Two Topics from Pairings and Towers

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Multilinear Pairings

First Part
Overview

Have seen recent breakthrough in construction of multilinear pairings ...

In first part of this talk:

Security aspects in a generic model:
  ▶ Simplify ideas of Verheul, Galbraith, Karabina-Knapp-Menezes.
  ▶ Generalise to multilinear pairings.
  ▶ Joint work with Uzunkol.

Number theoretic aspects of pairing inversion:
  ▶ Observe that there is a community that has researched a closely related problem for a long time.
  ▶ But here no actual improvement for pairing inversion.
Multilinear Pairings:

- $G_1, \ldots, G_{r+1}$ cyclic groups of squarefree order $n$.
- $e : G_1 \times \cdots \times G_r \to G_{r+1}$ non-deg multilinear map.
- $G_1 = \cdots = G_r$: symmetric pairing.
- $G_1 = \cdots = G_r = G_{r+1}$: self pairing.
- $r = 1$: homomorphism, $r = 2$: bilinear map.

Generic model:

- Oracle access to compute $a = b$, $a + b$ and $-a$.
- Oracle access to compute $e(a_1, \ldots, a_r)$.
- Unit cost for oracle access and storage of $a$. 
Inversion

We say all pairings on the $G_i$ are efficiently invertible when given by oracle access, iff:

- Consider a multilinear $e$ on a subset of the $G_i$ with oracle access.
- Can solve $e(a_1, \ldots, a_d) = a_{d+1}$ efficiently, given $a_{d+1}$ and all but one argument, for the missing argument.
- Can do this for every such $e$ and parameters.

We say all pairings on the $G_i$ are efficiently computable when given by pointwise definition iff:

- Consider a multilinear $e$ on a subset of the $G_i$, defined by $e(u_1, \ldots, u_d) = u_{d+1}$ for generators $u_j$.
- Can compute $e(a_1, \ldots, a_d)$ given the $u_j$ and $a_j$.
- Can do this for every such $e$ and parameters.
Inversion

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- Consider a multilinear $e$ on a subset of the $G_i$, defined by $e(u_1, \ldots, u_d) = u_{d+1}$ for generators $u_j$.
- Can solve $e(a_1, \ldots, a_d) = a_{d+1}$ efficiently, given $a_{d+1}$ and all but one argument, for the missing argument.
- Can do this for every such $e$ and parameters.

This includes the computational Diffie-Hellman problem or variants on all $G_i$:

- Given a pointwise defined homomorphism $f : G_i \mapsto G_i$, $f : P \mapsto aP$, compute $f(bP) = abP$.
- Given a non-deg pairing $G_i \times G_i \mapsto G_i$ and $Q = e(P, P)$, compute $e(aP, bP) = abQ$. 
Bilinear Self Pairings

Key observation: Given cyclic \( G \) or order \( n \), \( e \) self pairing on \( G \). Then

\[
(G, +, e) \cong (\mathbb{Z}/n\mathbb{Z}, +, \cdot).
\]

Remarks:

- Can suppose \( G = \mathbb{Z}/n\mathbb{Z} \).
- Let \( w = e(1, 1) \). Then \( e(a, b) = wab \).
- \( f : \mathbb{Z}/n\mathbb{Z} \to G, x \mapsto w^{-1}x \) is ring isomorphism.
- Specifically \( 1_e = w^{-1} \).

Note

- similarity to Montgomery multiplication.
- is generalisation of Boneh-Lipton black box fields via CDH oracle, since self pairing gives only variant of CDH and \( n \) might be composite.
Bilinear Self Pairings: Theorem

Let $G$ be a cyclic group of order $n$ in the generic group model with oracle access to a bilinear self-pairing $\cdot$ of $G$.

If $\phi(n)$ is known then all pairings of $r$ arguments on $G$ are efficiently invertible when given by oracle access, and are efficiently computable and efficiently invertible when given by pointwise definition, for every $r \geq 1$.

Remarks:

- One shows that 1 and $a^{-1}$ can be efficiently computed from a generator of $G$ via powering.
- The rest is linear algebra.
- Thm applies also to isomorphisms defined by their images on a generator ($r = 1$) and provides „existence“ of multilinear pairings with arbitrary $r$.
- Thm encompasses Verheul and other results in a very easy and compact way.
Application to Asymmetric Pairings

Let $G_1, G_2, G_3$ be cyclic groups of order $n$ in the generic
group model with oracle access to a bilinear pairing

$$e : G_1 \times G_2 \rightarrow G_3.$$ 

Suppose $\phi(n)$ and generators of $G_1$ and $G_2$ are known, and
that oracle access to isomorphisms

$$G_3 \rightarrow G_2, \ G_2 \rightarrow G_1 \ \text{or} \ G_3 \rightarrow G_1, \ G_3 \rightarrow G_2$$

exists.

Then all pairings of $r$ arguments on $G_1, G_2, G_3$ are efficiently
invertible when given by oracle access, and are efficiently
computable and efficiently invertible when given by pointwise
definition, for every $r \geq 1$. 
Discussion

- Feeding back values of multilinear pairings into arguments potentially dangerous and should not be possible!
- Proof nicely relates to encoding of elements $x \mapsto w^{-1}x$ in the recent schemes.
Number Theoretic Aspect of Inversion

Consider inversion of Tate paring function:

- $t$ Tate pairing on $E(\mathbb{F}_q)[r]$, $t(P, Q) = f_Q(P)^{(q-1)/r}$.
- Given $Q, z$ find $P$ with $f_Q(P)^{(q-1)/r} = z$.
- This is equivalent to finding a $\mathbb{F}_q$-rational point on the curve $C$ defined as a cover of $E$ via
  \[ wy^r = f_Q, \]
  where $w$ is easily computed satisfying $w^{(q-1)/r} = z$.
- $C \rightarrow E$ is unramified, so $C$ is an elliptic curve $r$-isogenous to $E$.
- Can we compute the $j$-invariant and the dual isogeny?

This computation is the same as is carried out when computing $E(\mathbb{Q})$ using descent. But there $r \leq 10$, say.
Towers of Curves

Second Part
Towers of Curves

A tower of curves $\mathcal{C}$ is a sequence of surjective morphisms

$$\cdots \rightarrow C_3 \rightarrow C_2 \rightarrow C_1 \rightarrow X$$

of regular, complete and absolutely irreducible curves $C_i$ and $X$ over a finite field $\mathbb{F}_q$.

Uses of towers of curves with specific properties:

- Construction of error-correcting codes
- Secret sharing schemes and secure multi-party computation.
- Bilinear complexity of multiplication
- Codices ...
Specific properties

Main goal:

Construct curves $C$ with

- $\#C(\mathbb{F}_q)$ large,
- $g(C)$ small (i.e. $C$ has small defining equations).

Interpolation shows these are opposing requirements.

The tower is supposed to provide such $C_i$ with

$$\#C_i(\mathbb{F}_q), g(C_i) \rightarrow \infty.$$
First invariant

Let $d_i$ be the degree of $C_i \to X$.

Then

- $d_i$ grows exponentially in $i$.
- $\#C_i(\mathbb{F}_q) = O(d_i)$ for $i \to \infty$.
- $g(C_i) = \Omega(d_i)$ for $i \to \infty$.
- $\lim_{i \to \infty} \frac{\#C_i(\mathbb{F}_q)}{g(C_i)}$ exists.

Define

$$\beta_1(C) = \lim_{i \to \infty} \frac{\#C_i(\mathbb{F}_q)}{g(C_i)}.$$ 

The goal is to find $C$ such that

$$\beta_1(C) > 0.$$ 

Then $C$ is called asymptotically good (in degree one).
Some known facts

Upper bounds:
- \( A_1(q) := \lim \sup_{C} \beta_1(C) \).
- Then \( A_1(q) \leq \sqrt{q} - 1 \) (Drinfeld-Vladut).

Some lower bounds:
- \( q \) square then \( A_1(q) = \sqrt{q} - 1 \) (Ihara, Tsfasman-Vladut-Zink).
- \( A_1(q^n) = \Omega((n \log(q))^2/(n + \log(q))) \) any \( q, n \) (Serre, Temkine)
- \( A(2) \geq 0.316999, A(3) \geq 0.492876 \) (Duursma-Mak).
- \( A(p^{2m+1}) \geq 2(p^{m+1} - 1)/(p + 1 + (p - 1)/(p^m - 1)) \) for \( p^{2m+1} \) with \( 2m + 1 \geq 3 \) (Garcia-Stichtenoth-Bassa-Beelen)
Constructions:

- Modular curves.
- Class field towers.
- Recursive towers.

Lower bounds for $A_1(q)$ proved by these constructions.

Non-square $q$ by class field towers.

Non-square $q = p^{2m+1}$ with $2m + 1 \geq 3$ by recursive towers only recently. Yield much better lower bound than class field towers.
Higher Invariants: Definitions and Facts

Higher invariants of degree $r$:

- $B_r(C) = \# \{ P \in C \mid \deg(P) = r \}$
- $\beta_r(C) = \lim_{i \to \infty} B_r(C_i)/g(C_i)$.
- $A_r(q) = \limsup C \beta_r(C)$.

Some facts:

- $\sum_{r=1}^{\infty} r \beta_r(C)/(q^{r/2} - 1) \leq 1$ (Serre, Tsfasman).
- $A_r(q) \leq (q^{r/2} - 1)/r$.
- Exists $C$ with finitely many prescribed $\beta_r(C) > 0$ using class field towers (Hasegawa, Lebacque).
- Exists $C$ with $\beta_4(C) = A_4(2)$ (Ballet-Rolland).
- Exists $C$ with $\beta_2(C) = A_2(q)$ for any $q$ (Ballet-Rolland).
Joint work with Stichtenoth and Tutdere.

Our results:

- Explicit construction for $C$ with finitely many prescribed $\beta_r(C) > 0$.
- Explicit construction for $C$ with at least one positive $\beta_r(C)$ and certain prescribed $\beta_s(C) = 0$.
- Examples of $C$ with all but one $\beta_r(C) = 0$.
- Exists $C$ with $\beta_r(C) = A_r(q)$ if $q^r$ is square.
- $A_1(q^r)/r \geq A_r(q)$ and some lower bounds on $A_r(q)$. 
Main Idea

The main idea behind the paper is as follows:

Start with $C$ and $\beta_1(C) > 0$.

Construct $E \to X$ such that

- $E \times_X C_i$ is irreducible.
- Define $E_i$ to be the normalisation of $E \times_X C_i$.
- The preimages in $E_i$ of points of degree one of $C_i$ that are split completely in $C$ consist of points of $E_i$ of prescribed degrees.
- The maps $C_i \to C_{i-1}$ extend to $E_i \to E_{i-1}$.

We then get a tower of curves $\mathcal{E}$ where the many points of degree one of $C$ have been distributed in higher degrees and the genus does not increase too much.
Recursive towers: Definition

Provide explicit constructions. The most common form is:
Let \( f \in \mathbb{F}_q[X, Y] \). Then \( X = \mathbb{P}^1 \) and \( C_i \) is defined by

\[
\begin{align*}
    f(x_0, x_1) &= 0, \\
    f(x_2, x_3) &= 0, \\
    \ldots
    \end{align*}
\]

\[
f(x_{i-1}, x_i) = 0.
\]

The maps \( C_i \rightarrow C_{i-1} \) are \( (x_0, \ldots, x_i) \mapsto (x_0, \ldots, x_{i-1}) \).

\( K(C_i) = K(x_0, x_1, \ldots, x_i) \) with \( f(x_{i-1}, x_i) = 0 \).

Example:

- \( f = Y^qX^{q-1} + Y - X^q \in \mathbb{F}_{q^2}[X, Y] \),
- yields \( \beta_1(C) = A_1(q) = q - 1 \).
Very nice recent study by Hallouin-Perret.

- Consider properties of a graph associated to a recursive tower.
- Use intersection theory of correspondences, in particular to count cycles in the graph.
- Combine algebraic graph theory and Frobenius-Perron theory to deduce structure of the graph.
- $d$-regular strongly connected components of the graph are precisely responsible for many points.
- Under some mild hypotheses, there is at most one such component and hence at most one $\beta_r(C) > 0$. 
Challenge

Open problem:

Construct asymptotically good recursive tower over $\mathbb{F}_p$.

Construct $f$ with nice graphs over $\mathbb{F}_p$:

- Need $d$-regular strongly connected component.
- The singular part of the graph must allow/imply small genus.