Combinatorics lecture notes

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Preliminary remarks

These notes roughly summarize the material of my 4 lectures. This is a draft version that probably contains many errors (in particular in Section 4): do not hesitate to write me if you have doubts or questions: `gilles.schaeffer@lix.polytechnique.fr`.

1 First lecture. Topological and combinatorial maps

1.1 Summary of the lecture

The aim of the lecture was to present topological maps (also known as cellular decompositions of compact surfaces, or proper embeddings of graphs on surfaces) and their combinatorial representations using permutations.

In the first part of the lecture I described *maps* as surfaces constructed by identifying 2-by-2 the sides of a finite set of polygons. A good introduction to these topological maps can be found in the lecture notes of Eric Colin de Verdière for MPRI course 2-14-2:
`http://www.di.ens.fr/~colin/index.html.en#teaching`
In particular read Sections 1.1, 1.2, 2.1, 2.2 and 2.3.

In the second part of the lecture I discussed the combinatorial encodings of these maps by permutations, which led me to the definition of combinatorial maps as pairs of permutations (α, σ) such that

- σ is a permutation on $\{1, \ldots, 2n\}$
- α is a fix point free involution on $\{1, \ldots, 2n\}$ (all cycles of α have length 2)
- the subgroup $\langle \alpha, \sigma \rangle$ generated by α and σ acts transitively on $\{1, \ldots, 2n\}$.

The correspondance between topological and combinatorial maps can be described in two ways:

- With labels on the sides of the polygons: The cycles of σ describe labellings of the sides of the polygons, α indicates the pairs of sides that should be glued together.
- With labels on half-edges: The permutation σ describes the cyclic sequence of half-edges around each vertices, and α indicates the pairs of half-edges that are matched to form edges.

I presented the *dual* of a map M as the map M^* obtained by putting a new vertex of M^* in each face of M, and a new edge e^* dual to each edge e of M such that e^* joins the vertices in the two faces on each side of e. Given a combinatorial map (α, σ), we then observed that the two correspondences above respectively produce a topological map and its dual.

The Euler characteristic formula $\#\{vertices\} + \#\{faces\} = \#\{edges\} + 2 - 2g$, valid for any topological map, led me to the definition of the *genus* of a combinatorial map (α, σ) as the integer g such that $c(\sigma) + c(\alpha\sigma) = c(\alpha) + 2 - 2g$, where $c(\pi)$ is the number of cycles of a permutation π.

Finally I made some observations on automorphisms of topological and combinatorial maps. The conclusion of this lecture was that for the purpose of enumeration one can concentrate on rooted maps.

For the purpose of the course, we restrict our attention to *rooted planar maps*, for which the following equivalent definitions are sufficient:

- A *planar map* is an embedding of a connected graph G in the sphere. Two maps are considered equivalent if there is an orientation preserving homeomorphism[1] of the sphere that sends one onto the other.
- A planar map is *rooted* if one edge is distinguished and oriented. For two rooted maps to be equivalent, the homeomorphism must send the root edge of the first map onto the root edge of the second map.

A planar map can be faithfully represented in the plane by stereographic projection. Observe that a stereographic projection maps one point of the sphere to infinity: this point should be taken in a face. Two projections that use points that are not in the same face lead to two representations of the same map which are not equivalent up to homeomorphism *of the plane.* By convention, to represent a rooted planar map in the plane, we always choose the unbounded face to be the face on the right hand side of the root edge, which we call the *root face.* With this convention, two rooted planar maps are equivalent if and only if there exists an orientation preserving homeomorphism of the plane mapping one onto the other and preserving the roots.

2 Second lecture. Catalan structures

The second lecture was dedicated to a bestiary of some classical Catalan structures.

- Dyck path; decomposition at first return to the axis and bijection $\mathcal{D} \equiv x\times(1+D)\times\bar{x}\times(1+D)$; translation into a functional equation for generating series:

  $$D(z) = \sum_{d\in D} z^{|d|} = \sum_{(d_1,d_2)} z^{1+|d_1|+|d_2|} = z\Big(\sum_{d_1\in\{\varepsilon\}\cup D} z^{|d_1|}\Big)\Big(\sum_{d_2\in\{\varepsilon\}\cup D} z^{|d_2|}\Big) = z(1+D(z))^2$$

  Interpretation of the up and down steps as opening and closing parentheses of a balanced parenthesis word.

- Chord diagrams: definition as non intersecting chords of a $2n$-gon; decomposition and functional equation; bijection with Dyck words, open the polygon and straighten its border, chords become arches joining points on a line, which are immediately interpreted as matched opening and closing parentheses.
- Plane trees, bijection with Dyck words via contour traversal; bijection with chord diagrams via duality; decomposition by root edge deletion. Construction of plane trees from chord diagrams viewed as descriptions of the matchings of the sides of a $2n$-gon.
- The degree encoding of plane trees: write arity of vertices during a prefix left-to-right traversal of the tree. Words that are codes of trees are *Lukasiewicz words*, graphically represented by Łukasiewicz paths: paths from $(0,0)$ to $(n,-1)$ with steps $(1,i)$ with $i\geq -1$, that have the Łukasiewicz property, that is they never visit the lower half plane ($y<0$) before the endpoint.

  The cycle lemma saying that a fraction $1/n$ of the paths from $(0,0)$ to $(n,0)$ with arbitrary steps $(1,i)$, $i\in\mathbb{Z}$ have the Łukasiewicz property. Extension to count the concatenations of k codes of plane trees, as k/n among paths ending at level $-k$ with steps $(1,i)$, $i\geq -1$.

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[1] an homeomorphism is a continuous one-to-one mapping whose inverse is also continuous: in the case of the sphere, equivalence up to homeomorphisms can be viewed as equivalence up to continuous deformations

- Specialization of the degree encoding to binary trees: bijection with Dyck paths again (be careful with the definition of the size: length n for Łukasiewicz words, half-length n for Dyck words, number n of edges for plane trees, number n of inner vertices for binary trees, number n of chords of a chord diagram...); specialization to other degree distributions (ternary trees);
- Triangulations of a polygon and binary trees (by duality).

At some point I also discussed the enumeration of chord diagrams by insersion of the last arc, leading to a linear functional equation with one catalytic variable for $C(z,u)$, the bivariate generating function of Chord diagrams according to the size and the number of blocks:

- Represent the chord diagram linearly by n arches matching $2n$ points of a line. The arches are naturally ordered by inclusion: an arch (i,j) is maximal if there is no arch (k,ℓ) with $k<i<j<\ell$. Let $|c|$ and $m(c)$ denote respectively the number of arches and number of maximal arches of a chord diagram.
- Let ϕ be the application that takes a chord diagram of size $n\geq 1$ and remove the first arch to produce a diagram of size $n-1$; a diagram of size $n-1$ with m maximal arches has $m+1$ preimages by ϕ; more precisely these preimages have respectively 1, 2,... m, or $m+1$ arches.
- The previous remark translate into a functionnal equation: (C is the set of non empty chord diagrams and ε denotes the chord diagram with 0 chords).

$$\begin{aligned}
C(z,u) &= \sum_{c\in C} z^{|c|}u^{m(c)}\\
&= \sum_{c'\in\varepsilon\cup C}\ \sum_{c\in\phi^{-1}(c)} z^{|c|}u^{m(c)}\\
&= \sum_{c'\in\varepsilon\cup C} z^{|c'|+1}(u+u^2+\ldots+u^{m(c)+1})\\
&= zu + zu\sum_{c'\in C} z^{|c'|}\frac{1-u^{m(c)+1}}{1-u}\\
&= zu+\frac{zu}{1-u}\left(\sum_{c'\in C} z^{|c'|} - u\sum_{c'\in C} z^{|c'|}u^{m(c')}\right)\\
C(z,u) &= zu+\frac{zu}{1-u}\left(C(z,1)-uC(z,u)\right)
\end{aligned}$$

- The net result is a functional equation for the series $C(z,u)$ and $C(z,1)$. This functional equation defines a unique power series $C(z,u)$, as can be checked by computing the coefficient of z^n in both sides of the equation:

  $$[z^n]C(z,u) = \delta_{n=1}u + [z^{n-1}]\frac{u}{1-u}(C(z,1)-C(z,u)).$$

  Indeed this last equation gives a well funded recurrence for the polynomials $c_n(u)=[z^n]C(z,u)$. The equation for $C(z,u)$ is linear in the main unknown function $C(z,u)$ but involves evaluation of this function at specific value of the auxiliary variable u (here at $u=1$). Because the variable u was introduced to help writing the equation, this is called a *linear equation with one catalytic variable.*
- Linear equation with one catalytic variable can be solved by the kernel method: Consider an equation of the form

  $$K(z,u)F(z,u) = P(f(z),z,u)$$

  where $F(z,u)$ and $f(z)$ are the unknown functions and $K(z,u)$ and $P(f,z,u)$ are polynomials, and assume this equation defines a unique solution $F(z,u)$ which is a power series in z with polynomial coefficient in u.

Then if there exists a power series $U(z)$ such that $K(z, U(z)) = 0$, then the series $f(z)$ is solution of the equation $P(f(z), z, U(z)) = 0$. In particular algebraic elimination between the two polynomials allows to eliminate $U(z)$ to obtain a polynomial $Q(f, z)$ such that $Q(f(z), z) = 0$: $f(z)$ is an algebraic series.

- Applying this to the equation for $C(z, u)$ allows to recover the Catalan series: the equation rewrites
$$(1 - u + zu^2)C(z, u) = zu(1 - u) - zuC(z, 1)$$
so that $U(z)$ is identified as the Catalan series $1 + D(z)$, and $C(z, 1) = U(z) - 1 = D(z)$.

This method was also applied to Dyck paths: the natural linear construction of Dyck paths consists in adding one step after the other.

- The catalytic parameter which we need to describe this construction is the height of the intermediary paths: we consider therefore the set D^+ of left factors of Dyck paths, with lenght $\ell(d)$ and final height $h(d)$, and introduce the generating series $D^+(t, u) = \sum_{d\in D^+} t^{\ell(d)}u^{h(d)}$. The deletion of the last step immediately leads to the equation:
$$D^+(t, u) = 1 + tuD(t, u) + \frac{t}{u}(D(t, u) - D(t, 0))$$
since the only difficult point is that a down step should not be added when the path already ends at level 0: the series of these paths is $D^+(t, 0)$ (and this is exactly $D(t^2)$, since these are Dyck paths counted by length instead of half-length).
- Solving the equation with the kernel method leads again to Catalan series.

3 Third lecture. Rational and algebraic series

3.1 Rational series

In the first part of this lecture I discussed methods that lead to rational generating series. Recall that a formal power series $f(z)$ is said to be rational if there exists two polynomials $N(z)$ and $D(z)$ such that $f(z) = \frac{N(z)}{D(z)}$.

The methods I discussed are:

- the enumeration of words of rational languages with respect to the length:
  - A rational language can be always be presented by a non ambiguous regular expression; a non ambiguous regular expression immediately translates into a combinatorial decomposition of the type introduced by Fusy to discuss Boltzmann sampling: disjoint union is translated by sum, concatenation by cartesian product, and the kleene operator $*$ by the Seq construction.

    As a consequence, from a non ambigous regular expression, one immediately deduce an explicit rational generating series for the language.
  - A rational language can be described as the language recognized by a deterministic finite automaton. Given a deterministic finite automaton (S, A, ϕ, I, F) (S the set of states, A the alphabet, ϕ the transitions, I the initial states and F the final states), define for each state s the language L_s of words that are recognized by the automaton $(S, A, \phi, I, \{s\})$. Then the generating series $L_s(z)$, $s \in S$ satisfies a system of linear equations
  $$L_s(z) = \delta_{s\in I} + z \sum_{(t,a)|\phi(t,a)=s} L_t(z)$$
  which determines all the series $L_s(z)$. In particular these series are rational and so is $\sum_{s\in F} L_s(z)$ the series of the initial language.

In particular the last equation can be refined to write multivariate generating series that count words of rational languages according to the number of occurrences of each letter of the alphabet: for an alphabet $\{1, \dots, n\}$,

$$L_s(z, x_1, \dots, x_n) = \delta_{s \in I} + z \sum_{(t,i) | \phi(t,i)=s} x_i L_t(z, x_1, \dots, x_n)$$

- Counting words recognized by a deterministic automaton is essentially equivalent to counting paths of length n in a fixed finite oriented graph. Let G be an oriented graph given by its adjacency matrix M, and I be a set of initial vertices given by its characteristic vector v (the ith entry is 1 if the ith vertex is in I, 0 otherwise). Then ith entry of the vector $v \cdot M^n$ is the number of paths in G of length n from a vertex of I to vertex i. The generating series of paths according to their length can be written

  $$v \cdot (\sum_{n \geq 0} (zM)^n) \cdot w = v \cdot \frac{1}{1 - zM} \cdot w$$

  where w is the characteristic vector of final vertices.

- For instance if M is the k by k matrix with $M_{i,i+1} = M_{i+1,i} = 1$, for all $i = 1, \dots, k-1$ and $M_{i,j} = 0$ otherwise, check that $(1, 0, \dots, 0)\frac{1}{1-zM}(1, 0, \dots, 0)^t$ is the generating function of Dyck paths of height at most k.

- There is no need that the matrix M be of finite dimension as long as the products are well defined: M can for instance be an infinite band matrix. Check for instance that Dyck paths are counted by the infinite version of the previous matrix.

- We applied this method first to domino tilings of a band: the domino tiling is grown column by column: the state of the "generation automata" are the configurations in the last column (with vertical dominos and horizontal left or right half-dominoes) and the transitions connect each column to all the column that can follow it.

- We applied then the method to polyominos in a band: a polyomino is a set of cells of the grid that are connected through edges. Again polyominoes are grown column by column and the transitions connect the current column to all the column that can appear right after it. However in order to deal with the connectivity condition, the state of the generation automata must be more complex than just the last column: we add the number of connected component and more precisely a description of the connected component to which all cells of the last column belong.

The conclusion of this discussion is that a class of combinatorial structures that are grown linearly using a finite set of states will have a rational generating function. Conversely when a class of combinatorial structures has a rational generating function, a natural question is to find such a linear growth mecanism. Depending on the combinatorial structures this problem can be more or less challenging: a feasible example is the class of vertically convex polyominoes (polyominoes whose intersection with any vertical line is connected).

3.2 Algebraic series

In the second part of this third lecture I started to discuss methods that lead to algebraic generating functions.

Recall that a formal power series $f(z)$ is said to be algebraic if there exists a bivariate polynomial $P(f, z)$ such that $P(f(z), z) = 0$. We say that $f(z)$ is algebraic of degree k if P has degree k in the variable f.

- The dictionnary between constructions for unlabelled structures and operations on ordinary generating functions immediately shows that classes of combinatorial structures that are recursively defined using only the operation +, × and SEQ have algebraic generating functions. The architypical example is of course again Catalan structures: the description $D \equiv z \times (1+D) \times (1+D)$ leads to the equation $D(z) = z(1+D)^2$.
- A natural way to get an algebraic generating function is thus in particular to enumerate words of an algebraic (aka context-free) language described by a non ambiguous algebraic grammar (remark that, as opposed to the case of rational languages which always admit non ambiguous descriptions, there are algebraic languages that are inherently ambiguous). Indeed the same translation rules as for rational langages (disjoint union by sum, concatenation by cartesian product) allows to translate an algebraic grammar into a standard recursive definition of a combinatorial class. This recursive definition in turn yields a system of algebraic equations defining the generating functions of the languages associated to each non terminal symbol of the grammar.

Why am I so excited about algebraic generating functions? Because once I have a polynomial equation for $f(z)$ I (almost) automatically get:

- A linear recurrence with polynomial coefficient for the coefficients f_n of the series: for exemple the Catalan number satisfy the linear recurrence $(n+1)c_n = 2(2n-1)c_{n-1}$. (The general case follows from the fact that algebraic series are "D-finite", *ie* they satisfy a linear differential equation; a proof of this result can be found in Flajolet-Sedgewick).
- The asymptotic behavior of the coefficient, via singularity analysis (cf Flajolet-Sedgewick again).
- Explicit formulas as summation of multinomial coefficients.

To illustrate the last point above I gave its simplest instance, Lagrange inversion formula:

- Let $\phi = \sum_n \phi_n x^n$ be a formal power series with $\phi(0) \neq 0$. Then the equation $f(z) = z\phi(f(z))$ has a unique solution that is a formal power series in z and for all $n \geq 1$,

$$f_n = [z^n]f(z) = \frac{1}{n}[x^{n-1}]\phi(x)^n$$

- more generally, $[z^n]f(z)^k = \frac{k}{n}[x^{n-k}]\phi(x)^n]$, and, upon summing,

$$[z^n]g(f(z)) = \frac{1}{n}[x^{n-1}]g'(x)\phi(x)^n \qquad \text{for any } g(x) = \sum_{k\geq 0} g_k x^k.$$

The proof of this result was already given in the second lecture: indeed if one view ϕ as a generating function for vertices with respect to the arity, then $f(z)$ is the generating function of rooted plane trees made with these vertices, while $\phi(x)^n$ is the generating functions for words of length n on the vertex alphabet and $[x^{n-1}]\phi(x)^n$ the number of such words whose corresponding Łukasiewicz path ends at level -1. Lagrange inversion formula is then just a mere rewriting of the cycle lemma.

Lagrange inversion formula thus naturally yields back the same formulas as the cycle lemma for k-ary trees. For trees with mixed degrees it leads to summations of binomial coefficients:

- consider 1-2-trees, that is rooted plane trees with nodes of arity 1 or 2 and leaves: their generating function satisfies

$$f(z) = z(1 + f(z) + f(z)^2)$$

so that the number of 1-2 trees with n vertices is

$$f_n = \frac{1}{n}[x^{n-1}](1+x+x^2)^n = \frac{1}{n}[x^{n-1}] \sum_{i+j+k=n} \binom{n}{i,j,k} x^{j+2k} = \frac{1}{n}\sum_{k=0}^{n/2-1} \binom{n}{k+1, k, n-2k-1}.$$

- consider rooted planar trees such that each inner vertex is incident to exactly 2 leaves. The siblings of each inner vertex form three sequences of subtrees separated by the two leaves, so that we have the decomposition:

  $$L \equiv \text{root} \times \textsc{Seq}(L) \times \text{leaf} \times \textsc{Seq}(L) \times \text{leaf} \times \textsc{Seq}(L)$$

  which yields the equation $L(z) = z/(1-L(z))^3$. Therefore the number of rooted plane trees with n inner nodes each incident to two leaves is

  $$[z^n]L(z) = \frac{1}{n}[x^{n-1}](1-x)^{-3n} = \frac{1}{n}\binom{4n-1}{n-1}.$$

  It will turn out to be useful to count planted plane trees with n inner nodes each incident to two leaves, where a tree is planted if it has a distinguished leaf instead of a root. The vertex incident to the marked leaf is incident to only one other leaf so that its other neighbors form two sequences of rooted plane trees (instead of 3): $L^\bullet(z) = z/(1-L(z))^2$ and the number of these planted plane trees with n inner nodes is

  $$[z^n]\frac{z}{(1-L(z))^2} = [x^{n-2}]2(1-x)^{-3} \cdot (1-x)^{-3n} = 2[x^{n-2}](1-x)^{-3(n+1)} = 2\binom{4n}{n-2}.$$

As for the rational case, when a class of combinatorial structures has an algebraic generating series, a natural question is to find an encoding by an algebraic grammar, or more directly an algebraic decomposition (Schützenberger methodology).

A source of algebraic generating functions that do not clearly come from algebraic decompositions is the resolution of linear (or polynomial) equations with one catalytic variable. Let us know illustrate this on the example of planar maps.

4 Fourth lecture. The enumeration of rooted planar maps

4.1 Triangulations of a polygon with interior points

A triangulation of a polygon with interior vertices is a rooted planar map such that: the root face has degree $k \geq 3$ and all other faces have degree 3; there are no multiple edges or loops; the border of the exterior face is a simple polygon (with k vertices).

Assume the triangulation has n interior vertices and m internal triangles. Then the total number of vertices is $n+k$, and in view of the degrees of faces, the number of edges is $\frac{1}{2}(3m+k)$, and Euler's formula reads: $(n+k)+(m+1) = (3m+k)/2+2$ or $2n+2k+2m+2 = 3m+k+4$, or $m = 2n+k-2$.

We already dealed with the case of triangulations without interior points: upon deleting the root edge the triangulation separates into two smaller triangulations, and we obtained a Catalan equation: $T_0 \equiv \text{root} \times (1+T_0)^2$. We now want to compute the generating function of triangulations of a k-gon with n internal vertices: it will prove convenient to use the generating function $T(z,u)\sum_t x^{m(t)}u^{k(t)-3}$ with respect to the number of internal triangles and to the degree of the polygon minus 3.

In order to generalize the previous decomposition, let v denote the third vertex of the interior triangle adjacent to the root edge. There are two cases depending on the position of v:

- Triangulations of type A: v is a vertex of the polygon. Then the deletion of the root edge breaks the triangulation into two (possibly empty) triangulations of smaller polygons; the number of inner triangles and degree of the polygon are additive parameters of this decomposition. Conversely any two triangulations can be attached together by a triangle to form a bigger triangulation of type A. We thus get as contribution for this case:

  $$zu^2(1+T(z,u))^2$$

- Triangulations of type B: v is an internal vertex of the triangulation. Then the deletion of the root edge produces a proper triangulation of a polygon of degree incremented by one (v is no more internal). Conversely given any triangulation of a polygon with degree at least 4, adding an edge between the origin of the root the extremity of the next edge around the polygon creates a triangulation of type B. We thus get as contribution for this case

  $$\frac{z}{u}(T(z,u)-T(z,0)).$$

Summarizing, and noting that the initial case of the triangle is of type A, we get the equation

$$T(z,u) = zu^2(1+T(z,u))^2 + \frac{z}{u}(T(z,u)-T(z,0))$$

This is an equation in $T(z,u)$ and $T(z,0)$, which mixes a term akin to polynomial equations and a term involving a substitution $u=0$ of a catalytic variable. We call this type of equations *polynomial equations with one catalytic variable.*

A general theorem of Bousquet-Mélou and Jehanne shows that under quite general condition the polynomial equations with one catalytic variable that can be obtained from combinatorial decompositions have power series solutions that are algebraic series. Here is an idea of the proof:

- Assume that there exists a polynom $P(F,f,z,u)$ such that $F(z,u)$ and $f(z)$ are the unique formal power series such that $P(F(z,u),f(z),z,u)=0$. After derivation with respect to the variable u the equation reads

  $$P'_F(F(z,u),f(z),z,u)F'_u(z,u) + P'_u(F(z,u),f(z),z,u) = 0.$$

  If one can find a series $U(z)$ such that $P'_F(F(z,U(z)),f(z),z,U(z)) = 0$, then the series $G(z) = F(z,U(z))$, $f(z)$ and $U(z)$ will be solutions of the algebraic system:

  $$\begin{array}{rcl} P(G(z),f(z),z,U(z)) & = & 0 \\ P'_F(G(z),f(z),z,U(z)) & = & 0 \\ P'_u(G(z),f(z),z,U(z)) & = & 0 \end{array}$$

  Under some mild technical conditions, that are almost automatically satisfied for equations that arise from combinatorial decomposition, the series $U(z)$ exists and the system is non degenerate so that $f(z)$ (and $U(z)$, $G(z)$ and $F(z,u)$) is algebraic. In particular a single polynomial equation for $f(z)$ can be obtained from the system by elimination.

Applying this programm to our equation for triangulation leads [2] to the following result:

$$T(z,0) = U(z) - 2U(z)^2, \qquad \text{where } U(z) = \frac{z}{(1-U(z))^3}.$$

Using the Lagrange inversion formula the number of triangulations of a 3-gon with n internal triangles is found to be $\frac{2(4n+1)!}{(n+1)!(3n+2)!}$. This formula is due to Tutte in 1963.

4.2 Rooted planar maps

The root edge deletion seems to be a powerful method. Let us try to apply it to the general case of rooted planar maps. Let $M(z,u) = \sum_m z^{\#\{edges\}}u^{\#\{exteriordegree\}}$ where the sum is taken over all rooted planar maps with at least one edge.

There are two cases again:

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[2] in fact the equation we wrote is not correct because our decomposition may create double edges: one has to forbid this by removing a term $zT(z,0)T(z,u)$ in the right hand side of the equation

- Case A: the deletion of the root disconnects the map. These maps are uniquely constructed by attaching two rooted planar maps by a new root edge. The resulting contribution is

$$zu^2(1+M(z,u))^2$$

- Case B: the deletion of the root does not disconnect the map. These maps are uniquely reconstructed by inserting a root edge in the root face in all possible ways: the number of possible ways depends on the position of the chosen endpoints and each choice results in a different exterior degree. The contribution is therefore

$$\sum_{m\in\{\varepsilon\}\cup M} z^{\#\{edges\}+1}(u+u^2+\ldots+u^{\#\{exteriordegree\}+1}) = zu+z\frac{u}{1-u}(uM(z,u)-M(z,1)).$$

We thus obtain the equation

$$M(z,u) = zu + \frac{zu}{1-u}(uM(z,u)-M(z,1)) + zu^2(1+M(z,u))^2$$

which is amenable to the same method as before. The computation yields

$$M(z,1) = U(z)-U(z)^3, \qquad \text{where } U(z) = 3z(1+U(z))^2.$$

Lagrange inversion formula then gives the number of rooted planar maps with n edges as

$$\frac{2\cdot 3^n(2n)!}{(n+2)!n!}$$

This formula is again due to Tutte in 1964

A lot of similar results can be obtained for various families of rooted planar maps (2-connected, 3-connected, well colored maps, etc) and even for maps of genus g: in more complex cases there is no more simple explicit formula but in many cases there still is a simple algebraic expression for the generating series. A natural question is thus to find simple algebraic decompositions that explain these results.

4.3 Tree decompositions to explain algebraic generating series

The case of rooted planar maps: the series

$$U(z) = 3z(1+U(z))^2$$

is the series of planted[3] binary trees with n inner vertices with 3 types of inner vertices: in other terms, the number of these trees is

$$3^n\cdot\frac{1}{n+1}\binom{2n}{n}$$

The factor 3 for an inner vertex of a binary tree naturally correspond to the 3 possibles ways to attach a pending half edge to each inner vertex.

Observe then that the number of leaf (including the root leaf) is $n+2$, while the number of pending half edges is n. Leaves and pending half edge thus form around the tree a circular sequence which can be interpreted as made of n opening parentheses (the pending half edges) and $n+2$ closing parentheses (the leaves), or n up steps and $n+2$ down steps: the cycle lemma tells us that there are exactly two leaves such that starting to read the sequence of up and down steps in clockwise order will give a product of two Łukasiewicz words: let us call *free* leaves these two leaves, since each one of the other leaves is naturally matched to a pending half edge by the parenthesis interpretation of the two Dyck factors.

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[3] recall that a planted plane tree is an unrooted plane tree with a distinguished leaf, which we call the root leaf of the tree

- A tree with pending half edges as above is said to be *balanced* if its root leaf is one of the two free edges.
- The number of balanced trees with n inner vertices is then
$$\frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$$
which is also the number of rooted planar maps.

As already mentionned the non free leaves of a balanced tree are naturally matched to pending edges: more interestingly each matching pair of a leave and a pending edge can be joined by an arch in a such way that the picture remains planar with the two free leaves still in the unbounded face (we are just drawing two arch systems associated to two Dyck words). Upon drawing a final root edge between the two free leaves one gets a rooted planar map, with n vertices of degree 4.

Theorem The previous construction is a one-to-one correspondance between balanced trees with n inner nodes and rooted 4-regular planar maps with n vertices.

Corollaries By duality this is also the number of rooted planar quadrangulations as stated at the end of the first lecture. By another simple bijection between 4-regular planar maps with n vertices and general planar maps with n edges, this is also the number of rooted planar maps with n edges.

4.4 Further results along these lines

Observe the similarity between the generating series of rooted plane trees with two leaves per vertex (as computed during lecture 2 and 3) and of rooted triangulations. A bijective proof of the Tutte's formula above for triangulations can be given using a similar kind of idea.

The quest of simple algebraic explanations for algebraic generating functions has been led to many beautiful and unexpected bijections by Dominique Poulalhon (Paris 7), Eric Fusy (CNRS), Olivier Bernardi (MIT), or more recently Marie Albenque (CNRS), Guillaume Chapuy (postdoc in Vancouver), Axel Bacher (PhD student in Bordeaux) and many others.

These constructions are praised not only because they are elegant but also because they usually allow to understand new properties of the structures: for instance the bijection for planar maps allows to prove that the diameter of a uniform random planar map with n edges is $\Theta(n^{1/4})$ a result that had been conjectured more than 15 years ago by physicists.

Several important open problems of this type are waiting for explanations, some regarding maps, other regarding polyominos, etc. Experience shows that these problems are often solved by PhD students, maybe because finding a new bijection requires to bring new, “different” ideas.

5 About the exam

For the part of the exam regarding my lectures you are expected to be able to:

- manipulate basic Catalan structures to create simple bijections
- prove rationality via encodings with regular langage, or construction of generation automata
- find simple linear or algebraic bijective decompositions, possibly using a catalytic parameter, and write their correct translation into functional equations; solve these equations if it does not require long computations
- perform simple computations using Lagrange inversion or the cycle lemma
- decompose simple families of rooted planar maps and write corresponding equations
- propose a family of trees whose generating function satisfies a given functional equation.

There might also be questions combining the material of my lectures with that of Eric Fusy's lectures on random generation.