Miklos Bonà editor

Handbook of enumerative combinatorics

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Gilles Schaeffer

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Part I

This is a Part

Chapter 1

Planar maps

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The enumerative theory of planar maps is born in the early sixties with the seminal work of William T. Tutte on the enumeration of planar triangulations. Over 50 years it has led several parallel lives in combinatorics, statistical physics, quantum gravity, enumerative topology and probability theory, that have started to interact intensely only in the last ten to fifteen years. Writing a fair survey of this whole story appears to be a great challenge. We concentrate instead in this text on the combinatorial point of view on map enumeration, and only review very tangentially the physics, topology and probability literature.

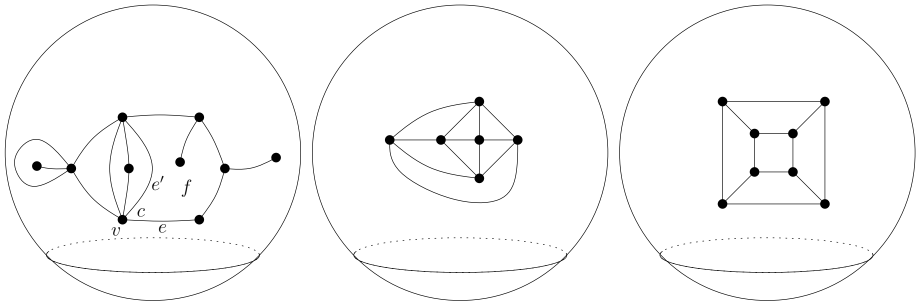

Figure 1.1
Three planar maps. The first has 10 vertices, 14 edges and 6 faces. The second is a tetravalent triangulation with 8 triangles. The third is a trivalent quadrangulation with 6 quadrangles, it is usually called the *cube*. On the first map the corner c is incident to the vertex $v = v(c)$, to the face $f = f(c)$, and the edges $e = cw(c)$ and $e' = ccw(c)$.

1.1 What is a map?

1.1.1 A few definitions

Combinatorial maps usually arise from one of two settings: either the study of some planar graph drawings, or the construction of surfaces via polygon gluings. Accordingly we give two definitions of maps and discuss how duality reconciles them.

The graph drawing point of view. An embedding (or drawing) of a graph $G = (V,E)$ on the oriented sphere $\mathbb{S}$, is *proper* if the vertices are represented by distinct points and the edges are represented by arcs that only intersect at their endpoints and in agreements with the incidence relation of G.

Definition 1.1 A *planar map M* is a proper embedding of a connected graph G in the sphere $\mathbb{S}$, considered up to orientation preserving homeomorphisms of $\mathbb{S}$.

Loops and multiple edges are allowed, and the map is instead said to be *simple* if it contains nor multiple edges or loops. A *face* is a connected component of $\mathbb{S} \setminus G$. A *corner* is the angular sector delimited by two consecutive edges around a vertex. Each corner c is incident to a vertex $v(c)$, to a face $f(c)$, and to two edges: in counterclockwise direction around $v(c)$, let $cw(c)$ denote the edge preceding c and $ccw(c)$ denote the edge following c. The *degree* of a vertex or face is the number of incident corners. A map is *Eulerian* if all its vertices have even degrees. It is *m-valent* if all its vertices have degree m, it is a *m-angulation* if all its faces have degree m. In the special cases $m = 3,4$ we use the standard terminology trivalent maps and tetravalent maps, triangulations and quadrangulations. Observe that with these definitions, triangulations and quadrangulations are allowed to have multiple edges or loops. Some

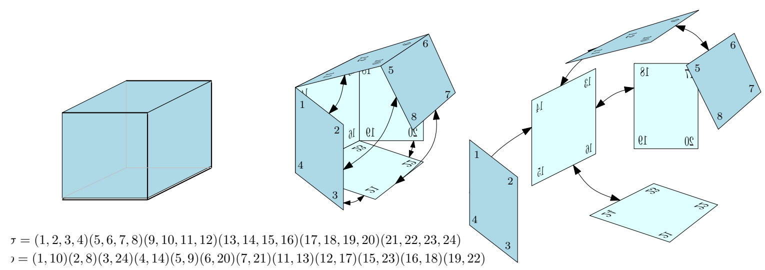

Figure 1.2
The cube as a cellular decomposition of the sphere. To make the picture more readable, the side identifications are performed in two stages.

examples are given in Figure 1.1. Observe that the leftmost map is not simple because the edge e' has a parallel edge.

The polygon gluing point of view. A *cellular decomposition* of the sphere is a collection of oriented polygons with labeled corners, and a complete set of coherent side identifications such that the resulting surface is the sphere. An example of cellular decomposition of the sphere is given by Figure 1.2. More precisely, a cellular decomposition can be given by the associated *rotation system* (σ, ρ), consisting of a permutation σ whose cycles describe the clockwise arrangement of corner labels around polygons and a matching (or fix point free involution) ρ describing side identifications: if $\rho(i) = j$ then the polygon side $(i, \sigma(i))$ is identified with the polygon side $(j, \sigma(j))$.

Definition 1.2 A *planar map* is a cellular decompositions of the sphere considered up to relabeling of the corners of the polygons.

Equivalence between Definitions 1.1 and 1.2 follows from standard results in surface topology: the faces of a proper embedding of a connected graph G in $\mathbb{S}$ are simply connected, so that the components of $\mathbb{S} \setminus G$ can be identified with polygons, and conversely, in a cellular decomposition the sides of the original polygons define a proper embedding of a connected graph on the resulting surface. In particular the numbers $v(M)$ of vertices, $f(M)$ of faces and $e(M)$ of edges of a planar map M satisfy Euler's formula:

$$\mathrm{v}(M) + \mathrm{f}(M) = \mathrm{e}(M) + 2. \tag{1.1}$$

The dual of a map. There is a natural symmetry between the role of vertices in Definition 1.1 and the role of polygons in Definition 1.2. This observation directly leads to the fundamental idea of *duality*, illustrated by Figure 1.3. The *dual* of a map M, denoted M^*, is the map obtained by drawing a vertex f^* of M^* in each face f of M and an edge e^* of M^* across each edge e of M (see Figure 1.3). By construction each face of M^* then contains exactly one vertex of M. The superimposition of a map

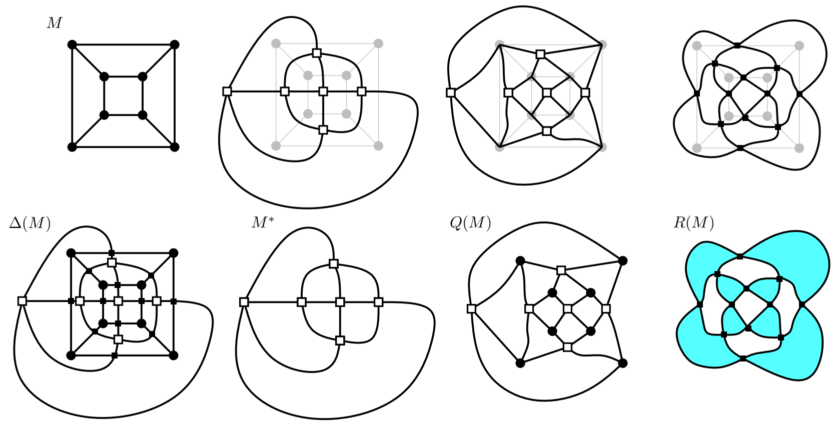

Figure 1.3
The cube map M, and in the first line, the construction of its dual, incidence map and edge map. The resulting maps $\Delta(M)$, M^*, $Q(M)$ and $R(M)$ appear on the second line. The underlying spheres are omitted.

M and its dual M^* (with tetravalent vertices created at the intersection of dual edges) is a quadrangulation $\Delta(M)$ which is called the *derived map* of M. Observe that faces of $\Delta(M)$ are in one-to-one correspondence with corners of M.

Theorem 1 *Duality is an involution on the set of planar maps. It preserves the number of edges, and exchanges the numbers of vertices and faces:* $M^{**} = M$, $\mathrm{e}(M^*) = \mathrm{e}(M)$*, and* $\mathrm{v}(M^*) = \mathrm{f}(M)$.

Let M be a map with vertex set V and edge set E, and let M^* be its dual with vertex set V^* and edge set E^*. The *incidence map* (or quadrangulation) $Q(M)$ of the map M is the map whose vertex set is $V \cup V^*$ and with one edge per corner c connecting $v(c)$ and $f(c)^*$. The *edge map* $R(M)$ of M is instead the map with vertex set E and with one edge per corner c connecting $cw(c)$ and $ccw(c)$. The mapping Q and R are bijections from maps with n edges respectively onto vertex-bicolored quadrangulations with n faces, and onto face-bicolored tetravalent maps with n vertices. Moreover $Q(M) = \overline{Q(M^*)}$, $R(M) = \overline{R(M^*)}$ and $Q(M) = R(M)^*$, where the bar denotes the exchange of colors.

The transformation between M, M^*, $\Delta(M)$, $Q(M)$ and $R(M)$ should be viewed as mere changes of representations for a same underlying object: In the language of computer science, one would say that they represent different data structures that can be used to encode a same cellular decomposition of the sphere. Alternatively this statement can be made precise using the language of topology, and in particular branched covers of the sphere (see [127, Chapter 1] for an exposition, which requires too many definitions to be reproduced here).

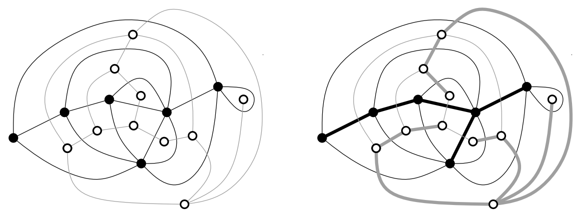

Figure 1.4
The superimposition of a map M and its dual M^*, and two dual spanning trees.

Duality, spanning trees and Euler's formula revisited. A *spanning tree* of a planar map M is a subset T of the set of edges of M that forms a tree and that is incident to every vertex of M. Figure 1.4.b illustrates the following fundamental property of duality and spanning trees for planar maps:

Theorem 2 *Let (T_1, T_2) be a partition of the edges of a planar map M. Then T_1 is a spanning tree of M if and only if T_2^* is a spanning tree of M^*.*

The trees T_1 and T_2^* are usually called *dual spanning trees*, although this terminology is somewhat improper since the edges of T_2^* are not the duals of the edges of T_1, but rather the duals of the edges *not* in T_1. The proposition can be viewed as a consequence of the characterization of the sphere by Jordan's lemma ($\mathbb{S} \setminus T_1$ is connected if and only if there is no simple cycle in T_1), together with the fact that $\mathbb{S} \setminus T_1$ is connected if and only if T_2^* is connected.

Observe that the above proposition, together with the facts that every map admits a spanning tree, and that any tree with v vertices has $v-1$ edges, gives an interpretation of Euler's formula (1.1): the $e(M)$ edges of a map are partitioned into the $v(M)-1$ edges of a spanning tree and the $f(M)-1$ edges of the dual spanning tree.

1.1.2 Plane maps, rooted maps and orientations

Rather than drawing maps on the sphere, we usually draw maps on the plane. This naturally leads to the notion of rooted maps, and to the discussion of orientations.

Plane maps, rooted planar maps. In order to represent a planar map M in the plane, we choose a point x_0 of $\mathbb{S}$ in a face of M and identify the punctured sphere $\mathbb{S}^2 \setminus \{x_0\}$ with the plane, sending x_0 at infinity. In such a representation, all faces are homeomorphic to discs, except for the face containing x_0, which is usually called the *exterior* or *outer* face. Depending on the choice of x_0 we *a priori* get different drawings, but up to homeomorphisms of the plane only the choice of the face in which x_0 is taken matters. Accordingly, let a *plane map* (M, f) be a planar map M with a marked face f, so that plane maps are in one-to-one correspondence with equivalence classes of proper embeddings of connected graphs in the plane up to homeomorphisms of the oriented plane.

Given a planar map M, the choice of a marked face does however not fix in

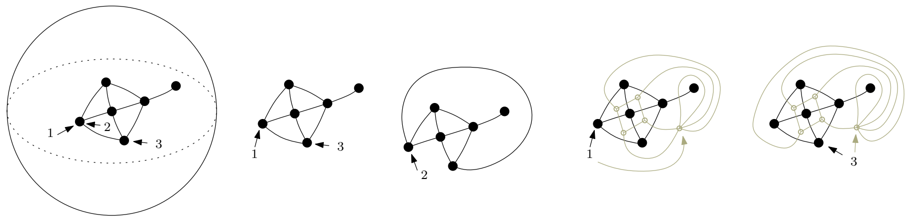

Figure 1.5
Three roots for a planar map: the corresponding plane maps, and the plane embedding of the derived map of the first and third one.

general the embedding of the dual map M^* and of the derived map ΔM up to homeomorphisms of the plane. As illustrated by Figure 1.5, what is needed for this is the choice of a face of ΔM, or equivalently of a corner of M. Accordingly, let a *rooted planar map* (M,c) be a planar map with a marked corner c. The *root face*, *root vertex* and *root edge* of (M,c) are then defined to be respectively $f(c)$ and $v(c)$ and $ccw(c)$. (In the literature, a rooted map is sometimes defined as a map with a marked and oriented edge, or with a marked half-edge. This definition is equivalent to ours: to each oriented edge $\vec{e}$ is associated the unique corner c such that $v(c)$ is the origin of $\vec{e}$ and $ccw(c) = e$.)

Upon setting $(M,c)^* = (M^*,c)$, duality extends into an involution on rooted planar maps. The derived map of a rooted map (M,c) is instead naturally endowed with a marked face (the face of $\Delta(M)$ that corresponds to c). The incidence map of a rooted map (M,c) is a bicolored quadrangulation with a marked edge (the edge e of $Q(M)$ that corresponds to c), or equivalently a rooted quadrangulation (taking as root the unique corner c' incident to the root vertex of M and such that $ccw(c') = e$). Similarly the edge map of a rooted map can be considered as a rooted tetravalent map.

Proposition 1 *The incidence map Q and edge map R are one-to-one correspondences between rooted planar maps with n edges and respectively rooted planar quadrangulations with n faces, and rooted tetravalent planar maps with n vertices.*

Orientations. Let (M,c) be a rooted map, and $\mathscr{O}$ an orientation of the edges of M. A *circuit* is a directed cycle of oriented edges, it is *simple* if it visits each vertex at most once. Any simple circuit C divides the sphere into a *left component*, which borders the left hand side of every edge of C, and a *right component*, which borders the right hand side of every edge of C.

Let us say that a simple circuit C in a rooted planar map is *clockwise* if the root corner lies in its left component, *counterclockwise* otherwise. With the convention that the outer face of a rooted plane map is its root face, these notions of clockwise and counterclockwise circuits coincide with the usual definitions: a simple circuit is clockwise if the unbounded component it defines is on its left hand side, counterclockwise otherwise.

A function $\alpha : V \to \mathbb{N}$ is *feasible on the planar map M* if there exists an orienta-

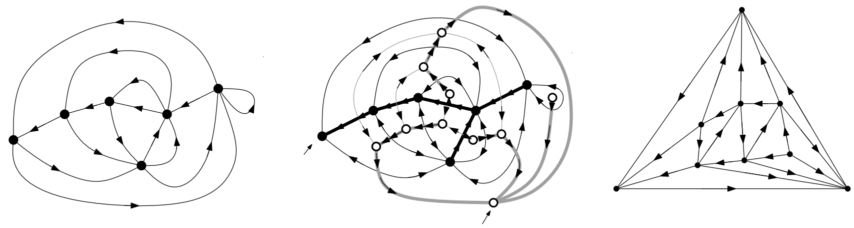

Figure 1.6
A Eulerian orientation, the t-orientation induced by the dual spanning trees of Figure 1.4, and a s-orientation of a triangulation.

tion $\mathscr{O}$ of M such that for every vertex v of M, $\alpha(v)$ is the out-degree of v. Such an orientation is then called an α-orientation.

A classical example of feasible function is the half-degree function $h(v) = \deg(v)/2$ on a Eulerian map (recall that a map is Eulerian if all its vertices have even degree). The h-orientations are precisely *Eulerian orientations*, that is, orientations such that the in- and out-degree are equal on each vertex. Another example is given by the orientation induced by spanning trees on derived maps: let (M,c) be a rooted planar map and for each vertex v of the derived map $\Delta(M)$ let $t(v) = 0$ if v is the root vertex of M or M^*, $t(v) = 1$ if v is a non-root vertex of M or M^*, and $t(v) = 3$ if v is a dual edge intersection vertex. Then t is feasible since each pair of dual spanning trees on M and M^* induces a t-orientation of $\Delta(M)$ by selecting as unique out-going edge on each non-root vertex of M or M' the edge going toward its father in the tree it belongs to. Finally a third example is Schnyder's orientations of triangulations [161]: Let T be a plane triangulation and $s(v) = 1$ if v is incident to the outer-face, $s(v) = 3$ otherwise. Then s is feasible and the s-orientations are called *3-orientations*.

Let α be a feasible function on a plane map (M, f), and let $\mathscr{O}$ be an α-orientation of M. Observe that simultaneously changing the orientation of all the edges of a circuit in $(M,\mathscr{O})$ yields another orientation $\mathscr{O}'$ that is still an α-orientation. Moreover two α-orientations $\mathscr{O}$ and $\mathscr{O}'$ of a planar map M always differ on a set of edges that form a collection of Eulerian submaps: since any Eulerian graph admits a Eulerian tour (that is, a circuit that visit every edge once), it is possible to go from one α-orientation to any other by a sequence of circuit reversals.

The clockwise or counterclockwise orientation of simple circuits can now be used to endow the set of α-orientations of a plane map (M, f) with an even nicer structure: let us say that a circuit reversal is *increasing* if it consists in returning a ccw-circuit into the opposite cw-circuit. We admit the following theorem, which gives a first illustration of the rich combinatorial properties enjoyed by planar maps.

Theorem 3 (Felsner [102]) *Let α be a feasible function on a planar map (M, f). Then increasing circuit reversals endow the set of α-orientation of a plane map (M, f) with a lattice structure. In particular there exists a unique minimal α-orientation in this lattice, which is the unique α-orientation without cw-circuit.*

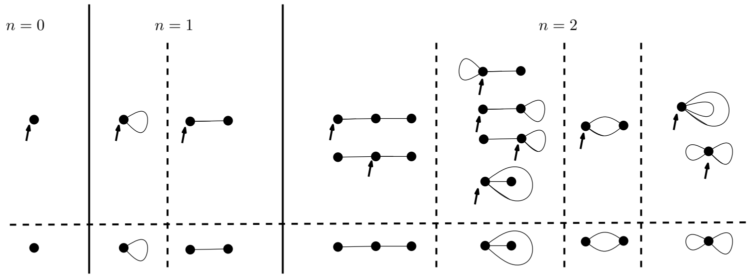

Figure 1.7
The 1, 2 and 9 rooted planar maps with 0, 1 and 2 edges, and the corresponding 1, 2 and 4 unrooted planar maps

1.1.3 Which maps shall we count?

Let $\mathscr{M}^u$, $\mathscr{M}^r$ and $\mathscr{M}^\ell$ denote the sets of (unrooted) planar maps, rooted planar maps and corner labeled planar maps respectively. Then the following four counting problems are the most commonly considered:

1. Count rooted planar maps with n edges, or compute the ordinary generating function (gf)
$$M^r(z) = \sum_{M\in\mathscr{M}^r} z^{e(M)} = 1+2z+9z^2+O(z^3)$$
2. Count planar maps with $2n$ labeled corners, or compute the exponential generating function
$$M^\ell(z) = \sum_{M\in\mathscr{M}^\ell} \frac{z^{e(M)}}{(2e(M))!} = 1+2\frac{z}{2!}+9\cdot 3!\frac{z^2}{4!}+O(z^3)$$
3. Count unrooted planar maps with n edges with a weight $1/|\mathrm{aut}(M)|$ per map M with automorphism group $\mathrm{aut}(M)$, or compute
$$M^a(z) = \sum_{M\in\mathscr{M}^u} \frac{z^{e(M)}}{|\mathrm{aut}(M)|} = 1+(\tfrac{1}{2}+\tfrac{1}{2})z+(\tfrac{1}{2}+\tfrac{1}{1}+\tfrac{1}{4}+\tfrac{1}{2})z^2+O(z^3)$$
4. Count unrooted planar maps with n edges, or compute
$$M^u(z) = \sum_{M\in\mathscr{M}^u} z^{e(M)} = 1+2z+4z^2+O(z^3)$$

Observe that we classify maps according to their number of edges (or corners, equivalently): in view of duality this is the most natural size parameter to consider, but we shall see that most results can later be refined to take into account the numbers of vertices and faces.

Automorphisms of maps.

Let (M, r) be a rooted map and assume once again that its corners are labeled with $\{1, \ldots, 2n\}$, with (σ, ρ) the corresponding rotation system. By definition of rotation systems, two corners are neighboring in M if their labels can be mapped one onto the other by σ or ρ: as a consequence, the connexity of M implies that for any two corners (c, c') there exists a sequence $(\tau_1, \ldots, \tau_k)$ of elements of $\{\sigma, \rho\}$ such that the labels i and i' of c and c' satisfy $i' = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k(i)$.

By definition an *automorphism* of the map M is a relabeling $\pi : \{1, \ldots, 2n\} \to \{1, \ldots, 2n\}$ of its corners that preserves the map M, or equivalently, that commutes with σ and ρ: $\sigma \circ \pi = \pi \circ \sigma$, and $\rho \circ \pi = \pi \circ \rho$. An automorphism π of the rooted map (M, c) is an automorphism of M that also preserves the label i_0 of the root corner c: $\pi(i_0) = i_0$. Now according to the previous discussion, $\pi(i) = \pi \circ \tau_1 \circ \cdots \circ \tau_k(i_0)$ for some sequence $(\tau_1, \ldots, \tau_k)$ in $\{\sigma, \rho\}^*$. Using the commutation relations between π and σ or ρ, and the equality $\pi(i_0) = i_0$, this shows that $\pi(i) = i$: the only automorphism of a rooted map is the trivial one.

An important consequence of this discussion is that the number of different labeling of the corners of a rooted map with $2n$ edges is always $(2n)!$, and the number of such labeling where the root corner has label 1 is $(2n-1)!$. A more detailed discussion of map automorphisms can be found in [90], or [146].

The first three problems are in fact essentially equivalent: On the one hand, rooted maps have no nontrivial automorphisms (see above), so that each rooted map admits $(2e(M) - 1)!$ distinct labeling such that the root corner has label 1. On the other hand, each unrooted map corresponds by definition of $\mathrm{aut}(M)$ to $\frac{(2e(M))!}{|\mathrm{aut}(M)|}$ labeled maps. Therefore:

$$M^r(z) = \frac{2zd}{dz} M^\ell(z) = \frac{2zd}{dz} M^a(z). \tag{1.2}$$

As illustrated by Figure 1.7 the fourth problem is not equivalent to the other three. Observe that we could have considered a fifth problem, namely to count plane maps (*i.e.* unrooted planar maps with a marked face). This problem is less advertised in the literature but it arises as an intermediary step while counting classes of unrooted maps. The literature overwhelmingly concentrates on the rooted (or labeled) case: it is technically simpler, and most of the results for rooted maps can *a posteriori* be transferred to unrooted maps.

1.2 Counting tree-rooted maps

We start this exposition with rooted planar maps with a marked spanning tree, called tree-rooted maps: these maps are the main characters of some simple extensions of classical bijections between rooted plane trees, balanced parenthesis words and non-crossing (*aka* planar) arch diagrams. There are at least two reasons to discuss at once these bijections: First it is a gentle way to get the reader used to manipulating planar maps, by building on the more standard combinatorics of plane trees. Second the bijections for tree-rooted planar maps are useful tools to simplify the later description of bijections for rooted planar maps. This section is largely inspired by the seminal work [28].

1.2.1 Mullin's decomposition

Our first series of results dates back to the work of Tutte and Mullin in the 60's.

Walking around a spanning tree. Let a *tree-rooted planar map* be a rooted planar map (M,c) with n edges, endowed with a spanning tree T_1. The *counterclockwise walk* around (T_1,c) induces a total order on the $2n$ corners of M given by the order in which these corner are visited by a 2d little ant traveling on the border of the tree T_1 in counterclockwise direction. This process is illustrated by Figure 1.8.

Each edge is visited twice during the walk and four symbols can be used to record the four types of moves of the ant during the walk: $\bullet$ and $\bar{\bullet}$ for the first and second time it goes across an edge, and $\circ$ and $\bar{\circ}$ for the first and second time it goes along an edge. The *counterclockwise contour code* is then the word w on the alphabet $\{\circ,\bar{\circ},\bullet,\bar{\bullet}\}$ whose ith letter is the type w_i of the ith move of the ant.

The restriction of the ccw contour code w to the letters $\{\circ,\bar{\circ}\}$ is the standard counterclockwise contour code of the rooted plane tree (T_1,c) (see [infra, Chapter on Trees]): in particular it is a *balanced parenthesis word*, that is, $|w|_\circ = |w|_{\bar{\circ}}$ and for all prefix w' of w, $|w'|_\circ \geq |w'|_{\bar{\circ}}$, where $|w|_x$ denotes the number of letters x in the word w. The restriction of the ccw contour code w to the letters $\{\bullet,\bar{\bullet}\}$ is instead a direct encoding of the arch diagram formed around T_1 by the edges not in T_1: it is in particular again a balanced parenthesis word, $|w|_\bullet = |w|_{\bar{\bullet}}$ and for all prefix w' of w, $|w'|_\bullet \geq |w'|_{\bar{\bullet}}$.

The word w is thus composed of the shuffle of two balanced parenthesis words on the alphabet $\{\circ,\bar{\circ}\}$ and $\{\bullet,\bar{\bullet}\}$ respectively. Mullin's result is essentially that this encoding is one-to-one.

Theorem 4 (Mullin's encoding [147]) *The contour code is a bijection between*

- *tree-rooted planar maps (M,c) with n edges, and*
- *shuffles of balanced words on $\{\circ,\bar{\circ}\}$ and $\{\bullet,\bar{\bullet}\}$ of length $2n$.*

As a consequence, the number of tree-rooted planar maps with n edges derives from the number of ways to shuffle two balanced words of respective length i and j with $i+j=n$.

Corollary 1 *The number of tree-rooted planar maps with n edges is*

$$\sum_{\substack{i+j=n\\ i,j\geq 0}} \binom{2n}{2i} C_i C_j = C_n C_{n+1}$$

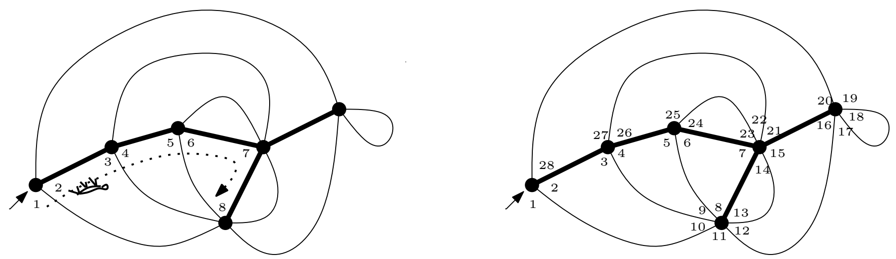

Figure 1.8
The eight first corners visited during a ccw walk around a spanning tree, and the full induced numbering of corners. The transition from one corner to the next are

$$1 \xrightarrow{\bullet} 2 \xrightarrow{\circ} 3 \xrightarrow{\bullet} 4 \xrightarrow{\circ} 5 \xrightarrow{\bullet} 6 \xrightarrow{\circ} 7 \xrightarrow{\circ} 8 \xrightarrow{\bar\bullet} 9 \xrightarrow{\bar\bullet} 10 \xrightarrow{\bar\bullet} 11 \xrightarrow{\bullet} 12 \xrightarrow{\bullet} 13 \xrightarrow{\bar\circ} 14 \xrightarrow{\bar\bullet} 15 \xrightarrow{\circ} 16$$
$$\xrightarrow{\bar\bullet} 17 \xrightarrow{\bullet} 18 \xrightarrow{\bar\bullet} 19 \xrightarrow{\bullet} 20 \xrightarrow{\bar\circ} 21 \xrightarrow{\bullet} 22 \xrightarrow{\bullet} 23 \xrightarrow{\bar\circ} 24 \xrightarrow{\bar\bullet} 25 \xrightarrow{\bar\circ} 26 \xrightarrow{\bar\bullet} 27 \xrightarrow{\bar\circ} 28 \xrightarrow{\bar\bullet}$$

so that the contour code is: $\bullet \circ \bullet \circ \bullet \circ \circ \bar\bullet \bar\bullet \bar\bullet \bullet \bullet \bar\circ \bar\bullet \circ \bar\bullet \bullet \bar\bullet \bullet \bar\circ \bullet \bullet \bar\circ \bar\bullet \bar\circ \bar\bullet \bar\circ \bar\bullet$

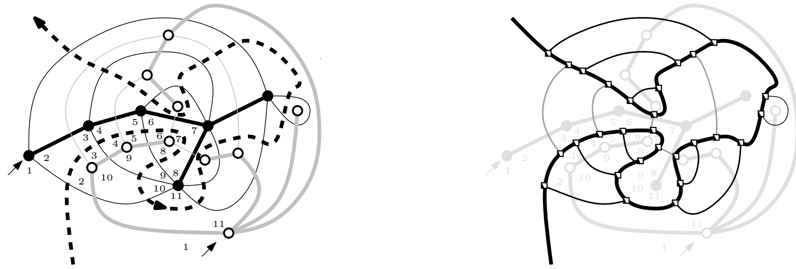

Figure 1.9
(i) The contour walk between a spanning tree and its dual. (ii) The associated cubic map with a rooted Hamiltonian cycle.

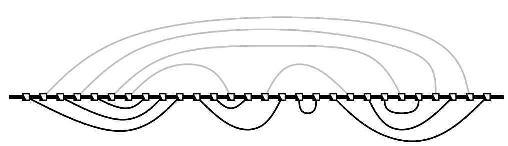

Figure 1.10
A shuffle of two arc diagrams

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the nth Catalan number, and the second expression follows from the Chu-Vandermonde identity [15]

Duality and cubic maps with a rooted Hamiltonian cycle. It is worth observing that the contour code is compatible with duality. Let (M,c) be a rooted planar map, T_1 be a spanning tree of M and T_2^* the dual spanning tree of M^*. Then, as illustrated by Figure 1.9, an ant performing a counterclockwise walk around T_1 simultaneously performs a clockwise walk around T_2^*. Using the same symbol as for the ccw contour code, one can define the clockwise contour code of the rooted map (M^*,c) endowed with the tree T_2^*, and state the following proposition:

Proposition 2 *The total order induced by the counterclockwise walk around T_1 starting from c in M is identical to the total order induced by the clockwise walk around T_2^* starting from c in M^*. In particular the counterclockwise contour code of (M,c) endowed with T_1 and the clockwise contour code of (M^*,c) endowed with T_2^* are mapped one onto the other by the exchanges $\circ \leftrightarrow \bullet$, $\bar{\circ} \leftrightarrow \bar{\bullet}$.*

From this primal/dual point of view, it is natural to draw the contour walk as a curve $\mathscr{C}$ traveling between the spanning tree T_1 and the dual spanning tree T_2^* in the superimposition of M and M^*, as illustrated by Figure 1.9. The intersections of the curve $\mathscr{C}$ with edges of $M \setminus T_1$ and $M^* \setminus T_2^*$ create tetravalent vertices in the middle of every half-edge of M and M^* that is not in T_1 or T_2^*. In the superimposition of $M \setminus T_1$, $M^* \setminus T_2^*$ and $\mathscr{C}$, each of these new vertices is adjacent to three others new vertices (two along $\mathscr{C}$ and one along an edge of $M \setminus T$ or $M^* \setminus T_2^*$) and to one vertex of M or M^*. Upon keeping only the new vertices and the induced map, one obtains a cubic planar map endowed with a rooted *Hamiltonian cycle* $\mathscr{C}$ (that is a rooted cycle that visit every vertex exactly once). Again this construction is bijective. Upon straightening the rooted cycle $\mathscr{C}$ it yields a direct geometric interpretation of the contour code as the shuffle of two arch diagrams above and below $\mathscr{C}$.

Theorem 5 (Mullin [147]) *There is a bijection between tree-rooted planar maps with n edges and cubic planar maps endowed with a rooted Hamiltonian cycle of length $2n$.*

Corollary 2 (Tutte [163]) *The number of cubic planar maps endowed with a rooted Hamiltonian cycle of length $2n$ is $\sum_{i+j=n}\binom{2n}{2i}C_iC_j = C_nC_{n+1}$.*

Breaking the symmetry, balanced trees. The previous constructions were symmetric with respect to duality. Let us now consider an alternative construction that breaks this symmetry. The idea is to break during the ccw walk the edges that are crossed into pairs of half-edges. More precisely, in view of the previous discussion, it is natural to distinguish for each edge the half-edge that corresponds to the first visit and the half-edge that corresponds to the second visit: this is done by using outgoing half-edges for the first ones, and incoming half-edges for the second ones. The resulting operation, called the *opening*, turns a tree-rooted map M into a *rooted decorated plane tree*, that is, a rooted plane tree with two types of dangling half-edges, as

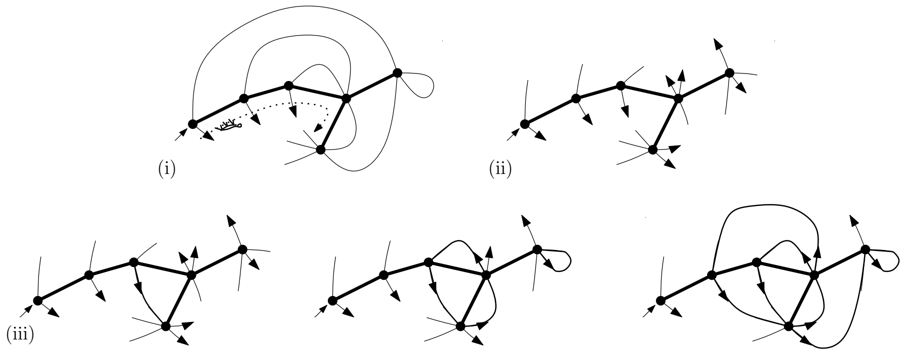

Figure 1.11
(i) The opening of the tree-rooted map of Figure 1.8. (ii) The resulting balanced tree. (iii) The first few steps of the closure of the previous balanced tree. .

illustrated by Figure 1.11. In view of the previous discussion, the decorated tree that are produced in this way are characterized by the fact that the number of outgoing and incoming half-edges are equal, and that the sequence of outgoing and incoming half-edges that are met during a counterclockwise walk around the tree forms a balanced parenthesis word. These rooted decorated trees are called *balanced trees*.

Conversely the fact that the outgoing and incoming half-edges of a balanced tree form by definition a balanced parenthesis word ensures that there is a unique way to *close* any balanced tree into a tree-rooted planar map such that the root corner stays in the outer face. In particular this can be done by repeating the following local closure rule: Starting from the root corner and walking counterclockwise around the current map, let h be the first outgoing half-edge that is followed by an incoming half-edge h', and glue the pair (h,h') into an edge in the unique way that leaves the root corner in the outer face.

Theorem 6 (Walsh and Lehman [172]) *Opening and closure are inverse bijections between tree-rooted planar maps with n edges, v vertices and f faces, and balanced trees with v vertices and* $f+1$ *outgoing half edges.*

A balanced tree should be viewed as a rooted plane tree that carries on its border the contour code of another tree. We will reuse this idea several times.

1.2.2 Spanning trees and orientations

Mullin's encoding explains his convolution formula for the number of tree-rooted planar maps, but not the simple product form C_nC_{n+1} arising from the Chu-Vandermonde formula. In preparation of an explanation of this formula, we first "forget about the spanning trees" and reformulate the result in terms of orientations.

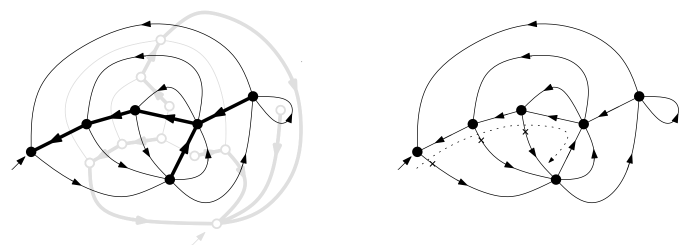

Figure 1.12
The orientation induced by a spanning tree, and the ccw-exploration.

The orientation induced by a spanning tree. Let (M,c) be a rooted map, and let T be a spanning tree of M. The *orientation of* (M,c) *induced by* T is the orientation $\mathscr{O}_T$ such that each edge of T is oriented toward the root, and each edge e of $M \setminus T$ *turns counterclockwise around* T, that is, the unique simple circuit formed by e and edges of T is counterclockwise (see Figure 1.12). Equivalently, the map (M,c) endowed with $\mathscr{O}_T$ can be constructed using the closure of Theorem 6, upon orienting the edges of the balanced tree toward the root and keeping the orientation of the matched half-edges.

It is instructive to perform the closure incrementally: Starting from the oriented balanced tree, close one edge at a time, and observe that at each step no cw-circuit can be created. Finally in view of the orientation of the edges of T, the orientation $\mathscr{O}_T$ is *root-accessible*, *i.e.* there is an oriented path from each vertex to the root vertex.

Proposition 3 *Let T be a spanning tree of a rooted map (M,c). Then the orientation $\mathscr{O}_T$ induced by T is root-accessible and has no cw-circuit.*

The ccw-exploration of an oriented map. A key observation now is that on the rooted oriented map $(M,c,\mathscr{O}_T)$, the ccw walk around T can be performed without knowing T. Indeed in the ccw walk, each edge is reached for the first time from its endpoint if it is an edge of T, and from its origin if is not an edge of T. This implies that the orientation alone allows us to decide at the first visit of each edge whether it belongs to T or not, and whether the walk should follows the edge or cross it. The resulting ccw walk around the initially unknown tree T is called the *ccw-exploration* of the oriented map M.

Conversely an elementary case analysis allows us to characterize the rooted oriented maps on which ccw-exploration produces a spanning tree.

Proposition 4 *The ccw exploration of a rooted planar map (M,c) endowed with an orientation $\mathscr{O}$ without cw-circuit outputs a tree T, and this tree is a spanning tree of M if and only if the orientation $\mathscr{O}$ is root-accessible.*

Summarizing these results yields the following theorem, first explicitly stated in this general form in Bernardi's PhD thesis.

Theorem 7 (Bernardi [29]) *Let (M,c) be a rooted planar map. The ccw-exploration and induced orientation are inverse bijections between accessible orientations without cw-circuit of (M,c) and spanning trees of (M,c).*

Earlier instances of the result in the special case of triangulations and 3-connected planar maps appears in [151, 107], extensions were proposed in [29, 6]. Observe that ccw-exploration is just leftmost oriented depth first search, it might thus be the case that this result is known already in other contexts.

Corollary 3 *The number of rooted planar map with n edges endowed with a root-accessible orientation without cw-circuit is* $\sum_{i+j=n}\binom{2n}{2i}C_iC_j = C_nC_{n+1}$.

Again in this corollary only the convolution formula is obtained bijectively.

In view of Theorem 3, it is tempting to define a *root-accessible feasible function* on the set V of vertices of a rooted planar map (M,c) as a feasible function $\alpha : V \to \mathbb{N}$ such that α-orientations are root-accessible (observe that if one α-orientation is accessible then all are since circuit reversal preserves accessibility).

Corollary 4 *The number of rooted planar maps with n edges endowed with a root-accessible feasible function is* $\sum_{i+j=n}\binom{2n}{2i}C_iC_j = C_nC_{n+1}$.

Further reformulations and extensions of these results in terms for instance of recurrent sand pile configurations, and relations to Tutte polynomials are given in [29].

Ccw-exploration and opening. As already observed, the orientation induced by a spanning tree of a map coincides with the orientation induced by the closure of Theorem 6. As a consequence, Theorems 6 and 7 can be combined to obtain the following corollary, which will be useful when we turn to the problem of enumerating rooted planar maps.

Corollary 5 *Ccw-exploration followed by opening is a bijection between rooted planar maps with i vertices and j faces endowed with an accessible orientation without cw-circuit and balanced trees with i vertices and j outgoing half-edges. Moreover, this bijection preserves in- and out-degrees of every vertex.*

1.2.3 Vertex blowing and polyhedral nets.

The last result we shall present in this section is a construction that was first used in a particular case by Cori and Vauquelin in the 80's [92] and that was shown by Bernardi 30 years later to finally explain the product formula C_nC_{n+1} [28].

Vertex blowing and split trees. Let (M,c) be a rooted planar map and $\mathscr{O}$ an orientation of M. The *blowing* of a vertex v with out-degree k is the operation replacing v by a white polygon with k nodes each carrying one outgoing edge and the incoming edges that precede it the cw direction, as illustrated by Figure 1.13. The *complete blowing* $\Sigma(M,\mathscr{O})$ of an oriented rooted map $(M,\mathscr{O})$ consists in blowing each vertex

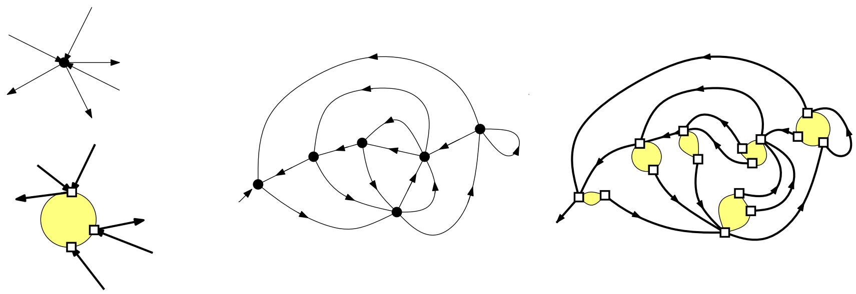

Figure 1.13
The blowing of a vertex, a map M, its complete blowing with the split tree.

of M independently (the blowing of the root vertex is performed as if there was an outgoing half-edge at the root corner). The map $\Sigma(M,\mathscr{O})$ naturally inherits from $\mathscr{O}$ a partial orientation which we continue to denote $\mathscr{O}$: the edges inherited from M are oriented while the edges of the white polygons created by blowings are not.

A key observation here is that, if $(M,\mathscr{O})$ is accessible, then the oriented edges of $\Sigma(M,\mathscr{O})$ cannot form ccw circuits: Indeed if they would form a ccw circuit then this circuit would also be a ccw circuit in M. In view of the blowing rule, the vertices of M on this circuit would only have outgoing edges on the left hand side of the circuit. But in a ccw circuit the root lies in the right hand side of the circuit, so this contradicts accessibility.

Assuming moreover that $\mathscr{O}$ is an orientation without cw-circuit, then $\Sigma(M,\mathscr{O})$ is an accessible oriented map without circuit, that is a tree.

Proposition 5 *Let (M,c) be a rooted planar map and $\mathscr{O}$ a root-accessible orientation without cw-circuit. Then the oriented edges of the complete blowing $\Sigma = \Sigma(M,\mathscr{O})$ form a spanning tree of Σ oriented toward c, called the* split tree *of M.*

Polyhedral nets. Let us use the split tree to cut the sphere on which the map Σ is drawn. As illustrated by Figure 1.14, this *splitting* construction yields a polyhedral net. In order to be able to reconstruct the map Σ from the polyhedral net one needs to specify the way edges should be glued together. This can be done by explicitly recording a planar arch diagram as in Figure 1.2, but it turns out to be more convenient to just keep track of the orientation of the edges of the split tree on the sides of the polyhedral net, as shown in Figure 1.14. The *folding* of the polyhedral net then just consist in iteratively gluing pairs of oriented border edges that originate from a same vertex until all boundary edges have been matched.

Let us describe more precisely the oriented polyhedral nets that arise from this construction: they consist of an outface and a collection of black and white simple faces called *polygons*, such that the sides of white polygons are incident only to black polygons, and the boundary of each black polygon consists of an alternating sequence of sides incident to white polygons and *super-edges* formed of a ccw edge incident

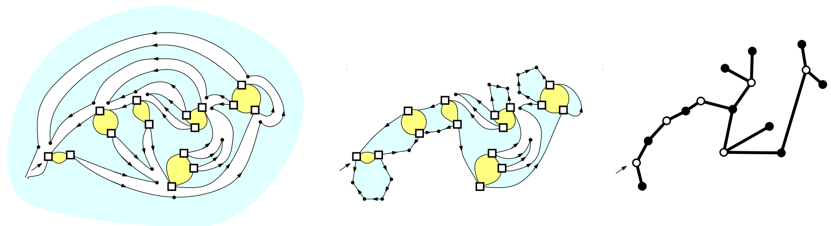

Figure 1.14
(i) Cutting along the split tree to get a polyhedral net, another representation of the same polyhedral net, and its skeleton.

to the outerface, followed by a (possibly empty) sequence of cw edges incident to the outerface.

An oriented polyhedral net is *rooted* if a corner incident to a white node in the outerface is marked, and *balanced* if moreover, starting from the root corner and turning clockwise around the polyhedral net, its boundary edges form the contour code of a planted planar tree (equivalently if there are always strictly more already seen ccw edges than cw edges during the tour).

Theorem 8 *Splitting and folding are inverse bijections between rooted planar maps with i vertices and j faces and $n = i + j - 2$ edges endowed with an accessible orientation without cw-circuit, and balanced polyhedral nets with i white bounded faces and j black bounded faces and $2n+2$ boundary edges.*

Let the *skeleton* of a balanced polyhedral net be the bicolored tree obtained by putting a (black or white) vertex in each (black or white) polygon and joining vertices corresponding to adjacent polygons by an edge. In particular the degree of a black (resp. white) vertex in the skeleton corresponds to the out-degree of the associated vertex of the initial map (resp. to the number of ccw oriented edges around the associated face). Now observe that in a balanced polyhedral net with $2n+2$ boundary edges, the $2n+2$ symbols forming the contour code of the planted split tree are written sequencially on the $2n+2$ boundary edges of the polyhedral net in a deterministic way: the kth opening parenthesis of the code is preceded by ℓ_k closing parenthesis if and only if, in cw order around the polyhedral net, the kth super-edge consists of a ccw edge preceded by ℓ_k cw edges. As a consequence a balanced polyhedral net $2n+2$ boundary edges is determined by the pair formed by its skeleton and its split tree. Conversely any pair formed of a rooted bicolored plane tree with $n+2$ vertices and a rooted plane tree with n edges yields a balanced polyhedral net with $2n+2$ boundary edges.

Corollary 6 (Bernardi [28]) *There is a bijection between tree-rooted maps M with i vertices and j faces and $n = i + j - 2$ edges, and pairs (t_1, t_2) where t_1 is a (bipartite) rooted plane tree with i black and j white vertices and $n+1$ edges, and t_2 is a rooted plane tree with n edges. In particular the number of tree-rooted maps with n edges is $C_n C_{n+1}$.*

A nice feature of the above result is that it not only explains the univariate product formula $C_{n+1}C_n$, but also give an alternate proof of the bivariate formula as $N_{i,j}C_n$, where $N_{i,j} = \frac{n!(n+1)!}{i!(i+1)!j!(j+1)!}$ is the Narayana number of rooted bipartite trees with i black and j white vertices. Or conversely, combined with the formula $\binom{2n}{2i}C_iC_j$ obtained by Mullin's encoding, it allows us to recover Naranaya's formula.

1.2.4 A summary and some observations

In this section we have successively shown that tree-rooted maps are in one-to-one correspondence with shuffle of parenthesis words, cubic maps endowed a rooted Hamiltonian cycle, balanced trees, root-accessible oriented maps without cw-circuit, balanced polyhedral nets and finally pairs of rooted plane trees.

On the one hand, these results form a coherent and nice chapter of bijective combinatorics that relies on and extends several extremely classical results of the *Catalan garden* [168]. These constructions may seem intrinsically planar: for instance on a surface of genus g, the dual of the edges that are not in a spanning tree of a map do not form a spanning tree of the dual, and the combinatorics of higher genus tree-rooted maps is not very satisfying. Instead a remarkable observation of Bernardi and Chapuy is the fact that many of these constructions extend to higher genus maps upon considering *covered maps* rather than tree-rooted maps: a *covering* of a map is a spanning subset of edges C such that the local cyclic arrangements of edges around T forms a map with one face (recall that a *planar* map is a tree if and only if it has one face). This approach leads to various elegant enumerative consequences, as explained in [36].

On the other hand, these bijections are some of the most fundamental ingredients in the search of bijections for rooted planar maps: indeed a natural approach to the enumeration of rooted planar maps is to endow each map with a canonical spanning tree such that the family of balanced trees arising from Theorem 6 or the family of balanced polyhedral net arising from Theorem 8 are simple to describe and enumerate.

1.3 Counting planar maps

1.3.1 The exact number of rooted planar maps

Rooted planar maps. The most striking result about planar maps is certainly the fact that the number of rooted planar maps with n edges has a simple closed formula

$$|\mathscr{M}_n^r| = \frac{2\cdot 3^n \cdot (2n)!}{n!(n+2)!} = \frac{2}{n+2}\cdot\frac{3^n}{n+1}\binom{2n}{n}, \qquad (1.3)$$

and in particular satisfies the following linear recurrence

$$(n+2)|\mathscr{M}_n^r| = 6(2n-1)|\mathscr{M}_{n-1}^r|, \quad \text{with} \quad |\mathscr{M}_0^r| = 1. \tag{1.4}$$

Equivalently, the ordinary generating function (gf) of rooted planar maps with respect to the number of edges $M^r(z) = \sum_{M\in\mathscr{M}^r} z^{e(M)}$ satisfies

$$M^r(z) = T(z) - zT(z)^3 \tag{1.5}$$

where $T(z)$ is the unique power series solution of

$$T(z) = 1 + 3zT(z)^2. \tag{1.6}$$

(The equivalence of Formula (1.3) and Equations (1.5)-(1.6) follows from a direct application of Lagrange inversion formula [infra, Chapter on Tree].) Another way to write the relation between $M^r(z)$ and $T(z)$ is

$$\frac{\partial}{\partial z}(z^2 M^r(z)) = 2T(z). \tag{1.7}$$

More blandly, $M^r(z)$ is the unique power series root of the polynomial

$$P(M,z) = 1 - 16z + 18zM - 27z^2M^2. \tag{1.8}$$

In particular $M^r(z)$ is an algebraic function over the field $\mathbb{Q}(z)$.

Formula (1.3) was first discovered by W. T. Tutte in a series of papers written between January 1961 and February 1962 where he deals with the enumeration of various subfamilies of planar maps. Using little more than the simple idea of root-edge deletion, Tutte established in [164] a recurrence for the number $E(d_1,d_2,\ldots)$ of rooted Eulerian planar maps with d_i vertices of degree $2i$ and $e = \sum_{i\geq 1} id_i$ edges, then guessed that these numbers admit the following simple form

$$E(d_1,d_2,\ldots) = \frac{2(e!)}{(e-\sum_{i\geq 1} d_i + 2)!}\prod_{i\geq 1}\frac{1}{d_i!}\binom{2i-1}{i}^{d_i}. \tag{1.9}$$

He then checked that this formula satisfies his recursion by an incredible computational *tour de force*. Quickly after this, Tutte observed in [165] that the special case $d_2 = n$ and $d_i = 0$ for all $i \neq 2$ of Formula (1.9), that counts rooted tetravalent planar maps with n vertices, gives Formula (1.3) for the number of rooted planar maps with n edges in view of the edge map transformation (see page 6).

A few years later, in [166], Tutte proposed a streamlined version of the root edge deletion method he uses to establish recursive decompositions and to translate them into functional equation for gfs: he is literally applying there what is now known as the symbolic method for combinatorial enumeration [104, Chapter 1]. Let $M_j(z,\mathbf{y})$ denote the gf of rooted planar maps with a root face of degree j, counted by number of edges (variable z) and by number of non-root faces with degree i (variable y_i) for all $i \geq 1$, and let $M(z,\mathbf{y};u) = \sum_{j\geq 0} u^j M_j(z,\mathbf{y})$. Then Tutte's equation reads

$$M(z,\mathbf{y};u) = 1 + zu^2M(z,\mathbf{y};u)^2 + z\sum_{i\geq 1} y_i \frac{M(z,\mathbf{y};u) - \sum_{j=0}^{i-2} u^j M_j(z,\mathbf{y})}{u^{i-2}}, \tag{1.10}$$

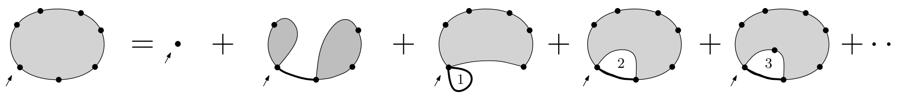

Figure 1.15
Tutte's root edge deletion.

where each terms has a clear interpretation in terms of a decomposition by deletion of the root edge, as illustrated by Figure 1.15:

- The quadratic term accounts for planar maps whose root edge is a separating edge: such a map can be uniquely obtained from a pair of rooted planar maps by joining their root corners with a new root edge.
- The *i*th term in the sum instead corresponds to planar maps whose root edge has a face of degree i on its left-hand side: such a map can be uniquely obtained from a rooted planar map whose outerface has degree at least $i-1$. Hence the truncated M series.

Iterating Equation (1.10) as a fixed point equation in the space of formal power series, one obtains the first coefficients:

$$\begin{aligned}
M(z,\mathbf{y},u) &= 1+zu^2(1+O(z))^2+zy_1\tfrac{1+O(z)}{u^{-1}} = 1+z(u^2+uy_1)+O(z^2)\\
&= 1+zu^2(1+z(u^2+uy_1))^2+zy_1\tfrac{1+z(u^2+uy_1)}{u^{-1}}\\
&\quad +zy_2\frac{1+z(u^2+uy_1)-1}{u^0}+zy_3\frac{1+z(u^2+uy_1)-(1+zuy_1)}{u}+O(z^3)\\
&= 1+z(u^2+uy_1)+z^2(2u^4+3u^3y_1+u^2y_1^2+u^2y_2+uy_2y_1+uy_3)+O(z^3)
\end{aligned}$$

One can check that these first terms agree with what can be seen on Figure 1.7, and for $u=1$ and $y_i=1$ for all i, with the first values of Formula (1.3).

In the case $y_i=0$ for all odd i, Equation (1.10) is essentially equivalent to the recurrence used by Tutte to conjecture and prove Formula (1.9), but no such simple formula is known to enumerate generic (*i.e.* non necessarily Eulerian) rooted planar maps with a fixed vertex degree distribution, and Tutte was unable in the 70's to guess the general solution. Instead he observed that for $y_i=y$, his equation rewrites as

$$\begin{aligned}
M(z,y;u) &= 1+zu^2M(z,y;u)^2+zyu\sum_{j\geq 0}\left(\sum_{i=0}^{j}u^{j-i}\right)M_j(z,y)\\
&= 1+zu^2M(z,y;u)^2+zyu\frac{M(z,y;1)-uM(z,y;u)}{1-u}, \qquad (1.11)
\end{aligned}$$

and using again a guess and check approach, he was able to solve this equation to count maps by number of edges and faces. The resulting number of rooted planar maps with k vertices and ℓ faces and $n=k+\ell-2$ edges is a not particularly appealing triple summation of binomial coefficients [13] but the bivariate gf

$M^r(x,y) = \sum_{M \in \mathscr{M}^r} x^{v(M)} y^{f(M)} = xy(1 + M(x, y/x; 1))$ satisfies an elegant variant of Formula (1.5):

$$M^r(x,y) = N(x,y)P(x,y)(1 - 2N(x,y) - 2P(x,y)) \tag{1.12}$$

where $N(x,y)$ and $P(x,y)$ are given by the bivariate analog of (1.6)

$$\begin{cases} N(x,y) &= x + N(x,y)^2 + 2N(x,y)P(x,y), \\ P(x,y) &= y + P(x,y)^2 + 2N(x,y)P(x,y). \end{cases} \tag{1.13}$$

Equivalently, the bivariate analog of Equation (1.7) reads

$$\frac{\partial}{\partial x} M^r(x,y) = P(x,y). \tag{1.14}$$

Solving Tutte's equations. The difficulty in deriving the explicit expressions (1.3)-(1.6) or their refinement (1.12)-(1.13) from Equation (1.11) lies in the particular rôle played by variable u: observe indeed that variable y can be considered as an optional parameter (setting $y = 1$ yields a valid equation that allows us to compute $M(z,1;u)$ order by order in z by iteration as above), whereas variable u appears to be necessary to get a non trivial equation (setting $u = 1$ yields instead an *a priori* useless equation involving $M(z,y;u)$ and $\frac{d}{du}M(z,y;u)|_{u=1}$ because of the discrete derivative appearing in the last summand). Following Zeilberger, the variable u is called a *catalytic variable* (in analogy with the *catalytic* ingredients that are sometime added to allow for a chemical reaction to take place), and Equations (1.10) and (1.11) are *equations with one catalytic variable*.

Starting with Brown [77, 76], methods to solve quadratic equation with one catalytic variable without guessing were developed, that allowed in particular Bender and Canfield [21], almost 30 years later, to give an essentially complete solution to Tutte's equation. Brown's method was later turned into a systematic approach to polynomial equations with one catalytic variable by Bousquet-Mélou and Jehanne [56]. Let us quote here their main general statement.

Let $\mathbb{K}$ be a field of characteristic 0. Let $F(u) \equiv F(z;u)$ be an element of $\mathbb{K}[u][[z]]$, that is, a power series in z with coefficients that are polynomial in u over $\mathbb{K}$. Then the following divided difference is well-defined:

$$\Delta F(u) = \frac{F(u) - F(0)}{u},$$

and $\lim_{u\to 0} \Delta F(u) = F'(0)$ (from now on in this section derivatives are taken with respect to the variable u unless explicitly mentioned). More generally the iterated application of Δ yields

$$\Delta^{(i)} F(u) = \frac{F(u) - F(0) - uF'(0) - \ldots - \frac{u^{i-1}}{(i-1)!} F^{(i-1)}(0)}{u^i}$$

Then the main result of [56] is the following theorem (see also [91] for an earlier derivation of the case $k = 1$).

Theorem 9 *Let $Q(w_0, w_1, w_2, \ldots, w_k; z)$ be a polynomial in u, z and the w_i with coefficients in a field $\mathbb{K}$ with characteristic 0. Then the unique formal power series solution $F(u)$ of the equation*

$$F(u) \equiv F(z,u) = zQ\left(F(u), \Delta F(u), \Delta^{(2)} F(u), \ldots, \Delta^{(k)} F(u), u; z\right) \qquad (1.15)$$

is algebraic over $\mathbb{K}(z,u)$.

For any $i_0 \geq 2$, this theorem directly applies to the specialization $y_i = 0$, $i > i_0$ of Equation (1.10) and proves that the corresponding series are algebraic.

The proof of the theorem is in fact constructive, and in principle gives a method to derive a system of algebraic equations that determines $F(u)$ and all the $\Delta^i F(u)$, as well as the specializations $F^{(i-1)}(0)$. We content ourselves here with the case $k = 1$, which is sufficient to derive Equation (1.8) from Equation (1.11), and refer to [54] for a gentle introduction to the general case and to [56] for the full details (see also [35, Section 9] for a slight extension allowing F and Q to be rational in u).

A derivation of Formula (1.3). The first observation is that up to a slight change of variable $F(u) = (1+u)(M(1+u)-1)$, the simplified Tutte's equation (1.11) can be recast into the form (1.15) with $k = 1$, and

$$Q(w_0, w_1, w_2; z) = (1+w_2)(1+w_2+w_0)^2 + (1+w_2)^2(1+w_1).$$

The initial trick is to differentiate Equation (1.15) with respect to u to get

$$\begin{aligned} F'(u) \quad &= \quad zF'(u)Q_0'(F(u), \Delta F(u), u; z) + \frac{zF'(u)}{u} Q_1'(F(u), \Delta F(u), u; z) \\ &\quad - \frac{z\Delta F(u)}{u^2} Q_1'(F(u), \Delta F(u), u; z) + zQ_2'(F(u), \Delta(F(u), u; z) \end{aligned} \qquad (1.16)$$

Now observe that since $Q(w_0, w_1, w_2; z)$ is a polynomial and $F(u)$ is a power series in z with polynomial coefficients in u, the equation

$$U = zUQ_0'(F(U), \Delta F(U), U; z) + zQ_1'(F(U), \Delta F(U), U; z)) \qquad (1.17)$$

is contracting on the space of power series in z and has a unique power series solution $U(z) = zQ_1'(0,0,0;0) + O(z^2)$. Upon multiplying by u and substituting $u = U(z)$ in Equation (1.16), all terms in the first line cancel and we obtain a second equation

$$\Delta F(U) Q_1'(F(U), \Delta F(U), U; z) = U^2 Q_2'(F(U), \Delta F(U), U; z)$$

which can be rewritten, using Equation (1.17) to eliminate Q_1', as

$$\Delta F(U)(1 - zQ_0'(F(U), \Delta F(U), U; z)) = zUQ_2'(F(U), \Delta F(U), U; z).$$

Summarizing, the triple of unknown functions $(F(U(z)), \Delta F(U(z)), U(z))$ is a solution of the system of algebraic equations in the variables w_0, w_1 and w_2:

$$\left\{ \begin{array}{lcl} w_0 &=& zQ(w_0, w_1, w_2; z) \\ w_1 &=& z(w_1 Q_0'(w_0, w_1, w_2; z) + Q_2'(w_0, w_1, w_2; z)) \\ w_2 &=& z(w_2 Q_0'(w_0, w_1, w_2; z) + Q_1'(w_0, w_1, w_2; z)). \end{array} \right. \qquad (1.18)$$

In particular the series $F(U(z))$, $\Delta F(U(z))$ and $U(z)$ are algebraic series over $\mathbb{K}(z)$, and so is

$$F(0) = F(U(z)) - U(z)\Delta F(U(z)). \tag{1.19}$$

Finally, returning to Equation (1.15), $F(u)$ is seen to be algebraic on $\mathbb{K}(z,u)$. Observe moreover that if Q has positive coefficients then the system (1.18) is $\mathbb{N}$-algebraic in the sense of [54].

Eliminating the unknown functions $F(V(z))$, $\Delta F(V(z))$ and $V(z)$ in the system of equations (1.18)-(1.19) result in an algebraic equation for $F(0)$ which directly yields Equation 1.8 for $M(z,0) = 1 + F(0)$.

The complete solution of Tutte's equation. Let us finally state the complete solution of Equation (1.10), as taken from [21, 60, 56]: the gf $M(z,\mathbf{y}) = M(z,\mathbf{y},1)$ of rooted planar maps with respect to the number of edges (variable z), and the number of faces of degree i (variable y_i) for all $i \geq 1$ satisfies

$$\frac{\partial}{\partial z}(zM(z,\mathbf{y})) = \frac{1}{z^2} S_2(9S_2 - 4S_1) \tag{1.20}$$

where S_1 and S_2 are the unique formal power series in z with coefficients in $\mathbb{Q}[y_1, y_2, \ldots]$ solutions of the system

$$\left\{ \begin{array}{rcl} S_1 & = & t[v^0] \sum_{i \geq 1} y_i (v + S_1 + S_2/v)^{i-1} \\ S_2 & = & t + t[v^{-1}] \sum_{i \geq 1} y_i (v + S_1 + S_2/v)^{i-1} \end{array} \right. \tag{1.21}$$

These expressions may look complicated at first sight, and maybe one could be tempted to argue that Tutte's equation (1.10) is more compact. However Equations (1.20)-(1.21) should be considered as much more explicit: They have a clear tree-gf structure, directly amenable for instance to bivariate Lagrange inversion, or to singularity analysis, and they imply that for any non-degenerate finite set $\mathscr{D}$ of allowed degrees, the gf of rooted planar maps with vertex degrees in $\mathscr{D}$ with respect to the number of edges (variable z) are given by the specialization $y_i = \delta_{i \in \mathscr{D}}$ of the above system, and in particular this gf is algebraic over $\mathbb{Q}(z)$.

Tutte equations and matrix integrals. Remarkably Tutte's results were reproduced in the physics literature [74, 42] using an ingenious representation of the all-genus map gf using Hermitian matrix integrals. We shall not discuss here this alternative approach because various accessible texts already exists. We only stress the fact that there are (at least) two point of views regarding matrix integral representations.

On the one hand they can be used as a convenient short-hand notation for the all genus gf of maps viewed as a formal power series, and this point of view leads to the derivation of *loop equations* or *Schwinger-Dyson equations* that are essentially equivalent to Tutte equations (see [98]). These equations are dealt with using clever changes of variables and more or less detailed discussions of the possible singularities of the resulting expressions. These approaches appear to be analytic variants of

Brown's more algebraic approach (a rigorous discussion of this is still missing in the literature however). From there many explicit formulas have appeared in the physics literature, culminating with Eynard and Orantin's topological recurrence [97, 101]. All these results are closely related to the present enumerative approach and we refer to [127, Chapter Matrix] for an elementary introduction, and [99, 59] for more detailed discussions.

On the other hand one can try to give a *bona fide* matrix integral representation to the all genus gf viewed as an analytic function. This is the original point of view of [74, 42], which allows them to use more analytic tools to derive explicit expressions, like saddle point approximations and orthogonal polynomials. However the validity of these representations requires careful discussion [96, 120, 154, 123] and involve deep relations with matrix integral theory which makes this approach less elementary to follow from an enumerative point of view. Finally let us refer to older but very complete surveys [8, 94] for discussions of the motivations for the study of maps in physics.

1.3.2 Unrooted planar maps

For unrooted maps on the oriented sphere, the formulas are still explicit, although somewhat more involved. In particular the number of unrooted planar maps with n edges is

$$|\mathscr{M}_n^u| = \frac{1}{2n}|\mathscr{M}_n^r| + \sum_{\substack{d<n \\ d|n}} \frac{\phi(\frac{n}{d})}{2n}\binom{d+2}{2}|\mathscr{M}_d^r| + \left(\frac{n+1}{4} - \frac{(-1)^n}{2}\right)|\mathscr{M}_{\lfloor \frac{n}{2} \rfloor}^r| \qquad (1.22)$$

where $\phi(x)$ is Euler's totient function, giving the number of non-trivial divisors of x.

The form of this formula admits a nice explanation. Generically there are $2n$ choices of root corner for an unrooted map, hence the first term in the right hand side. But in this first attempt, maps with non trivial automorphisms are undercounted since they correspond to less than $2n$ rooted maps each: according to Burnside lemma, to obtain the correct formula, one has to introduce a correction term for each pair (M,ω) where ω is an automorphism of M.

The exact correction terms to be used arise from a classification of the possible automorphisms of planar maps. This classification is best understood in geometric terms, using the (non-trivial, but very appealing) fact that any symmetric map admits a drawing on standard unit sphere in $\mathbb{R}^3$ that realizes its automorphisms as isometries of the oriented sphere, *i.e.* rotations. Rotations acting on maps are then simply classified according to their order, and to the type of cells (vertex, edge, or face) that they leave invariant (a rotation fixes exactly the two cells intersecting its axis).

Once the classification is established, the idea is that each pair (M,ω) with ω a rotation of order n/d can be constructed from its *quotient map* M/ω which is identified as a map with d edges and 2 marked cells as illustrated by Figure 1.16. The construction slightly differs depending whether the rotation fixes an edge or not: if not, the two fixed cells are chosen among the $v(M)+f(M)=e(M)+2$ vertices or faces of M, and this yields the first correcting term; otherwise, depending on the

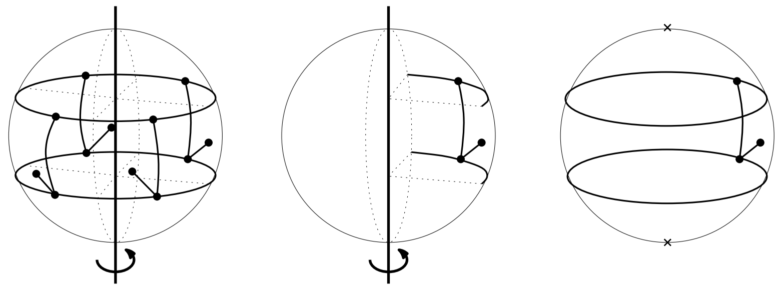

Figure 1.16
A symmetric map and the construction of the quotient map associated with the rotation of order 4 with vertical axis.

parity of n either one or two fixed cells are edges, and this explains the alternating second correcting term.

This elegant approach of unrooted map enumeration via the study of quotient maps was pioneered by Liskovets [134] (see also [169]) and extends to count planar maps up to sense reversing automorphisms [135] (see also the more recent [137, 139] for a streamlined presentation). An alternative approach based on multiply rooted maps is due to Wormald [174, 175], building on earlier work of Brown [75] but yields less attractive formulas.

1.3.3 Two bijections between maps and trees

The series $T(z)$ given by Formula (1.6) is closely related to the classical Catalan numbers gf, and as a consequence it admits dozens of combinatorial interpretations in terms for instance of plane trees, lattice paths or non crossing arch diagrams. A natural question that was raised very soon after the publication of Tutte's results is to use such a classical interpretation of $T(z)$ to explain Formulas (1.3), (1.5) or (1.7). Two main interpretations have lead to particularly elegant answers: Cori and Vauquelin's *well labeled trees* [92] and my *blossoming trees* [156].

Curiously, neither of these two interpretations directly deal with rooted planar maps, but rather with quadrangulations and tetravalent maps: recall indeed again that, thanks to the incidence map and edge map constructions of Page 6, all the above formulas apply to rooted planar quadrangulations with n faces, or to rooted tetravalent maps with n vertices.

The blossoming tree approach. This first approach builds on a variant of binary trees to interpret Formula (1.6) and to arrive at a direct explanation of Formulas (1.3) and (1.5) in terms of tetravalent maps.

More specifically, let a *blossoming tree* of size n be a rooted planar map with one face (*i.e.* a plane tree) with n vertices of degree 4 (the *nodes*) and $2n+2$ vertices of degree one (the *leaves*) that are colored in black and white in such a way that every node is incident to exactly one black leaf. As illustrated by Figure 1.17, the standard

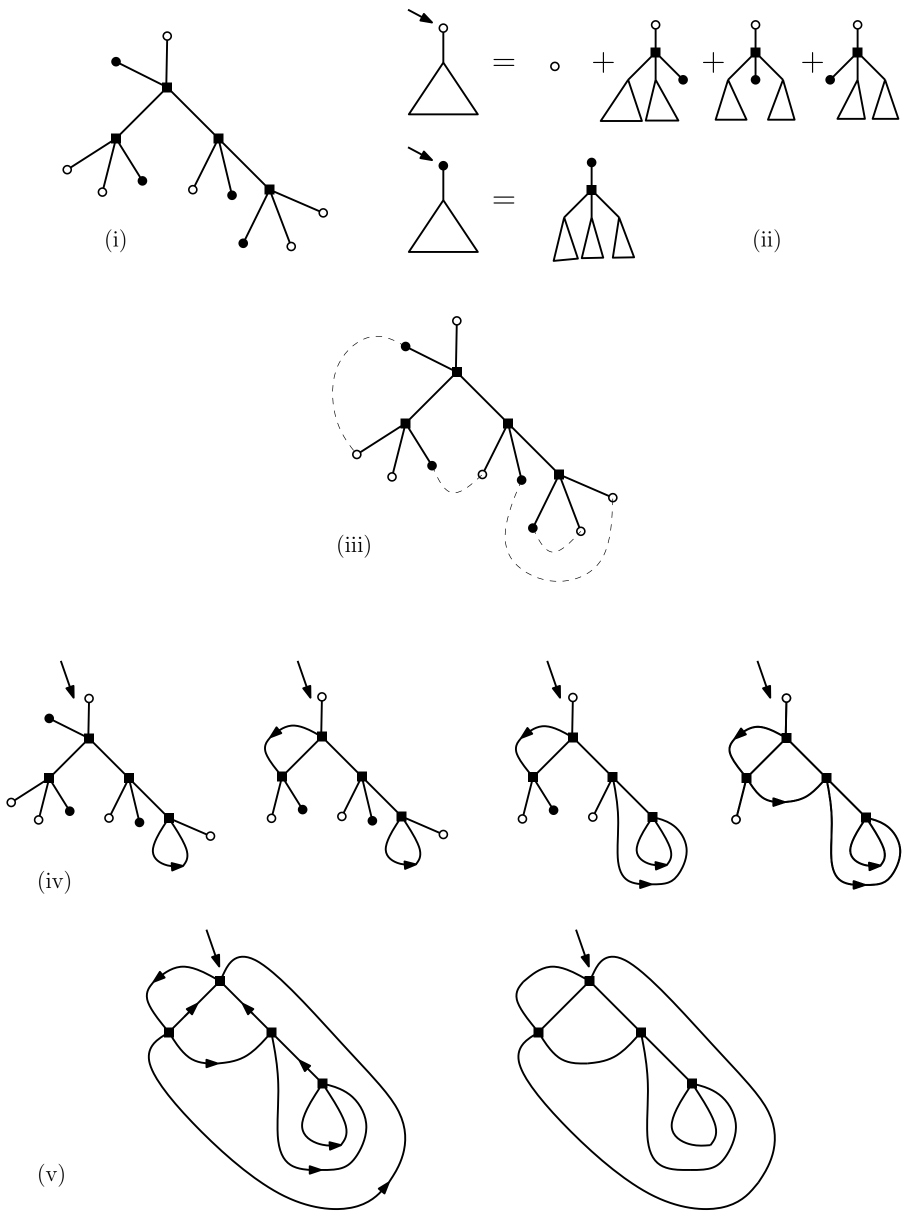

Figure 1.17
(i) An unrooted blossoming tree T. (ii) The decomposition of white and black trees. (iii) The canonical matching of the black and white leaves of T. (iv) The iterative closure of a balanced blossoming tree into a 2-leg tetravalent map. (v) The associated 2-oriented tetravalent maps, and the underlying non-oriented rooted tetravalent map.

decomposition of rooted plane trees applied to blossoming trees rooted at a white leaf matches Equation (1.6), so that the gf of these *white trees* is $T(z)$. Similarly, the gf of *black trees* (blossoming trees rooted at a black leaf) is $zT(z)^3$.

Now the right hand side $T(z) - zT(z)^3$ of Formula (1.5) can be interpreted as the gf of *balanced blossoming trees*: select for each unrooted blossoming tree with n nodes a canonical way to match its n black leaves to n of its $n+2$ white leaves, and declare *balanced* the white trees that are rooted on an unmatched leaf of the underlying unrooted blossoming tree. The gf of balanced blossoming trees is then the difference of the gf of white and black trees. Equivalently the right hand side of Formula (1.3) can be interpreted as follows: a fraction $\frac{2}{n+2}$ of the $\frac{3^n}{n+1}\binom{2n}{n}$ white trees with n nodes are balanced because among the $n+2$ leaves of any unrooted blossoming tree, exactly 2 are unmatched.

The bottom line of this approach is that the requested canonical matching can be performed in a greedy iterative way that preserves planarity and maintains unmatched leaves in the outer face, as illustrated by Figure 1.17: in particular the closure of Theorem 6 describes how to do this. In view of Corollary 5, closure yields in fact a one-to-one correspondence between balanced blossoming trees and some oriented almost tetravalent maps with 2 vertices of degree 1 in the outer face. Upon gluing these two vertices to form a root edge, the correspondence is easily recast as a bijection between balanced blossoming trees with n nodes and rooted tetravalent planar maps with n vertices endowed with a Eulerian orientation without cw-circuit.

Now according to Theorem 3, each rooted tetravalent planar map has a unique Eulerian orientation without cw-circuit, and any Eulerian orientation is accessible. The bijection is therefore between balanced blossoming trees with n nodes and rooted tetravalent planar maps with n vertices. This proves that Formulas (1.3) and (1.5) count rooted tetravalent planar maps with n vertices, and, via the edge map construction, rooted planar maps with n edges.

The well labeled tree approach. This second approach builds instead on some labeled plane trees to interpret Formula (1.6) and involves a double pointing argument to explain Formula (1.7) in terms of rooted planar quadrangulations.

Let a plane tree be *well labeled* if its vertices carry positive integer labels that differ at most by one along each edge, and the minimal label is 1 (see Figure 1.18(i)). Rooted well labeled trees are in one-to-one correspondence with *rooted embedded trees* that have integer labels that differ at most by one and root label 0: given a rooted well labeled tree with root label k, decrease by k all labels to get a rooted embedded tree, and vice-versa, given a rooted embedded tree with minimal label $-\ell \leq 0$, add $\ell+1$ to all labels to get a rooted well labeled tree with root label $\ell+1$.

As illustrated by Figure 1.18(ii), $T(z)$ is the gf of rooted embedded trees counted by their number of edges, or, equivalently, since there are 3 possible variations of labels along each edge, the number of rooted embedded trees with n edges is 3^n times the nth Catalan number $\frac{3^n}{n+1}\binom{2n}{n}$. Consequently, this is also the number of rooted well labeled trees with n edges.

Consider now a planar quadrangulation Q with a marked vertex v_0 and label each vertex v of Q by the number of edges in a shortest path from v to v_0 (see Fig-

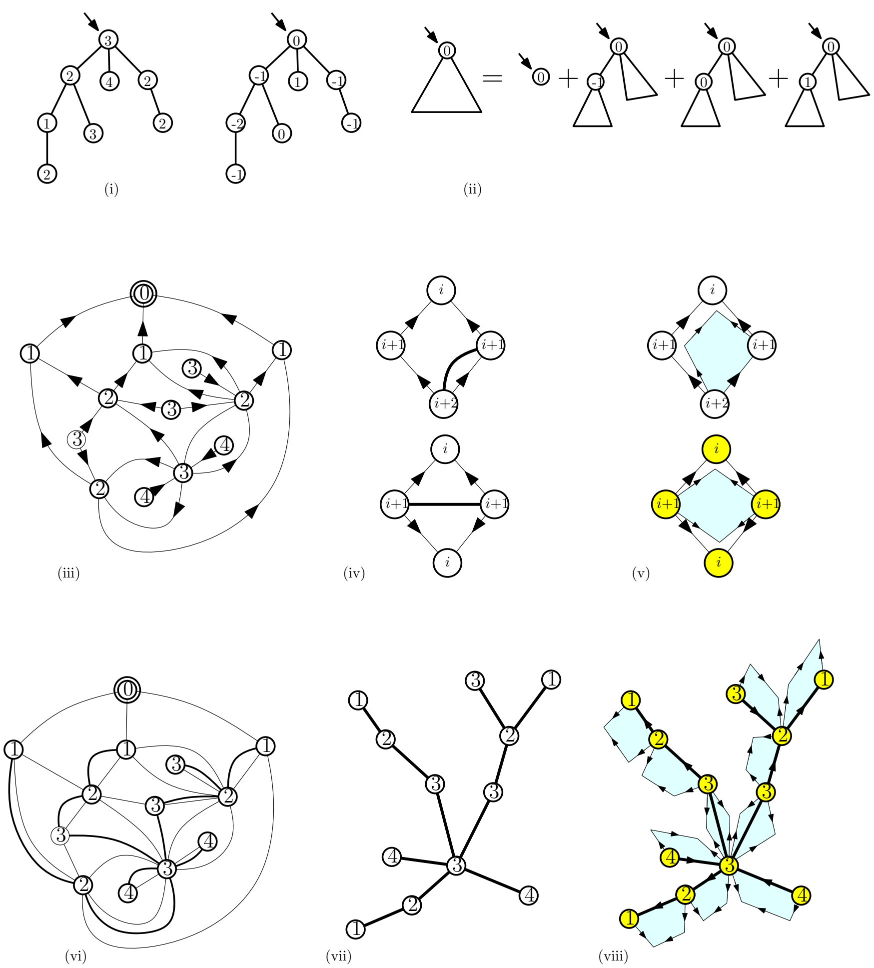

Figure 1.18
(i) A rooted well labeled tree and the associated rooted embedded tree. (ii) The decomposition of rooted embedded trees. (iii) A planar quadrangulation Q with a marked vertex endowed with distance labels, and the associated geodesic orientation. (iv) The local rules defining $\mathscr{T}$. (v) Local configurations in the geodesic splitting. (vi) The construction of $\mathscr{T}(Q, v_0)$. (vii) The resulting tree $\mathscr{T}(Q, v_0)$. (viii) The polyhedral net encoded by $\mathscr{T}(Q, v_0)$.

ure 1.18(iii)). In each face of this labeled quadrangulation, draw a new edge according the rules of Figure 1.18(iv). As suggested by the example in Figure 1.18(vi)-(vii), these new edges form a well labeled tree $\mathscr{T}(Q)$ spanning all vertices of Q except v_0: the fact that these edges form a tree follows immediately from the facts that the rules are not compatible with the existence of a cycle of new edges (because v_0 can only be on one side of the cycle), and that there are n new edges and $n+1$ non-root vertices in Q.

Theorem 10 *The construction $\mathscr{T}$ is a bijection between planar quadrangulations with n faces and a marked vertex and well labeled trees with n edges.*

The fact that construction $\mathscr{T}$ is a bijection can be deduced from Theorem 8. More precisely, when (Q,c) is endowed with the geodesic orientation (which is obviously acyclic and accessible), the splitting of Theorem 8 yields a polyhedral net with quadrangular black polygons, each of which is incident to exactly two white polygons (see Figure 1.18(viii)). The local rules of Figure 1.18(iv) are seen to describe the two possible local configurations in the construction of the polyhedral net, as shown in Figure 1.18(v). These observations are sufficient to prove that $\mathscr{T}$, as a special case of splitting, is injective. To conclude the proof of Theorem 10, one needs to check that folding the polyhedral net described by any well labeled tree yields a quadrangulation endowed with the geodesic orientation from one of its vertices. This follows from the fact that the labels of the vertices of the well labeled trees coincide with the natural geodesic labels on the nodes of the folding tree, as shown by Figure 1.18(vi)-(v).

Since each edge of $\mathscr{T}(Q)$ is drawn in a different face of Q, any local convention allows us to map the $4n$ corners of Q onto the $2n$ corners of $\mathscr{T}(Q)$ canonically to obtain:

Corollary 7 *There is a* 2*-to-*1 *correspondence between rooted planar quadrangulations with n faces and a marked vertex and rooted well labeled trees with n edges. In particular this proves Formula (1.7).*

If v_0 is taken instead to be the root vertex of the rooted quadrangulation Q, then the degree of v_0 is equal to the number of corners with label 1 in $\mathscr{T}(Q)$, and

Corollary 8 *There is a bijection between rooted planar quadrangulations with n faces and rooted well labeled trees with n edges and root label 1 (or non-negative embedded trees).*

This implies that the number of rooted well labeled trees with n edges and root label 1 is given by Formula (1.3), but it is not immediate to use the corollary the other way round because counting directly these rooted well labeled trees with root label 1 is not trivial (a direct proof appears however along with the original description of the bijection in [92]). Our presentation follows the reformulation of [157].

Going further with bijections. The *blossoming tree* and *well labeled tree* approaches have grown from the status of nice bijective alternatives to Tutte's computational method into powerful tools that have led for instance to results about distances

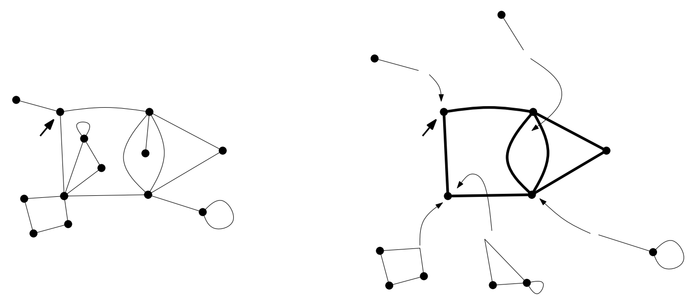

Figure 1.19
A rooted planar map and its decomposition into its non-separable core and submaps attached to corners.

in maps that are currently out of reach of the other methods: see Section 1.3.6 for these developments, and Section 1.4.2 for a further discussion of the combinatorial properties underlying these bijections.

In the meantime, let us conclude this paragraph with the remark that the recurrence formula (1.4) also admits a elegant bijective proof [44], akin to Rémi's bijective proof [152] of the recurrence $(n+1)C_n = 2(2n-1)C_{n-1}$ for the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$.

1.3.4 Substitution relations

The non-separable core of a map and 2-connected planar maps. Following Tutte [165], let us call a planar map *non-separable* if it is either reduced to a single edge (which can be a bridge or a loop), or if it is loopless and 2-connected. Let $\mathscr{C}$ denote the set of rooted 2-connected planar maps. The *non-separable core* of a rooted planar map is the largest non-separable submap containing the root. As illustrated by Figure 1.19, any rooted planar map decomposes bijectively into its non-separable core and a collection of rooted submaps attached to each corner of the core. This immediately yields the substitution equation

$$M^r(z) = 1 + C(zM^r(z)^2), \tag{1.23}$$

where $C(t)$ is the gf of rooted non-separable planar maps counted by edges with variable t (recall that the number of corners in a map is twice the number of edges, hence the substitution $t \to zM^r(z)^2$).

Equation (1.23) allows us to determines $C(t)$ from our knowledge of $M(z)$: Recall that $M(z)$ is an algebraic function, and consider the polynomial P such that $P(M(z),z) = 0$ as given by Formula (1.8). Then the series $H(z) = zM(z)^2$ is a root of the polynomial $Q(t,z) = \text{Resultant}_x(P(x,z), t - zx^2)$. Moreover, $H(z) = z + O(z^2)$ clearly admits a compositional inverse $Y(t)$ in the space of formal power series (that is, the unique formal power series $Y(t)$ such that $H(Y(t)) = t$). This series is algebraic as well as it satisfies $Q(t,Y(t)) = 0$. Finally letting $z = Y(t)$ in $P(M(z),z) = 0$

yields $P(C(t),Y(t)) = 0$, from which we deduce that $C(t)$ is a root of the polynomial $R(x,t) = \text{Resultant}_z(P(x,z),Q(t,z),z)$: explicit computations give $R(C(t),t) = 0$ with

$$R(C,t) = C^3 + 2C^2 + (1-18t)C + 27t^2 - 2t$$

Alternatively the computation can be made using the parametrization (1.5)-(1.6) to obtain directly a parametrization of $C(t)$ as

$$C(t) = 2B(t) - 3B(t)^2 \quad \text{where } B(t) = 1/(1-B(t))^2. \tag{1.24}$$

The number of rooted non-separable planar maps with n edges follows using Lagrange inversion formula,

$$|\mathscr{C}_n| = \frac{4}{2n} \cdot \frac{1}{2n-1}\binom{3n-3}{n-1}.$$

The previous discussion is easily adapted to take into account the number of vertices, and this yields the refined formula

$$|\mathscr{C}_{i+1,j+1}| = \frac{(2i+j-2)!(i+2j-2)!}{(2i-1)!(2j-1)!i!j!},$$

for the number of rooted non-separable planar maps with $i+1$ vertices and $j+1$ faces.

Polyhedral graphs and 3-connected maps. Following Tutte's steps [165], one can go further and define the 3-connected core of a map: this requires however some extra care to classify maps without a 3-connected core. The outcome of his analysis, which we do not reproduce here (see also [infra, Chapter on Planar Graphs, Section 2]), is that any 2-connected map belongs to one of the following three subsets $\mathscr{S}$, $\mathscr{P}$, or $\mathscr{H}$, where

- the set $\mathscr{S}$ of maps that are serial product of maps of $\mathscr{P}$ or $\mathscr{H}$,
- the set $\mathscr{P}$ of maps that are parallel product of maps of $\mathscr{S}$ or $\mathscr{H}$
- the set $\mathscr{H}$ of maps that have a non-trivial 3-connected core.

Then, by definition of $\mathscr{S}$ and $\mathscr{P}$, the gfs $S(t)$, $P(t)$ and $H(t)$ are determined as rational functions of $C(t)$ by the system of equations

$$\left\{\begin{array}{rcl} C(t) &=& S(t)+P(t)+H(t) \\ S(t) &=& \frac{P(t)+H(t)}{1-P(t)-H(t)} \\ P(t) &=& \frac{S(t)+H(t)}{1-S(t)-H(t)}. \end{array}\right. \tag{1.25}$$

Finally, rooted non-separable maps that have a non-trivial 3-connected core can be related to rooted 3-connected maps by a substitution scheme: each such map is indeed uniquely obtained from a rooted 3-connected map in which each non-root edge is

Ten steps to planar graphs

By substitution we went from 1-connected to 2-connected and to 3-connected maps. The enumeration of 3-connected planar maps is a great achievement from the point of view of enumerative graph theory because of Whitney's theorem: a vertex labeled 3-connected planar graph essentially has only one embedding as a vertex labeled 3-connected planar map. We already argued that the exponential generating series of edge labeled planar maps is, up to a derivative, the ordinary generating series of rooted planar maps. In the 3-connected case, a similar relation can be devised for vertex-labeled planar maps so that Equation (1.28) essentially yields an exponential generating series for labeled 3-connected planar graphs.
As shown in [infra, Chapter on Planar Graphs], substitution relations, now written for planar graphs instead of planar maps, can be used the other way round to obtain successively from (1.28) the generating series of labeled 2-connected planar graphs, of labeled connected planar graphs, and of planar graphs.
This up and down approach to the enumeration of planar graphs was first explicitly proposed by Liskovets and Walsh [138] in the form of a ten step program to count unlabeled planar graphs.

replaced by a rooted 2-connected map (the replacement operation of an oriented edge e by a rooted map N in a map M consists in identifying the endpoints of e and of the root of N and removing these two edges). If $G(z)$ denotes the gf of rooted planar 3-connected maps counted by non-root edges (variable z), then the resulting substitution equation reads

$$H(t) = G(C^\dagger(t)) \tag{1.26}$$

where $C^\dagger(t) = \frac{1}{t}C(t) - 2$ is the generating function of rooted planar 2-connected maps counted by non-root edges, $C(t)$ given by Formula (1.24) and $H(t)$ by Equations (1.25). As for Equation (1.23), this substitution equation determines $G(z)$ in terms of $C(z)$. Again, as shown by Mullin and Schellenberg [148], the system of equations can be refined to take into account the number of vertices, and we directly state the bivariate result as

Theorem 11 (Mullin-Schellenberg [148]) *Let $U(z,x)$ and $V(z,x)$ denote the unique pair of formal power series solutions of the system*

$$U = x(1+V)^2 \quad \text{and} \quad V = y(1+U)^2. \tag{1.27}$$

Then

$$G(x,y) = x^2y^2\left(\frac{1}{1+x} + \frac{1}{1+y} - 1 - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\right) \tag{1.28}$$

1.3.5 Asymptotic enumeration and uniform random planar maps

Asymptotic formulas. From the asymptotic counting perspective, the exact results of Sections 1.3.1 and 1.3.4 have direct simple consequences: when n goes to infinity, the number of rooted planar maps with n edges satisfies

$$|\mathscr{M}_n| = c \cdot \rho_0^n n^{-5/2} \cdot (1 + O(1/n)) \tag{1.29}$$

where $\rho_0 = 12$ and c is an explicit positive constant, and for any fixed $\delta \in (-\frac{1}{2}, \frac{1}{2})$, the number of rooted planar maps with n edges and k vertices satisfies

$$|\mathscr{M}_{n,k}| = c_\delta \rho_\delta^n n^{-3}(1 + o(1)), \quad \text{for} \quad k = \lceil n(\tfrac{1}{2} - \delta) \rceil \tag{1.30}$$

where again c_δ and ρ_δ are positive constants that have explicit expressions in term of δ [22], with $\rho_\delta < \rho_0 = 12$ for all $\delta \neq 0$. Asymptotic enumerative results on rooted planar maps are intimately related to the study of the random variable $\mathbf{M}_n$ with uniform distribution on the set of rooted planar maps with n edges. From this point of view Formula (1.30) can be restated as a Gaussian local limit law with linear variance for the number of vertices of $\mathbf{M}_n$, using the expansion $\ln \rho_\delta / \rho_0 = -\alpha \delta^2 (1 + o(1))$ near $\delta = 0$, where α is a positive constant (see [infra, Chapter on Planar Graphs, Section 3] for similar results in the case of planar graphs).

Random maps and face degrees. The root face degree is a natural parameter of $\mathbf{M}_n$ to study: as shown by Equations (1.10)-(1.11), the root face degree $d_f(M)$ plays a distinguished role in Tutte's decomposition, and it follows from his approach that the series

$$M^r(z,u) = \sum_{M \in \mathscr{M}^r} z^{e(M)} u^{d_f(M)},$$

is algebraic and in fact fairly explicit. As a consequence $d_f(\mathbf{M}_n)$, and by duality $d_v(\mathbf{M}_n)$, are quite well known: as n goes to infinity, the degree has a discrete limit law, $D_f(k) = \lim_{n\to\infty} \Pr(d_f(\mathbf{M}_n) = k) > 0$, with exponential decay with k. More precisely, as k goes to infinity,

$$D_f(k) \underset{k\to\infty}{\sim} c'(k/\pi)^{1/2}(5/6)^k$$

where $c' = \sqrt{10}/20$ [110]. Upon selecting uniformly a second root corner at random in $\mathbf{M}_n$, one realizes that the root face degree distribution is the distribution of the degree of the face bordering any random edge in $\mathbf{M}_n$. This rerooting trick makes it possible to derive results about the general local structure of large uniform random maps: for instance [113], the maximum face degree satisfies for large n,

$$\mathbb{E}(\text{max\_deg}(\mathbf{M}_n)) = \frac{\ln n - \frac{1}{2} \ln \ln n}{\ln(6/5)} + O(1)$$

with a variance of order $O(\ln n)$. More precisely, for values of k close to the above expected value, the numbers of faces of degree k behave like independent Poisson random variables.

Submaps and asymmetry. Another example is the number $n_H(M)$ of induced copies of a plane submap H in $\mathbf{M}_n$ (by an induced copy of H in M we mean a simple cycle C such the vertices, faces and edges inside and on C form exactly a copy of

H). In general for a given H there exists a constant c such that the probability that $n_H(\mathbf{M}_n) < cn$ is exponentially small: in other terms, the number of copies of H in $\mathbf{M}_n$ is almost surely linear. Under further technical restrictions on H, the random variable $n_H(\mathbf{M}_n)$ is expected to linear expectation and variance and Gaussian fluctuation (but this is only proved for restricted classes of maps like 3-connected triangulations [114]). The almost sure linearity property implies in turn by a remarkably elegant argument [153] that among rooted planar maps with n edges, the proportion of maps having a non trivial automorphism (of unrooted planar map) is exponentially small (essentially because an automorphism must fix the relative orientations of a large number of copies of any given asymmetric submap H). In particular this allows us to recover independently of Formula (1.22) the asymptotic number of unrooted planar maps as

$$|\mathscr{M}_n^u| = |\mathscr{M}_n^r|/(2n) \cdot (1 + o(\varepsilon^n)) \tag{1.31}$$

for some positive constant $\varepsilon < 1$. It implies furthermore that the random variable $\mathbf{U}_n$ with uniform distribution on unrooted planar maps with n edges behaves asymptotically like $\mathbf{M}_n$ at least for parameters that are independent of the root and polynomially bounded. More precisely if p is an integer-valued parameter of unrooted planar maps such that $p(M) < c \cdot e(M)^\alpha$ for some $\alpha > 0$ and $c > 0$, then the total variation distance between the distribution of $p(\mathbf{U}_n)$ and $p(\mathbf{M}_n)$ is small:

$$\sum_k |\Pr(p(\mathbf{U}_n) = k) - \Pr(p(\mathbf{M}_n) = k)| \underset{n\to\infty}{\longrightarrow} 0.$$

This gives an *a posteriori* further justification of the fact that the literature focuses on the slightly easier problem of enumeration of rooted maps.

Another interesting consequence of the submap density results is the possibility to prove a 0-1 law for the first order logic on planar maps, as pioneered by Bender, Compton and Richmond [23]: roughly speaking one proves that any property expressible in this framework has a probability to be true on $\mathbf{M}_n$ which tends to 0 or to 1 as n goes to infinity.

Separation properties. Continuing with properties that are accessible via explicit bivariate enumeration, another fundamental problem is that of the existence of small cuts in random maps. This problem was first raised in mathematical physics, where $\mathbf{M}_n$ appears as a relevant model of 2d quantum geometry: in this context, the random map is viewed as a random discretized surface, a *2-dimensional quantum universe* [8], and a natural question is whether this random universe is expected to branch into several almost independent parts. One way to formalize the question (with two variants) is to ask about the existence of a vertex v in $\mathbf{M}_n$ whose removal produces at least two components and *(i)* one of the components is a tree of size at least k, or *(ii)* both components have size at least k.

In the first variant *(i)* the probability that there is such a tree-cutting vertex is bounded by the expected number of tree-cutting vertices, itself bounded by

$$\sum_{i=k}^{n-1} \frac{2nT_iM_{n-i}}{M_n} \approx \sum_{i=k}^{n-1} \frac{n^{7/2}}{i^{5/2}(n-i)^{5/2}} \frac{4^i 12^{n-i}}{12^i} \approx 3^{-k},$$

where we have used the asymptotic formula (1.29) and the classic approximation $C_n \approx 4^n n^{-3/2}$ for the number of rooted plane trees with n edges. In particular the above bound is exponentially decreasing with k and one can prove that the size of the largest induced subtree in $\mathbf{M}_n$ is $\Theta(\ln n)$. The precise analysis is made possible by the existence of a substitution scheme analog to those of Section 1.3.4: Removing all induced subtrees from a map M (or equivalently removing iteratively all vertices of degree one that are not incident to the root edge) yields its *1-core* $C^1(M)$, which is a rooted planar maps without non-root vertex of degree 1. This decomposition yields the gf of rooted planar maps counted by number of edges (variable z) and by number of edges in the 1-core (variable u)

$$M_T(z,u) = 1 + C^1(uzT(z)^2), \tag{1.32}$$

where $C^1(z)$ is the gf of rooted planar maps without non-root vertex of degree 1, and this equation for $u = 1$ determines the gf $C^1(z)$. Then the singular analysis of the bivariate substitution scheme (1.32) in the sense of [104] makes it possible to show that $C^1(\mathbf{M}_n)$ has expected size αn for some constant α with Gaussian fluctuations in the range $n^{1/2}$. In the quantum gravity literature, this situation received the evocative description of that of a *unique mother quantum universe* from which logarithmic size *tree-like baby universes* emerge.

In the second variant *(ii)* the probability that the separation is possible is bounded by

$$\sum_{i=k}^{n-k} \frac{2nM_iM_{n-i}}{M_n} \approx \sum_{i=k}^{n/2} \frac{n^{7/2}}{i^{5/2}(n-i)^{5/2}} \approx n \cdot k^{-3/2}.$$

This suggests, and it can actually be proved [112], that at most one 2-connected component of $\mathbf{M}_n$ has linear size (the *mother universe*) and the second largest component (the *largest baby universe*) has size $k = \Theta(n^{2/3})$. The substitution Equation (1.23) refines into the bivariate equation

$$M(z,u) = 1 + C^{\mathrm{ns}}(uzM(z)^2) \tag{1.33}$$

whose analysis show more precisely [16] that the largest non-separable component $C^{\mathrm{ns}}(\mathbf{M}_n)$ has expected size $n/3$ with fluctuations in the range $n^{2/3}$ given by a stable law of index $2/3$.

The qualitative difference between variants *(i)* and *(ii)* is directly explained by the fact that plane trees have a strictly lower growth constant than planar maps. Equivalently, at the technical level, the difference is between subcritical and critical compositions of gfs (see [104, Chapter VI.9]). In both cases, the existence of a bivariate substitution scheme to describe the decomposition of $\mathbf{M}_n$ into a mother universe and baby universes, implies that the mother universe, conditionally to its size, is itself uniformly chosen among the corresponding set of maps: $C^{\mathrm{ns}}(\mathbf{M}_n)$ is a uniform random 2-connected map with $\mathbf{m}$ edges, where the number $\mathbf{m}$ of edges is a concentrated random variable.

The above discussion concentrate on submaps that can be separated by a unique cut vertex, but it can be extended to consider 2-vertex separators, and, qualitatively, the results is expected to hold for fixed separating cycles of fixed length (although in that case there is some ambiguity in the definition of the *mother universe*). A natural next step is to ask about the length of a shortest cycle allowing to separate a random planar map into two big regions (*i.e.* so that there are at least αn edges on each side of the cycle). The answer to this question requires a detour via the study of the intrinsic geometry of random planar maps.

1.3.6 Distances in planar maps

The intrinsic geometries of planar quadrangulations and planar maps. Given two vertices x and y in a map M, let $d_M(x,y)$ denote their distance, that is the minimal number of edges in a path from x to y. The *intrinsic geometry* of M is the metric space (V_M, d_M), where V_M denotes the set of vertices of M. In order to study the intrinsic geometry of $\mathbf{M}_n$ a natural first step is to consider the distances between two random vertices, or between the root vertex and a random vertex.

This can be done thanks to the two above mentioned bijections between maps and trees. In particular our presentation of the bijection $\mathscr{T}$ of Theorem 10 clearly shows that all the distances to the marked point in a rooted quadrangulation with a marked vertex are encoded in the labels of the associated rooted well labeled tree. This gives access to the intrinsic geometry of $\mathbf{Q}_n$, the r.v. with uniform distribution on the set of rooted planar quadrangulations with n faces. As a consequence most of the results on distances in the literature are stated for this related but *a priori* different model of random map. Let us temporarily ignore this problem and discuss the results in terms of $\mathbf{Q}_n$.

From a probabilistic point of view the sole fact that labels of rooted well labeled trees can be interpreted in terms of distances in quadrangulations is already extremely fruitful: let $\mathbf{T}_n$ denote a random variable taken uniformly at random in the set of rooted well labeled trees with n edges. The random well labeled tree $\mathbf{T}_n$ is obtained by shifting the labels of a random rooted embedded tree $\mathbf{E}_n$ by their minimum. The tree $\mathbf{E}_n$ can be constructed in two steps, first taking a uniform random plane tree with n edges, and then choosing the label increment on each edge in $\{-1,0,+1\}$ uniformly and independently. As a consequence a uniform random vertex $\mathbf{v}$ in $\mathbf{E}_n$ is typically at distance $\ell = \Theta(n^{1/2})$ of the root in the tree (by standard results on the height profile of trees) and its label $\lambda_{\mathbf{T}_n}(\mathbf{v})$ is a sum of ℓ i.i.d. random variables taken uniformly in $\{-1,0,1\}$: it is thus almost surely of order $n^{1/4}$, as n goes to infinity. It can be proved that this is also almost surely the case for the minimal label, so that the statement remains true for $\mathbf{T}_n$ [87]. In terms of distances in $\mathbf{Q}_n$, we conclude that the typical distance between two random vertices is of order $n^{1/4}$. This combinatorial derivation of the typical distance in $\mathbf{Q}_n$, proposed in [87], confirms earlier semi-rigorous prediction of the physics literature (see [9] and reference therein).

Exact counting results for labeled trees and distances in quadrangulations. At the enumerative level, results can be made even much more precise. For all $i \geq 0$, let T_i be the gf of rooted embedded trees with label strictly larger than $-i$. Then the standard recursive decomposition of rooted plane trees can be refined to write the following system of equations for the infinite family of gfs $(T_i)_{i\geq 0}$:

$$T_i = 1 + zT_i(T_{i-1} + T_i + T_{i+1}) \text{ for } i \geq 0, \quad \text{and} \quad T_0 = 0. \tag{1.34}$$

A true miracle is that the system of equations (1.34) can actually be solved exactly. As first shown by Bouttier, Di Francesco and Guitter in [63],

$$T_i(z) = T\,\frac{(1-Y^i)(1-Y^{i+3})}{(1-Y^{i+1})(1-Y^{i+2})} \tag{1.35}$$

where $Y \equiv Y(z)$ and $T \equiv T(z)$ are the unique power series solutions of

$$Y = zT^2(1+Y+Y^2) \quad \text{and} \quad T = 1+3zT^2. \tag{1.36}$$

Observe that the series T in this expression is the same as in Equation (1.6): accordingly the limit when i goes to infinity of T_i is just T. While Formula (1.35) was guessed and checked in [63], it was recently observed by Eynard and Guitter that the system of equations (1.34) belongs to a family of equations known to admit explicit computable solution in a different context [100].

The bijection between rooted well labeled trees with root label 1 (or non negative rooted embedded trees) and rooted quadrangulations implies that $M(z) = T_1(z)$ and

$$M(z) = T\,\frac{(1-Y)(1-Y^4)}{(1-Y^2)(1-Y^3)} = T - \frac{Y}{1+Y+Y^2}T = T - zT^3.$$

in agreement with Formula (1.5). The general explicit expression (1.35) has far reaching implications for distance statistics on quadrangulations, as discussed in [63, 64, 67, 69, 68, 122] (see also the results mentioned below for distances in $\mathbf{M}_n$).

Finally, let us mention that the combinatorial structure of Formula (1.35) is still not completely clear: from the point of view of quadrangulations, an appealing interpretation of Y and $T_i(z)$ was given in [71] (see also [3]) in terms of *slices* of quadrangulations, from which the explicit expressions can be understood. However we still miss a direct explanation of Formula (1.35) in terms the natural interpretation of Y/z as gf of well labeled trees with a marked branch whose labels form an excursion.

Exact results for distances in planar maps. Quite unexpectedly, many of the exact results for quadrangulations can be easily transferred to planar maps: as already indicated above, the classical incidence map transformation does not help much for this, but the solution is provided by another bijection between marked quadrangulations and marked planar maps that was recently discovered by Ambjørn and Budd [7] (see also [66] for a broader discussion of the relation between this bijection and well labeled tree approach). As illustrated by Figure 1.20 this bijection consists in applying to a quadrangulation Q with a marked vertex v_0 the rules opposite to those of Figure 1.18(iv).

The two key properties of this bijection for our purpose are that (i) it preserves the distance to v_0 of the common vertices of the two maps, namely if $(M, \bar{v}_0) = AB(Q, v_0)$ then $d_Q(v, v_0) = d_M(\bar{v}, \bar{v}_0)$, and (ii) the vertices of Q that do not appear in M are local

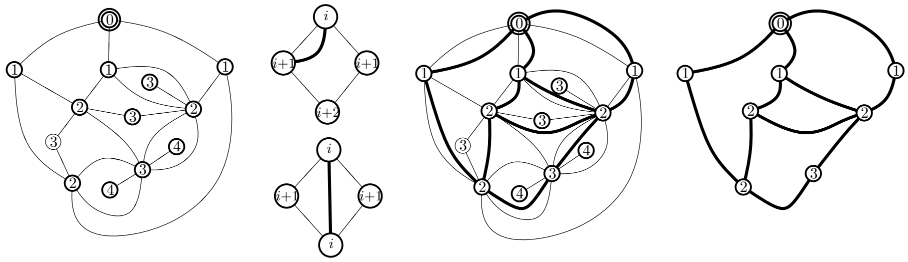

Figure 1.20
A quadrangulation Q with distances to a marked v_0, the Ambjørn-Budd rules, and their applications to (Q, v_0).

maxima of the distance to v_0 in Q, and they correspond to vertices whose labels are local maxima in the associated well labeled tree. As a consequence of these two properties the labels of a random well labeled tree that are not local maxima describe the distances to a random marked vertex in a random planar map.

Let us illustrate by an example the very explicit results on the intrinsic geometry of random planar maps that are derived in [66]. Let us say that the root edge of a vertex-pointed rooted planar map M has type (i, j) if the root vertex is at distance i of the pointed vertex, and the other extremity of the root edge is at distance j of the pointed vertex (by definition $|i-j| \leq 1$). Let then $R_i(z)$ denote the gf of pointed rooted maps with a root edge of type $(j-1, j)$ for $j \leq i$. Upon reverting the root, $R_{j+1}(z)$ is the gf of such maps with a root edge of type $(j+1, j)$ for $j \geq i$, and let $S_j^2(z)$ denote the gf of such maps with a root edge of type (j, j) for $j \leq i$ (for consistency with the literature this gf is written as a square). Then the combination of the well labeled tree approach and the AB bijection is that R_i and S_i^2 can be written for all $i \geq 0$ as

$$R_i = zT_iT_{i+1} \quad \text{and} \quad S_i^2 = tT_i^2,$$

where the T_i are given by the system of equation (1.35). For completeness let us state the explicit expressions for the R_i and S_i,

$$R_i(z) = (1+z(1+T)^2)\frac{(1-Y^{i+1})(1-Y^{i+3})}{(1-Y^{i+2})^2} \tag{1.37}$$

$$S_i^2(z) = z(1+T)^2\left(\frac{(1-Y^i)(1-Y^{i+3})}{(1-Y^{i+1})(1-Y^{i+2})}\right)^2 \tag{1.38}$$

In particular these explicit expressions allow to compute distance statistics for $\mathbf{M}_n$: for instance, for any fixed $i \geq 1$, the expected number of vertices at distance i of a uniformly chosen vertex in $\mathbf{M}_n$ is

$$\lim_{n\to\infty} \mathbb{E}(|\{v \mid d_{\mathbf{M}_n}(v, \mathbf{v_0})\} = i|) = \frac{3}{280}(2i+3)(10i^2+30i+9). \tag{1.39}$$

Similarly the limit for large n of the expected number of vertices in the ball of radius i around a uniform random vertex of $\mathbf{M}_n$ grows like i^4.

1.3.7 Local limit, continuum limit

Local limits. A nice way to subsume the above results on the local structure of random planar maps is the following statement: Large uniform random rooted planar maps have a local limit, the *uniform infinite planar map*, which has Hausdorff dimension 4. The first part of this statement means that, as n goes to infinity, for any fixed k, the submap of radius k around the root of $\mathbf{M}_n$ converges in distribution to a random planar map $\mathbf{P_k}$ with radius k, and the family of random planar map $(R_k)_{k\geq 1}$ coherently defines an random infinite (but locally finite) planar map $\mathbf{P}$. The second part means that in the random infinite planar map $\mathbf{P}$ the number of vertices at distance at most i of the root roughly grows like i^4. This statement was first stated and proved in the case of random triangulations and quadrangulations [11, 10, 86], leading to the definition of the Uniform Infinite Planar Triangulation and Quadrangulation (UIPT, UIPQ), but again, the Ambjørn-Budd bijection makes it possible to transfer the results to general planar maps to define the UIPM.

A natural way to continue the study of the local properties of large planar maps is to concentrate on these limit uniform infinite planar maps. We will however not discuss further this direction for at least two reasons: many of the methods involved there escape from the strict range of enumerative combinatorics, and the topic is currently evolving at a very fast pace. We refer the reader to the elegant presentations of Nicolas Curien.

The continuum limit. Instead of concentrating on local properties, one can return to the observation that the distance between the root and a random vertex of $\mathbf{M}_n$ is of order $n^{1/4}$. The correspondence between quadrangulations and well labeled trees makes it possible to show more precisely that the *rescaled profile* (average number of vertices at distance k of the pointed vertex) and *radius* (maximal distance to the pointed vertex) of uniform random rooted planar quadrangulation with a random marked vertex converge upon renormalizing the distances by a factor $n^{-1/4}$ to functionals of a well studied continuum random process called the *Integrated Super-Brownian Excursion (ISE)* or of its variant the *Brownian snake* [87].

The example of the convergence of rescaled simple random walks to the Brownian motion and that of rescaled simple trees to the Continuum Random Tree then suggest that after rescaling distances by such a factor $n^{-1/4}$, one should look for a continuum limit of random planar quadrangulations. This question has attracted a lot of attention in the last few years, that has culminated with the proof of the existence and uniqueness of such a continuum limit, the *Brownian planar map* [131, 142]. A key ingredient underlying these results is the fact that the bijective correspondence between well labeled trees and pointed rooted maps is robust enough to go through the process of taking a continuum limit: the only currently known explicit construction of the *Brownian planar map* consists indeed in starting from the continuum limit of rescaled uniform random well labeled trees of size n when $n \to \infty$, the above mentioned *Brownian snake*, and applying a continuum analog of the discrete bijection to define a metric on this embedded continuum tree. Again the initial constructions of [131, 142] were done for quadrangulations, but the Ambjørn-Budd correspondence makes it possible to transfer most of these results to the intrinsic geometry of uniform random planar maps with n edges [45].

Remarkably it has been proved that the Brownian planar map is a random metric space with the topology of the sphere [133, 140]: a consequence of this last statement is the fact that for any positive $\alpha < 1/2$, the shortest cycle on $\mathbf{M}_n$ with at least αn edges on each side has almost surely length $\Theta(n^{1/4})$ when n goes to infinity. Although, as discussed above, the enumerative results and in particular the bijections between maps and trees are fundamental ingredients of construction of the Brownian planar map, these consequences about shortest α-separating cycles currently appears to be out of reach with purely enumerative methods (see however [68] for a tractable variant of the problem). Further discussion of these results is thus largely out of scope here and we refer to the [132, 143, 144] for a survey of these developments, or to [129, 130] for short reviews.

1.4 Beyond planar maps, an even shorter account

At this point we are more or less done with general planar maps, and there are (at least) four ways to go further: universality, master theorems, maps on surfaces, and decorated maps.

1.4.1 Patterns and universality

A first direction to explore is the observation that literally all of the above results admit variants for various natural subfamilies of planar maps. In the combinatorial literature, *patterns* in the asymptotic behavior have been observed, and equivalently, in the physics and probabilistic literature some *critical exponents* are expected to be *universal*.

Pattern in the asymptotic behavior and universality. A first example is the fact that for a large collection of subfamilies $\mathscr{F}$ of maps for which the asymptotic number of rooted planar maps with n edges in $\mathscr{F}$ is known, the polynomial growth exponent takes the same value $\gamma = -\frac{5}{2}$: let $\mathscr{M}_n^{\mathscr{F}}$ denote the set of rooted planar maps with n edges in the subfamily $\mathscr{F}$, then

$$|\mathscr{M}_n^{\mathscr{F}}| \underset{n\to\infty}{\sim} c_{\mathscr{F}} \rho_{\mathscr{F}}^n n^{-5/2} \tag{1.40}$$

for all admissible values of n, where $c_{\mathscr{F}}$ and $\rho_{\mathscr{F}}$ are constants depending on $\mathscr{F}$. This asymptotic behavior holds in particular for various families of planar maps defined by combinations of a finite restriction on the allowed vertex or face degrees (but not restrictions on both), a finite restriction on the girth (length of the shortest simple cycle), or an irreducibility condition (a map is irreducible it the length of the shortest non-facial cycle is strictly larger than its girth), and for some of these families, with a further bipartiteness constraint [26, 27, 18, 109, 81, 31]. Similar patterns or universal exponents hold for each of the asymptotic/probabilistic statements of the previous section about degree distribution [136], absence of non-trivial automorphisms [24], separation properties [155, 112, 16], distances [70]...

A quite natural way to unify these results is to state the universality of the Brownian planar map as a continuum limit: for all "reasonable" family $\mathscr{F}$ of planar maps, there should exists a constant $\alpha_{\mathscr{F}}$ and a definition of the size such that uniform random planar maps of size n in $\mathscr{F}$ with distances rescaled by a factor $\alpha_{\mathscr{F}} n^{-1/4}$ converge to the Brownian map in the sense of [131, 142]. The collection of families of "reasonable" maps for which this statement has actually been proved is more restricted than for the previous ones but it has recently grown fastly to include planar quadrangulations and more generally $2p$-angulations, general, simple, and bipartite planar maps, and simple triangulations [1, 2, 4, 45], and all the above mentioned families are expected to belong to the same *universality class*.

A common point of these families is that they are defined by constraints that can be checked locally (at finite distance in the derived map). From this point of view examples of "non-reasonable" subfamilies are outerplanar maps because the existence of a large outerface is not a local constraint, series-parallel maps or stack triangulations whose characterization needs to be checked recursively. And indeed these families lead to other continuum limits [5, 78]. A different example is that of planar maps with a global rotation or reflection symmetry whose continuum limit is expected to conserve the initial symmetry: they do not even satisfy the initial asymptotic pattern because, in view of Section 1.3.2, they are in bijection quotient maps, that are multiply rooted planar maps. Finally let us observe that too stringent restrictions on both vertex and face degrees cannot either lead to the same universality class due to rigidity constraints: for instance triangulations with too many vertices of degree 6 have to be constituted of patches of regular triangular lattices. The question of whether a family of maps belongs or not to some universality class is a key issue in the physics literature on maps, and in particular on decorated maps: accordingly we shall return briefly to this question in Section 1.4.4.

Parametric families and meta theorems. A different kind of recurrent statements in the theory of maps is illustrated by the observation that many families of rooted planar maps defined by local restrictions have algebraic gfs. Here the universality is more questionable and it is in fact not difficult to design "reasonable" families of rooted planar maps that most likely do not have algebraic gfs and yet will satisfy most other universality properties: the family of rooted planar maps with only prime vertex degrees seems an obvious candidate, and several families of non-critical decorated maps, as discussed in Section 1.4.4, provide other examples.

One way to circumvent the difficulty of defining universality is to obtain general parametric results of the form of Equation (1.20)-(1.21). Tutte's equation can be generalized to bicolored maps counted by number of faces of degree i of each color [56]. Similar parametric equations have also been written for families of rooted non-separable planar maps [173]. However for 3-connected maps and other families of maps defined by girth restrictions, equations with catalytic variables are harder to establish from the root edge deletion approach (see for instance [46] for the already quite intricate case of 3-connected maps counted by edges and root face degree).

General parametric results have instead been obtained by Bouttier and Guitter via a far reaching and beautiful extension of the substitution schemes of Section 1.3.4 to the gfs of maps with girth g and d-irreducible maps with respect to the distribution of face degrees [73, 72]. Through standard transformations like duality, incidence map, edge map, and a few others, their theorem encompass all the known critical substitution schemes summarized in [16] and allows us to recover easily the enumeration of rooted planar 5-connected triangulations of [111].

Another way to assert some form of universality could be to resort on logic and prove a *meta theorem*: Let f be a formula of the first order logic on maps (with quantifiers on vertices, faces, edges and corner, and adjacency/incidence predicates). It is tempting to conjecture that the family $\mathscr{M}^f$ of rooted planar maps M such that $f(M)$ is true has an algebraic gf, and that the number of such maps satisfy the universal asymptotic pattern above. (The second part of the statement is actually more likely to be true than the first, but probably hard to prove independently.) This approach is reminiscent of the 0-1 laws briefly discussed in Page 36, but appears to requires new technical ingredients.

1.4.2 The bijective canvas and master bijections

The initial motivation for looking for bijective proofs for the formulas of Tutte was closely related to Schützenberger's methodology according to which combinatorial structures with algebraic gfs should admit natural encodings by words of context free languages (or equivalently, they should be in natural correspondence with simple families of trees). Accounts of early attempts in this directions are [88, 89]. A recent discussion (and partial refutation) of the statement that algebraic gfs should be the trace of such natural encodings can be found in [53].

In the case of planar maps, most of the known gf algebraicity results do admit derivations by bijections with some trees. Rather than universality results, master bijections have been proposed [6, 38, 39, 40], that allow to derive these bijections in a unified way. A beautiful outcome of these results is that they make explicit the structural properties of maps that make the existence of such bijections possible.

Blossoming trees and well labeled trees made parametric. The first extensions of the original two bijections of Section 1.3.3 that were proposed were rather adhoc variants for specific subclasses like bipartite maps [12, 157], loopless, simple or irreducible triangulations or quadrangulations [157, 150, 151, 107, 105, 106] (see also [93] for an independently found recursive variant for non-separable maps). In all these cases, known formulas were instrumental to help guessing an adequate family of balanced trees, which was then used to design a bijection.

A first parametric extension of the blossoming tree approach was to the case of planar constellations in [57]: although no previous enumerative results were available for this case, conjecturing a formula was again a preliminary to the design of the right family of trees. It is yet a first example of a result that is actually much harder to derive from Tutte's root deletion approach [56] than from bijections.

Similarly, while basic blossoming trees and well labeled trees are natural combinatorial interpretations of Equation (1.6), parametric extensions were build as interpretations of the system (1.21). This approach was then extended further to bicolored maps counted by number of faces of degree i of each color [58], building on the enumerative indications given by partial enumerative results for trivalent and tetravalent maps [61]. With this example the paradigm started to change and the bijective approach yields new formulas that were not even conjectured before.

A structural result about orientations. This shift of paradigm lead to the question of understanding under what conditions bijections between maps and trees can work. In view of Section 1.2, the bijections of Section 1.3.3 are seen to rely on identifying an easy to enumerate family of decorated trees whose closure yields the expected family of maps by a specialization of the bijections of Theorems 6 and 8. Conversely, given a family of planar maps, these bijections rely on identifying the right notion of canonical spanning tree for which these theorems lead to a simple family of decorated trees. A first step is Theorem 7 which suggests that in order to find the right canonical spanning trees, one should look for accessible orientations without cw-circuit. This in turn is made easier by Theorem 3, that assert that it is sufficient to look for an accessible feasible function on our family of maps.

The key structural result is now the fact that the existence of feasible and accessible α-functions is a natural graph theoretic property, expressible in terms of girth conditions. A most general result of this type is given by Bernardi and Fusy in [40] for what they call *fittingly charged hypermaps*. We only quote here a subcase of their classification taken from [38, 39]:

Theorem 12 *A planar d-angulation has girth d if and only if it admits an accessible $(d+2)/d$-fractional orientation.*

The exact definition of fractional orientation is out of the scope of this text, but the key point is that the theory of α-orientations extends to these fractional orientations, and in particular the fact that there is a unique such orientation without cw-circuit. It suggests that, at least for those classes of maps, the quest is over: instead of guessing the right family of tree from enumerative formulas, one can start from the naturally associated feasible α function and work the other way round.

The bijective canvas. The resulting canvas for bijection between maps and trees is the following: start with a subfamily $\mathscr{F}$ of rooted planar maps defined by degree and girth conditions.

1. Reformulate the degree and girth conditions in terms of the existence of some generalized α-orientations (using *e.g.* Theorem 12).

2. For a given α, the set of generalized α-orientations on a given map has a lattice structure, and in particular there is a unique such orientation without cw-circuit (Theorem 3).

At this point there are two options: The first option generalizes the blossoming tree approach and was formalized in [6] (see also [158]).

3a. Use ccw-exploration as in Section 1.2.2 to obtain a canonical spanning tree T of the map M, and check that external edges can be encoded as decorations of this tree T in such a way that the resulting set of decorated trees can be described by local rules, directly inherited from the degree and α-constraints.

In the case of tetravalent maps, we have already seen that this approach directly yields the blossoming trees of Section 1.3.3: a planar map is tetravalent if and only if it admits a 2-orientation (as particular case of Euler characterization of Eulerian maps); in particular a tetravalent map admits a unique 2-orientation without cw-circuit; the cw-exploration of this orientation yields a spanning tree whose opening has vertices of degree 4 with in and out-degree 2; by construction, apart from the root vertex, each vertex has one outgoing edge toward the root, so that the other outgoing edge must be a dangling half-edge, and each vertex has two incoming edges or half-edges: this provides, without guessing, the definition of blossoming trees of Section 1.3.3.

Another example is the case of simple triangulations (that is, triangulations with girth three): as first shown by Schnyder [161] a planar map is a simple triangulation if and only if it admits a 3-orientation ; in particular a simple triangulation admits a unique 3-orientation without cw-circuit; the cw-exploration of this orientation yields a spanning tree whose opening has two dangling outgoing half-edges per vertices; the face degree condition implies that incoming half-edges are not necessary to perform the closure; the resulting family of blossoming tree is exactly the family of plane trees such that each inner vertex is incident to two leaves and this allows us to recover the bijection first proposed in [149].

The second option generalizes the well labeled approach and was formalized in [38, 39, 40].

3b. Use vertex blowing and splitting as in Section 1.2.3 to obtain a balanced polyhedral net M, and again, check that, apart from the fact of being balanced, all the constraint on the resulting set of polyhedral nets can be described by local rules inherited from the degree and α-constraints.

In the case of quadrangulations, we have already seen that using the geodesic distance to orient the map directly yields the bijection of Section 1.3.3 with well labeled trees. The extension of the bijection to general bicolored maps, known as Bouttier, Di Francesco, Guitter's bijection [60] is recovered as well using the orientation induced by oriented geodesics and a similar local analysis of the possible configurations around black and white polygons: the mobiles of [60] naturally arise as simplified descriptions of the resulting polyhedral nets.

This second approach yields the currently most general master theorem [40]: It extends in particular to hypermaps with *ingirth d* (that naturally generalize maps with girth d) and in this context gives a very general set of parametric equations for hypermaps with ingirth d and outerface of degree d counted by degrees of black and white faces, that extends Equations (1.20)-(1.21). As already mentioned the most general statement involves some *fittingly charged hypermaps* that arise as the natural setting to which the techniques of proof underlying Theorem 12 apply.

Finally we should mention that a third bijective canvas arises from the extension of the substitution scheme approach proposed in [73]: this last approach has the advantage that it yields non-trivial but directly $\mathbb{N}$-algebraic decompositions for planar maps with a boundary, as first proposed in [71, 3].

Bijections with other combinatorial structures. Tutte's formula (1.3) for the number of rooted planar maps with n edges, and its variants for rooted non-separable planar or triangulations have such a simple closed form that one should expect *a priori* unrelated combinatorial structures to be counted by the same numbers. This is indeed the case and bijections have been devised to explain some of these coincidences, in particular in the study of permutations with forbidden pattern (see [119] and references therein). A higher level "explanation" is that pattern avoiding permutation tend to admit natural decompositions leading to polynomial equations with catalytic variables, and we have seen that the simplest of those equations count maps... Exhibiting structurally identical recursive decompositions for two equinumerous families of combinatorial structures is a natural way to get a (recursive) bijection and this approach has been fruitful in this context (see [47] and ref therein). Another remarkable example is that of Baxter permutations and intervals in the Catalan and Tamari orders (see [108] and reference therein).

1.4.3 Maps on surfaces

The third way to go is to move to *maps on surfaces*, that is, map on the torus, or more generally on an oriented or non oriented surface of genus g. The main idea is that, *mutatis mutandis*, many results have analogs for maps on oriented and also on non-oriented surfaces but actual statements are more complicated, and proofs much more technical.

Exact and asymptotic counting. A particularly nice result is that Formula (1.29) is replaced for rooted map on orientable surfaces of genus g by the more general

$$|\mathscr{M}_n^g| = c \cdot \tau_g \cdot 12^n n^{\frac{5}{2}(g-1)} \cdot (1 + O(1/\sqrt{n})), \tag{1.41}$$

where the limit is to be taken at g fixed and n going to infinity [19] (for similar results in the physics literature see references in the survey [94]). This result is a consequence of a more technical statement giving the general form of the gf of these maps [19, 20, 14]: there exists a family of polynomials $P_g(x)$ such that

$$M^g(z) = \frac{P_g(T)}{(2 - T(z))^{3g}}, \quad \text{where} \quad T(z) = 1 + 3zT(z)^2 \tag{1.42}$$

As usual, this statement can be refined to take into account the number of vertices [22]. Earlier statements for small values of g were already given in [74, 42], and probably the general form of the result was known to the physics community in the early 80's as well but we could not trace the reference yet. Recently, Eynard and Orantin have proposed a more explicit expression of the series $M^g(z)$ as a finite sum of resultants of rational functions in $T(z)$ indexed by some simple trivalent graphs, which they derive from an explicit *topological recurrence* [97, 101]. The statement of this results requires however too many notations to be reproduced here.

But maybe the most surprising result is the following simple quadratic recurrence recently discovered by Carrell and Chapuy [80]:

$$\begin{aligned}(n+1)M_n^g = 4(2n-1)M_{n-1}^g + \frac{(2n-3)(2n-2)(2n-1)}{2}M_{n-2}^{g-1} \\ +3\sum_{\substack{k+\ell=n\\ k,\ell\geq 1}}\sum_{\substack{i+j=g\\ i,j\geq 0}}(2k-1)(2\ell-1)M_{k-1}^{i}M_{\ell-1}^{j},\end{aligned} \tag{1.43}$$

for $n \geq 0$, $g \geq 0$, with initial condition $M_0^0 = 1$, and $M_n^g = 0$ for $n < 0$ or $g < 0$. This recurrence even admits a barely more complicated refinement for the number $M_{i,j}^g$ of rooted planar maps with i vertices and j faces on a surface of genus g:

$$\begin{aligned}(n+1)M_{i,j}^g = 2(2n-1)(M_{i-1,j}^g + M_{i,j-1}^g) + \frac{(2n-3)(2n-2)(2n-1)}{2}M_{i,j}^{g-1} \\ +3\sum_{\substack{i_1+i_2=i\\ i_1,i_2\geq 1}}\sum_{\substack{j_1+j_2=j,\\ j_1,j_2\geq 1}}\sum_{\substack{g_1+g_2=g\\ g_1,g_2\geq 0}}(2n_1-1)(2n_2-1)M_{i_1,j_1}^{g_1}M_{i_2,j_2}^{g_2},\end{aligned} \tag{1.44}$$

for $i, j \geq 1$, with the initial conditions that $M_{i,j}^g = 0$ if $i+j+2g < 2$, that if $i+j+2g = 2$ then $M_{i,j}^g = \mathbf{1}_{\{(i,j)=(1,1)\}}$, and where we use the notation $n = i+j+2g-2$, $n_1 = i_1+j_1+2g_1-1$, and $n_2 = i_2+j_2+2g_2-1$.

While Formula (1.41) arises from (non-trivial) refinements of the gf techniques traditionally used to study planar maps, it is worth observing that the proof of Formula (1.43) requires more algebraic ingredients based on the encoding of maps in terms of permutations briefly used in formalizing Definition 1.2 (see [116, 80], the earliest reference we could found is [125] in the combinatorial literature and [42, Appendix 6] in the physics literature, both in the case of unicellular maps). Another remarkable outcome of the approach of [116, 80] is the possibility to derive from it detailed asymptotics for the constant τ_g [25]. Analogous results for non-orientable surfaces have also appeared [19, 109, 79].

Unrooted maps in higher genus surfaces have also been considered, see [170] and reference therein for exact results, and asymptotic asymmetry results like Formula (1.31) are in fact valid on higher genus surfaces as well [24].

Universality again. As in the planar case, the asymptotic result are valid not only for the family of all rooted planar maps but the pattern holds for various subfamilies:

$$|\mathscr{M}_{g;n}^{\mathscr{F}}| = c_{\mathscr{F}} \cdot \tau_g \cdot (\rho_{\mathscr{F}})^n \cdot (\alpha_{\mathscr{F}} n)^{\frac{5}{2}(g-1)} \cdot (1+O(1/n)) \tag{1.45}$$

where $c_{\mathscr{F}}$, $\rho_{\mathscr{F}}$ and $\alpha_{\mathscr{F}}$ depend on the family, but the dependency in the genus is entirely controlled by the constants τ_g and $\gamma_g = \frac{5}{2}(g-1)$ that already appear in Formula (1.41). All the families of maps satisfying the pattern (1.40) are expected to satisfy (1.45) as well on higher genus surfaces, but this is currently proved only for fewer families (in particular for those defined by one of the aforementioned restrictions: degrees, girth, or irreducibility) [109, 81].

Another general observation is the claim by Eynard, that map gfs should in general satisfy a *topological recurrence* parameterized by the so-called *spectral curve* [99]. This statement is more general than the algebraicity and asymptotic pattern (1.45) since the topological recurrence can be written for gfs of critical decorated maps as well as for series arising from algebraic geometry problems. For the numerous developments around this recurrence we refer to the forthcoming book [99].

Bijection for maps on surfaces. Regarding bijective proofs, only the approach via well labeled trees currently extends to higher genus maps: it yields in particular a correspondence between rooted maps on a surface of genus g and well labeled unicellular maps of genus g (a map is *unicellular* if it has only one face). Combined with decompositions *à la Wright*, it yields a combinatorial explanation [85] of the occurrence of the exponent $\frac{5}{2}(g-1)$ in Formula (1.41). A closer relation with well labeled trees with several marked points was given by Chapuy in [82] using a remarkable bijective decomposition of unicellular maps [83]. Another remarkable extension to higher genus is Miermont's result for maps with several marked points [141]: in particular as discussed in [66], this result unifies the well labeled tree and the Ambjørn-Budd results.

As in the planar case, the bijective approach allows us to keep track of distances in quadrangulations. Until now exact distance enumeration results like Formulas (1.37)-(1.38) have been obtained only in the case $g = 1$, [121], but the approach has allowed to show that in general typical distances remain in the order $n^{1/4}$, [85, 82], and it has opened the way to the construction of continuum limits of random quadrangulations on surfaces [43].

It is instead an open problem to give combinatorial interpretation of the Eynard-Orantin topological recurrence, or of the Carrell-Chapuy recurrences (1.43) or (1.44). The later problem is particularly intriguing: each term in the recurrence has a clear combinatorial interpretation, and the special case $j = 1$ is the celebrated Harer-Zagier recurrence for the number $\varepsilon_g(n) = M^g_{n+1-2g,1}$ of rooted unicellular maps with n edges and genus g, for which an elegant bijective proof was recently proposed by Chapuy, Feray and Fusy in [84], building on the earlier construction of Chapuy [83].

It is worth mentioning the fact that there is a parallel story of unicellular maps, starting with the root deletion method [171, 172, 173], continuing with characters of the symmetric group [125, 118, 149] and matrix integrals [124] to arrive to more combinatorial methods [128, 117, 159, 41, 33, 83, 32].

1.4.4 Decorated maps

Finally the fourth direction to discuss is the study of decorated maps. The decoration of a map can be a coloring of its vertices, a spanning tree, a spin or loop configuration, an orientation, etc. Of interest is then the gf of maps endowed with such a structure, or more generally the weighted gf where each decoration is given a weight and the weights are summed over all pairs (map,decoration). We briefly discuss at the end of this section some motivations for this study arising from physics and probability, but we concentrate on counting results that have been obtained in the combinatorial literature, either in the form of explicit counting formula or of equations for gfs.

Exact counting results by direct decompositions. As discussed in Section 1.2, probably the first model to have been dealt with is that of cubic maps endowed with a Hamiltonian cycle, which were enumerated by Tutte in [163] even before counting planar maps. Spanning trees on general maps were counted later by Mullin [147] who realized that there is a simple correspondence between these two models (see also Theorem 5). Other models like bipolar orientations [17, 49, 108], realizers [48, 34, 108] or Schnyder decompositions [37] share with spanning trees and Hamiltonian cycles the property that they enjoy quite explicit formulas and bijective proofs, even though their generating series are non algebraic. We include in this list the bijection of [30] although it deals *a priori* with the family of cubic planar map without decoration but the proof here uses the fact that each such a map admits 2^n spanning trees and then deals with tree-rooted maps. We believe that all these results await for a unifying *master bijection* mapping rooted planar maps with an α-orientation onto some bidimensional walks, maybe extending the master bijection between rooted planar maps endowed with a *minimal* α-orientation onto decorated trees as discussed in Section 1.4.2.

Trees and Tutte's equations for decorated maps. Another family of results is that of dimer models, hard particles, and Ising models on planar maps which admit bijective proofs based on the blossoming or well labeled approach [62, 58, 65], and enjoy algebraic generating series. Most of these results were first obtained via the matrix model formalism, and several have been rederived via Tutte root-deletion method before getting bijective proof with trees.

The enumeration of planar maps weighted by the Tutte polynomial (also known as the Potts model on random lattices) plays a particular role in this list, as it contains many of the above previously solved model: as first shown by Tutte for the special case of well q-colored triangulations in a long series of 10 papers spanning 20 years of research ([167] and ref. therein), this model is still amenable to the root-deletion approach but the equation is amazingly difficult to solve. The resulting gf is non-algebraic but it satisfies an explicit differential algebraic equation of order three. Tutte's solution remained as a isolated artifact for almost 25 years until Bernardi and Bousquet-Mélou were able to distillate and extend his approach to the general Tutte polynomial on triangulations and to deal with non-trivial variants of the problem on rooted planar maps (see [35, 54] and reference therein). As opposed to the now well understood polynomial equations with one catalytic variables that are associated to standard Tutte equation (see Section 1.3.1), much is still to be understood on the class of polynomial equations with two catalytic variables that are involved in these decorated models.

Exact results from substitution schemes. Other models have been dealt with by variants and generalizations of Tutte's substitution schemes that we discussed in Section 1.3.4. Using a refinement of this approach Sundberg and Thistlewaite obtained the gf of diagrams of open prime alternating links [162] and the asymptotic number of prime alternating links then follows from an asymmetry analysis [126]. Zinn-Justin and Zuber [176] extended the approach to 2-colored links, in an attempt to attach the problem of enumerating prime alternating knots (see also [160]). Tutte's substitution scheme allows us also to deal with the Ising model on non-separable maps [35] or with orientations on maps with girth or connectivity constraints [103]. In all those examples, the main idea is that the decomposition follows the structures of the underlying non-decorated map, and the interactions between the decoration and the decomposition is local and relatively well controlled.

In the case of the so-called $O(n)$ of colored loops on maps, Borot, Bouttier and Guitter [52, 51, 50] apply instead the idea of decomposing along the maximal loops of the model. By a careful analysis of the possible interface between the core (here renamed *gasket*) and the submaps along these maximal loops, they obtain a functional equation that they are not able to solve exactly in general but which is sufficient to derive remarkably precise non-trivial asymptotic results.

Finally let us quote the recent work of Bousquet-Mélou and Courtiel [55], which starts this time from a direct, non-recursive, substitution scheme: rooted planar map with a spanning forest can indeed be obtained from non-decorated maps upon inserting spanning trees in vertices with proper weights, as already observed in [65]. This approach yields almost directly an equation, but its study is quite difficult and leads to remarkable developments involving differentially algebraic gf and to non standard asymptotic expansions for a particular interpretation of the model in terms of forested maps weighted by the external activity of the spanning forest.

Decorated maps in physics and probability. To conclude, let us give a Boeotia's motivation for the study of decorated map in physics. From the physics point of view, we are considering the *annealed* partition function of a toy model of statistical physics coupled to a random lattice. Toy models of statistical physics are usually defined on a fixed regular lattice (typically the triangular, square or hexagonal lattice) and are viewed as microscopic models of matter. The idea to couple such a toy model with an irregular lattice arises from quantum geometry: the regular *classical euclidean geometry* is to be replaced by a *quantum geometry*, that is a distribution of probability on a set of irregular geometries: in the pair (map, decoration), the map plays the role of the random lattice and it is decorated by the configuration of the model. Accordingly, non-decorated maps are considered as a model of *pure 2d quantum geometry*, while decorated maps correspond to *2d quantum geometry with matter*. We refer to the survey [94] and the book [8] for an introduction to the motivations for the study of these toy models on random lattices in physics.

The physics community has developed a remarkable expertise in classifying the toy models into universality classes, and in predicting which universality classes and critical exponents are possible. From this point of view the pattern and universality statements of Section 1.4.1 are just the tip of the iceberg, corresponding as already mentioned to the simplest case of pure 2d quantum geometry without coupling to matter. The more general situation of quantum geometry with matter is quite comparable to that of the study of statistical physics conformally invariant toy models on regular 2d lattices, and in fact a tight relation exists between these two areas, first discovered by Knitchik, Polyakov and Zamolochikov and now referred to as the KPZ relation. Again the interested reader is directed to [94], and to the work of Duplantier and Sheffield [95] and Miller and Sheffield [145] for recent developments on these interactions, see [115] for a survey.

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