EPIT 2016, *Draft version*

Gilles Schaeffer

May 9, 2016

Contents

Preliminary remarks

These notes covers material related to the first lecture. It is an evolving draft probably with many errors...

These notes are intended to be read by someone who has read Eric Colin de Verdière's lecture (Chapters 1 and 2): basic definitions and properties of surfaces and cellular graph embeddings are not recalled.

In particular Eric discussed duality, Euler's formula, and explained how to view these embeddings topologically or combinatorially rather than geometrically. In the present notes this more precisely means that we work up to homeomorphisms of $\mathcal{S}$:

> A *map on $\mathcal{S}$ the orientable surface of genus g with ℓ boundary* (or simply a *map with genus g and ℓ boundaries*) is a cellular graph embedding on $\mathcal{S}$ considered up to homeomorphisms of $\mathcal{S}$.

Eric presented the flag representation of combinatorial maps: this representation shows that, up to relabelling of the flags, the set of maps with say n edges ($4n$ flags) is finite. Alternatively one can label the $4n$ flags with integers $\{1,\dots,4n\}$, and the flag representation shows that the number of these *labeled maps* is (very roughly) bounded by the number of triples of fix point free involutions on $\{1,\dots,4n\}$.

At this point the reader should pause and wonder whether he feels like he is able to produce in a relatively efficient way a list of all maps with say 4 edges (loops and multiple edges allowed), or of all triangulations with 4 triangles. Constructing all labeled maps should clearly appear as a waste of time. This motivates the introduction of (unlabelled) rooted maps:

> A map is *rooted* if one of the corners of the map is marked. The vertex and face incident to the root corner are called the *root-vertex* and *root-face* of the map.

Once a map is rooted it can be viewed as implicitely labeled in a canonical way starting from the root (for instance using a clockwise depth first search traversal of the underlying graph). In other terms, the number of ways to label the flags of a rooted map so that the root flag has label 1 is $(4n-1)!$, or to put it again in an other way, the number of labeled maps with a given underlying rooted map is $(4n-1)!$. We have thus already saved a factor $(4n-1)!$.

Another advantage of rooted maps is that they are easy to compare: again the reader should convince himself that given two rooted maps, he can decide if they are the same in linear time (hint: cw dfs wins again).

The aim of this first lecture is to introduce the reader to the enumerative theory of maps. Rather than surveying the most advanced results we concentrate on a few results that can be made completely explicit in one lecture.

1 Boundary triangulations and duality

Boundary maps, dissections and triangular maps Let us consider a surface $\mathcal{S}$ of genus g with $\ell \geq 1$ boundaries. Recall that the Euler characteristic of $\mathcal{S}$ is $\chi = 2-2g-\ell$ and that in a map on $\mathcal{S}$ boundaries must be simple cycles. A map $\mathfrak{m}$ is a *boundary map on $\mathcal{S}$* if all the vertices of $\mathfrak{m}$ are on the boundary. From now on, *all the boundary maps we consider are rooted on a boundary corner.*

A map is a *dissection of $\mathcal{S}$*, or a *dissection with genus g and ℓ boundaries*, if every face of degree k is incident with k distinct vertices, and the intersection of two faces consists of at most one vertex or one edge. A map is *triangular* if it only has triangular faces. It is a *triangulation* if it is also a dissection. Consequently, to summarize, a boundary triangulation of $\mathcal{S}$ is a map on $\mathcal{S}$ with no internal vertex, no loops, nor multiple edges and whose faces have all degree 3

Boundary triangulations of genus 0 with 1 boundary and n non-root vertices correspond to the popular "triangulations of a convex $(n+1)$-gon": the number of these objects is well known to be the $(n-1)$th Catalan number C_{n-1}, where $C_n = \frac{1}{n+1}\binom{2n}{n}$. Our aim in this lecture is to prove this result in a way that can be extended to deal with the general case of boundary triangulations.

Proposition 1 *Let $\mathcal{S}$ be the surface of genus g with $\ell > 1$ boundaries, and $\mathfrak{m}$ be a boundary triangular map on $\mathcal{S}$ with n non-root vertices. Then $\mathfrak{m}$ has $n + 2\ell + 4g - 3$ faces and $2n + 3\ell + 6g - 4$ edges.*

Proof. Euler's formula reads $(n+1)+(f+\ell) = e+2-2g$ and the hand shaking lemma gives $2e-(n+1) = 3f$ so that $2(n+1)+2(f+\ell) = n+1+3f+4-4g$ and $f = n+2\ell+4g-3$, and returning to Euler, $(n+1)+n+3\ell+4g-3 = e+2-2g$. ◻

Duality and pre-cubic maps Duality for map on surfaces without boundary is classically defined as follows. Given a map $\mathfrak{m}$ create a map $\mathfrak{m}^*$ having a vertex in each face of $\mathfrak{m}$ and an edge e^* dual to every edge e of $\mathfrak{m}$: the edge e^* crosses e to join the vertices of $\mathfrak{m}^*$ in the faces of $\mathfrak{m}$ incident to e.

Let us adapt this definition to rooted maps on a surface with boudaries: Given such a map $\mathfrak{m}$ on $\mathcal{S}$, fill the boundaries of $\mathcal{S}$ with discs to get a map $\bar{\mathfrak{m}}$ on a surface $\bar{\mathcal{S}}$ without boundaries, take the dual $(\bar{\mathfrak{m}})^*$ and split the dual vertices which corresponds to boundaries of $\mathfrak{m}$: each such vertex of degree k is split into k leaves. The *modified dual of* $\mathfrak{m}$, still denoted $\mathfrak{m}^*$, is resulting map on $\bar{\mathcal{S}}$, considered as rooted on the leaf on the left hand side of the root corner of $\mathfrak{m}$.

Clearly the modified dual of a rooted triangular map $\mathfrak{m}$ has one vertex of degree 3 for each triangle of $\mathfrak{m}$ and one leaf for each boundary edge of $\mathfrak{m}$. Moreover as already observed, the genus is preserved and the number of faces of $\mathfrak{m}^*$ is the number of boundaries of $\mathfrak{m}$.

Proposition 2 *Modified duality is a bijection between*

- *boundary triangular maps with genus g, $\ell \geq 1$ boundaries, n non-root vertices and $n + 2\ell + 4g - 3$ faces,*
- *and leaf-rooted maps with genus g, ℓ faces, n non-root leaves and $n + 2\ell + 4g - 3$ vertices of degree 3, such that each face contains at least one leaf.*

Proof. This duality operation is not an involution anymore but it is clearly reversible: the leaves arising from a given boundary of $\mathfrak{m}$ all lie together in an associated face of $\mathfrak{m}^*$, and upon gluing all these leaves to form a mark vertex, one reform the dual of $\bar{\mathfrak{m}}$ and recover $\mathfrak{m}$.

The non empty face condition ensures that the resulting triangular map is correctly drawn on the surface with genus g and ℓ boundaries (that is each boundary is a cycle of the map). ◻

Observe that it is *a priori* non trivial to characterize the dual maps that are associated to boundary triangulations. Recall that, in order to be a boundary triangulation, a triangular map must contain no loops or double edges: in standard duality this corresponds to the dual map having no face incident to both sides of an edge, or a pair of doubly adjacent faces, but this is not true any more in our modified duality due to the vertices that are cut. This difficulty is discussed further in Section 4.

2 Pre-cubic maps and their cores

Let us temporarily forget about the non-empty-face condition and consider the resulting maps: A map is *pre-cubic* if it has only vertices of degree 1 or 3 and it is rooted on a leaf. Observe that planar pre-cubic maps with one face are exactly (leaf-rooted) binary trees, with the convention that the rooted map reduced to one edge is considered as a binary leaf-rooted binary tree with one non-root leaf (in fact the only one). For later purpose, let a *doubly leaf-rooted binary tree* be a leaf-rooted binary tree with a secondary root leaf. Observe that there are two doubly leaf-rooted binary trees with one non-root leaf.

The core decomposition. The *1-core* $\mathfrak{c}(\mathfrak{m})$ of a rooted map $\mathfrak{m}$ is obtained from $\mathfrak{m}$ by recursively deleting all non-root vertices of degree 1. If $\mathfrak{m}$ is a tree then $\mathfrak{c}(\mathfrak{m})$ is reduced to a single vertex, otherwise it is a rooted map without non-root vertices of degree 1.

The *2-core* (also called *scheme*) of a map is the map $\mathfrak{s}(\mathfrak{m})$ obtained from $\mathfrak{c}(\mathfrak{m})$ by smoothing all vertices of degree 2, that is, replacing iteratively each non-root vertex of degree 2 and the two incident edges by a unique edge: $\mathfrak{s}(\mathfrak{m})$ is a rooted map without non-root vertices of degree 1 or 2.

In the case of a pre-cubic map $\mathfrak{m}$ with genus $g \geq 1$, the 2-core is by construction itself a pre-cubic map with genus g, with vertices of degree 3 but no non-root leaf: as already discussed in the dual setting Euler's formula implies that the number of these vertices of degree 3 is then $v = 2\ell + 4g - 3$ and the number of edges $k = 3\ell + 6g - 4$.

Now let us decompose a pre-cubic map $\mathfrak{m}$ with genus at least 1 along its 2-core.

Theorem 3 *There is a bijection between*

- *pre-cubic maps with genus g, ℓ faces and n non-root leaves,*
- *and pairs $(\mathfrak{s}, (\mathfrak{a}_1, \ldots, \mathfrak{a}_k))$ made of*
  - *a pre-cubic map $\mathfrak{s}$ with genus g, ℓ faces and $k = 3\ell + 6g - 4$ edges but without non-root leaves,*
  - *and a k-uple $(\mathfrak{a}_1, \ldots, \mathfrak{a}_k)$ of doubly leaf-rooted binary trees with a total of n non-root leaves.*

Proof. Given a leaf-rooted map, endow its 2-core with an arbitrary canonical ordering and orientation of the edges (using for instance a clockwise depth first search traversal of the

2-core). Then each edge (u,v) of the 2-core corresponds to a subgraph of $\mathfrak{m}$ which can be obtained from the edge (u,v) by a sequence of edge subdivisions, followed by a sequence of insertions of pending edges at existing vertices. These operations do not create cycles so that this subgraph must be a tree attached to u and v. In view of the degree constraints it is a doubly rooted binary tree.

Conversely given a pre-cubic map $\mathfrak{c}$ without non-root leaves and a list of doubly leaf-rooted binary trees we first endow $\mathfrak{c}$ with its canonical ordering and orientation and then use the orientation of the ith edge to replace the ith tree. We obtain a pre-cubic map whose decomposition clearly gives back $\mathfrak{c}$ and the same list of trees. $\square$

3 The core decomposition and generating functions

3.1 Binary trees

The standard recursive decomposition of binary tree at their root can be interpreted as a size preserving bijection

$$\phi : \mathcal{B} \to \left\{ \substack{| \\ \circ} \right\} \cup \left\{ \substack{| \\ \bullet \\ /\backslash} \right\} \times \mathcal{B} \times \mathcal{B}.$$

Size preserving means that if we denote by $|t|$ the size of a binary tree t, say its number of non-root leaves $\circ$, then if $t \equiv (\bullet, t_1, t_2)$ then $|t| = |t_1| + |t_2|$, and if $t \equiv \circ$ then $|t| = 1$.

In the rest of the text we use the short hand notation

$$\mathcal{B} \equiv \circ + \bullet \times \mathcal{B} \times \mathcal{B}$$

which could be considered as a recursive definition of binary trees.

Then consider the *generating function* (gf) of $\mathcal{B}$ according to the size, that is the formal power series obtained by summing over all $t \in B$ a monomial $x^{|t|}$:

$$B(x) = \sum_{t \in \mathcal{B}} x^{|t|} = \sum_{n \geq 1} |\mathcal{B}_n| x^n.$$

Then $B(x)$ satisfies

$$B(x) = x^{|\bullet|} + \sum_{(t_1,t_2) \in \mathcal{B} \times \mathcal{B}} x^{|t_1|+|t_2|} = x + \left(\sum_{t \in \mathcal{B}} x^{|t|} \right)^2 = x + B(x)^2$$

Solving the quadratic equation $B = x + B^2$ yields two possible solutions $B_{\pm}(x) = \frac{1 \pm \sqrt{1-4x}}{2}$, having respective Taylor expansion $B_-(x) = x + O(x^2)$ and $B_+(x) = 1 + O(x)$. In view of the definition of $B(x)$ we identify it as $B(x) = B_-(x)$. The generalized binomial theorem,

$$(1-x)^{-\alpha} = \sum_{n \geq 0} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)}{n!} x^n,$$

then give an explicit expression for the coefficients of $B(x)$: $|\mathcal{B}_{n+1}| = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number.

The number of doubly leaf-rooted binary trees with n non-root leaves is just $n+1$ times that of leaf-rooted binary trees with $n+1$ non-root leaves: $|\mathcal{D}_n| = (n+1)|\mathcal{B}_n| = \binom{2n}{n}$. We will need the generating function $D(x) = \sum_{n\geq 0}(n+1)|\mathcal{B}_n| = B'(x) = \frac{1}{4}(1-4x)^{-1/2}$.

3.2 Pre-cubic maps

We can now use Theorem 3 to rewrite the gf for the set $\mathcal{C}_{\mathfrak{s}}(x)$ of pre-cubic map with a given 2-core $\mathfrak{s}$: indeed

$$\begin{aligned} C_{\mathfrak{s}}(x) &= \sum_{\mathfrak{m}\in\mathcal{C}_{\mathfrak{s}}} x^{|\mathfrak{m}|} = \sum_{\mathfrak{a}_1,\ldots,\mathfrak{a}_{k(\mathfrak{s})}} x^{|\mathfrak{a}_1|+\cdots+|\mathfrak{a}_{k(\mathfrak{s})}|} = \left(\sum_{\mathfrak{a}_1} x^{|\mathfrak{a}_1|}\right)\cdots\left(\sum_{\mathfrak{a}_{k(\mathfrak{s})}} x^{|\mathfrak{a}_{k(\mathfrak{s})}|}\right) \\ &= (D(x))^{k(\mathfrak{s})} = \left(\frac{16}{1-4x}\right)^{\frac{1}{2}k(\mathfrak{s})}. \end{aligned} \tag{1}$$

Upon summing over all possible cores we get:

Proposition 4 *The gf of pre-cubic maps with genus g and ℓ faces with respect to non-root leaves is*

$$C_{g,\ell}(x) = \sum_{s\in\mathcal{C}_{g,\ell,0}} (D(x))^{3\ell+6g-4} = C_{g,\ell,0}\cdot\left(\frac{16}{1-4x}\right)^{\frac{3}{2}\ell+3g-2} \tag{2}$$

where $C_{g,\ell,0}$ is the number of pre-cubic maps with genus g and f faces without non-root leaves.

Using again the generalized binomial theorem we get:

Corollary 5 *The number $C_{g,\ell,n}$ of pre-cubic maps with genus g, f faces, n non-root leaves and $n+k$ edges ($k = 3\ell+6g-4$) satisfies:*

$$C_{g,\ell,n} = C_{g,\ell,0}\cdot 4^k\frac{k(k+2)\cdots(k+2n-2)}{2^n n!}$$

At this point to get an explicit formula for a given surface $\mathcal{S}$ we need to compute $C_{g,\ell}$: for $g+\ell = 1$ (torus with one boundary or cylinder) there is only one possible core, but already for $g+\ell = 2$ constructing the list of core is tedious. We will return to this problem in Section 5. While the exact formula above is nice, it also interesting to look at the asymptotic behavior:

Corollary 6 *When n goes to infinity with g and ℓ fixed*[1]

$$C_{g,\ell,0} = c_{g,\ell}\cdot n^{\frac{3}{2}\ell+3g-3}\cdot 4^n\cdot(1+O(1/\sqrt{n})),$$

where the constant $c_{g,\ell}$ can be computed explicitely in terms of $C_{g,\ell,0}$.

[1]For two functions $f(n)$ and $g(n)$ of n, $g(n) = O(f(n))$ means there exists a constant c such that $g(n) \leq f(n)$ for all n large enough.

Proof. In this case one can directly apply Stirling formula $n! = \Theta((n/e)^{n+1/2})$ to the explict expression, dealing separately with the cases k even and k odd. A more systematic approach is to rely on principles of singularity analysis of complex functions, (see Flajolet-Sedgewick 2009 [Thm VI.1 p381]):in that context the *asymptotic* binomial expansion says that for $\alpha \in \mathbb{R} \setminus \{-1, -2, \ldots\}$,

$$[x^n](1 - x/\rho)^{-\alpha} = \frac{n^{\alpha-1}}{\Gamma(\alpha)} \cdot \rho^{-n} \cdot (1 + O(1/\sqrt{n})).$$

This immediately gives the result with $c_{g,\ell} = C_{g,\ell,0}/\Gamma(\frac{3}{2}\ell + 3g - 2)$. □

4 Counting boundary triangular maps and triangulations

4.1 Exact counting of triangular maps

According to Proposition 2 pre-cubic maps that correspond to triangular maps must satisfy the no-empty-face condition. The non-empty-face condition is in fact compatible with the 2-core decomposition:

Proposition 7 *A face F of a pre-cubic map is empty if and only if all the doubly rooted binary trees that are incident to it in the 2-core decomposition have no leaf on the side of F.*

A doubly leaf-rooted tree has no leaf on the right hand side of its roots if and only if the secondary root is the rightmost leaf. This implies that these 1-sided doubly rooted trees can be viewed as usual leaf-rooted binary trees with the rightmost leaf implicitely marked: their gf is thus simply $D_{\dashv}(x) = B(x)/x$. In particular one also obtain that the gf of doubly leaf-rooted trees that are not 1-sided is

$$D_{+}(x) = D(x) - x - 2(B(x)/x - 1)$$

The main decomposition of Theorem 3 can be refined to distinguish edges of the core that remain untouched, those that get a 1-sided tree and those that get a tree with leaves on both sides. Let a *decorated core* be a pre-cubic map without non-root leaf with a coloration on each edge indicating the type of tree it is expected to carry: each core can be decorated in 3^k ways. A decorated core is *valid* it ensures that no face is empty (this can be easily checked by a traversal). Let $\mathcal{V}$ denote the set of valid decorated cores, and given a core $\mathfrak{c} \in \mathcal{V}$ let $k_{\dashv}$ and k_{+} be its number of edges expecting 1-sided trees and two sided trees respectively.

Theorem 8 *The gf of triangular maps with genus g and f boundaries is*

$$T_{g,\ell}(x) = \sum_{\mathfrak{c} \in \mathcal{V}_{g,\ell}} D_{\dashv}(x)^{k_{\dashv}} D_{+}(x)^{k_{+}}.$$

For any fixed value of g and f one can in principle compute the generating function and obtain an explicit formula for the number of boundary triangular maps of genus g and f, but this is extremely tedious already for small values of g and f.

4.2 Asymptotic counting

Instead one can observe that in the asymptotic binomial theorem, if $\alpha_1 > \alpha_2$ then the coefficients of $(1 - x/\rho)^{-\alpha_2}$ are asymptotically neglegible in front of those of $(1 - x/\rho)^{-\alpha_2}$: this indeed can be used to compare coefficients of

$$D_+(x) = \frac{1}{4}(1-4x)^{-\frac{1}{2}} + Pol((1-4x)^{\frac{1}{2}})/x,$$

and

$$D_{\dashv}(x) = \frac{1}{2x} - \frac{1}{2x}(1-4x)^{\frac{1}{2}}$$

and more generally it implies that the valid cores that give the dominant asymptotic contribution to the number of boundary triangular maps are those such that k_+ is maximal, that is having k double sided edges. The asymptotic contribution of these terms is then exactly the same as that of non decorated cores in Corollary 6, since every non decorated core become a valid core when all edges are marked as double sided:

Corollary 9 *The number of boundary triangular maps with genus g, b boundary and n vertices satisfies*

$$T_{g,\ell,n} = C_{g,\ell,n} \cdot (1 + O(1/\sqrt{n})) = c_{g,\ell} \cdot n^{\frac{3}{2}\ell+3g-3} \cdot 4^n \cdot (1 + O(1/\sqrt{n})).$$

4.3 Boundary triangulations

The strategy to deal with boundary triangulations follows the same line as for boundary triangular maps, with details slightly more involved:

- Consider the number of *turns* on the branch between the two root leaves of a doubly leaf-rooted trees. Observe that 1-sided trees are exactly doubly leaf-rooted trees without such turn. The analysis requires to distinguish between doubly rooted trees with 1 turn, 2 turns, 3 turns and at least 4 turns.

- Decorate cores according to the type of trees that will be substituted in it: the number of such decorated cores is still finite.

- The key point is that one can check whether a boundary triangular map is a boundary triangulation by inspecting its decorated core, without looking at the actual trees: this implies that one can again define a finite set of *valid decorated core* and restrict attention to these.

- The gf of trees in with 2 or 3 turns can be computed in terms of $B(x)$, and we obtain an analog of Theorem 8.

Again one obtains a messy exact counting result, but the main byproduct of this analysis is the following:

Theorem 10 *The number of boundary triangulation with genus g, ℓ boundaries and n vertices is*

$$\bar{T}_{g,\ell,n} = C_{g,\ell,n} \cdot (1 + O(1/\sqrt{n})) = c_{g,\ell} \cdot n^{\frac{3}{2}\ell+3g-3} \cdot 4^n \cdot (1 + O(1/\sqrt{n})).$$

It is interesting to see that the growth rate (=4) does not depend on the genus or number of faces, while the degree of the polynomial correction ($n^{6g+3f-4} = n^{-\frac{3}{2}\chi(\mathcal{S})}$) depends linearly on the Euler characteristic of the surface.

5 A bijective recurrence for pre-cubic maps.

The constants $C_{g,\ell,0}$. While the dependancy in n at fixed g and ℓ of the number of pre-cubic maps is clearly given by Corollary 6, one would like if possible to know more about the constants $C_{g,\ell,0}$. By definition these numbers count pre-cubic without non-root leaves having genus g and ℓ faces. Upon absorbing the root leaf, these maps are exactly rooted cubic maps with genus g with ℓ faces and $v = 2\ell + 4g - 4$ vertices (including the root).

Using generating function technics the approach of Section 3.1 can be extended to give an explicit formula for rooted planar cubic maps with ℓ faces (the case $g = 0$), and for unicellular cubic maps of genus g (the case $\ell = 1$), and a complicated recurrence for the general case. More recently, using completely different algebraic technics, the following much simpler recurrence was given for the number $F(v,g) = C_{g,v-2g+2}$ of cubic maps with $2v$ cubic vertices, $v = \ell + 2g - 2$:

$$(v+1) \cdot F(v,g) = 4v(3v-2)^2 F(v-2, g-1) + \sum_{\substack{k+h=g \\ i+j=v-2}} (3i+2)(3j+2)F(i,k)F(j,k)$$

A fascinating open problem is to give a combinatorial proof of this recurrence through some decomposition of cubic maps.

A recurrence. We present instead the known beautiful recurrence for pre-cubic unicellular maps with genus g and n leaves ($\ell = 1$). The idea is to identify $2g$ special vertices among the set of vertices of degree 3 of the map, and break one of these special vertices to get a map of smaller genus.

In order to identify the special vertices, let us label the $2e$ corners of the map from 1 to $2e$ in clockwise order starting from the root. Then half-edges can be classified into *ascent* and *descent* depending whether the two incident corners strictly increase in counterclockwise direction around their origin vertex or not. Each edge is incident to a minimal label i and, its half-edges are labeled $(i, j+1 \mod e)$ and $(j, i+1)$ respectively with $j \geq i$, that is one ascent and one descent, except if $j = e$ where this gives two descents. Since the case $j = e$ occurs only once, the map has $e + 1$ descents and $e - 1$ ascents.

Next observe that each vertex has at least one descent, right after its maximal incident label. A descent which is not of this type is called *special*, and in view of the number of vertices and descents there are $2g$ special descents. Since a vertex of degree $i > 1$ can have at most $i - 1$ descents, hence at most $i - 2$ special descents, our $2g$ descents must occur on $2g$ different vertices of degree 3, the *special* vertices of the map.

Now consider a pre-cubic unicellular map $\mathfrak{m}$ with genus g and with a marked special vertex, and let $\phi(\mathfrak{m})$ be the map obtained by cutting the marked special vertex into 3 marked leaves. Then $\phi(\mathfrak{m})$ has three new marked leaves and one vertex of degree 3 less, and it is still a unicellular map (because the vertex was special, make a picture), so it has genus $g - 1$. Conversely observe that, among the two ways to glue three marked non-root leaves of a pre-cubic unicellular map into a marked vertex of degree 3, only one is such that the resulting map is unicellular, and the resulting map has genus g and the marked vertex is special. This bijection ϕ implies that the number of pre-cubic unicellular maps of genus g satisfies

$$2g \cdot C_{g,1,n} = \binom{n+3}{3} C_{g-1,1,n+3}.$$

Iterating we get

$$C_{g,1,n} = \frac{1}{2^g g!} \binom{n+3g}{3} \binom{n+3g-3}{3} \cdots \binom{n+3}{3} C_{0,1,n+3g},$$

where $C_{0,1,n+3g} = C_{n+3g-1}$ the $(n + 3g - 1)$th Catalan number. Equivalently:

$$C_{g,1,n} = \frac{(3g)!}{12^g g!} \binom{n+3g}{3g} \cdot C_{n+3g-1}.$$

6 What about triangulations with interior points?

In the planar case explicit results can in fact be obtained for many families of planar maps using recursive decompositions and generating function manipulations. Let us quote for instance the case of planar triangulations with one boundary:

Theorem 11 *The number of rooted planar triangulations of a $(n + 1)$-gon with i interior vertices is*

$$T_{0,n,i} = \frac{2(2n-1)!(4i+2n-3)!}{n!(n-2)!i!(3i+2n-1)!} \tag{3}$$

Observe that for $i = 0$ these are triangulations of a polygon with $T_{0,n,0} = C_{n-1}$ as expected.

For surfaces of genus $g \geq 1$ we have the following analog of Corollary 9:

Theorem 12 *There are positive constants $c_{\mathcal{T}}$, $\rho_{\mathcal{T}}$ and $(t_g)_{g\geq 0}$ such that for any fixed g, the number of rooted triangulations of a surface of genus g without boundary is*

$$c_{\mathcal{T}} \cdot t_g \cdot n^{\frac{5}{2}(g-1)} \cdot (\rho_{\mathcal{T}})^n \cdot (1 + O(\sqrt{n}))$$

Since the planar case admits an explicit formula and higher genus triangulations satisfy an asymptotic pattern analog to Corollary 9 one could hope to be able to prove these results via using cores: this is actually possible in some cases but significantly more involved than the boundary triangulation case. In particular it requires as a first step to make explicit the link between planar triangulations and trees that is suggested by Formula (3).

7 Notes

- *All the computations of Sections 3.1 and 3.2 could have been done without generating functions, using explicit encodings of binary trees with binary words (parenthesis strings). See Lothaire 2005 [Chapter 9] for an introduction to these technics.*

  On the other hand the gf approach is more systematic: it allows to extend the results to the case of boundary dissections with various polygon sizes (instead of only triangles). Indeed in these more general cases explicit formulas become involved but generating functions still satisfy simple functional equations and the asymptotic behavior of their coefficients can be derived by singularity analysis. For this powerful approach, called analytic combinatorics*, we refer to the book Flajolet-Sedgewick 2009: it is not only the ultimate technical reference but it also contains starts with a very evocative introductory chapter.*

  Returning to boundary dissection, the generalization of Corollary 9 is

  $$\mathcal{D}_n^{\mathcal{S},\Delta} = c_{g,\ell} \cdot n^{-\frac{3}{2}\chi(\mathcal{S})} \cdot \alpha_\Delta \cdot (\rho_\Delta)^n \cdot (1 + O(1/\sqrt{n})) \qquad (4)$$

  where $\mathcal{S}$ can be any surface with boundary (orientable or not) and Δ is the set of allowed vertex degrees. See Bernardi-Rué 2012.

- *The derivation of the previous sections can be summarized as follows:*

  - *Solve the enumeration problem for objects of type 0 (the planar case!).*
  - *Show that that for each fixed k there exists a finite set of cores of type k such that the objects of type k can be decomposed into a core of type k and a finite collection of objects of type 0 with a finite number of marks.*

  This core paradigm*, if it applies, proves that the growth rate for objects of type k remains the same as that of objects of type* 0*. It can actually be used for a variety of structures, we list a few examples where in fact it yields the full asymptotic behavior using generating functions:*

  - *Connected labeled graphs with fixed excess (that is, with n edges and $n+1-k$ vertices for k fixed). Connected graphs were studied in Wright 1977 and this is one of the first example of use of this approach that I am aware of.*

  - *Partitions with few crossings (Pilaud-Rué 2014)*
  - *Non-crossing partitions on surfaces (Rué-Sau-Thilikos 2013)*
  - *Well colored simplicial complexes of bounded embedding degree in higher dimension (Rugau-Schaeffer 2016).*

  This list is certainly not exhaustive and the core paradigm is quite versatile.

- *The bijective recurrence of Section 5 is due to Chapuy 2009 (see also Feray-Fusy-Chapuy 2013). The more general linear recurrences is from Goulden-Jackson 2008 (see also Chapuy-Carrel 2015).*

- *Theorem 3 is just one instance (due to Brown 1963) of a family of results obtained by W. T. Tutte and his collaborators in the early 60's. Similarly Theorem* **??** *is one of a collection obtained by Bender and collaborators in the 80's. A detailed introduction to the enumerative theory of maps can be found in Schaeffer 2015 [Chapter 5 in Bonà ed. 2015].*

  In particular Formula (4) should be compared to the analogous result for maps with vertex degrees in Δ*: (see Bender-Canfield 1994, Gao 1993)*

$$|\mathcal{M}_{g,n}^{\Delta}| = t_g \cdot n^{\frac{5}{2}(g-1)} \cdot \alpha'_{\Delta} \cdot (\rho'_{\Delta})^n \cdot (1 + O(1/\sqrt{n}))$$

  where $(t_g)_{g\geq 0}$ *is a family of universal constants (independant of* Δ*) and* ρ'_{Δ} *depends on* Δ *but not on* g*.*

  Observe in particular the exponent $5/2$ *instead of* $3/2$*.*

- *The fact that the growth constant* ρ'_{Δ} *depends only on* Δ *but not on* g *while the* critical exponent $\mu_g = 5/2(g-1)$ *depends linearly on the genus independantly of* $\mathcal{M}$ *has an appealing interpretation in terms of random maps: Given* g *and* n *and let* $X_{g,n}$ *denote a random map taken uniformly among rooted maps of* $\mathcal{M}_{n,g}$*. Then, when* n *is large while* g *is positive and fixed, the map* $X_{g,n}$ *is expected to be locally planar, with edge-width* $\Theta(n^{1/4})$*: on the one hand the local structure of* $X_{n,g}$ *is essentially the same as that of* $X_{n,0}$ *(hence* ρ *is independant of* g *since it measures the diversity of possible local structures), and on the other hand the large scale properties of* $X_{n,g}$ *(like the shape of the distribution of the edge-width, or of the distance between two random points) do not depend on the exact local structure of these maps,* ie *on* $\mathcal{M}$*.*

  The interpretation of the existence of the family of universal multiplicative constants $(t_g)_g$ *correspond to even subtler properties of random maps on surfaces, that raises fascinating open questions (see Chapuy 2016).*

  An introduction to the relation between enumerative results and probabilistic properties of random maps can be found in Schaeffer 2015 [Chapter 5 in Bonà ed. 2015].