Laplacian matrix of a graph

Basic results
Algebraic connectivity
Resistance distance matrix

Sukanta Pati
pati@iitg.ernet.in

IIT Guwahati, INDIA.

\textsuperscript{a}Talk prepared for the Indo-French workshop, 2011.
Which one has better connectivity?
Connectivity of the sunflowers

- Imagine rotating them (around their centers).
Connectivity of the sunflowers

- Imagine rotating them (around their centers).
Connectivity of the sunflowers

- Imagine rotating them (around their centers).
Connectivity of the sunflowers

- Imagine rotating them (around their centers).
- Imagine rotating them (around their centers).
Connectivity of the sunflowers

- Imagine rotating them (around their centers).

- Which one will ‘break’ first?
• Simple graphs only: no loops, no parallel edges.
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$. 
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$:
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 \end{cases}$
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ (i is adjacent to } j) \\ \end{cases}$
The Laplacian matrix

- Simple graphs only: no loops, no parallel edges.

- Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

- Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ (i is adjacent to j)} \\ 0 & \text{otherwise} \end{cases}$
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \ (i \text{ is adjacent to } j) \\ 0 & \text{otherwise} \end{cases}$
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \ (i \text{ is adjacent to } j) \\ 0 & \text{otherwise.} \end{cases}$

• Laplacian matrix $L$ of $G$:
The Laplacian matrix

- Simple graphs only: no loops, no parallel edges.

- Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

- Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ (i is adjacent to j)} \\ 0 & \text{otherwise.} \end{cases}$

- Laplacian matrix $L$ of $G$: $L = D - A$,

  where $D$ is the diagonal degree matrix.
The Laplacian matrix

- Simple graphs only: no loops, no parallel edges.
- Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

- Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \ (i \text{ is adjacent to } j) \\ 0 & \text{otherwise.} \end{cases}$

- Laplacian matrix $L$ of $G$: $L = D - A$,
  
  where $D$ is the diagonal degree matrix.

- Take $H$;
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \ (i \text{ is adjacent to } j) \\ 0 & \text{otherwise.} \end{cases}$

• Laplacian matrix $L$ of $G$: $L = D - A$,
  where $D$ is the diagonal degree matrix.

• Take $H$; $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$.
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \ (i \text{ is adjacent to } j) \\ 0 & \text{otherwise.} \end{cases}$

• Laplacian matrix $L$ of $G$: $L = D - A$,

where $D$ is the diagonal degree matrix.

• Take $H$; $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$; $L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: 
  
  \[ a_{ij} = \begin{cases} 
  1 & \text{if } i \sim j \text{ (} i \text{ is adjacent to } j \text{)} \\
  0 & \text{otherwise.} 
  \end{cases} \]

• Laplacian matrix $L$ of $G$: 
  \[ L = D - A, \]
  where $D$ is the diagonal degree matrix.

• Take $H$; 
  \[
  A = \begin{bmatrix} 
  0 & 1 & 0 & 0 & 0 & 0 \\
  1 & 0 & 1 & 0 & 1 & 0 \\
  0 & 1 & 0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 1 \\
  0 & 1 & 0 & 0 & 0 & 1 \\
  0 & 0 & 0 & 1 & 1 & 0 
  \end{bmatrix} \quad \text{and} \quad
  L = \begin{bmatrix} 
  1 & -1 & 0 & 0 & 0 & 0 \\
  -1 & 3 & -1 & 0 & -1 & 0 \\
  0 & -1 & 2 & -1 & 0 & 0 \\
  0 & 0 & -1 & 2 & 0 & -1 \\
  0 & -1 & 0 & 0 & 2 & -1 \\
  0 & 0 & 0 & -1 & -1 & 2 
  \end{bmatrix} \]
Simple graphs only: no loops, no parallel edges.

Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ (i is adjacent to j)} \\ 0 & \text{otherwise.} \end{cases}$

Laplacian matrix $L$ of $G$: $L = D - A$, where $D$ is the diagonal degree matrix.

Take $H$; $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$; $L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \text{ (i is adjacent to j)} \\ 0 & \text{otherwise.} \end{cases}$

• Laplacian matrix $L$ of $G$: $L = D - A$,

where $D$ is the diagonal degree matrix.

• Take $H$; $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix}$, \[ L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix} \]
- Simple graphs only: no loops, no parallel edges.

- Let \( G \) be a simple graph on vertices \( 1, 2, \ldots, n \).

- Adjacency matrix \( A \) of \( G \): \( a_{ij} = \begin{cases} 1 & \text{if } i \sim j \quad (i \text{ is adjacent to } j) \\ 0 & \text{otherwise.} \end{cases} \)

- Laplacian matrix \( L \) of \( G \): \( L = D - A \),

where \( D \) is the diagonal degree matrix.

- Take \( H \); \( A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \); \( L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix} \)
• Simple graphs only: no loops, no parallel edges.

• Let $G$ be a simple graph on vertices $1, 2, \ldots, n$.

• Adjacency matrix $A$ of $G$: $a_{ij} = \begin{cases} 1 & \text{if } i \sim j \ (i \text{ is adjacent to } j) \\ 0 & \text{otherwise.} \end{cases}$

• Laplacian matrix $L$ of $G$: $L = D - A$, where $D$ is the diagonal degree matrix.

• Take $H$; $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$; $L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$
• Laplacian matrix is also known as Kirchhoff matrix.

• Take $H$; $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$; $L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{bmatrix}$
• Laplacian matrix is also known as Kirchhoff matrix.

• G. Kirchhoff; Ann. Phys. Chem; 1847.

• Take $H$; $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$; $L = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 3 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 & 0 \end{bmatrix}$
• Laplacian matrix is also known as Kirchhoff matrix.


\[ L = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 \\
0 & -1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & -1 & -1 & 2 \\
\end{bmatrix} \]
• Laplacian matrix is also known as Kirchhoff matrix.

• G. Kirchhoff; Ann. Phys. Chem; 1847. Delete $i$th row and $j$th column of $L$. Call it $L(i|j)$.

\[ L = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 \\
0 & -1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & -1 & -1 & 2
\end{bmatrix} \]
• Laplacian matrix is also known as Kirchhoff matrix.

• G. Kirchhoff; Ann. Phys. Chem; 1847. Delete $i$th row and $j$th column of $L$. Call it $L(i|j)$.

\[
L(1|6) = \begin{bmatrix}
-1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
• Laplacian matrix is also known as Kirchhoff matrix.

• G. Kirchhoff; Ann. Phys. Chem; 1847. Delete $i$th row and $j$th column of $L$. Call it $L(i|j)$. Then $(-1)^{i+j}|L(i|j)| = \text{number of spanning trees in } G.$

\[
L(1|6) = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 \\
0 & -1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & -1 & -1 & 2
\end{bmatrix}
\]
• Laplacian matrix is also known as Kirchhoff matrix.

• G. Kirchhoff; Ann. Phys. Chem; 1847. Delete \(i\)th row and \(j\)th column of \(L\). Call it \(L(i|j)\). Then 

\[
(-1)^{i+j} |L(i|j)| = \text{number of spanning trees in } G.
\]

\[
L(1|6) = \begin{bmatrix}
1 & -1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 3 & -1 & 0 & -1 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & 0 & -1 & 0 \\
0 & -1 & 0 & 0 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & -1 & 2 & 0
\end{bmatrix}
\]
Known properties of Laplacian matrix
• Give any orientations to the edges.
• Give any orientations to the edges.
• Give any orientations to the edges.
• Give any orientations to the edges. Label the edges.
Known properties of Laplacian matrix

- Give any orientations to the edges. Label the edges.
- Give any orientations to the edges. Label the edges.
- The vertex edge incidence matrix $Q$: 
- Give any orientations to the edges. Label the edges.

- The vertex edge incidence matrix $Q$:

$$ q_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ starts from } i \\
-1 & \text{if } e_j \text{ ends at } i \\
0 & \text{otherwise.}
\end{cases} $$
• Give any orientations to the edges. Label the edges.

• The vertex edge incidence matrix \( Q \):

\[
q_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ starts from } i \\
-1 & \text{if } e_j \text{ ends at } i \\
0 & \text{otherwise.}
\end{cases}
\]

• For this graph we have \( Q = \)
- Give any orientations to the edges. Label the edges.

- The vertex edge incidence matrix $Q$:
  \[
  q_{ij} = \begin{cases} 
  1 & \text{if } e_j \text{ starts from } i \\
  -1 & \text{if } e_j \text{ ends at } i \\
  0 & \text{otherwise.}
  \end{cases}
  \]

- For this graph we have $Q = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- Give any orientations to the edges. Label the edges.

- The vertex edge incidence matrix $Q$:

\[
q_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ starts from } i \\
-1 & \text{if } e_j \text{ ends at } i \\
0 & \text{otherwise.}
\end{cases}
\]

- For this graph we have $Q = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix}$
• Give any orientations to the edges. Label the edges.

• The vertex edge incidence matrix $Q$:

$$ q_{ij} = \begin{cases} 1 & \text{if } e_j \text{ starts from } i \\ -1 & \text{if } e_j \text{ ends at } i \\ 0 & \text{otherwise.} \end{cases} $$

• For this graph we have $Q =$

$$
\begin{bmatrix}
1 & 0 & 0 \\
-1 & 1 & 0 \\
0 & -1 & -1 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
$$
• Give any orientations to the edges. Label the edges.

• The vertex edge incidence matrix $Q$:

$$q_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ starts from } i \\
-1 & \text{if } e_j \text{ ends at } i \\
0 & \text{otherwise}.
\end{cases}$$

• For this graph we have $Q =$

$$\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0
\end{bmatrix}$$
• Give any orientations to the edges. Label the edges.

• The vertex edge incidence matrix $Q$:

\[
q_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ starts from } i \\
-1 & \text{if } e_j \text{ ends at } i \\
0 & \text{otherwise.}
\end{cases}
\]

• For this graph we have $Q =$

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & -1 & 0
\end{bmatrix}
\]

• $L = QQ^t$. 
• Give any orientations to the edges. Label the edges.

• The vertex edge incidence matrix $Q$:

$$q_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ starts from } i \\
-1 & \text{if } e_j \text{ ends at } i \\
0 & \text{otherwise.}
\end{cases}$$

• For this graph we have $Q$ =

$$Q = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & -1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & -1 & -1 & 0
\end{bmatrix}$$

• $L = QQ^t$. Does not depend on the orientations.
• Give any orientations to the edges. Label the edges.

The vertex edge incidence matrix $Q$:

$$q_{ij} = \begin{cases} 
1 & \text{if } e_j \text{ starts from } i \\
-1 & \text{if } e_j \text{ ends at } i \\
0 & \text{otherwise.}
\end{cases}$$

For this graph we have $Q = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 0 & 0 & -1 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 0
\end{bmatrix}$

$L = QQ^t$. Does not depend on the orientations.

So $L$ is positive semidefinite:
• Give any orientations to the edges. Label the edges.

The vertex edge incidence matrix $Q$:

$$q_{ij} = \begin{cases} 1 & \text{if } e_j \text{ starts from } i \\ -1 & \text{if } e_j \text{ ends at } i \\ 0 & \text{otherwise.} \end{cases}$$

For this graph we have $Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & -1 & 0 \end{bmatrix}$

$L = QQ^t$. Does not depend on the orientations.

• So $L$ is positive semidefinite: Hermitian, eigenvalues $\geq 0$. 
• Give any orientations to the edges. Label the edges.

• The vertex edge incidence matrix $Q$:
  
  $q_{ij} = \begin{cases} 
  1 & \text{if } e_j \text{ starts from } i \\
  -1 & \text{if } e_j \text{ ends at } i \\
  0 & \text{otherwise.}
\end{cases}$

• For this graph we have $Q = \begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 \\
  -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
  0 & -1 & -1 & 0 & 0 & -1 & \\
  0 & 0 & 1 & 1 & 0 & 0 & \\
  0 & 0 & 0 & 0 & 1 & 1 & \\
  0 & 0 & 0 & -1 & -1 & 0 & \\
\end{bmatrix}$

• $L = QQ^t$. Does not depend on the orientations.
  
  • So $L$ is positive semidefinite: Hermitian, eigenvalues $\geq 0$.

• Also $x^t L(G^t)x = \sum_{i \sim j} (x_i - x_j)^2$. 

Laplacian matrix of a graph – p.4/28
Known properties of Laplacian matrix
• It is also an $M$-matrix:
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho(B)$. 
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho (B)$.

- Smallest eigenvalue $\lambda_1 (L)$ is 0;
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0], \alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbf{1} = [1, \ldots, 1]^T$. 
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbb{1} = [1, \ldots, 1]^T$.
- Fiedler, 73. Second smallest eigenvalue $\mu = \lambda_2(L) > 0$ iff $G$ is connected.
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbf{1} = [1, \ldots, 1]^T$.
- Fiedler, 73. Second smallest eigenvalue $\mu = \lambda_2(L) > 0$ iff $G$ is connected.
- Call: $\mu$ the algebraic connectivity;
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbf{1} = [1, \ldots, 1]^T$.
- Fiedler, 73. Second smallest eigenvalue $\mu = \lambda_2(L) > 0$ iff $G$ is connected.
- Call: $\mu$ the algebraic connectivity; an eigenvector for $\mu$ a Fiedler vector.
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbf{1} = [1, \ldots, 1]^T$.
- Fiedler, 73. Second smallest eigenvalue $\mu = \lambda_2(L) > 0$ iff $G$ is connected.
- Call: $\mu$ the algebraic connectivity; an eigenvector for $\mu$ a Fiedler vector.

\[ S_n, n \geq 3 \]

\[ \mu = 1 \]
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbf{1} = [1, \ldots, 1]^T$.
- Fiedler, 73. Second smallest eigenvalue $\mu = \lambda_2(L) > 0$ iff $G$ is connected.
- Call: $\mu$ the algebraic connectivity; an eigenvector for $\mu$ a Fiedler vector.

\begin{align*}
\text{For } S_n, n \geq 3 & \quad \mu = 1 \\
\text{For } C_n, n \geq 4 & \quad 0 < \mu \leq 2
\end{align*}
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0]$, $\alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbf{1} = [1, \ldots, 1]^T$.
- Fiedler, 73. Second smallest eigenvalue $\mu = \lambda_2(L) > 0$ iff $G$ is connected.
- Call: $\mu$ the algebraic connectivity; an eigenvector for $\mu$ a Fiedler vector.

\[ S_n, n \geq 3 \quad \begin{array}{c} \mu = 1 \\ \text{C}_n, n \geq 4 \end{array} \quad \begin{array}{c} 0 < \mu \leq 2 \\ \text{K}_n, n \geq 3 \end{array} \]
Known properties of Laplacian matrix

- It is also an $M$-matrix: $L = \alpha I - B$, where $B = [b_{ij} \geq 0], \alpha > \rho(B)$.
- Smallest eigenvalue $\lambda_1(L)$ is 0; an eigenvector $\mathbb{1} = [1, \ldots, 1]^T$.
- Fiedler, 73. Second smallest eigenvalue $\mu = \lambda_2(L) > 0$ iff $G$ is connected.
- Call: $\mu$ the algebraic connectivity; an eigenvector for $\mu$ a Fiedler vector.

\begin{align*}
\text{Fidler, 73.} & \quad \mu = 1 \\ & \text{for } S_n, n \geq 3 \\
\text{Fidler, 73.} & \quad 0 < \mu \leq 2 \\ & \text{for } C_n, n \geq 4 \\
\text{Fidler, 73.} & \quad \mu = n \\ & \text{for } K_n, n \geq 3 \\
\text{Fidler, 73.} & \quad 0 < \mu \leq 1 \\ & \text{for } T_n, n \geq 3
\end{align*}
Fiedler vector
• Fiedler, 73. Take $G$ connected and a Fiedler vector of $Y$. Then
• Fiedler, 73. Take $G$ connected and a Fiedler vector of $Y$. Then
  • the subgraph induced by $\{v \mid Y(v) \geq 0\}$ is connected;
Fiedler, 73. Take $G$ connected and a Fiedler vector of $Y$. Then

- the subgraph induced by $\{v | Y(v) \geq 0\}$ is connected;
- the subgraph induced by $\{v | Y(v) \leq 0\}$ is connected.
• Fiedler, 73. Take $G$ connected and a Fiedler vector of $Y$. Then
  
  • the subgraph induced by $\{v | Y(v) \geq 0\}$ is connected;
  
  • the subgraph induced by $\{v | Y(v) \leq 0\}$ is connected.
• Fiedler, 73. Take $G$ connected and a Fiedler vector of $Y$. Then
  
  • the subgraph induced by $\{v | Y(v) \geq 0\}$ is connected;
  
  • the subgraph induced by $\{v | Y(v) \leq 0\}$ is connected.
• Take $G$ connected and a Fiedler vector of $Y$. 


• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$. 
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$. 

Laplacian matrix of a graph – p.7/28
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $\mathcal{C}(G,Y)$: all characteristic vertices and edges.

• Case B: It is possible that $\mathcal{C}(G,Y)$ is a singleton vertex.
Characteristic vertex, edge and set

- Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

- An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

- Characteristic set $C(G, Y)$: all characteristic vertices and edges.

- Case B: It is possible that $C(G, Y)$ is a singleton vertex.

- Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. 
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $C(G, Y)$ is a singleton vertex.

• Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the characteristic block (ch.block).
Characteristic vertex, edge and set

• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $C(G, Y)$ is a singleton vertex.

• Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the characteristic block (ch.block).
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $\mathcal{C}(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.

• Case A: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the characteristic block (ch.block).
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $C(G, Y)$ is a singleton vertex.

• Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the characteristic block (ch.block).
Characteristic vertex, edge and set

- Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.
- An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.
- Characteristic set $C(G, Y)$: all characteristic vertices and edges.
- Case B: It is possible that $C(G, Y)$ is a singleton vertex.
- Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the characteristic block (ch.block).

Characteristic vertex

Characteristic edge

Laplacian matrix of a graph – p.7/28
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $C(G, Y)$ is a singleton vertex.

• Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the characteristic block (ch.block).
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a **characteristic vertex** if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a **characteristic edge** if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $C(G, Y)$ is a singleton vertex.

• Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the **characteristic block** (ch.block).

**CASE A**
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a **characteristic vertex** if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a **characteristic edge** if $Y(v)Y(u) < 0$.

• **Characteristic set** $\mathcal{C}(G, Y)$: all characteristic vertices and edges.

• **Case B**: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.

• **Case A**: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the **characteristic block** (ch.block).

CASE B
Characteristic vertex, edge and set

• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $\mathcal{C}(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.

• Case A: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the characteristic block ($\text{ch.block}$).

• Known. Let $G$ be a connected graph.
Characteristic vertex, edge and set

- Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.
- An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.
- Characteristic set $\mathcal{C}(G, Y)$: all characteristic vertices and edges.
- Case B: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.
- Case A: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the characteristic block (ch.block).

Known. Let $G$ be a connected graph. Assume that Case B happens and $v$ is a characteristic vertex w.r.t one Fiedler vector.
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $C(G, Y)$ is a singleton vertex.

• Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the characteristic block (ch.block).

**Known.** Let $G$ be a connected graph. Assume that Case B happens and $v$ is a characteristic vertex w.r.t one Fiedler vector. Then Case B happens for each Fiedler vector and $v$ remains the characteristic vertex.
Characteristic vertex, edge and set

- Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a **characteristic vertex** if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

- An edge $uv$ is a **characteristic edge** if $Y(v)Y(u) < 0$.

- Characteristic set $\mathcal{C}(G, Y)$: all characteristic vertices and edges.

- **Case B**: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.

- **Case A**: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the **characteristic block** (ch.block).

- **known**. Let $G$ be a connected graph.
Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

Characteristic set $C(G,Y)$: all characteristic vertices and edges.

Case B: It is possible that $C(G,Y)$ is a singleton vertex.

Case A: If $C(G,Y)$ is not a single vertex, then there is a unique block which contains $C(G,Y)$. This is the characteristic block (ch.block).

known. Let $G$ be a connected graph. Assume that Case A happens and $B$ is a characteristic block w.r.t one Fiedler vector.
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $C(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $C(G, Y)$ is a singleton vertex.

• Case A: If $C(G, Y)$ is not a single vertex, then there is a unique block which contains $C(G, Y)$. This is the characteristic block (ch.block).

known. Let $G$ be a connected graph. Assume that Case A happens and $B$ is a characteristic block w.r.t one Fiedler vector. Then Case A happens for each Fiedler vector and $B$ remains the characteristic block.
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $\mathcal{C}(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.

• Case A: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the characteristic block (ch.block).

known. Let $G$ be a connected graph and $Y$ a Fiedler vector.
Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

- An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

- Characteristic set $\mathcal{C}(G, Y)$: all characteristic vertices and edges.

- Case B: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.

- Case A: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the characteristic block (ch.block).

- known. Let $G$ be a connected graph and $Y$ a Fiedler vector. Assume that $f_1, f_2$ are two characteristic elements.
• Take $G$ connected and a Fiedler vector of $Y$. Vertex $v$ is a characteristic vertex if $Y(v) = 0$ and $Y(w) \neq 0$ for some $w \sim v$ i.e. $vw \in G$.

• An edge $uv$ is a characteristic edge if $Y(v)Y(u) < 0$.

• Characteristic set $\mathcal{C}(G, Y)$: all characteristic vertices and edges.

• Case B: It is possible that $\mathcal{C}(G, Y)$ is a singleton vertex.

• Case A: If $\mathcal{C}(G, Y)$ is not a single vertex, then there is a unique block which contains $\mathcal{C}(G, Y)$. This is the characteristic block (ch.block).

Known. Let $G$ be a connected graph and $Y$ a Fiedler vector. Assume that $f_1, f_2$ are two characteristic elements. Then there is a simple cycle which contains these two elements and no other characteristic elements.
Fiedler’s Monotonicity theorem: for trees
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
  - Merris, 87. If a tree has a characteristic vertex w.r.t one Fiedler vector,
A tree can either have a characteristic vertex or a characteristic edge.

Merris, 87. If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
  - Merris, 87. If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.
- If a tree has a characteristic edge then $\mu$ is simple. Fiedler 75.
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
  - Merris, 87. If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.
- If a tree has a characteristic edge then $\mu$ is simple. Fiedler 75.

- Fiedler, 75; Kirkland, Neumann, Shader, 96. $T$ a tree, $Y$ a Fiedler vector.
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
  - Merris, 1987. If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.
  - If a tree has a characteristic edge then $\mu$ is simple. Fiedler 75.

- Fiedler, 1975; Kirkland, Neumann, Shader, 1996. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex.
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
  - Merris, 87. If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.
  - If a tree has a characteristic edge then $\mu$ is simple. Fiedler 75.

- Fiedler, 75; Kirkland, Neumann, Shader, 96. $T$ a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
  - Merris, 87. If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.
  - If a tree has a characteristic edge then $\mu$ is simple. Fiedler 75.

- Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.  

A tree can either have a characteristic vertex or a characteristic edge.

- **Merris, 87.** If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.

- If a tree has a characteristic edge then $\mu$ is simple. **Fiedler 75.**

**Fiedler, 75; Kirkland, Neumann, Shader, 96.** $T$ a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then

- Either $Y(v_i) > 0$, increase and concave down along $P$.
- Or $Y(v_i) < 0$, decrease and concave up along $P$. 
Fiedler’s Monotonicity theorem: for trees

- A tree can either have a characteristic vertex or a characteristic edge.
  - Merris, 87. If a tree has a characteristic vertex w.r.t one Fiedler vector, then the same vertex is a characteristic vertex for each Fiedler vector.
  - If a tree has a characteristic edge then $\mu$ is simple. Fiedler 75.

- Fiedler, 75; Kirkland, Neumann, Shader, 96. $T$ a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.
  - Or $Y(v_i) < 0$, decrease and concave up along $P$.
  - Or $Y(v_i) = 0$, along $P$. 

Laplacian matrix of a graph – p.8/28
Fiedler’s Monotonicity theorem: for trees

• Fiedler, 75; Kirkland, Neumann, Shader, 96. \( T \) a tree, \( Y \) a Fiedler vector. Let \( k \) be a characteristic vertex. Let \( P \) be a path that starts from \( k \). Then

- Either \( Y(v_i) > 0 \), increase and concave down along \( P \).
- Or \( Y(v_i) < 0 \), decrease and concave up along \( P \).
- Or \( Y(v_i) = 0 \), along \( P \).
Fiedler’s Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.
  - Or $Y(v_i) < 0$, decrease and concave up along $P$.
  - Or $Y(v_i) = 0$, along $P$. 

Fiedler’s Monotonicity theorem: for trees

Fiedler, 75; Kirkland, Neumann, Shader, 96. \( T \) a tree, \( Y \) a Fiedler vector. Let \( k \) be a characteristic vertex. Let \( P \) be a path that starts from \( k \). Then

- Either \( Y(v_i) > 0 \), increase and concave down along \( P \).
- Or \( Y(v_i) < 0 \), decrease and concave up along \( P \).
- Or \( Y(v_i) = 0 \), along \( P \).
Fiedler’s Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.
  - Or $Y(v_i) < 0$, decrease and concave up along $P$.
  - Or $Y(v_i) = 0$, along $P$.

Laplacian matrix of a graph – p. 8/28
Fiedler’s Monotonicity theorem: for trees

Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then

- Either $Y(v_i) > 0$, increase and concave down along $P$.
- Or $Y(v_i) < 0$, decrease and concave up along $P$.
- Or $Y(v_i) = 0$, along $P$. 

![Laplacian matrix of a graph](image-url)
Fiedler's Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.
  - Or $Y(v_i) < 0$, decrease and concave up along $P$.
  - Or $Y(v_i) = 0$, along $P$.
Fiedler’s Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.
  - Or $Y(v_i) < 0$, decrease and concave up along $P$.
  - Or $Y(v_i) = 0$, along $P$. 

Laplacian matrix of a graph – p.8/28
Fiedler’s Monotonicity theorem: for trees

Fiedler, 75; Kirkland, Neumann, Shader, 96. Let \( T \) be a tree, \( Y \) a Fiedler vector. Let \( k \) be a characteristic vertex. Let \( P \) be a path that starts from \( k \). Then

- Either \( Y(v_i) > 0 \), increase and concave down along \( P \).
- Or \( Y(v_i) < 0 \), decrease and concave up along \( P \).
- Or \( Y(v_i) = 0 \), along \( P \).
Fiedler's Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. \( T \) a tree, \( Y \) a Fiedler vector. Let \( k \) be a characteristic vertex. Let \( P \) be a path that starts from \( k \). Then
  - Either \( Y(v_i) > 0 \), increase and concave down along \( P \).
  - Or \( Y(v_i) < 0 \), decrease and concave up along \( P \).
  - Or \( Y(v_i) = 0 \), along \( P \).
Fiedler’s Monotonicity theorem: for trees

• Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then

  • Either $Y(v_i) > 0$, increase and concave down along $P$.
  • Or $Y(v_i) < 0$, decrease and concave up along $P$.
  • Or $Y(v_i) = 0$, along $P$.

Laplacian matrix of a graph – p.8/28
Fiedler's Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. \( T \) a tree, \( Y \) a Fiedler vector. Let \( k \) be a characteristic vertex. Let \( P \) be a path that starts from \( k \). Then
  - Either \( Y(v_i) > 0 \), increase and concave down along \( P \).
  - Or \( Y(v_i) < 0 \), decrease and concave up along \( P \).
  - Or \( Y(v_i) = 0 \), along \( P \).
Fiedler’s Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. Let $T$ be a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.
  - Or $Y(v_i) < 0$, decrease and concave up along $P$.
  - Or $Y(v_i) = 0$, along $P$.

- A similar statement holds when we have a characteristic edge, only that there are no 0 vertices.

Laplacian matrix of a graph – p.8/28
Fiedler’s Monotonicity theorem: for trees

- Fiedler, 75; Kirkland, Neumann, Shader, 96. $T$ a tree, $Y$ a Fiedler vector. Let $k$ be a characteristic vertex. Let $P$ be a path that starts from $k$. Then
  - Either $Y(v_i) > 0$, increase and concave down along $P$.
  - Or $Y(v_i) < 0$, decrease and concave up along $P$.
  - Or $Y(v_i) = 0$, along $P$.

These results are used to give a powerful graph partitioning algorithm. Pothen, Simon, Liou; SIAMAX, 1990.
• Take two copies of the same graph.
• Take two copies of the same graph.

\[ \mu \simeq 0.14 \]
• Take two copies of the same graph.

\[ \mu \simeq 0.14 \]

• Let us move a branch in the second picture.
• Take two copies of the same graph.

\[ \mu \simeq 0.14 \]

• Let us move a branch in the second picture.
• Take two copies of the same graph.

\[ \mu \simeq 0.14 \]

• Let us move a branch in the second picture.
• Take two copies of the same graph.

\[ \mu \simeq 0.14 \]

\[ \mu \simeq 0.12 \]

• Let us move a branch in the second picture.
• Take two copies of the same graph.

\[ \mu \simeq 0.14 \]

\[ \mu \simeq 0.12 \]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.
• Take two copies of the same graph.

\[
\mu \approx 0.14
\]

\[
\mu \approx 0.12
\]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex $k$ from the LHS tree.
• Take two copies of the same graph.

\[ \mu \approx .12 \]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex \( k \) from the LHS tree. We have three branches.
• Take two copies of the same graph.

\[ \mu \approx .12 \]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex $k$ from the LHS tree. We have three branches. The resp. principal submatrices of $L$ are
• Take two copies of the same graph.

\[ B_1 \]

\[ \mu \approx 0.12 \]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex \( k \) from the LHS tree. We have three branches. The resp. principal submatrices of \( L \) are

\[
B_1 = \begin{bmatrix}
3 & -1 & 0 & -1 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix}
\]
• Take two copies of the same graph.

\[
\begin{align*}
B_1 & : \begin{bmatrix}
3 & -1 & 0 & -1 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & -1 & 1
\end{bmatrix} \\
B_2 & : \begin{bmatrix}
3 & -1 & 0 \\
-1 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\end{align*}
\]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex \( k \) from the LHS tree. We have three branches. The resp. principal submatrices of \( L \) are

\[
\begin{align*}
\mu & \simeq .12
\end{align*}
\]
• Take two copies of the same graph.

\[
\begin{array}{c}
\begin{array}{c}
B_1 \\
B_2 \\
B_3
\end{array}
\end{array}
\]

\[
\mu \approx 0.12
\]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex \( k \) from the LHS tree. We have three branches. The resp. principal submatrices of \( L \) are

\[
B_1 : \begin{bmatrix}
3 & -1 & 0 & -1 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 \\
1
\end{bmatrix}
\]

\[
B_2 : \begin{bmatrix}
3 & -1 & -1 \\
-1 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\]

\[
B_3 : \begin{bmatrix}
2 & -1 & 0 & 0 \\
-1 & 3 & -1 & -1 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]
• Take two copies of the same graph.

![Graph Diagram]

\[ \mu \simeq 0.12 \]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex \( k \) from the LHS tree. We have three branches. The resp. principal submatrices of \( L \) are

\[
B_1 : \begin{bmatrix} 3 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \quad B_2 : \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \quad B_3 : \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 3 & -1 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}
\]

Laplacian matrix of a graph – p.9/28
• Take two copies of the same graph.

\[
\begin{align*}
B_1 & \quad B_2 \\
B_3 & \quad B_4
\end{align*}
\]

\[\mu \simeq .12\]

• Let us move a branch in the second picture. Connectivity of the network decreases if we move branches away from the characteristic set.

• Delete the vertex \( k \) from the LHS tree. We have three branches. The resp. principal submatrices of \( L \) are

\[
B_1 : \begin{bmatrix}
3 & -1 & 0 & -1 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 2 & -1 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
B_2 : \begin{bmatrix}
3 & -1 & -1 \\
-1 & 1 & 0 \\
-1 & 0 & 1
\end{bmatrix}
\]

\[
B_3 : \begin{bmatrix}
3 & -1 & 0 & 0 \\
-1 & 3 & -1 & -1 \\
0 & -1 & 1 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\]

\[\lambda_1 = \mu\quad \lambda_1 > \mu\quad \lambda_1 = \mu\]

Laplacian matrix of a graph – p.9/28
• Take two copies of the same graph.

• Let look at the tree on the RHS.
• Take two copies of the same graph.

\[
\begin{array}{c}
B_1 \\
B_2 \\
B_3
\end{array}
\]

\[
\mu \simeq 0.12
\]

• Let look at the tree on the RHS. Look at the branches at \( k \) and the smallest eigenvalues of the resp principal submatrices of \( L \).
• Take two copies of the same graph.

• Let look at the tree on the RHS. Look at the branches at $k$ and the smallest eigenvalues of the resp principal submatrices of $L$. 

$\lambda_1 \simeq 0.27$

$\lambda_1 \simeq 0.14$

$\mu \simeq 0.12$
• Take two copies of the same graph.

![Graph Diagram]

• Let look at the tree on the RHS. Look at the branches at $k$ and the smallest eigenvalues of the resp principal submatrices of $L$.

• Connectivity (from $k$) is least on this branch.
• Take two copies of the same graph.

$\mu \simeq .12$

• Let look at the tree on the RHS. Look at the branches at $k$ and the smallest eigenvalues of the resp principal submatrices of $L$.

• Now look at the branches at $l$ and the smallest eigenvalues of the resp principal submatrices of $L$. 
• Take two copies of the same graph.

\[ B_1 \]
\[ B_2 \]
\[ B_3 \]

\[ \lambda_1 \simeq 0.17 \]
\[ \mu \simeq 0.12 \]

• Let look at the tree on the RHS. Look at the branches at \( k \) and the smallest eigenvalues of the resp principal submatrices of \( L \).

• Now look at the branches at \( l \) and the smallest eigenvalues of the resp principal submatrices of \( L \).
• Take two copies of the same graph.

\[ \lambda_1 \approx 0.17 \]
\[ \mu \approx 0.12 \]

• Let look at the tree on the RHS. Look at the branches at \( k \) and the smallest eigenvalues of the resp principal submatrices of \( L \).

• Now look at the branches at \( l \) and the smallest eigenvalues of the resp principal submatrices of \( L \).

• Connectivity (from \( l \)) is least on this branch.
- Take two copies of the same graph.

- Let look at the tree on the RHS. Look at the branches at $k$ and the smallest eigenvalues of the resp principal submatrices of $L$.

- Now look at the branches at $l$ and the smallest eigenvalues of the resp principal submatrices of $L$.

- That is why the red edge is the characteristic edge for the RHS tree.
• Take two copies of the same graph.

\[ B_1 \quad \mu \simeq .12 \]

\[ B_2 \]

\[ B_3 \]

• Let look at the tree on the RHS. Look at the branches at \( k \) and the smallest eigenvalues of the resp principal submatrices of \( L \).

• Now look at the branches at \( l \) and the smallest eigenvalues of the resp principal submatrices of \( L \).

• That is why the red edge is the characteristic edge for the RHS tree. We probably should place the headquarter somewhere on this edge.
\lambda_1\ of\ the\ perturbed\ Laplacian
• We are taking a connected graph.
We are taking a connected graph. Taking its Laplacian matrix.
• We are taking a connected graph. Taking its Laplacian matrix. Adding $+1$ to the $i$th diagonal entry, intending it to be a root.
\( \lambda_1 \) of the perturbed Laplacian

- We are taking a connected graph. Taking its Laplacian matrix. Adding \(+1\) to the \(i\)th diagonal entry, intending it to be a root.

- Then the smallest eigenvalue \( \lambda_1 \) of this perturbed Laplacian is an index of the connectivity.
• We are taking a connected graph. Taking its Laplacian matrix. Adding $+1$ to the $i$th diagonal entry, intending it to be a root.

• Then the smallest eigenvalue $\lambda_1$ of this perturbed Laplacian is an index of the connectivity.

• Adding more roots increases the index.
\( \lambda_1 \) of the perturbed Laplacian

- We are taking a connected graph. Taking its Laplacian matrix. Adding \(+1\) to the \(i\)th diagonal entry, intending it to be a root.

- Then the smallest eigenvalue \( \lambda_1 \) of this **perturbed Laplacian** is an index of the connectivity.

- Adding more roots increases the index.

- Adding new edges increases the index.
\( \lambda_1 \) of the perturbed Laplacian

- We are taking a connected graph. Taking its Laplacian matrix. Adding \(+1\) to the \(i\)th diagonal entry, intending it to be a root.

- Then the smallest eigenvalue \( \lambda_1 \) of this perturbed Laplacian is an index of the connectivity.

- Adding more roots increases the index.

- Adding new edges increases the index.

- Moving a branch away from the roots will decrease the index.
We are taking a connected graph. Taking its Laplacian matrix. Adding $+1$ to the $i$th diagonal entry, intending it to be a root.

Then the smallest eigenvalue $\lambda_1$ of this perturbed Laplacian is an index of the connectivity.

Adding more roots increases the index.

Adding new edges increases the index.

Moving a branch away from the roots will decrease the index.

For a background on Laplacian refer to R. Merris, LAA, 96.
We are taking a connected graph. Taking its Laplacian matrix. Adding $+1$ to the $i$th diagonal entry, intending it to be a root.

Then the smallest eigenvalue $\lambda_1$ of this *perturbed Laplacian* is an index of the connectivity.

- Adding more roots increases the index.
- Adding new edges increases the index.
- Moving a branch away from the roots will decrease the index.

For a background on Laplacian refer to [R. Merris, LAA, 96](#).

The characteristic set identifies a ‘middle’ of the graph.
More on perturbed Laplacian
Let $G$ be connected, $D$ be any diagonal matrix.
• Let $G$ be connected, $D$ be any diagonal matrix. We define $\bar{P} := D - A(G)$. 
• Let $G$ be connected, $D$ be any diagonal matrix. We define $\bar{D} := D - A(G)$.

• When $D = \text{diagonal degree matrix}$, $\bar{D} = L$ and when $D = 0$, $\bar{D} = -A$. 
• Let $G$ be connected, $D$ be any diagonal matrix. We define $\overset{\rightarrow}{\mathcal{P}} := D - A(G)$.

• When $D = \text{diagonal degree matrix}$, $\overset{\rightarrow}{\mathcal{P}} = L$ and when $D = 0$, $\overset{\rightarrow}{\mathcal{P}} = -A$.

• For a connected graph $\lambda_1(\overset{\rightarrow}{\mathcal{P}})$ is simple.
• Let $G$ be connected, $D$ be any diagonal matrix. We define $\tilde{P} := D - A(G)$.

• When $D = \text{diagonal degree matrix}$ $\tilde{P} = L$ and when $D = 0$, $\tilde{P} = -A$.

• For a connected graph $\lambda_1(\tilde{P})$ is simple. A corresponding eigenvector $Z$ is entrywise positive.
More on perturbed Laplacian

- Let $G$ be connected, $D$ be any diagonal matrix. We define $\overline{P} := D - A(G)$.
  - When $D$ = diagonal degree matrix $\overline{P} = L$ and when $D = 0$, $\overline{P} = -A$.
  - For a connected graph $\lambda_1(\overline{P})$ is simple. A corresponding eigenvector $Z$ is entrywise positive.
  - Let $Y$ be a Fiedler vector (eigenvector for $\lambda_2(\overline{P})$).
• Let $G$ be connected, $D$ be any diagonal matrix. We define $\tilde{P} := D - A(G)$.

• When $D =$ diagonal degree matrix $\tilde{P} = L$ and when $D = 0$, $\tilde{P} = -A$.

• For a connected graph $\lambda_1(\tilde{P})$ is simple. A corresponding eigenvector $Z$ is entrywise positive.

• Let $Y$ be a Fiedler vector (eigenvector for $\lambda_2(\tilde{P})$). Then the subgraph induced by $\{v : Y(v) \geq 0\}$ is connected and the subgraph induced by $\{v : Y(v) \leq 0\}$ is connected.
• Let $G$ be connected, $D$ be any diagonal matrix. We define $\mathcal{P} := D - A(G)$.

• When $D = \text{diagonal degree matrix}$ $\mathcal{P} = L$ and when $D = 0$, $\mathcal{P} = -A$.

• For a connected graph $\lambda_1(\mathcal{P})$ is simple. A corresponding eigenvector $Z$ is entrywise positive.

• Let $Y$ be a Fiedler vector (eigenvector for $\lambda_2(\mathcal{P})$). Then the subgraph induced by $\{v : Y(v) \geq 0\}$ is connected and the subgraph induced by $\{v : Y(v) \leq 0\}$ is connected.

• The vector $\frac{Y}{Z} := \left[ \frac{Y(1)}{Z(1)} \cdots \frac{Y(n)}{Z(n)} \right]$ has the monotonicity property.
More on perturbed Laplacian

- Let $G$ be connected, $D$ be any diagonal matrix. We define $\bar{D} := D - A(G)$.
  - When $D = \text{diagonal degree matrix}$, $\bar{D} = L$ and when $D = 0$, $\bar{D} = -A$.
  - For a connected graph $\lambda_1(\bar{D})$ is simple. A corresponding eigenvector $Z$ is entrywise positive.
  - Let $Y$ be a Fiedler vector (eigenvector for $\lambda_2(\bar{D}))$. Then the subgraph induced by $\{v : Y(v) \geq 0\}$ is connected and the subgraph induced by $\{v : Y(v) \leq 0\}$ is connected.
  - The vector $\frac{Y}{Z} := \begin{bmatrix} Y(1) \\ Z(1) \\ \vdots \\ Y(n) \\ Z(n) \end{bmatrix}$ has the monotonicity property.
  - We can define characteristic set.
More on perturbed Laplacian

- Let $G$ be connected, $D$ be any diagonal matrix. We define $\overline{D} := D - A(G)$.
  - When $D =$ diagonal degree matrix $\overline{D} = L$ and when $D = 0$, $\overline{D} = -A$.
  - For a connected graph $\lambda_1(\overline{D})$ is simple. A corresponding eigenvector $Z$ is entrywise positive.
  - Let $Y$ be a Fiedler vector (eigenvector for $\lambda_2(\overline{D})$). Then the subgraph induced by $\{v : Y(v) \geq 0\}$ is connected and the subgraph induced by $\{v : Y(v) \leq 0\}$ is connected.
  - The vector $\frac{Y}{Z} := \left[ \frac{Y(1)}{Z(1)} \cdots \frac{Y(n)}{Z(n)} \right]$ has the monotonicity property.
  - We can define characteristic set.
  - The characteristic set is either a single vertex or in a unique block.
More on perturbed Laplacian

- Let $G$ be connected, $D$ be any diagonal matrix. We define $\hat{P} := D - A(G)$.
  - When $D =$ diagonal degree matrix $\hat{P} = L$ and when $D = 0$, $\hat{P} = -A$.
  - For a connected graph $\lambda_1(\hat{P})$ is simple. A corresponding eigenvector $Z$ is entrywise positive.
  - Let $Y$ be a Fiedler vector (eigenvector for $\lambda_2(\hat{P})$). Then the subgraph induced by $\{v : Y(v) \geq 0\}$ is connected and the subgraph induced by $\{v : Y(v) \leq 0\}$ is connected.
  - The vector $\frac{Y}{Z} := \left[ \frac{Y(1)}{Z(1)} \cdots \frac{Y(n)}{Z(n)} \right]$ has the monotonicity property.
  - We can define characteristic set.
  - The characteristic set is either a single vertex or in a unique block.
  - For more results refer to Bapat, Kirkland, Pati, LAMA 2001.
Relation with resistance distance
• Consider two graphs.
• Consider two graphs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{example.png}
\caption{Graphs with different edge weights.}
\label{fig:example}
\end{figure}
• Consider two graphs.

![Graph 1](image1)

![Graph 2](image2)

• The distance between \( a \) and \( b \) in both the graphs are the same,
• Consider two graphs.

![](image)

• The distance between $a$ and $b$ in both the graphs are the same, though these two vertices are better connected in the LHS graph.
• Consider two graphs.

- The distance between $a$ and $b$ in both the graphs are the same, though these two vertices are better connected in the LHS graph. It is reasonable that the distance between $a$ and $b$ should be lesser in the LHS graph.
Consider two graphs.

![Graph illustration](image)

- The distance between $a$ and $b$ in both the graphs are the same, though these two vertices are better connected in the LHS graph. It is reasonable that the distance between $a$ and $b$ should be lesser in the LHS graph.

- ‘Resistance distance’ captures it more appropriately.
• Consider two graphs.

- The distance between $a$ and $b$ in both the graphs are the same, though these two vertices are better connected in the LHS graph. **It is reasonable that the distance between $a$ and $b$ should be lesser in the LHS graph.**

- ‘Resistance distance’ captures it more appropriately.

- For a tree ‘classical distance’ and resistance distance coincide.
• Consider two graphs.

\[ \begin{array}{cc}
\text{a=1} & \text{b=5} \\
\end{array} \quad \begin{array}{cc}
\text{a=1} & \text{b=3} \\
\end{array} \]

• \( G \) be connected with vertices 1, \ldots, \( n \).
• Consider two graphs.

\[ a=1 \quad b=5 \quad a=1 \quad b=3 \]

• \( G \) be connected with vertices 1, \ldots, \( n \). Take \( M = L^+ \), the Moore-Penrose inverse of \( L \).
• Consider two graphs.

\[ a=1 \quad b=5 \]

\[ a=1 \quad b=3 \]

• \( G \) be connected with vertices 1, \ldots, \( n \). Take \( M = L^+ \), the Moore-Penrose inverse of \( L \). \( LML = L \), \( MLM = M \); \( ML \) and \( LM \) are symmetric.

Known: \( M \) is positive semidefinite.
Consider two graphs.

$\begin{align*}
&\text{a=1} \quad \text{b=5} \\
&\text{a=1} \quad \text{b=3}
\end{align*}$

$G$ be connected with vertices $1, \ldots, n$. Take $M = L^+$, the Moore-Penrose inverse of $L$.

Define $r(i, j) = m_{ii} + m_{jj} - 2m_{ij}$. 

Relation with resistance distance
• Consider two graphs.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{Graphs with vertices labeled.}
\end{figure}

• $G$ be connected with vertices $1, \ldots, n$. Take $M = L^+$, the Moore-Penrose inverse of $L$.

Define $r(i, j) = m_{ii} + m_{jj} - 2m_{ij}$.

• The resistance matrices for the above graphs:
• Consider two graphs.

![Graphs](image)

• $G$ be connected with vertices $1, \ldots, n$. Take $M = L^+$, the Moore-Penrose inverse of $L$.

Define $r(i, j) = m_{ii} + m_{jj} - 2m_{ij}$.

• The resistance matrices for the above graphs:

$$
\begin{bmatrix}
0 & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} \\
\frac{2}{3} & 0 & 1 & 1 & \frac{2}{3} \\
\frac{2}{3} & 1 & 0 & 1 & \frac{2}{3} \\
\frac{2}{3} & 1 & 1 & 0 & \frac{2}{3} \\
\frac{3}{3} & \frac{2}{3} & \frac{2}{3} & \frac{2}{3} & 0
\end{bmatrix}
$$

and

$$
\begin{bmatrix}
0 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 0
\end{bmatrix}
$$
Relation with resistance distance
known. \( r(i, j) \leq d(i, j); \)
Relation with resistance distance

- known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.
known. $r(i, j) \leq d(i, j)$; equality holds iff there is a unique $i$-$j$-path.

- (triangle inequality) $r(i, j) + r(j, k) \geq r(i, k)$.
• known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.
  
  • (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).

• \( G \): connected.
• known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.

  - (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).

• \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{1}\mathbb{1}^t \).
• known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.

• (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).

• \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{1}\mathbb{1}^t \). Then \( L^+ = X - \frac{J}{n} \).
Relation with resistance distance

- known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.
  - (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).
- \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{1} \mathbb{1}^t \). Then \( L^+ = X - \frac{J}{n} \).
- Put \( \tilde{X} = \text{diag}(x_{11}, \ldots, x_{nn}) \).
Relation with resistance distance

- known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.
  - (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).
- \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{1}_n^t \). Then \( L^+ = X - \frac{J}{n} \).
- Put \( \tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn}) \). Then \( R = \tilde{X}J + J\tilde{X} - 2X \).
Relation with resistance distance

- known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.
  
  - (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).
  
  - \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \text{1}\text{1}^t \). Then \( L^+ = X - \frac{J}{n} \).
  
  - Put \( \tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn}) \). Then \( R = \tilde{X} J + J \tilde{X} - 2X \).
  
  - Put \( \tau_i = 2 - \sum_{j \sim i} r(i, j) \).
known. $r(i, j) \leq d(i, j)$; equality holds iff there is a unique $i$-$j$-path.

- (triangle inequality) $r(i, j) + r(j, k) \geq r(i, k)$.
- $G$: connected. Put $X = (L + \frac{J}{n})^{-1}$, where $J = 11^t$. Then $L^+ = X - \frac{J}{n}$.
- Put $\tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn})$. Then $R = \tilde{X}J + J\tilde{X} - 2X$.
- Put $\tau_i = 2 - \sum_{j \sim i} r(i, j)$. Then $\tau = L\tilde{X}1 + \frac{2}{n}1$. 

Relation with resistance distance
• known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.

  - (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).

• \( G \): connected. Put \( X = (L + \frac{1}{n}J)^{-1} \), where \( J = \mathbb{1} \mathbb{1}^t \). Then \( L^+ = X - \frac{1}{n}J \).

• Put \( \tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn}) \). Then \( R = \tilde{X} J + J \tilde{X} - 2X \).

• Put \( \tau_i = 2 - \sum_{j \sim i} r(i, j) \). Then \( \tau = L \tilde{X} \mathbb{1} + \frac{2}{n} \mathbb{1} \).

• \( \sum_i \sum_{j \sim i} r(i, j) = 2(n - 1) \).
• known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.
  
  • (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).
  
  • \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{1}\mathbb{1}^t \). Then \( L^+ = X - \frac{J}{n} \).
  
  • Put \( \tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn}) \). Then \( R = \tilde{X}J + J\tilde{X} - 2X \).
  
  • Put \( \tau_i = 2 - \sum_{j \sim i} r(i, j) \). Then \( \tau = L\tilde{X}\mathbb{1} + \frac{2}{n} \mathbb{1} \).
  
  • \( \sum_{i} \sum_{j \sim i} r(i, j) = 2(n - 1) \). So \( \mathbb{1}^t \tau = 2 \).
known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.

- (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).
- \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{1}\mathbb{1}^t \). Then \( L^+ = X - \frac{J}{n} \).
- Put \( \tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn}) \). Then \( R = \tilde{X}J + J\tilde{X} - 2X \).
- Put \( \tau_i = 2 - \sum_{j \sim i} r(i, j) \). Then \( \tau = L\tilde{X}\mathbb{1} + \frac{2}{n}\mathbb{1} \).
- \( \sum_i \sum_{j \sim i} r(i, j) = 2(n - 1) \). So \( \mathbb{1}^t\tau = 2 \).
- \( R^{-1} = -\frac{1}{2}L + \frac{1}{\tau^tR\tau} \tau \tau^t \). Generalizes inverse of distance matrix (tree).
Relation with resistance distance

- known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.
  - (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).
- \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{I} \mathbb{I}^t \). Then \( L^+ = X - \frac{J}{n} \).
- Put \( \tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn}) \). Then \( R = \tilde{X} J + J \tilde{X} - 2X \).
- Put \( \tau_i = 2 - \sum_{j \sim i} r(i, j) \). Then \( \tau = L \tilde{X} \mathbb{I} + \frac{2}{n} \mathbb{I} \).
- \( \sum_i \sum_{j \sim i} r(i, j) = 2(n - 1) \). So \( \mathbb{I}^t \tau = 2 \).
- \( R^{-1} = -\frac{1}{2} L + \frac{1}{\tau^t R \tau} \tau \tau^t \). Generalizes inverse of distance matrix (tree).
- \( G \): connected, \( \lambda_i := \lambda_i(L) \).
Relation with resistance distance

- known. $r(i, j) \leq d(i, j)$; equality holds iff there is a unique $i$-$j$-path.

- (triangle inequality) $r(i, j) + r(j, k) \geq r(i, k)$.

- $G$: connected. Put $X = (L + J/n)^{-1}$, where $J = \mathbb{1}\mathbb{1}^t$. Then $L^+ = X - J/n$.

- Put $\tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn})$. Then $R = \tilde{X}J + J\tilde{X} - 2X$.

- Put $\tau_i = 2 - \sum_{j \sim i} r(i, j)$. Then $\tau = L\tilde{X}\mathbb{1} + \frac{2}{n}\mathbb{1}$.

- $\sum_i \sum_{j \sim i} r(i, j) = 2(n - 1)$. So $\mathbb{1}^t \tau = 2$.

- $R^{-1} = -\frac{1}{2}L + \frac{1}{\tau^t R \tau} \tau \tau^t$. Generalizes inverse of distance matrix (tree).

- $G$: connected, $\lambda_i := \lambda_i(L)$. Then $\sum_i \sum_{j} r(i, j) = 2 \sum_{i=2}^{n} \frac{1}{\lambda_i}$. 

Laplacian matrix of a graph – p.13/28
Relation with resistance distance

• known. \( r(i, j) \leq d(i, j) \); equality holds iff there is a unique \( i-j \)-path.

  • (triangle inequality) \( r(i, j) + r(j, k) \geq r(i, k) \).

• \( G \): connected. Put \( X = (L + \frac{J}{n})^{-1} \), where \( J = \mathbb{1}\mathbb{1}^t \). Then \( L^+ = X - \frac{J}{n} \).

• Put \( \tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn}) \). Then \( R = \tilde{X}J + J\tilde{X} - 2X \).

• Put \( \tau_i = 2 - \sum_{j \sim i} r(i, j) \). Then \( \tau = L\tilde{X}\mathbb{1} + \frac{2}{n} \mathbb{1} \).

• \( \sum_{i} \sum_{j \sim i} r(i, j) = 2(n - 1) \). So \( \mathbb{1}^t\tau = 2 \).

• \( R^{-1} = -\frac{1}{2}L + \frac{1}{\tau^tR\tau}\tau\tau^t \). Generalizes inverse of distance matrix (tree).

• \( G \): connected, \( \lambda_i := \lambda_i(L) \). Then \( \sum_i \sum_j r(i, j) = 2 \sum_{i=2}^{n} \frac{1}{\lambda_i} \).

• \( R \) of a connected \( G \) has exactly one positive eigenvalue.
• **known.** $r(i, j) \leq d(i, j)$; equality holds iff there is a unique $i$-$j$-path.

  • (triangle inequality) $r(i, j) + r(j, k) \geq r(i, k)$.

• $G$: connected. Put $X = (L + \frac{J}{n})^{-1}$, where $J = \mathbb{I}\mathbb{I}^t$. Then $L^+ = X - \frac{J}{n}$.

• Put $\tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn})$. Then $R = \tilde{X}J + J\tilde{X} - 2X$.

• Put $\tau_i = 2 - \sum_{j \sim i} r(i, j)$. Then $\tau = L\tilde{X}\mathbb{I} + \frac{2}{n}\mathbb{I}$.

• $\sum_i \sum_{j \sim i} r(i, j) = 2(n - 1)$. So $\mathbb{I}^t\tau = 2$.

• $R^{-1} = -\frac{1}{2}L + \frac{1}{\tau^tR\tau}\tau\tau^t$. Generalizes inverse of distance matrix (tree).

• $G$: connected, $\lambda_i := \lambda_i(L)$. Then $\sum_i \sum_j r(i, j) = 2\sum_{i=2}^{n} \frac{1}{\lambda_i}$.

• $R$ of a connected $G$ has exactly one positive eigenvalue.

• $G$ be connected and $i \sim j$; $k(G) =$ number of spanning trees; $k'(G) =$ number of spanning trees containing $ij$. 
Relation with resistance distance

- **known.** $r(i, j) \leq d(i, j)$; equality holds iff there is a unique $i$-$j$-path.
  - (triangle inequality) $r(i, j) + r(j, k) \geq r(i, k)$.
  - $G$: connected. Put $X = (L + \frac{J}{n})^{-1}$, where $J = \mathbb{1}\mathbb{1}^t$. Then $L^+ = X - \frac{J}{n}$.
  - Put $\tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn})$. Then $R = \tilde{X}J + J\tilde{X} - 2X$.
  - Put $\tau_i = 2 - \sum_{j \sim i} r(i, j)$. Then $\tau = L\tilde{X}\mathbb{1} + \frac{2}{n}\mathbb{1}$.
  - $\sum_{i} \sum_{j \sim i} r(i, j) = 2(n - 1)$. So $\mathbb{1}^t\tau = 2$.
  - $R^{-1} = -\frac{1}{2}L + \frac{1}{\tau^tR\tau}\tau\tau^t$. Generalizes inverse of distance matrix (tree).
  - $G$: connected, $\lambda_i := \lambda_i(L)$. Then $\sum_{i} \sum_{j} r(i, j) = 2 \sum_{i=2}^{n} \frac{1}{\lambda_i}$.
  - $R$ of a connected $G$ has exactly one positive eigenvalue.
  - $G$ be connected and $i \sim j$; $k(G) =$ number of spanning trees; $k'(G) =$ number of spanning trees containing $ij$. Then $r(i, j) = \frac{k'(G)}{k(G)}$. 

Laplacian matrix of a graph – p.13/28
**known.** $r(i, j) \leq d(i, j)$; equality holds iff there is a unique $i$-$j$-path.

- (triangle inequality) $r(i, j) + r(j, k) \geq r(i, k)$.

- $G$: connected. Put $X = (L + J/n)^{-1}$, where $J = \mathbb{1}\mathbb{1}^t$. Then $L^+ = X - J/n$.

- Put $\tilde{X} = \text{DIAG}(x_{11}, \ldots, x_{nn})$. Then $R = \tilde{X}J + J\tilde{X} - 2X$.

- Put $\tau_i = 2 - \sum_{j \sim i} r(i, j)$. Then $\tau = L\tilde{X}\mathbb{1} + \frac{2}{n}\mathbb{1}$.

- $\sum_i \sum_{j \sim i} r(i, j) = 2(n - 1)$. So $\mathbb{1}^t\tau = 2$.

- $R^{-1} = -\frac{1}{2}L + \frac{1}{\tau^tR\tau}\tau\tau^t$. Generalizes inverse of distance matrix (tree).

- $G$: connected, $\lambda_i := \lambda_i(L)$. Then $\sum_i \sum_j r(i, j) = 2 \sum_{i=2}^n \frac{1}{\lambda_i}$.

- $R$ of a connected $G$ has exactly one positive eigenvalue.

- $G$ be connected and $i \sim j$; $k(G) = \text{number of spanning trees}$; $k'(G) = \text{number of spanning trees containing } ij$. Then $r(i, j) = \frac{k'(G)}{k(G)}$.

- For more refer to Book-‘Graphs and Matrices’- Bapat.
Thank You

\[ \lambda_2(L(G)) \] is a measure of the stability and the robustness of the network dynamic system” –

Y. Kim and M. Mesbahi,

2005 American control conference.
Fiedler’s Monotonicity theorem

- Call a path pure if it has at most two points of articulation of each block.
Fiedler’s Monotonicity theorem

- Call a path pure if it has at most two points of articulation of each block.
- Fiedler, 75. \( G \) connected, \( Y \) a Fiedler vector. Assume Case A. Then
Call a path pure if it has at most two points of articulation of each block.

Fiedler, 75. \( G \) connected, \( Y \) a Fiedler vector. Assume Case A. Then

- Only the ch. block \( C \) has both positive and negative vertices (w.r.t \( Y \)).
Call a path pure if it has at most two points of articulation of each block.

Fiedler, 75. \( G \) connected, \( Y \) a Fiedler vector. Assume Case A. Then

- Only the ch. block \( C \) has both positive and negative vertices (w.r.t \( Y \)).
- Let \( k \in C \).
• Call a path **pure** if it has at most two points of articulation of each block.

• **Fiedler,75.** $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then
  • Only the ch.block $C$ has both positive and negative vertices (w.r.t $Y$).
  • Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$. 
Fiedler’s Monotonicity theorem

• Call a path pure if it has at most two points of articulation of each block.

• Fiedler, 75. $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then
  • Only the ch.block $C$ has both positive and negative vertices (w.r.t $Y$).
  • Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$.
  • Then the values at points of articulation along $P$ increase, or decrease, or remain zero according to whether $Y(k) > 0$, $Y(k) < 0$ or $Y(k) = 0$;
Fiedler’s Monotonicity theorem

- Call a path **pure** if it has at most two points of articulation of each block.

- Fiedler, 75. $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then
  - Only the ch. block $C$ has both positive and negative vertices (w.r.t $Y$).
  - Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$.
  - Then the values at points of articulation along $P$ increase, or decrease, or remain zero according to whether $Y(k) > 0$, $Y(k) < 0$ or $Y(k) = 0$; in the last case all vertices in $P$ have valuation zero.
Fiedler’s Monotonicity theorem

- Call a path **pure** if it has at most two points of articulation of each block.

- Fiedler, 75. $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then
  - Only the ch.block $C$ has both positive and negative vertices (w.r.t $Y$).
  - Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$.
  - Then the values at points of articulation along $P$ increase, or decrease, or remain zero according to whether $Y(k) > 0$, $Y(k) < 0$ or $Y(k) = 0$; in the last case all vertices in $P$ have valuation zero.
Fiedler's Monotonicity theorem

- Call a path **pure** if it has at most two points of articulation of each block.
- **Fiedler,75.** \( G \) connected, \( Y \) a Fiedler vector. Assume Case A. Then
  - Only the ch.block \( C \) has both positive and negative vertices (w.r.t \( Y \)).
  - Let \( k \in C \). Take a pure path \( P \) starting at \( k \) and leaving \( C \).
  - Then the values at points of articulation along \( P \) increase, or decrease, or remain zero according to whether \( Y(k) > 0 \), \( Y(k) < 0 \) or \( Y(k) = 0 \); in the last case all vertices in \( P \) have valuation zero.
Fiedler’s Monotonicity theorem

- Call a path **pure** if it has at most two points of articulation of each block.

- **Fiedler,75.** \( G \) connected, \( Y \) a Fiedler vector. Assume Case A. Then
  - Only the ch.block \( C \) has both positive and negative vertices (w.r.t \( Y \)).
  - Let \( k \in C \). Take a pure path \( P \) starting at \( k \) and leaving \( C \).
  - Then the values at points of articulation along \( P \) increase, or decrease, or remain zero according to whether \( Y(k) > 0 \), \( Y(k) < 0 \) or \( Y(k) = 0 \); in the last case all vertices in \( P \) have valuation zero.
• Call a path pure if it has at most two points of articulation of each block.

• Fiedler, 75. $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then
  • Only the ch. block $C$ has both positive and negative vertices (w.r.t $Y$).
  • Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$.
  • Then the values at points of articulation along $P$ increase, or decrease, or remain zero according to whether $Y(k) > 0$, $Y(k) < 0$ or $Y(k) = 0$; in the last case all vertices in $P$ have valuation zero.
• Call a path pure if it has at most two points of articulation of each block.

• **Fiedler, 75.** $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then
  
  • Only the ch.block $C$ has both positive and negative vertices (w.r.t $Y$).
  
  • Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$.
  
  • Then the values at points of articulation along $P$ increase, or decrease, or remain zero according to whether $Y(k) > 0$, $Y(k) < 0$ or $Y(k) = 0$; in the last case all vertices in $P$ have valuation zero.
Fiedler’s Monotonicity theorem

- Call a path pure if it has at most two points of articulation of each block.

- Fiedler, 75. \( G \) connected, \( Y \) a Fiedler vector. Assume Case A. Then
  - Only the ch.block \( C \) has both positive and negative vertices (w.r.t \( Y \)).
  - Let \( k \in C \). Take a pure path \( P \) starting at \( k \) and leaving \( C \).
  - Then the values at points of articulation along \( P \) increase, or decrease, or remain zero according to whether \( Y(k) > 0 \), \( Y(k) < 0 \) or \( Y(k) = 0 \); in the last case all vertices in \( P \) have valuation zero.
Call a path pure if it has at most two points of articulation of each block.

Fiedler, 75. $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then

- Only the ch.block $C$ has both positive and negative vertices (w.r.t $Y$).
- Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$.
- Then the values at points of articulation along $P$ increase, or decrease, or remain zero according to whether $Y(k) > 0$, $Y(k) < 0$ or $Y(k) = 0$; in the last case all vertices in $P$ have valuation zero.

A similar statement holds for Case B.
Call a path pure if it has at most two points of articulation of each block.

Fiedler, 75. $G$ connected, $Y$ a Fiedler vector. Assume Case A. Then

- Only the ch.block $C$ has both positive and negative vertices (w.r.t $Y$).
- Let $k \in C$. Take a pure path $P$ starting at $k$ and leaving $C$.
- Then the values at points of articulation along $P$ increase, or decrease, or remain zero according to whether $Y(k) > 0$, $Y(k) < 0$ or $Y(k) = 0$; in the last case all vertices in $P$ have valuation zero.

A similar statement holds for Case B.

Q. Can we have monotonicity along the path instead of monotonicity along the point of articulations on the path?
A little more
• No. Not for each \( z-u \)-path. Here \( z \) is the characteristic vertex.
No. Not for each $z$-$v$-path. Here $z$ is the characteristic vertex.
• No. Not for each $z$-$v$-path. Here $z$ is the characteristic vertex.

• However, there is a $z$-$v$-path.
• No. Not for each \( z-v \)-path. Here \( z \) is the characteristic vertex.

\[ \text{A branch at } v \text{ is a connected component of } G - v. \]
• No. Not for each \( z-v \)-path. Here \( z \) is the characteristic vertex.

• However, there is a \( z-v \)-path.

• A branch at \( v \) is a connected component of \( G - v \).

• Lemma. \( G \) be connected, \( Y \) be a Fiedler vector, Case A be true, \( C \) be the ch.block, and \( k \) be a vertex in \( C \) with \( Y(k) > 0 \).
• No. Not for each $z$-$v$-path. Here $z$ is the characteristic vertex.

• However, there is a $z$-$v$-path.

• A branch at $v$ is a connected component of $G - v$.

• Lemma. $G$ be connected, $Y$ be a Fiedler vector, Case A be true, $C$ be the ch.block, and $k$ be a vertex in $C$ with $Y(k) > 0$. Let $B$ be a branch at $k$ not containing a vertex of $C$. 

Laplacian matrix of a graph – p.16/28
• No. Not for each $z$-$v$-path. Here $z$ is the characteristic vertex.

• However, there is a $z$-$v$-path.

• A branch at $v$ is a connected component of $G - v$.

• Lemma. $G$ be connected, $Y$ be a Fiedler vector, Case A be true, $C$ be the ch.block, and $k$ be a vertex in $C$ with $Y(k) > 0$. Let $B$ be a branch at $k$ not containing a vertex of $C$. Let $u$ be a vertex in $B$. 

Laplacian matrix of a graph – p.16/28
• No. Not for each $z$-$v$-path. Here $z$ is the characteristic vertex.

• However, there is a $z$-$v$-path.

• A branch at $v$ is a connected component of $G - v$.

**Lemma.** $G$ be connected, $Y$ be a Fiedler vector, Case A be true, $C$ be the ch.block, and $k$ be a vertex in $C$ with $Y(k) > 0$. Let $B$ be a branch at $k$ not containing a vertex of $C$. Let $u$ be a vertex in $B$. Then there is a path $P = [k, v_1, \ldots, v_s = u]$ s.t. $Y(k) < Y(v_1) < \cdots < Y(u)$, where $v_i \in B$. 
A little more: in general
• **Theorem.** $G$ be connected, $Y$ be a Fiedler vector, Case A be true, $C'$ be the ch.block.
Theorem. $G$ be connected, $Y$ be a Fiedler vector, Case A be true, $C$ be the ch. block. Then there is a spanning subgraph $H$ of $G$. 
• **Theorem.** \( G \) be connected, \( Y \) be a Fiedler vector, Case A be true, \( C \) be the ch.block. Then there is a spanning subgraph \( H \) of \( G \), which does not contain a cycle outside \( C \).
Theorem. $G$ be connected, $Y$ be a Fiedler vector, Case A be true, $C$ be the ch.block. Then there is a spanning subgraph $H$ of $G$, which does not contain a cycle outside $C$ such that every path $P$ which starts from a vertex $k$ of $C$ and does not contain any other vertex of $C$ has the following property:
• **Theorem.** Let $G$ be connected, $Y$ be a Fiedler vector, Case A be true, $C$ be the ch.block. Then there is a spanning subgraph $H$ of $G$, which does not contain a cycle outside $C$ such that every path $P$ which starts from a vertex $k$ of $C$ and does not contain any other vertex of $C$ has the following property:

• If $Y(k) > 0$, then the valuation along $P$ is strictly increasing.
Theorem. \( G \) be connected, \( Y \) be a Fiedler vector, Case A be true, \( C \) be the ch.block. Then there is a spanning subgraph \( H \) of \( G \), which does not contain a cycle outside \( C \) such that every path \( P \) which starts from a vertex \( k \) of \( C \) and does not contain any other vertex of \( C \) has the following property:

- If \( Y(k) > 0 \), then the valuation along \( P \) is strictly increasing.
- If \( Y(k) < 0 \), then the valuation along \( P \) is strictly decreasing.
Theorem. \( G \) be connected, \( Y \) be a Fiedler vector, Case A be true, \( C \) be the ch.block. Then there is a spanning subgraph \( H \) of \( G \), which does not contain a cycle outside \( C \) such that every path \( P \) which starts from a vertex \( k \) of \( C \) and does not contain any other vertex of \( C \) has the following property:

- If \( Y(k) > 0 \), then the valuation along \( P \) is strictly increasing.
- If \( Y(k) < 0 \), then the valuation along \( P \) is strictly decreasing.
- If \( Y(k) = 0 \), then the valuation along \( P \) is identically zero.
Theorem. \( G \) be connected, \( Y \) be a Fiedler vector, Case A be true, \( C \) be the ch.block. Then there is a spanning subgraph \( H \) of \( G \), which does not contain a cycle outside \( C \) such that every path \( P \) which starts from a vertex \( k \) of \( C \) and does not contain any other vertex of \( C \) has the following property:

- If \( Y(k) > 0 \), then the valuation along \( P \) is strictly increasing.
- If \( Y(k) < 0 \), then the valuation along \( P \) is strictly decreasing.
- If \( Y(k) = 0 \), then the valuation along \( P \) is identically zero.
• **Theorem.** \( G \) be connected, \( Y \) be a Fiedler vector, Case A be true, \( C \) be the ch.block. Then there is a spanning subgraph \( H \) of \( G \), which does not contain a cycle outside \( C \) such that every path \( P \) which starts from a vertex \( k \) of \( C \) and does not contain any other vertex of \( C \) has the following property:

- If \( Y(k) > 0 \), then the valuation along \( P \) is strictly increasing.
- If \( Y(k) < 0 \), then the valuation along \( P \) is strictly decreasing.
- If \( Y(k) = 0 \), then the valuation along \( P \) is identically zero.
A little more: in general

Laplacian matrix of a graph

Laplacian matrix of a graph – p.18/28
A little more: in general

- A similar statement holds for Case B. (spanning tree)
A similar statement holds for Case B. (spanning tree)

- We may have more choices for the spanning subgraph in both cases.
• A similar statement holds for Case B. (spanning tree)
  
  • We may have more choices for the spanning subgraph in both cases.
  
  • There are graph classes for which we have essentially one choice.
• A similar statement holds for Case B. (spanning tree)
  • We may have more choices for the spanning subgraph in both cases.
  • There are graph classes for which we have essentially one choice.
A similar statement holds for Case B. (spanning tree)

- We may have more choices for the spanning subgraph in both cases.
- There are graph classes for which we have essentially one choice.
A little more: in general

- A similar statement holds for Case B. (spanning tree)
  - We may have more choices for the spanning subgraph in both cases.
  - There are graph classes for which we have essentially one choice.
• A similar statement holds for Case B. (spanning tree)
  • We may have more choices for the spanning subgraph in both cases.
  • There are graph classes for which we have essentially one choice.

• About this graph:
A similar statement holds for Case B. (spanning tree)

- We may have more choices for the spanning subgraph in both cases.
- There are graph classes for which we have essentially one choice.

About this graph:

- Blocks are ‘path bundles’ (internally vertex disjoint paths of same length with common end points).
A little more: in general

- A similar statement holds for Case B. (spanning tree)
  - We may have more choices for the spanning subgraph in both cases.
  - There are graph classes for which we have essentially one choice.

- About this graph:
  - Blocks are ‘path bundles’ (internally vertex disjoint paths of same length with common end points).
  - ‘Restricted blocks’ (each block has at most two points of articulations).
Minimizing algebraic connectivity of graphs with restricted blocks
Consider a connected graph with restricted blocks. The ‘block structure’ of such a graph is a tree.
Consider a connected graph with restricted blocks. The ‘block structure’ of such a graph is a tree.
Consider a connected graph with restricted blocks. The ‘block structure’ of such a graph is a tree.

- View it: take a tree, replace each edge by a ‘suitable’ restricted block.
• Consider a connected graph with restricted blocks. The ‘block structure’ of such a graph is a tree.

• View it: take a tree, replace each edge by a ‘suitable’ restricted block.

• Let $G$ be connected, $B$ a branch at $v$ and
Consider a connected graph with restricted blocks. The ‘block structure’ of such a graph is a tree.

- View it: take a tree, replace each edge by a ‘suitable’ restricted block.
- Let $G$ be connected, $B$ a branch at $v$ and $\hat{L}(B)$ the corresponding principal submatrix of $L(G)$. 
Consider a connected graph with restricted blocks. The ‘block structure’ of such a graph is a tree.

- View it: take a tree, replace each edge by a ‘suitable’ restricted block.
- Let $G$ be connected, $B$ a branch at $v$ and $\hat{L}(B)$ the corresponding principal submatrix of $L(G)$. Note: the bottleneck matrix $B[B] = \hat{L}(B)^{-1}$ is entrywise positive.
Consider a connected graph with restricted blocks. The ‘block structure’ of such a graph is a tree.

- View it: take a tree, replace each edge by a ‘suitable’ restricted block.

- Let $G$ be connected, $B$ a branch at $v$ and $\hat{L}(B)$ the corresponding principal submatrix of $L(G)$. Note: the bottleneck matrix $B[B] = \hat{L}(B)^{-1}$ is entrywise positive. A branch $B$ at $v$ for which the spectral radius $\rho(B[B]) \geq \frac{1}{\mu}$ is a Perron branch (component).
Known results involving Perron branch
Known results involving Perron branch

- Fallat & Kirkland, 98. \( G \) be connected. Then at each point of articulation \( v \) there is at least one Perron branch (component).
Known results involving Perron branch

- Fallat & Kirkland, 98. \( G \) be connected. Then at each point of articulation \( v \), there is at least one Perron branch (component).
- If there are more than one Perron components at \( v \), then Case B holds.
Known results involving Perron branch

- **Fallat & Kirkland, 98.** Let $G$ be connected. Then at each point of articulation $v$ there is at least one Perron branch (component).

- If there are more than one Perron components at $v$, then Case B holds and $v$ is the characteristic vertex for each Fiedler vector.
Known results involving Perron branch

- Fallat & Kirkland, 98. \( G \) be connected. Then at each point of articulation \( v \) there is at least one Perron branch (component).

- If there are more than one Perron components at \( v \), then Case B holds and \( v \) is the characteristic vertex for each Fiedler vector.

- [Replacement lemma] Fallat & Kirkland, 98. \( G \) be connected, \( v \) be a point of articulation with branches \( C_1, \ldots, C_k \) at \( v \).
Known results involving Perron branch

• Fallat & Kirkland, 98. $G$ be connected. Then at each point of articulation $v$ there is at least one Perron branch (component).

• If there are more than one Perron components at $v$, then Case B holds and $v$ is the characteristic vertex for each Fiedler vector.

• [Replacement lemma] Fallat & Kirkland, 98. $G$ be connected, $v$ be a point of articulation with branches $C_1, \ldots, C_k$ at $v$. Assume that $C = \bigcup_{i=1}^{j} C_i$ misses vertices of some Perron branch at $v$. 
Known results involving Perron branch

- **Fallat & Kirkland, 98.** Let $G$ be connected. Then at each point of articulation $v$ there is at least one Perron branch (component).
  - If there are more than one Perron components at $v$, then Case B holds and $v$ is the characteristic vertex for each Fiedler vector.

- **[Replacement lemma]Fallat & Kirkland, 98.** Let $G$ be connected, $v$ be a point of articulation with branches $C_1, \ldots, C_k$ at $v$. Assume that $C = \bigcup_{i=1}^{j} C_i$ misses vertices of some Perron branch at $v$. Form $\tilde{G}$ by replacing $C$ with a single connected component $\tilde{C}$ at $v$. 

Laplacian matrix of a graph – p.20/28
Known results involving Perron branch

• Fallat & Kirkland, 98. Let $G$ be connected. Then at each point of articulation $v$ there is at least one Perron branch (component).

• If there are more than one Perron components at $v$, then Case B holds and $v$ is the characteristic vertex for each Fiedler vector.

• [Replacement lemma] Fallat & Kirkland, 98. Let $G$ be connected, $v$ be a point of articulation with branches $C_1, \ldots, C_k$ at $v$. Assume that $C = \bigcup_{i=1}^{j} C_i$ misses vertices of some Perron branch at $v$. Form $\tilde{G}$ by replacing $C$ with a single connected component $\tilde{C}$ at $v$. If $B[\tilde{C}] \geq B[C]$, then $\mu(\tilde{G}) \leq \mu(G)$. 
• Fallat & Kirkland, 98. \( G \) be connected. Then at each point of articulation \( v \) there is at least one Perron branch (component).

• If there are more than one Perron components at \( v \), then Case B holds and \( v \) is the characteristic vertex for each Fiedler vector.

• [Replacement lemma] Fallat & Kirkland, 98. \( G \) be connected, \( v \) be a point of articulation with branches \( C_1, \ldots, C_k \) at \( v \). Assume that \( C = \bigcup_{i=1}^{j} C_i \) misses vertices of some Perron branch at \( v \). Form \( \tilde{G} \) by replacing \( C \) with a single connected component \( \tilde{C} \) at \( v \). If \( B[\tilde{C}] \geq B[C] \), then \( \mu(\tilde{G}) \leq \mu(G) \).

• In above, if \( B[\tilde{C}] \leq B[C] \), then \( \mu(\tilde{G}) \geq \mu(G) \).
Known results involving Perron branch

- Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. 
Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. Then the Laplacian matrix of a graph is given by

$$B[G - g]_{ij} = \frac{i(g - j)}{g} \text{ if } i \leq j.$$
Known results involving Perron branch

- Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. Then the Laplacian matrix of a graph $G$ is

$$B[G - g]_{ij} = \frac{i(g - j)}{g} \text{ if } i \leq j.$$
Known results involving Perron branch

- Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. Then the

$$B[G-g]_{ij} = \frac{i(g-j)}{g} \text{ if } i \leq j.$$  

![Graph](image)

$$B[G - 5] = \begin{bmatrix}
\frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\
\frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\
\frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5}
\end{bmatrix}$$
Known results involving Perron branch

- Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. Then the

$$B[G - g]_{ij} = \frac{i(g - j)}{g} \text{ if } i \leq j.$$

![Diagram of a cycle graph](image)

$$B[G - 5] = \begin{bmatrix}
4 & 3 & 2 & 1 \\
3 & 5 & 5 & 5 \\
2 & 4 & 6 & 3 \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5}
\end{bmatrix}$$
Known results involving Perron branch

- Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. Then the Laplacian matrix of a graph

\[
B[G - g]_{ij} = \frac{i(g - j)}{g} \text{ if } i \leq j.
\]

\[
B[G - 5] = \begin{bmatrix}
4 & 3 & 2 & 1 \\
3 & 6 & 4 & 5 \\
2 & 4 & 6 & 3 \\
1 & 2 & 3 & 4 \\
\end{bmatrix}
\]
• Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. Then the

$$B[G - g]_{ij} = \frac{i(g - j)}{g} \text{ if } i \leq j.$$
Known results involving Perron branch

- Fallat, Kirkland & Pati, 02. Let $G$ be the cycle $[1, 2, \ldots, g, 1]$. Then the Laplacian matrix of a graph $G$ is

$$B[G - g]_{ij} = \frac{i(g - j)}{g}$$

if $i \leq j$.

\[ B[G - 5] = \begin{bmatrix}
4 & 3 & 2 & 1 \\
\frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\
\frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\
\frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5}
\end{bmatrix} \]

\[
\begin{array}{c}
\frac{2}{5} \\
\frac{3}{5} \\
\frac{4}{5}
\end{array}
\]

\[ \text{girth} \]

$G$
Known results involving Perron branch
 Lemma[*FK98]. $G, H$ be connected on vertices $\{1, \ldots, n\}$ and $\{n, n + 1, \ldots, k\}$, respectively. Put $\delta = B[G - 1]_{nn}$. Then

$$B[G \cup H - 1] = \begin{bmatrix}
B[G - 1] & B[G - 1](; n) \mathbb{1}^T \\
\mathbb{1}B[G - 1](n, :) & \delta J + B[H - n]
\end{bmatrix}.$$
Known results involving Perron branch

- **Lemma[*FK98]**. Let $G, H$ be connected on vertices $\{1, \ldots, n\}$ and $\{n, n+1, \ldots, k\}$, respectively. Put $\delta = B[G - 1]_{nn}$. Then

$$B[G \cup H - 1] = \begin{bmatrix} B[G - 1] & B[G - 1](:, n)1^T \\ 1B[G - 1](n,:) & \delta J + B[H - n] \end{bmatrix}.$$
Lemma[*FK98]. Let $G$, $H$ be connected on vertices $\{1, \ldots, n\}$ and $\{n, n+1, \ldots, k\}$, respectively. Put $\delta = B[G - 1]_{nn}$. Then

\[
B[G \cup H - 1] = \begin{bmatrix}
B[G - 1] & B[G - 1](:, n) \mathbf{1}^T \\
\mathbf{1} B[G - 1](n, :) & \delta J + B[H - n]
\end{bmatrix}.
\]

\[
B[F - 0] = \begin{bmatrix}
\begin{array}{cccc|cccc}
4 & 3 & 2 & 1 & 3 & 3 & 3 & 3 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
3 & 6 & 4 & 2 & 6 & 6 & 6 & 6 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
2 & 4 & 6 & 3 & 4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
1 & 2 & 3 & 4 & 2 & 2 & 2 & 2 \\
5 & 5 & 5 & 5 & 5 & 5 & 5 & 5 \\
\end{array}
\end{bmatrix}.
\]

Laplacian matrix of a graph – p.22/28.
A crucial result
A crucial result

- $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.
• $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

• Lemma [shifting]. $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n + 1, \ldots, k\}$, $\{k, k + 1, \ldots, r\}$, and $\{n, r + 1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then

$$M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].$$
• \( A \ll B \) means: \( B \) dominates \( A \) entrywise; strict at some entry.

• Lemma [shifting]. \( G_0, G_1, G_2, F \) be connected, on vertices \( \{1, \ldots, n\}, \{n, n+1, \ldots, k\}, \{k, k+1, \ldots, r\}, \) and \( \{n, r+1, \ldots, s\} \), respectively. Let \( F^* \) be obtained from \( F \) by renaming \( n \) to \( k \). Then

\[
M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].
\]
A crucial result

- \( A \ll B \) means: \( B \) dominates \( A \) entrywise; strict at some entry.

**Lemma [shifting].** \( G_0, G_1, G_2, F \) be connected, on vertices \( \{1, \ldots, n\} \), \( \{n, n+1, \ldots, k\} \), \( \{k, k+1, \ldots, r\} \), and \( \{n, r+1, \ldots, s\} \), respectively. Let \( F^* \) be obtained from \( F \) by renaming \( n \) to \( k \). Then

\[
M = B[ G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[ G_0 \cup G_1 \cup G_2 \cup F^* - 1].
\]

- Put \( G = G_0 \cup G_1 \cup G_2 \). Take \( w \in F - n \). Then
• $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

• Lemma [shifting]. Let $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n+1, \ldots, k\}$, $\{k, k+1, \ldots, r\}$, and $\{n, r+1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then

$$M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].$$

• Put $G = G_0 \cup G_1 \cup G_2$. Take $w \in F - n$. Then

$$m_{ww} =$$

Laplacian matrix of a graph – p.23/28
A crucial result

- \( A \ll B \) means: \( B \) dominates \( A \) entrywise; strict at some entry.

- **Lemma [shifting].** \( G_0, G_1, G_2, F \) be connected, on vertices \( \{1, \ldots, n\}, \{n, n+1, \ldots, k\}, \{k, k+1, \ldots, r\}, \) and \( \{n, r+1, \ldots, s\} \), respectively. Let \( F^* \) be obtained from \( F \) by renaming \( n \) to \( k \). Then

\[
M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].
\]

\[ \begin{array}{c}
1 & \quad G_0 & \quad \quad n & \quad G_1 & \quad k & \quad G_2 \\
& F & \quad \quad w & \quad \quad u \\
\end{array} \]

- Put \( G = G_0 \cup G_1 \cup G_2 \). Take \( w \in F - n \). Then

\[
m_{ww} = B[G - 1]_{nn} + B[F - n]_{ww}
\]
A crucial result

- $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

- **Lemma [shifting].** $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n+1, \ldots, k\}$, $\{k, k+1, \ldots, r\}$, and $\{n, r+1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then

$$M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].$$

- Put $G = G_0 \cup G_1 \cup G_2$. Take $w \in F - n$. Then

$$m_{uw} = B[G - 1]_{nn} + B[F - n]_{uw} \leq B[G - 1]_{kk} + B[F^* - k]_{uw} = n_{uw}.$$
• $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

• Lemma [shifting]. Let $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n + 1, \ldots, k\}$, $\{k, k + 1, \ldots, r\}$, and $\{n, r + 1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then

$$M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].$$

• For $w \in G_0 - 1$, we have
• $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

• Lemma [shifting]. $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n+1, \ldots, k\}$, $\{k, k+1, \ldots, r\}$, and $\{n, r+1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then $M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1]$.

• For $w \in G_0 - 1$, we have $m_{wu} = m_{wn}$
- \( A \ll B \) means: \( B \) dominates \( A \) entrywise; strict at some entry.

- **Lemma [shifting].** \( G_0, G_1, G_2, F \) be connected, on vertices \( \{1, \ldots, n\}, \{n, n+1, \ldots, k\}, \{k, k+1, \ldots, r\}, \) and \( \{n, r+1, \ldots, s\} \), respectively. Let \( F^* \) be obtained from \( F \) by renaming \( n \) to \( k \). Then

\[
M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].
\]

- For \( w \in G_0 - 1 \), we have 
  \[
m_{wu} = m_{wn} = n_{wk}
\]
A crucial result

• $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

• Lemma [shifting]. Let $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n + 1, \ldots, k\}$, $\{k, k + 1, \ldots, r\}$, and $\{n, r + 1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then

$$M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].$$

For $w \in G_0 - 1$, we have $m_{wu} = m_{wn} = n_{wk} = n_{wu}$. 
• $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

• **Lemma [shifting].** $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n + 1, \ldots, k\}$, $\{k, k + 1, \ldots, r\}$, and $\{n, r + 1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then $M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1]$.

• For $w \in G_1 \cup G_2$, we have $m_{wu}$
• \( A \ll B \) means: \( B \) dominates \( A \) entrywise; strict at some entry.

• **Lemma [shifting].** \( G_0, G_1, G_2, F \) be connected, on vertices \( \{1, \ldots, n\}, \{n, n+1, \ldots, k\}, \{k, k+1, \ldots, r\}, \) and \( \{n, r+1, \ldots, s\} \), respectively. Let \( F^* \) be obtained from \( F \) by renaming \( n \) to \( k \). Then

\[
M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].
\]

- For \( w \in G_1 \cup G_2 \), we have \( m_{wu} = m_{nn} \)
A crucial result

- \( A \ll B \) means: \( B \) dominates \( A \) entrywise; strict at some entry.

- **Lemma [shifting].** Let \( G_0, G_1, G_2, F \) be connected, on vertices \( \{1, \ldots, n\} \), \( \{n, n+1, \ldots, k\} \), \( \{k, k+1, \ldots, r\} \), and \( \{n, r+1, \ldots, s\} \), respectively. Let \( F^* \) be obtained from \( F \) by renaming \( n \) to \( k \). Then

\[
M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].
\]

- For \( w \in G_1 \cup G_2 \), we have \( m_{wu} = m_{nn} < m_{wk} \).
A crucial result

- \( A \ll B \) means: \( B \) dominates \( A \) entrywise; strict at some entry.

- **Lemma [shifting].** \( G_0, G_1, G_2, F \) be connected, on vertices \( \{1, \ldots, n\} \), \( \{n, n + 1, \ldots, k\} \), \( \{k, k + 1, \ldots, r\} \), and \( \{n, r + 1, \ldots, s\} \), respectively. Let \( F^* \) be obtained from \( F \) by renaming \( n \) to \( k \). Then

\[
M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].
\]

- For \( w \in G_1 \cup G_2 \), we have \( m_{wu} = m_{nn} < m_{wk} = n_{wk} = n_{wu} \).
A crucial result

- $A \ll B$ means: $B$ dominates $A$ entrywise; strict at some entry.

-Lemma [shifting]. $G_0, G_1, G_2, F$ be connected, on vertices $\{1, \ldots, n\}$, $\{n, n+1, \ldots, k\}, \{k, k+1, \ldots, r\}$, and $\{n, r+1, \ldots, s\}$, respectively. Let $F^*$ be obtained from $F$ by renaming $n$ to $k$. Then

$$M = B[G_0 \cup G_1 \cup G_2 \cup F - 1] \ll N = B[G_0 \cup G_1 \cup G_2 \cup F^* - 1].$$

- Above statement is also valid when we move the branch from 1 to $n$ or from $k$ to a point $r$ of $G_2$. 
Minimizing $\mu$ in graphs with restricted blocks
Theorem. Let $G$ be connected with blocks having at most two points of articulations.
Theorem. Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that
Theorem. Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$, 
Minimizing $\mu$ in graphs with restricted blocks

• **Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a path,
**Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a path,

(iii) and $\mu(H) \leq \mu(G)$. 

Minimizing $\mu$ in graphs with restricted blocks
Minimizing $\mu$ in graphs with restricted blocks

- **Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that
  
  (i) blocks of $H$ are precisely that of $G$,
  
  (ii) the block structure of $H$ is a path,
  
  (iii) and $\mu(H) \leq \mu(G)$.

- **Proof sketch:**

Laplacian matrix of a graph – p.24/28
Theorem. Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a path,

(iii) and $\mu(H) \leq \mu(G)$.

Proof sketch: Assume Case A.
Minimizing $\mu$ in graphs with restricted blocks

- **Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that
  
  (i) blocks of $H$ are precisely that of $G$,
  
  (ii) the block structure of $H$ is a path,
  
  (iii) and $\mu(H) \leq \mu(G)$.

- **Proof sketch:** Assume Case A. Consider the ch.block.
• **Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that
(i) blocks of $H$ are precisely that of $G$,
(ii) the block structure of $H$ is a path,
(iii) and $\mu(H) \leq \mu(G)$.

• **Proof sketch:** Assume Case A. Consider the ch.block. Take a point of articulation $u$. 

Laplacian matrix of a graph – p.24/28
Theorem. Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,
(ii) the block structure of $H$ is a path,
(iii) and $\mu(H) \leq \mu(G)$.

Proof sketch: Assume Case A. Consider the ch.block. Take a point of articulation $u$. Consider the (union of) non-Perron branches $C$ at $u$. 

Laplacian matrix of a graph – p.24/28
Theorem. Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a path,

(iii) and $\mu(H) \leq \mu(G)$.

Proof sketch: Assume Case A. Consider the ch.block. Take a point of articulation $u$. Consider the (union of) non-Perron branches $C$ at $u$. Use ‘shifting’ and replace $C$ by $\tilde{C}$ s.t. $B[C] \leq B[\tilde{C}]$. 

Minimizing $\mu$ in graphs with restricted blocks

- **Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that
  (i) blocks of $H$ are precisely that of $G$,
  (ii) the block structure of $H$ is a path,
  (iii) and $\mu(H) \leq \mu(G)$.

- **Proof sketch:** Assume Case A. Consider the ch.block. Take a point of articulation $u$. Consider the (union of) non-Perron branches $C$ at $u$. Use ‘shifting’ and replace $C$ by $\tilde{C}$ s.t. $B[C] \leq B[\tilde{C}]$, the block structure $\tilde{C}$ is a path and blocks of $\tilde{C}$ are blocks of $C$. 
Minimizing $\mu$ in graphs with restricted blocks

**Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,
(ii) the block structure of $H$ is a path,
(iii) and $\mu(H) \leq \mu(G)$.

**Proof sketch:** Assume Case A. Consider the ch.block. Take a point of articulation $u$. Consider the (union of) non-Perron branches $C$ at $u$. Use ‘shifting’ and replace $C$ by $\tilde{C}$ s.t. $B[C] \leq B[\tilde{C}]$, the block structure $\tilde{C}$ is a path and blocks of $\tilde{C}$ are blocks of $C$. By ‘replacement lemma’ $\mu$ cannot increase.
Minimizing $\mu$ in graphs with restricted blocks

- **Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that
  (i) blocks of $H$ are precisely that of $G$,
  (ii) the block structure of $H$ is a path,
  (iii) and $\mu(H) \leq \mu(G)$.

- **Proof sketch:** Assume Case A. Consider the ch.block. Take a point of articulation $u$. Consider the (union of) non-Perron branches $C'$ at $u$. Use 'shifting' and replace $C$ by $\tilde{C}$ s.t. $B[C] \leq B[\tilde{C}]$, the block structure $\tilde{C}$ is a path and blocks of $\tilde{C}$ are blocks of $C$. By 'replacement lemma' $\mu$ cannot increase.

- If ch.block had only one point of articulation we are done.
Theorem. Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that 

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a path,

(iii) and $\mu(H) \leq \mu(G)$.

Proof sketch:

- Otherwise, let $v$ be the other point of articulation in ch.block.
Theorem. Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a path,

(iii) and $\mu(H) \leq \mu(G)$.

Proof sketch:

Otherwise, let $v$ be the other point of articulation in ch.block.

Note that the branch $B$ that contains $C$ was a Perron branch at $v$ in $G$. 

Laplacian matrix of a graph – p.24/28
Minimizing $\mu$ in graphs with restricted blocks

• **Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that
  
  (i) blocks of $H$ are precisely that of $G$,
  (ii) the block structure of $H$ is a path,
  (iii) and $\mu(H) \leq \mu(G)$.

  ![Diagram](Laplacian_matrix_of_a_graph.png)

• **Proof sketch:**

  • Otherwise, let $v$ be the other point of articulation in ch.block.

  Note that the branch $B$ that contains $C$ was a Perron branch at $v$ in $G$. So $\tilde{B}$ is a Perron branch at $v$ in $\tilde{G}$.
Minimizing $\mu$ in graphs with restricted blocks

**Theorem.** Let $G$ be connected with blocks having at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a path,

(iii) and $\mu(H) \leq \mu(G)$.

**Proof sketch:**

- Otherwise, let $v$ be the other point of articulation in ch.block.
  
  Note that the branch $B$ that contains $C$ was a Perron branch at $v$ in $G$. So $\tilde{B}$ is a Perron branch at $v$ in $\tilde{G}$.

- Hence in $\tilde{G}$, we can continue with ‘shifting’ and ‘replacement’ at $v$ for the remaining branches.
Maximizing $\mu$ in graphs with restricted blocks
• **Theorem.** Let $G$ be connected whose blocks have at most two points of articulations.
Theorem. Let $G$ be connected whose blocks have at most two points of articulations. Then there is a graph $H$ such that
Theorem. Let $G$ be connected whose blocks have at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$, 

Maximizing $\mu$ in graphs with restricted blocks
Theorem. Let $G$ be connected whose blocks have at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a tree of diameter 3,
Theorem. Let $G$ be connected whose blocks have at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a tree of diameter 3,

(iii) and $\mu(H) \geq \mu(G)$.
Theorem. Let $G$ be connected whose blocks have at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a tree of diameter 3,

(iii) and $\mu(H) \geq \mu(G)$.

A block graph is a graph with each block complete.

Maximizing $\mu$ in graphs with restricted blocks
Theorem. Let $G$ be connected whose blocks have at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,

(ii) the block structure of $H$ is a tree of diameter 3,

(iii) and $\mu(H) \geq \mu(G)$.

A block graph is a graph with each block complete.

Let $G$ be the complete graph on vertices $1, \ldots, n$. Then

$B[G - n]_{ij} = 1/n$ if $i \neq j$ and $2/n$ if $i = j$. 
Theorem. Let $G$ be connected whose blocks have at most two points of articulations. Then there is a graph $H$ such that

(i) blocks of $H$ are precisely that of $G$,
(ii) the block structure of $H$ is a tree of diameter 3,
(iii) and $\mu(H) \geq \mu(G)$.

A block graph is a graph with each block complete.

Let $G$ be the complete graph on vertices $1, \ldots, n$. Then

$B[G - n]_{ij} = 1/n$ if $i \neq j$ and $2/n$ if $i = j$.
Extremizing $\mu$ in block graphs with restricted blocks
Proposition. Let \( s_1 \leq s_2 \). Then \( B[G_{s_1,s_2,H-1}] \geq B[G_{s_2,s_1,H-1}] \).
Extremizing $\mu$ in block graphs with restricted blocks

- Proposition. Let $s_1 \leq s_2$. Then $B[G_{s_1,s_2,H-1}] \geq B[G_{s_2,s_1,H-1}]$.

- Proposition. End blocks of $G_{s_1,...,s_k}$, $k > 3$ are not characteristic blocks.
Extremizing $\mu$ in block graphs with restricted blocks

- **Proposition.** Let $s_1 \leq s_2$. Then $B[G_{s_1,s_2,H - 1}] \geq B[G_{s_2,s_1,H - 1}]$.

- **Proposition.** End blocks of $G_{s_1,...,s_k}$, $k > 3$ are not characteristic blocks.

- **Theorem.** Consider all connected block graphs made of restricted blocks $K_{s_1}, \ldots, K_{s_k}$. 

Laplacian matrix of a graph – p.26/28
Proposition. Let \( s_1 \leq s_2 \). Then \( B[G_{s_1,s_2,H} - 1] \geq B[G_{s_2,s_1,H} - 1] \).

Proposition. End blocks of \( G_{s_1,\ldots,s_k}, k > 3 \) are not characteristic blocks.

Theorem. Consider all connected block graphs made of restricted blocks \( K_{s_1}, \ldots, K_{s_k} \).

Then among all such graphs the algebraic connectivity is minimized for a graph \( H \) whose block structure is a path.
Extremizing $\mu$ in block graphs with restricted blocks

• Proposition. Let $s_1 \leq s_2$. Then $B[G_{s_1, s_2}, H - 1] \geq B[G_{s_2, s_1}, H - 1]$.

• Proposition. End blocks of $G_{s_1, \ldots, s_k}$, $k > 3$ are not characteristic blocks.

• Theorem. Consider all connected block graphs made of restricted blocks $K_{s_1}, \ldots, K_{s_k}$.
  • Then among all such graphs the algebraic connectivity is minimized for a graph $H$ whose block structure is a path.
  • Furthermore, the sizes of the blocks in $H$ increase as we move away from the characteristic set.
Proposition. Let $s_1 \leq s_2$. Then $B[G_{s_1,s_2},H - 1] \geq B[G_{s_2,s_1},H - 1]$.

Proposition. End blocks of $G_{s_1,...,s_k}$, $k > 3$ are not characteristic blocks.

Theorem. Consider all connected block graphs made of restricted blocks $K_{s_1}, \ldots, K_{s_k}$.

- Then among all such graphs the algebraic connectivity is minimized for a graph $H$ whose block structure is a path.
- Furthermore, the sizes of the blocks in $H$ increase as we move away from the characteristic set.
- The maximum algebraic connectivity is 1 and it is attained by the graphs whose block structure is a star.
Graphs with path bundles as blocks
• We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$. 

\[ G \quad \quad H \]
Graphs with path bundles as blocks

- We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

- Problem: there is no direct domination in the matrices.
• We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

$$B[G - 1]$$

$\lambda_1(\hat{L}(G - 1))$

• The Perron value of $B[G - 1]$ is $1/\lambda_1(\hat{L}(G - 1))$. 

$B[H - 1]$
We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

\[ \lambda_1(\hat{L}(G - 1)) \text{ is } \lambda_1 \text{ of } L_G = \begin{bmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -3 & 6 & -3 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -3 & 3 \\
\end{bmatrix} \text{ (compressed)}. \]
• We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

• $\lambda_1(\hat{L}(G - 1))$ is $\lambda_1$ of $L_G = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -3 & 6 & -3 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix}$ (compressed).

• Let $x$ be the corresponding eigenvector.

Graphs with path bundles as blocks
• We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

\[ \begin{bmatrix}
2 & -1 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 \\
0 & -3 & 6 & -3 & 0 \\
0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & -3 & 3
\end{bmatrix} \] (compressed).

• Let $x$ be the corresponding eigenvector. Note: $x(c) > x(b)$. 

Laplacian matrix of a graph -- p.27/28
We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$. 

\[ L_G = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -3 & 6 & -3 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix} \]

\[ L_H = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -3 & 6 & -3 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -3 & 3 \end{bmatrix} \]
We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

Their inverses:

$$
\begin{pmatrix}
1 & 1 & 1/3 & 1 & 1/3 \\
1 & 2 & 2/3 & 2 & 2/3 \\
1 & 2 & 1 & 3 & 1 \\
1 & 2 & 1 & 4 & 4/3 \\
1 & 2 & 1 & 4 & 5/3 \\
\end{pmatrix}
\quad \quad
\begin{pmatrix}
1 & 1/3 & 1 & 1 & 1/3 \\
1 & 2/3 & 2 & 2 & 2/3 \\
1 & 2/3 & 3 & 3 & 1 \\
1 & 2/3 & 3 & 4 & 4/3 \\
1 & 2/3 & 3 & 4 & 5/3 \\
\end{pmatrix}
$$
We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

Their inverses:

For $G$:

$$
\begin{bmatrix}
1 & 1 & 1/3 & 1 & 1/3 \\
1 & 2 & 2/3 & 2 & 2/3 \\
1 & 2 & 1 & 3 & 1 \\
1 & 2 & 1 & 4 & 4/3 \\
1 & 2 & 1 & 4 & 5/3
\end{bmatrix}
$$

For $H$:

$$
\begin{bmatrix}
1 & 1/3 & 1 & 1 & 1/3 \\
1 & 2/3 & 2 & 2 & 2/3 \\
1 & 2/3 & 3 & 3 & 1 \\
1 & 2/3 & 3 & 4 & 4/3 \\
1 & 2/3 & 3 & 4 & 5/3
\end{bmatrix}
$$

Right − Left:

$$
\begin{bmatrix}
0 & -2/3 & 2/3 & 0 & 0 \\
0 & -4/3 & 4/3 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0
\end{bmatrix}
$$

Laplacian matrix of a graph – p.27/28
• We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

Their inverses:

$$
\begin{pmatrix}
1 & 1 & 1/3 & 1 & 1/3 \\
1 & 2 & 2/3 & 2 & 2/3 \\
1 & 2 & 1 & 3 & 1 \\
1 & 2 & 1 & 4 & 4/3 \\
1 & 2 & 1 & 4 & 5/3
\end{pmatrix}
\quad
\begin{pmatrix}
1 & 1/3 & 1 & 1 & 1/3 \\
1 & 2/3 & 2 & 2 & 2/3 \\
1 & 2/3 & 3 & 3 & 1 \\
1 & 2/3 & 3 & 4 & 4/3 \\
1 & 2/3 & 3 & 4 & 5/3
\end{pmatrix}
\quad
\begin{pmatrix}
0 & -2/3 & 2/3 & 0 & 0 \\
0 & -4/3 & 4/3 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0
\end{pmatrix}
$$

• Now $x^T [L_H^{-1} - L_G^{-1}]x > 0$. 
We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

Their inverses:

$$
\begin{align*}
\begin{bmatrix}
1 & 1 & 1/3 & 1 & 1/3 \\
1 & 2 & 2/3 & 2 & 2/3 \\
1 & 2 & 1 & 3 & 1 \\
1 & 2 & 1 & 4 & 4/3 \\
1 & 2 & 1 & 4 & 5/3 \\
\end{bmatrix} & \quad \begin{bmatrix}
1 & 1/3 & 1 & 1 & 1/3 \\
1 & 2/3 & 2 & 2 & 2/3 \\
1 & 2/3 & 3 & 3 & 1 \\
1 & 2/3 & 3 & 4 & 4/3 \\
1 & 2/3 & 3 & 4 & 5/3 \\
\end{bmatrix} & \quad \begin{bmatrix}
0 & -2/3 & 2/3 & 0 & 0 \\
0 & -4/3 & 4/3 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
\end{bmatrix}
\end{align*}
$$

Now $x^T[L_H^{-1} - L_G^{-1}]x > 0$. So $\rho(L_H^{-1}) > \rho(L_G^{-1})$. 

Graphs with path bundles as blocks

Laplacian matrix of a graph – p.27/28
We want to compare the Perron values of $B[G - 1]$ and $B[H - 1]$.

Their inverses:

\[
\begin{bmatrix}
1 & 1 & 1/3 & 1 & 1/3 \\
1 & 2 & 2/3 & 2 & 2/3 \\
1 & 2 & 1 & 3 & 1 \\
1 & 2 & 1 & 4 & 4/3 \\
1 & 2 & 1 & 4 & 5/3
\end{bmatrix}
\begin{bmatrix}
1 & 1/3 & 1 & 1 & 1/3 \\
1 & 2/3 & 2 & 2 & 2/3 \\
1 & 2/3 & 3 & 3 & 1 \\
1 & 2/3 & 3 & 4 & 4/3 \\
1 & 2/3 & 3 & 4 & 5/3
\end{bmatrix}
\begin{bmatrix}
0 & -2/3 & 2/3 & 0 & 0 \\
0 & -4/3 & 4/3 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0 \\
0 & -4/3 & 2 & 0 & 0
\end{bmatrix}
\]

Now $x^T[L_H^{-1} - L_G^{-1}]x > 0$. So $\rho(L_H^{-1}) > \rho(L_G^{-1})$.

So $\rho(B[H - 1]) > \rho(B[G - 1])$. 
Which one has better connectivity?
Connectivity of the sunflowers

- Which one has better connectivity?

- Picture on the left has smaller algebraic connectivity!!