$ Prop*, and* $[\![\varphi]\!]^U = \{y \in U \mid \mathcal{X}, (y, U) \models \varphi\}$.

We say that a formula φ is *valid in a topo-model* $\mathcal{X} = (X, \tau, \nu)$, denoted $\mathcal{X} \models \varphi$, iff $\mathcal{X}, (x, U) \models \varphi$ for all $(x, U) \in EL(\mathcal{X})$, and that φ is *valid*, denoted $\models \varphi$, iff for all topo-models $\mathcal{X}$: $\mathcal{X} \models \varphi$. Soundness and completeness with respect to the above semantics are defined as usual.

[1] Equivalently, for any $A \subseteq X$, $Int(A) = \bigcup\{U \in \tau : U \subseteq A\}$.

Let us now have a closer look at the public announcement semantics from Definition 2. As given in the semantic clause for $[\varphi]\psi$, the precondition of an announcement is assumed to be $int(\varphi)$ which is a stronger requirement for being able to announce φ than φ simply being true at the state/epistemic scenario in question (see [6] for differences between these two requirements). Moreover, unlike the standard approach where the announcement of a formula is interpreted as a model restriction that leads to a 'global' change of the initial model, the effect of an announcement of φ in the setting of [6] is 'local': it is a shrinkage of the initial evaluation neighbourhood U to $Int([\![\varphi]\!]^U)$. Therefore, the effect of a public announcement is defined in such a way that it can be seen as information gain via a very *specific kind of effort*.

Theorem 3 ([6]) *The* epistemic logic EL_{int} *is axiomatized completely by the axioms and rules of propositional logic,* **S4** *for int,* **KD45** *for the knowledge modality and* $K\varphi \to int(\varphi)$*. The* logic of public announcements PAL_{int} *is axiomatized completely by the axioms of* EL_{int} *and the reduction axioms given in Proposition 5 (below).*

The system **KD45** for knowledge together with the axiom $K\varphi \to int(\varphi)$ yield $K\varphi \to \varphi$. Thus, the modality K in the logic EL_{int} unsurprisingly is of **S5**-type just like the one in SSL. We continue by reviewing some properties of these logics.

Proposition 4 *For any* $\varphi, \chi, \psi \in \mathcal{L}_{PAL_{int}}$,

1. $\models [\varphi]\psi \leftrightarrow [int(\varphi)]\psi$
2. $\models [\varphi][\psi]\chi \leftrightarrow [int(\varphi)][int(\psi)]\chi \leftrightarrow [int(\varphi) \wedge [int(\varphi)]int(\psi)]\chi \leftrightarrow [int(\varphi) \wedge [\varphi]int(\psi)]\chi$

Moreover, for any topo-model $\mathcal{X} = (X, \tau, \nu)$ *and* $(x, U) \in EL(\mathcal{X})$,

4. $[\![int(\varphi)]\!]^U = Int[\![\varphi]\!]^U$.
5. $[\![int(\psi) \wedge [\psi]int(\chi)]\!]^U = Int[\![\chi]\!]^{Int[\![\psi]\!]^U}$

Proof. The proofs have been removed from this presentation and can be found in [6].

Proposition 4.1 shows that there is no difference between announcing φ and $int(\varphi)$. In other words, $int(\varphi)$ constitutes the *core, essential* part of the information conveyed by the announcement of φ, that is, since $[\![int(\varphi)]\!]^U = Int[\![\varphi]\!]^U \subseteq [\![\varphi]\!]^U$ for any epistemic scenario (x, U), $Int[\![\varphi]\!]^U$ forms the set which represents exactly what an agent can learn from the announcement of φ.

We recall that in public announcement logic we have $[\varphi][\psi]\chi \leftrightarrow [\varphi \wedge [\varphi]\psi]\chi$. Hence, Prop. 4.2 shows that we have a similar principle of iterative announcements in PAL_{int}.

Proposition 5 ([6]) *The following* $\mathcal{L}_{PAL_{int}}$ *schemas are validities.*

1. $[\varphi]\bot \leftrightarrow \neg int(\varphi)$	4. $[\varphi](\psi \wedge \chi) \leftrightarrow [\varphi]\psi \wedge [\varphi]\chi$
2. $[\varphi]p \leftrightarrow (int(\varphi) \to p)$	5. $[\varphi]K\psi \leftrightarrow (int(\varphi) \to K[\varphi]\psi)$
3. $[\varphi]\neg\psi \leftrightarrow (int(\varphi) \to \neg[\varphi]\psi)$	6. $[\varphi][\psi]\chi \leftrightarrow [\langle\varphi\rangle int(\psi)]\chi$

Proof. The first four are straightforward to prove and the proof of (5) is given in [6]. We only prove (6). It has been proven in [6] that $[\varphi][\psi]\chi \leftrightarrow [int(\varphi) \wedge [\varphi]int(\psi)]\chi$ is valid. Hence, here we only prove that $(int(\varphi) \wedge [\varphi]int(\psi)) \leftrightarrow \langle\varphi\rangle int(\psi)$ is valid.

Let $\mathcal{X} = (X, \tau, \nu)$ be a topo-model and (x, U) be an epistemic scenario in $\mathcal{X}$. Then:

$$
\begin{aligned}
&(x, U) \models int(\varphi) \wedge [\varphi]int(\psi)\\
\text{iff } & x \in Int[\![\varphi]\!]^U \text{ and (if } x \in Int[\![\varphi]\!]^U \text{ then } (x, Int[\![\varphi]\!]^U) \models int(\psi))\\
\text{iff } & x \in Int[\![\varphi]\!]^U \text{ and } (x, Int[\![\varphi]\!]^U) \models int(\psi) \text{ (by tautology } p \wedge (p \to q) \leftrightarrow (p \wedge q))\\
\text{iff } & (x, U) \models \langle\varphi\rangle int(\psi)
\end{aligned}
$$

Therefore, by Proposition 4.2, we have that $[\varphi][\psi]\chi \leftrightarrow [\langle\varphi\rangle int(\psi)]\chi$.

3 The logic $APAL_{int}$

We now provide topological subset space semantics for the arbitrary announcement operator $\blacksquare\varphi$. We do so by modifying the public announcement semantics proposed in [6] in a natural way, as a generalization of public announcements. We thus aim to give a semantics for $\blacksquare$ which does not represent global model change, as in [2], but interprets the arbitrary announcement operator $\blacksquare$ in the same initial model, *locally*, in a given *epistemic scenario* in a similar way to the effort modality modelled on subset spaces. By doing so, we link it to the *effort modality* $\Box\varphi$ of [7].

3.1 Syntax and semantics

Syntax. We consider the language $\mathcal{L}_{APAL_{int}}$ obtained by extending $\mathcal{L}_{PAL_{int}}$ with the arbitrary announcement modality $\blacksquare$. In other words, $\mathcal{L}_{APAL_{int}}$ is defined by the following grammar

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K\varphi \mid int(\varphi) \mid [\varphi]\varphi \mid \blacksquare\varphi$$

where $p \in \text{Prop}$. The formulas in $\mathcal{L}_{PAL_{int}}$ are called $\blacksquare$-free formulas.

Recall that the arbitrary announcement modality $\blacksquare\varphi$ is read 'after any announcement, φ is true'. Its semantics is as follows. For the semantics of the other operators, we refer to Def. 2.

Definition 6 (Semantics of arbitrary announcement) *Given a topo-model $\mathcal{X} = (X, \tau, \nu)$, the semantic clause for the arbitrary announcement modality $\blacksquare$ reads*

$$\mathcal{X}, (x, U) \models \blacksquare\varphi \textit{ iff } (\forall\psi \in \mathcal{L}_{PAL_{int}})(\mathcal{X}, (x, U) \models [\psi]\varphi).$$

Proposition 7 (S4 character of $\blacksquare$) *For any $\varphi, \psi \in \mathcal{L}_{APAL_{int}}$,*

1. $\models \blacksquare(\varphi \wedge \psi) \leftrightarrow \blacksquare\varphi \wedge \blacksquare\psi$
2. $\models \blacksquare\varphi \to \varphi$
3. $\models \blacksquare\varphi \to \blacksquare\blacksquare\varphi$
4. $\models \varphi$ *implies* $\models \blacksquare\varphi$

Proof. We only show the third item. These validities demonstrate the similarity of $\blacksquare$ to the arbitrary announcement modality in [2], and their proofs are similar. Instead of proving $\models \blacksquare\varphi \to \blacksquare\blacksquare\varphi$, we will prove $\models \blacklozenge\blacklozenge\varphi \to \blacklozenge\varphi$, which is equivalent. Let

$\mathcal{X} = (X, \tau, \nu)$ be a topo-model and (x, U) be an epistemic scenario in $\mathcal{X}$. We omit $\mathcal{X}$ as it is obvious which model we are talking about.

$$
\begin{aligned}
& (x,U) \models \blacklozenge\blacklozenge\varphi \\
\text{iff } & \exists\psi \in \mathcal{L}_{PAL_{int}} : (x,U) \models \langle\psi\rangle\blacklozenge\varphi \\
\text{iff } & \exists\psi \in \mathcal{L}_{PAL_{int}} : (x,U) \models int(\psi) \text{ and } (x, Int[\![\psi]\!]^U) \models \blacklozenge\varphi \\
\text{iff } & \exists\psi \in \mathcal{L}_{PAL_{int}} : x \in Int[\![\psi]\!]^U \text{ and } (x, Int[\![\psi]\!]^U) \models \blacklozenge\varphi \\
\text{iff } & \exists\psi \in \mathcal{L}_{PAL_{int}} : x \in Int[\![\psi]\!]^U \text{ and } \exists\chi \in \mathcal{L}_{PAL_{int}} : (x, Int[\![\psi]\!]^U) \models int(\chi) \text{ and } (x, Int([\![\chi]\!]^{Int[\![\psi]\!]^U})) \models \varphi \\
\text{iff } & \exists\psi \in \mathcal{L}_{PAL_{int}} : x \in Int[\![\psi]\!]^U \text{ and } \exists\chi \in \mathcal{L}_{PAL_{int}} : x \in Int([\![\chi]\!]^{Int[\![\psi]\!]^U}) \text{ and } (x, Int([\![\chi]\!]^{Int[\![\psi]\!]^U})) \models \varphi \\
\text{iff } & \exists\psi,\chi \in \mathcal{L}_{PAL_{int}} : x \in Int([\![\chi]\!]^{Int[\![\psi]\!]^U}) \text{ and } (x, Int([\![\chi]\!]^{Int[\![\psi]\!]^U})) \models \varphi \\
\text{iff } & \exists\psi,\chi \in \mathcal{L}_{PAL_{int}} : x \in Int[\![int(\psi) \wedge [\psi]int(\chi)]\!]^U \text{ and } (x, Int[\![int(\psi) \wedge [\psi]int(\chi)]\!]^U) \models \varphi \\
\text{iff } & \exists\psi,\chi \in \mathcal{L}_{PAL_{int}} : (x,U) \models int(int(\psi) \wedge [\psi]int(\chi)) \text{ and } (x, Int[\![int(\psi) \wedge [\psi]int(\chi)]\!]^U) \models \varphi \\
\text{iff } & \exists\psi,\chi \in \mathcal{L}_{PAL_{int}} : (x,U) \models \langle int(\psi) \wedge [\psi]int(\chi)\rangle\varphi \\
\text{iff } & \exists\theta \in \mathcal{L}_{PAL_{int}} : (x,U) \models \langle\theta\rangle\varphi \quad (\text{where } \theta : int(\psi) \wedge [\psi]int(\chi)) \\
\text{iff } & (x,U) \models \blacklozenge\varphi
\end{aligned}
$$

3.2 Normal Forms for EL_{int}

In this section, we introduce normal forms for the logic EL_{int} and use the formulas in normal forms in order to provide the expressiveness results in Section 3.3. These normal forms are unique since they are based on subset space semantics and they are an extension of the normal form for basic epistemic logic given in [13] since we allow the modality *int* in our normal forms.

We denote the unimodal language having *int* as its only modality by $\mathcal{L}_{PL_{int}}$.

Definition 8 (Normal form for the language $\mathcal{L}_{EL_{int}}$) *We say a formula $\psi \in \mathcal{L}_{EL_{int}}$ is in* normal form *if it is a disjunction of conjunctions of the form*

$$\delta := \alpha \wedge K\beta \wedge \langle K\rangle\gamma_1 \wedge \cdots \wedge \langle K\rangle\gamma_n$$

where $\alpha, \beta, \gamma_i \in \mathcal{L}_{PL_{int}}$ for all $1 \leq i \leq n$.

Following the notation in [13], we call the formula δ *canonical conjunction* and the subformulas $K\beta$ and $\langle K\rangle\gamma_i$ *prenex formulas*.

Below we will prove that every formula in EL_{int} is equivalent to a formula in normal form, but first we need several results for this proof.

Lemma 9 *If $\psi \in \mathcal{L}_{EL_{int}}$ is in normal form and contains a prenex formula σ, then ψ can be written as $\pi \vee (\lambda \wedge \sigma)$ where π, λ and σ are all in normal form.*

The proof is similar to the proof of the same fact for epistemic logic found in[13, p.35].

Before stating the next propositions, it is important to note that 'local' evaluation of formulas in $\mathcal{L}_{EL_{int}}$ with respect to a neighbourhood of a given state is completely reflected in the interpretation of the knowledge modality:

Observation 10 *For any topological model $\mathcal{X} = (X, \tau, \nu)$, any epistemic scenario (x, U) of $\mathcal{X}$ and any $\varphi \in \mathcal{L}_{APAL_{int}}$, $[\![K\varphi]\!]^U = U$ or $[\![K\varphi]\!]^U = \emptyset$, and $[\![\langle K\rangle\varphi]\!]^U = U$ or $[\![\langle K\rangle\varphi]\!]^U = \emptyset$.*

This observation follows from the fact that if there is any $y \in U$ such that $(y, U) \not\models \varphi$, then for all $x \in U$, $(x, U) \not\models K\varphi$, and otherwise, for every $x \in U$, $(x, U) \models K\varphi$. Observation 10 thus expresses that the modality K behaves like *a universal modality within the given neighbourhood.*

Proposition 11 *We have the following equivalences in EL_{int}:*

1. $\vdash int(K\varphi) \leftrightarrow K\varphi$
2. $\vdash int(\langle K\rangle\varphi) \leftrightarrow \langle K\rangle\varphi$
3. $\vdash int(\varphi \vee K\beta) \leftrightarrow int(\varphi) \vee K\beta$
4. $\vdash int(\varphi \vee \langle K\rangle\beta) \leftrightarrow int(\varphi) \vee \langle K\rangle\beta$
5. $\vdash int(\varphi \vee (\sigma \wedge K\beta)) \leftrightarrow int(\varphi \vee \sigma) \wedge (int(\varphi) \vee K\beta)$
6. $\vdash int(\varphi \vee (\sigma \wedge \langle K\rangle\beta)) \leftrightarrow int(\varphi \vee \sigma) \wedge (int(\varphi) \vee \langle K\rangle\beta)$

Proof. We use a semantic argument for this proof since we can obtain the result by the completeness of EL_{int} with respect to all topological spaces [6, p.9]. Let $\varphi, \beta, \sigma \in \mathcal{L}_{EL_{int}}$, $\mathcal{X} = (X, \tau, \nu)$ be a topo-model and (x, U) be an epistemic scenario of $\mathcal{X}$.

1. ($\Rightarrow$) By the (T)-axiom for *int* .
   ($\Leftarrow$) Suppose $(x, U) \models K\varphi$. This means $x \in [\![K\varphi]\!]^U$, thus by Observation 10, $[\![K\varphi]\!]^U = U$. Then, as U is an open set, $Int[\![K\varphi]\!]^U = U$ and $x \in Int[\![K\varphi]\!]^U$. Therefore, by the semantics of *int*, $(x, U) \models int(K\varphi)$.
2. Similar to (1).
3. ($\Rightarrow$) Suppose $(x, U) \models int(\varphi \vee K\beta)$. We now have that $(x, U) \models int(\varphi \vee K\beta)$ iff $x \in Int[\![\varphi \vee K\beta]\!]^U$, and that $x \in Int[\![\varphi \vee K\beta]\!]^U$ iff $x \in Int([\![\varphi]\!]^U \cup [\![K\beta]\!]^U)$. Then, by Observation 10, we have two cases:
   - [Case 1:] $[\![K\beta]\!]^U = U$
     Then, $(x, U) \models K\beta$, and thus $(x, U) \models int(\varphi) \vee K\beta$.
   - [Case 2:] $[\![K\beta]\!]^U = \emptyset$
     Then, $Int([\![\varphi]\!]^U \cup [\![K\beta]\!]^U) = Int[\![\varphi]\!]^U$. Thus, $x \in Int[\![\varphi]\!]^U$, i.e., $(x, U) \models int(\varphi)$. Therefore, $(x, U) \models int(\varphi) \vee K\beta$.

   ($\Leftarrow$) Suppose $(x, U) \models int(\varphi) \vee K\beta$.
   - [Case 1:] $(x, U) \models int(\varphi)$
     $(x, U) \models int(\varphi)$ means $x \in Int[\![\varphi]\!]^U$. Since $[\![\varphi]\!]^U \subseteq [\![\varphi \vee K\beta]\!]^U$ and $Int[\![\varphi]\!]^U \subseteq Int[\![\varphi \vee K\beta]\!]^U$, we have that $x \in Int[\![\varphi \vee K\beta]\!]^U$. I.e., $(x, U) \models int(\varphi \vee K\beta)$.
   - [Case 2:] $(x, U) \models K\beta$
     This implies, by Observation 10, $[\![K\beta]\!]^U = U$. Thus, $Int[\![\varphi \vee K\beta]\!]^U = U$. Hence, $x \in Int[\![\varphi \vee K\beta]\!]^U$, i.e., $(x, U) \models int(\varphi \vee K\beta)$.
4. Similar to (3) as we have either $[\![\langle K\rangle\varphi]\!]^U = U$ or $[\![\langle K\rangle\varphi]\!]^U = \emptyset$.

$$\begin{aligned} int(\varphi \vee (\sigma \wedge K\beta)) &\leftrightarrow int((\varphi \vee \sigma) \wedge (\varphi \vee K\beta)) \\ &\leftrightarrow int(\varphi \vee \sigma) \wedge int(\varphi \vee K\beta) \\ &\leftrightarrow int(\varphi \vee \sigma) \wedge (int(\varphi) \vee K\beta) \quad \text{(by (3))} \end{aligned}$$

5. Similar to (5), by using (4).

Lemma 12 *The following equivalence is a propositional tautology:*

$$(\varphi_1 \vee \cdots \vee \varphi_n) \wedge (\psi_1 \vee \cdots \vee \psi_m) \leftrightarrow ((\varphi_1 \wedge \psi_1) \vee \ldots (\varphi_1 \wedge \psi_m)) \vee ((\varphi_2 \wedge \psi_1) \vee \ldots \cdots \vee (\varphi_2 \wedge \psi_m)) \vee \cdots \vee ((\varphi_n \wedge \psi_1) \vee \ldots (\varphi_n \wedge \psi_m)).$$

Theorem 13 *Every formula in EL_{int} is equivalent to a formula in normal form.*

Proof. The proof proceeds by induction on the complexity of φ.

– *Base Case $\varphi := p$:* In this case, as $p \in \mathcal{L}_{PL_{int}}$, φ is already in normal form.

  Now assume as an inductive hypothesis that ψ and χ can be written in an equivalent normal form.

– *Case $\varphi := \neg\psi$:* W.l.o.g. we can assume that ψ is in normal form. I.e., $\psi := \delta_1 \vee \cdots \vee \delta_m$ where each δ_i is a canonical conjunction. Thus, $\varphi = \neg\psi := \neg\delta_1 \wedge \cdots \wedge \neg\delta_m$. We can then distribute $\neg$ of each δ_i over the conjuncts. In other words, for each δ_i:

$$\neg\delta_i := \neg(\alpha \wedge K\beta \wedge \langle K\rangle\gamma_1 \wedge \cdots \wedge \langle K\rangle\gamma_n) = \neg\alpha \vee \langle K\rangle\neg\beta \vee K\neg\gamma_1 \vee \cdots \vee K\neg\gamma_n$$

  where $\alpha, \beta, \gamma_i \in \mathcal{L}_{PL_{int}}$ for all $1 \leq i \leq n$. Let us call $\neg\delta_i$ *canonical disjunction*. Notice that each disjunct of $\neg\delta_i$ is still in the required form, i.e., each disjunct is either a prenex formula or in $\mathcal{L}_{PL_{int}}$. By using Lemma 12 repeatedly, we can write φ in normal form, i.e., as disjunctions of canonical conjuncts.

– *Case $\varphi := \psi \wedge \chi$:* W.l.o.g. we can again assume that ψ and χ are in normal form. I.e., $\psi := \delta_1 \vee \cdots \vee \delta_m$ and $\chi := \delta'_1 \vee \cdots \vee \delta'_k$ where each δ_i and δ'_j is a canonical conjunct. Therefore, $\varphi := \psi \wedge \chi := (\delta_1 \vee \cdots \vee \delta_m) \wedge (\delta'_1 \vee \cdots \vee \delta'_k)$. Then, by Lemma 12, we easily obtain a formula in normal form.

– *Case $\varphi := int(\psi)$:* W.l.o.g. suppose ψ is in normal form. We also assume that ψ includes some prenex formulas, otherwise we are done. By Lemma 9, we can write $\psi := \pi \vee (\delta \wedge \sigma)$ where σ is a prenex formula occurring in ψ. Then, we have

$$\begin{aligned} int(\psi) &\leftrightarrow int(\pi \vee (\delta \wedge \sigma)) \\ &\leftrightarrow int(\pi \vee \delta) \wedge (int(\pi) \vee \sigma) \quad \text{(by (5) or (6))} \end{aligned}$$

  By repeating this procedure, we can push every prenex formula in the scope of *int* to the top level, hence, obtain a formula in normal form.

– *Case $\varphi := K\psi$:* Proof of this case is quite similar to the case for *int* and the argument can be found in [13, Theorem 1.7.6.4, p.37].

3.3 Expressiveness of $APAL_{int}$

This section includes the main result of this paper: we will prove that $APAL_{int}$ and EL_{int} are equally expressive and thus all $APAL_{int}$, PAL_{int} and EL_{int} are equally expressive. Moreover, this results yields the completeness of $APAL_{int}$.

Lemma 14 *For any $\varphi \in \mathcal{L}_{PL_{int}}$ and any topo-model $\mathcal{X} = (X, \tau, \nu)$ and any epistemic scenario (x, U) of $\mathcal{X}$, if $(x, V) \models \varphi$ for some $V \in \tau$ with $x \in V \subseteq U$, then $(x, U) \models \varphi$.*

Proof. It is elementary for propositional variables and boolean cases as their evaluation does not depend on the neighbourhood, but depends only on the evaluation state. Let us now by inductive hypothesis assume that the statement holds for χ.

Case $\varphi := int(\chi)$: Suppose $(x, V) \models int(\chi)$ for some $V \in \tau$ with $x \in V \subseteq U$. This means, $x \in Int[\![\chi]\!]^V$. By IH, $[\![\chi]\!]^V \subseteq [\![\chi]\!]^U$, and thus, $Int[\![\chi]\!]^V \subseteq Int[\![\chi]\!]^U$. Therefore $x \in Int[\![\chi]\!]^U$, i.e., $(x, U) \models int(\chi)$.

Lemma 15 *For any $\varphi \in \mathcal{L}_{APAL_{int}}$, $K\varphi \to K(int(\varphi))$ is valid.*

Proof. Let $\varphi \in \mathcal{L}_{APAL_{int}}$, $\mathcal{X} = (X, \tau, \nu)$ be a topo-model and (x, U) be an epistemic scenario of $\mathcal{X}$. Suppose $(x, U) \models K\varphi$. This means $[\![\varphi]\!]^U = U$. Thus, as U is open, $Int[\![\varphi]\!]^U = U$. Then, by Proposition 4.4, we have $[\![int\varphi]\!]^U = U$ meaning that for all $y \in U$, $(y, U) \models int(\varphi)$. Therefore, $(x, U) \models Kint\varphi$.

Lemma 16 *For any $\varphi \in \mathcal{L}_{PL_{int}}$, $int(\varphi) \to \langle\varphi\rangle K\varphi$ is valid.*

Proof. The proof proceeds by induction on the complexity of φ. Let $\mathcal{X} = (X, \tau, \nu)$ be a topological model and (x, U) be an epistemic scenario of $\mathcal{X}$. The cases for propositional variables and booleans are trivial since truth of those does not depend on the neighbourhood. The inductive hypothesis now is: $\models int(\psi) \to \langle\psi\rangle K\psi$.

Case $\varphi := int(\psi)$: Suppose $(x, U) \models int(\varphi)$, i.e., $(x, U) \models int(int(\psi))$. Thus, $(x, U) \models int(\psi)$. Then, by IH, $(x, U) \models \langle\psi\rangle K\psi$. This means, $(x, U) \models int(\psi)$ and $(x, Int[\![\psi]\!]^U) \models K\psi$. Then, by Lemma 15, $(x, Int[\![\psi]\!]^U) \models K(int(\psi))$. Thus, $(x, U) \models \langle int(\psi)\rangle Kint(\psi)$. Therefore, $(x, U) \models int(int(\psi)) \to \langle int(\psi)\rangle Kint(\psi)$, i.e., $(x, U) \models int(\varphi) \to \langle\varphi\rangle K\varphi$.

Proposition 17 *For any $\varphi \in \mathcal{L}_{PL_{int}}$, $\blacksquare\varphi \leftrightarrow \varphi$ is valid.*

Proof. Let $\mathcal{X} = (X, \tau, \nu)$ be a topological model and (x, U) be an epistemic scenario of $\mathcal{X}$. We will prove $\models \blacklozenge\varphi \leftrightarrow \varphi$.($\Leftarrow$) By Proposition 7-(2)
($\Rightarrow$) Suppose $(x, U) \models \blacklozenge\varphi$. This means, there is a $\blacksquare$-free ψ such that $(x, U) \models \langle\psi\rangle\varphi$. Thus, $x \in Int[\![\psi]\!]^U$ and $(x, Int[\![\psi]\!]^U) \models \varphi$. $(x, Int[\![\psi]\!]^U) \models \varphi$ implies $(x, U) \models \varphi$ by Lemma 14. Therefore, $(x, U) \models \blacklozenge\varphi \to \varphi$.

Proposition 18 *For any $\varphi, \varphi_i \in \mathcal{L}_{PL_{int}}$ for $1 \leq i \leq n$,*

$$\models \blacklozenge(\varphi \wedge K\varphi_0 \wedge \langle K\rangle\varphi_1 \wedge \langle K\rangle\varphi_2 \wedge \cdots \wedge \langle K\rangle\varphi_n) \leftrightarrow \varphi \wedge int(\varphi_0) \wedge \langle K\rangle(int(\varphi_0) \wedge \varphi_1) \wedge \cdots \wedge \langle K\rangle(int(\varphi_0) \wedge \varphi_n).$$

Proof. Let $\mathcal{X} = (X, \tau, \nu)$ be a topological model and (x, U) an epistemic scenario of $\mathcal{X}$.

W.l.o.g. we prove the required for $n = 1$. ($\Rightarrow$) Suppose $(x, U) \models \blacklozenge(\varphi \wedge K\varphi_0 \wedge \langle K\rangle\varphi_1)$. Let us first see what this means.

$(x, U) \models \blacklozenge(\varphi \wedge K\varphi_0 \wedge \langle K\rangle\varphi_1)$
iff there exists a $\blacksquare$-free ψ s.t. $(x, U) \models \langle\psi\rangle(\varphi \wedge K\varphi_0 \wedge \langle K\rangle\varphi_1)$
iff $x \in Int[\![\psi]\!]^U$ and $(x, Int[\![\psi]\!]^U) \models \varphi \wedge K\varphi_0 \wedge \langle K\rangle\varphi_1$
iff $x \in Int[\![\psi]\!]^U$ and $(x, Int[\![\psi]\!]^U) \models \varphi$ and $(x, Int[\![\psi]\!]^U) \models K\varphi_0$ and$(x, Int[\![\psi]\!]^U) \models \langle K\rangle\varphi_1$

For simplicity, we enumerate the conjuncts of the last line as: ① $x \in \mathrm{Int}[\![\psi]\!]^U$, ② $(x, Int[\![\psi]\!]^U) \models \varphi$, ③ $(x, Int[\![\psi]\!]^U) \models K\varphi_0$, and ④ $(x, Int[\![\psi]\!]^U) \models \langle K\rangle\varphi_1$. We want to show that $(x, U) \models \varphi \wedge int(\varphi_0) \wedge \langle K \rangle(int(\varphi_0) \wedge \varphi_1)$. Now ② and Lemma 14 imply $(x, U) \models \varphi$; and ③ implies that $(x, Int[\![\psi]\!]^U) \models int(\varphi_0)$, since in $\mathcal{L}_{PL_{int}}$, $K\varphi \to int(\varphi)$. Then, by Lemma 14, we have $(x, U) \models int(\varphi_0)$.

To show $(x, U) \models \langle K \rangle(int(\varphi_0) \wedge \varphi_1)$, we need to show that there is a $y \in U$ such that $(y, U) \models int(\varphi_0) \wedge \varphi_1$. ④ implies that there is a $z \in Int[\![\psi]\!]^U$ such that $(z, Int[\![\psi]\!]^U) \models \varphi_1$. Then, by Lemma 14, we have $(z, U) \models \varphi_1$. Moreover, ③ and Observation 10 imply that $[\![K\varphi_0]\!]^{Int[\![\psi]\!]^U} = Int[\![\psi]\!]^U$, and thus $(z, Int[\![\psi]\!]^U) \models K\varphi_0$. Hence, $(z, Int[\![\psi]\!]^U) \models int(\varphi_0)$. Then again by Lemma 14, $(z, U) \models int(\varphi_0)$. So, $(z, U) \models int(\varphi_0) \wedge \varphi_1$, and thus $(x, U) \models \langle K \rangle(int(\varphi_0) \wedge \varphi_1)$.

($\Leftarrow$) Suppose $(x, U) \models \varphi \wedge int(\varphi_0) \wedge \langle K \rangle(int(\varphi_0) \wedge \varphi_1)$. We unravel the assumption.

$$(x, U) \models \varphi \wedge int(\varphi_0) \wedge \langle K \rangle(int(\varphi_0) \wedge \varphi_1)$$
$$\text{iff } (x, U) \models \varphi \text{ and } (x, U) \models int(\varphi_0) \text{ and } \exists y \in U \text{ s.t. } (y, U) \models int(\varphi_0) \text{ and } (y, U) \models \varphi_1$$
$$\text{iff } (x, U) \models \varphi \text{ and } (x, U) \models int(\varphi_0) \text{ and } \exists y \in U \text{ s.t. } y \in Int[\![\varphi_0]\!]^U \text{ and } (y, U) \models \varphi_1$$
$$\text{iff } (x, U) \models \varphi \text{ and } (x, U) \models int(\varphi_0) \text{ and } \exists y \in Int[\![\varphi_0]\!]^U \text{ s.t. } (y, U) \models \varphi_1$$

We want to show $(x, U) \models \blacklozenge(\varphi \wedge K\varphi_0 \wedge \langle K \rangle\varphi_1)$, i.e., we want to show that there is a $\blacksquare$-free ψ such that $(x, U) \models \langle \psi \rangle(\varphi \wedge K\varphi_0 \wedge \langle K \rangle\varphi_1)$.

We now claim that $(x, U) \models \langle \varphi_0 \rangle(\varphi \wedge K\varphi_0 \wedge \langle K \rangle\varphi_1)$. To prove the claim, we need to show $x \in Int[\![\varphi_0]\!]^U$ and $(x, Int[\![\varphi_0]\!]^U) \models \varphi \wedge K\varphi_0 \wedge \langle K \rangle\varphi_1$. We have $(x, U) \models int(\varphi_0)$, i.e., $x \in Int[\![\varphi_0]\!]^U$, by assumption.

As $(x, U) \models \varphi$, we have $(x, U) \models \blacksquare\varphi$, by Prop. 17. This means, for all $\blacksquare$-free ψ if $x \in Int[\![\psi]\!]^U$ then $(x, Int[\![\psi]\!]^U) \models \varphi$. Therefore, as $x \in Int[\![\varphi_0]\!]^U$, we have $(x, Int[\![\varphi_0]\!]^U) \models \varphi$.

Since $(x, U) \models int(\varphi_0)$, by Lemma 16, $(x, U) \models \langle \varphi_0 \rangle K\varphi_0$. So $(x, Int[\![\varphi_0]\!]^U) \models K\varphi_0$.

Now suppose $(x, Int[\![\varphi_0]\!]^U) \not\models \langle K \rangle\varphi_1$, i.e., $(x, Int[\![\varphi_0]\!]^U) \models K\neg\varphi_1$. This means, for all $y \in Int[\![\varphi_0]\!]^U$, $(y, Int[\![\varphi_0]\!]^U) \models \neg\varphi_1$. Then, as $\neg\varphi \in \mathcal{L}_{PL_{int}}$, by Lemma 14, $(y, U) \models \neg\varphi_1$. This contradicts the main assumption, therefore, $(x, Int[\![\varphi_0]\!]^U) \models \langle K \rangle\varphi_1$.

Theorem 19 *Single agent $APAL_{int}$ and EL_{int} are equally expressive.*

Proof. We prove by induction on the number of occurrences of $\blacklozenge$ that every formula in $APAL_{int}$ is equivalent to a formula in EL_{int}. First of all, note that every formula in PAL_{int} is equivalent to formula in EL_{int} by Proposition 5. Hence, we do not need to consider this case: we can simply convert every subformula of a given formula in $APAL_{int}$ which includes a public announcement modality to a formula in EL_{int} by following the reduction axioms given in Proposition 5. Moreover, by Theorem 13, we can write every subformula of a given formula in EL_{int} in an equivalent normal form. Thus, put the epistemic formula in the scope of an innermost $\blacklozenge$ in normal form. Then, we can distribute $\blacklozenge$ over the disjunction, by Proposition 7-(1). We now get formulas of the form $\blacklozenge(\varphi \wedge K\varphi_0 \wedge \langle K \rangle\varphi_1 \wedge \langle K \rangle\varphi_2 \wedge \cdots \wedge \langle K \rangle\varphi_n)$ where $\varphi, \varphi_i \in \mathcal{L}_{PL_{int}}$ for all $0 \leq i \leq n$. Then, by Proposition 18, we can reduce these formulas to formulas of the form $\varphi \wedge int(\varphi_0) \wedge \langle K \rangle(int(\varphi_0) \wedge \varphi_1) \wedge \cdots \wedge \langle K \rangle(int(\varphi_0) \wedge \varphi_n)$. By repeating the same procedure as many times as the number of occurrences of $\blacklozenge$ in a given formula of $APAL_{int}$, we obtain an equivalent formula in EL_{int}.

Thus, we proved that every $APAL_{int}$ formula can be reduced to an EL_{int} formula. As EL_{int} and PAL_{int} are also equally expressive, we conclude by Theorem 19 that $APAL_{int}$,

EL_{int} and PAL_{int} have the same expressive power, hence, the completeness of $APAL_{int}$ follows directly from the completeness of EL_{int} or from the completeness of PAL_{int}.

4 Conclusions and future work

In this work, we proposed a topological subset space semantics for the arbitrary announcement modality as an extension of a proposal initially made by Wang and Ågotnes and later adopted by Bjorndahl [6]. By providing a topological semantics for the arbitrary announcement modality, we linked it to the effort modality. We then demonstrated the completeness of $APAL_{int}$ by proving that $APAL_{int}$ and EL_{int} have the same expressive power.

For future research we envisage investigating the syntactic characterization of *knowable/learnable formulas* in the setting where 'knowable' means 'known after an announcement'. More precisely, we would like to give a syntactic characterization of those $\varphi \in APAL_{int}$ such that $\models \varphi \rightarrow \blacklozenge K\varphi$ [12, 9, 2]. Last but not least, in the present paper we only focused on single $APAL_{int}$ and in the future we intend to generalize our logic to the *multi-agent* case [16, 4, 11, 10].

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