Random planar maps, alternating knots and links

Gilles Schaeffer – CNRS

Join work with Sébastien Kunz-Jacques
(LIX – Corps des télécoms)

An overview of the talk

The enumeration of maps

examples of algebraic functions

Random planar maps

almost sure properties

Enumerative knot theory ?

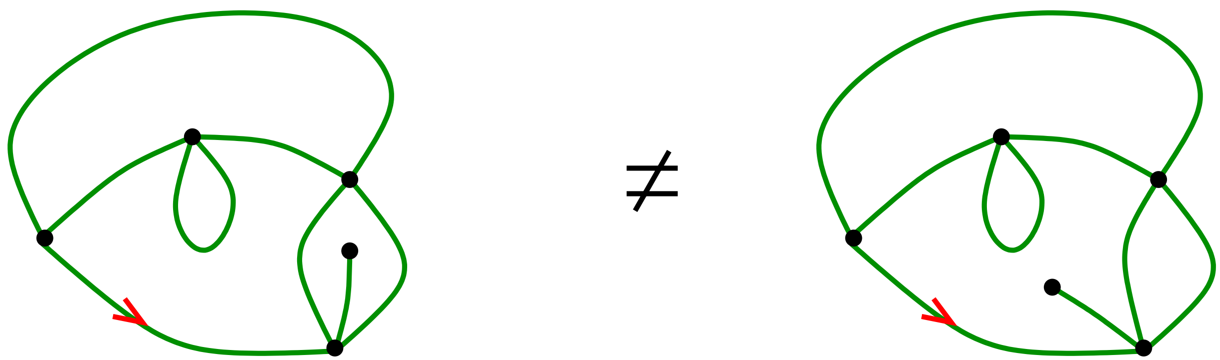
prime alternating links

Asymptotic enumeration of links

as an application of random maps

Rooted planar maps. Definition

a **planar map** = an embedding of a connected graph in the plane.

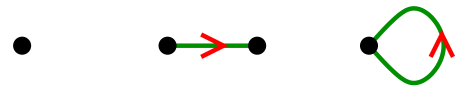


planar map = planar graph + cyclic order around vertices.

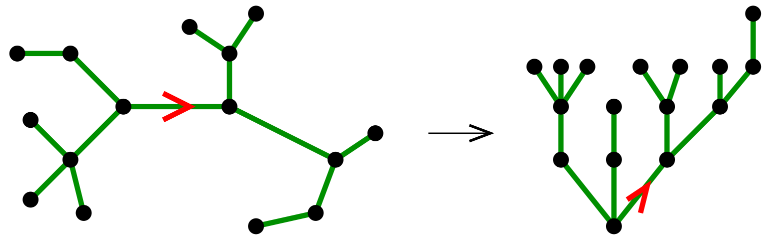
We consider *rooted* planar maps: a *root* edge is chosen around the infinite face and oriented counterclockwise.

Rooted planar maps. Examples

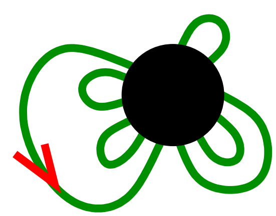
The smallest maps:



A planar map with only one face is a *plane tree*.

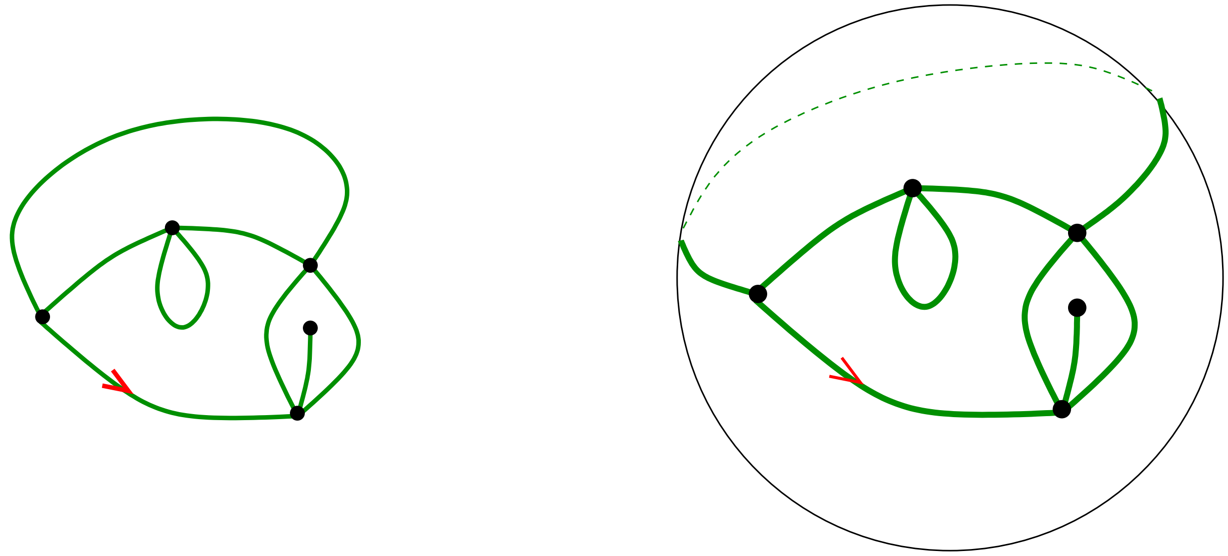


A planar map with only one vertex is a cycle of loops.



Rooted planar maps. On the sphere ?

Sometimes I like to replace the plane by a sphere . . .



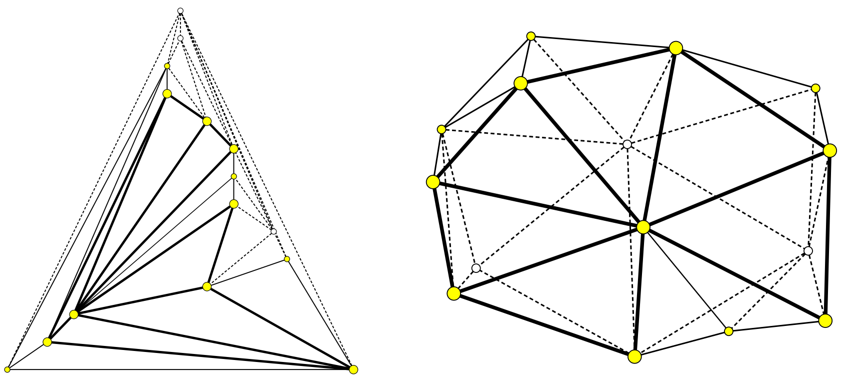
This is equivalent but looks more symmetric:
all faces are simply connected (=disc).

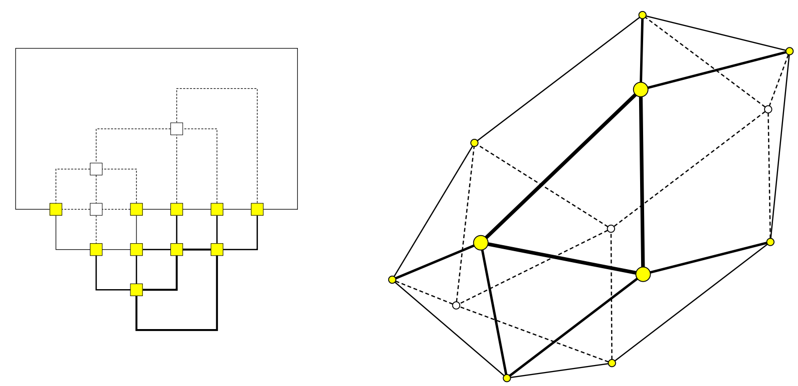
→ nicer pictures but that are more difficult to do . . .

Rooted planar maps. Example of subfamilies

Triangulations

4-regular maps





Such local restrictions should be irrelevant in the large size limit.

Compare to **simple trees**: m-ary trees, plane trees, 1-2 trees

$\Rightarrow$ they usually are all the same.

Enumeration of maps in combinatorics

as opposed to physics & enumerative topology

- Tutte (1962): *a census of triangulations*

  Originally to attack the four color theorem via enumeration.

- Counting planar maps (70's): Tutte, Brown, Mullin, Cori, Liu . . .

  Results for more than twenty subfamilies of planar maps.

  . . . Gao-Wormald (2001) *5-connected triangulations*

- Maps on surfaces (80's), random planar maps (90's):

  Bender, Canfield, Arquès, Gao, Richmond, Wormald, . . .

  For instance, Bender-Compton-Richmond (1999):

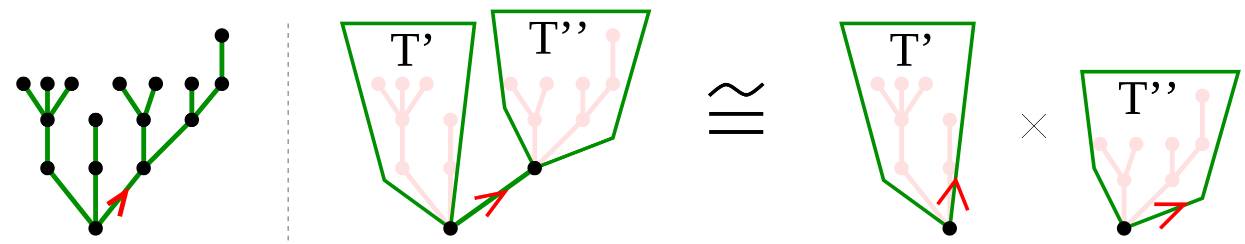
  0-1 laws for FO logic properties of random maps on surfaces.

Enumeration via generating functions

just one step away from plane trees...

Tutte's root deletion method. (i) plane trees

Usual *plane trees* are exactly maps with one face.



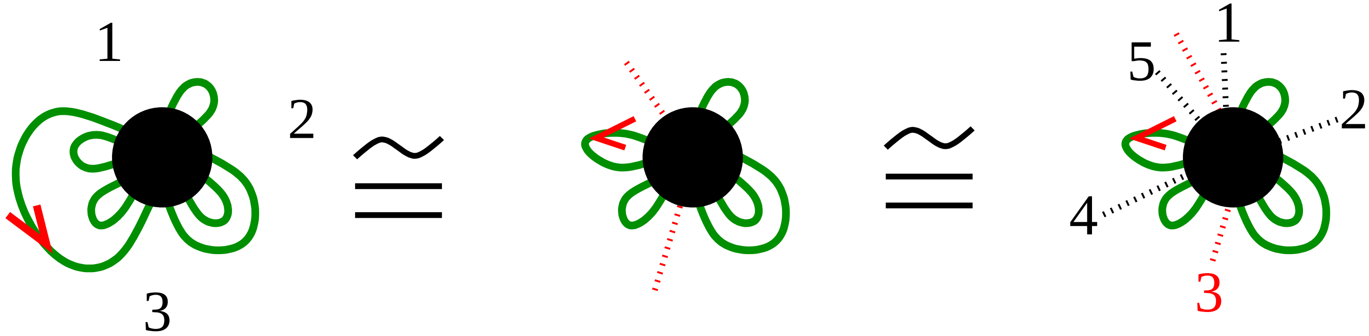
$$\sum_{T \neq \bullet} z^{|T|} \quad = \quad \sum_{T', T''} z^{1+|T'|+|T''|}$$

Thus the equation $t(z) - 1 \; = \; z t(z)^2$.

Plane trees (and in general simple trees) have algebraic GF.

Tutte's root deletion method. (ii) loops

A map with only one vertex is a *cycle of loops*.



$$\sum_{w \neq \bullet} z^{|w|} u^{d(w)} = \sum_{w} z^{1+|w|}(u + u^2 + \cdots + u^{d(w)+1})$$

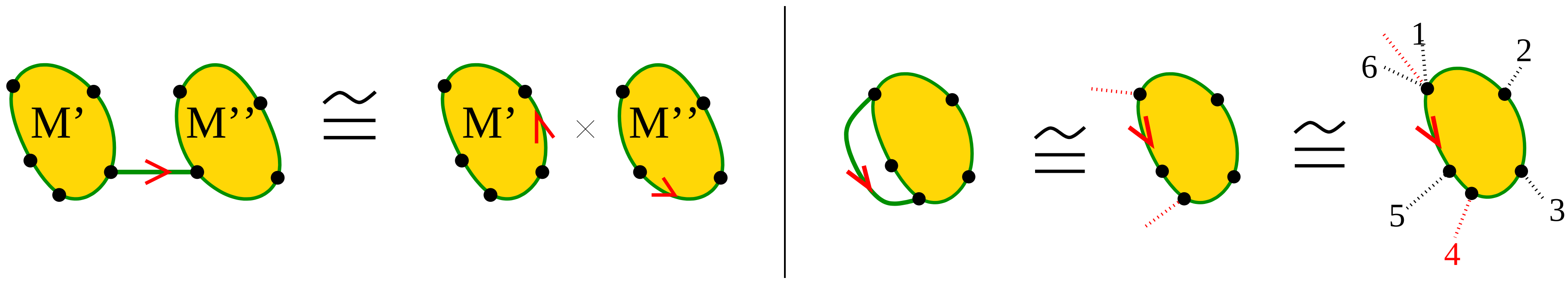
$$L(z,u) - 1 = \tfrac{zu^2}{u-1} L(z,u) - \tfrac{zu}{u-1} \ell(z).$$

A linear equation in $L(z,u)$ with polynomial coeffs in z, u and $\ell(z)$:

$$(u - 1 + zu^2) L(z,u) = u - 1 - zu\,\ell(z).$$

Tutte's root deletion method. (iii) all maps

The two previous cases generalize:



$$F(z,u) - 1 = zu^2 F(z,u)^2 + \frac{zu}{u-1}(uF(z,u) - f(z))$$

or equivalently *(dependences in z hidden)*

$$(u-1)zu^2F(u)^2 + (u-1+zu^2)F(u) + (u-1-zuf) = 0$$

a *quadratic* equation in $F(u)$ with polynomial coeffs in z, u and f.

Linear equations with a catalytic variable. The kernel method.

The kernel method for $\quad K(u)L(u) = q(z, u, \ell)$:

- *Look for a root u_0 of $K(u)$ such that $L(u_0)$ makes sense.*

Here $L(u) \in \mathbb{C}[u][[z]]$ and the roots of $K(u)$ are

$$u_1 = \frac{1+\sqrt{1-4z}}{2z} = 1/z + O(1) \text{ and } u_2 = \frac{1-\sqrt{1-4z}}{2z} = z + O(z^2).$$

$L(u_1)$ is not ok but $L(u_2)$ converges as a formal power series.

The substitution $u \leftarrow u_0$ in the linear equation gives

$$0 = u_2 - 1 + zu_2\,\ell, \qquad \text{so that } \ell(z) = \frac{1-\sqrt{1-4z}}{2z}$$

(See also Cyril Banderier's talk)

Polynomial equations with a catalytic variable.

Bousquet-Melou's method — (extends kernel & Tutte's quadratic methods)

$F(z,u)$ and $f(z)$ such that there is a polynomial $P(a,b,c)$ with

$$P(F(u),u,f)=0 \qquad \textit{(dependence in } z \textit{ hidden)}$$

- Differentiate with respect to u:

$$F'_u(u)P'_a(F(u),u,f)+P'_b(F(u),u,f)=0$$

- Suppose we find $u_0=u_0(z)$ such that $F(u_0)$ is well defined and

$$P'_b(F(u_0),u_0,f)=0.$$

Then $\quad P'_a(F(u_0),u_0,f)=0 \quad$ and $\quad P(F(u_0),u_0,f)=0.$

A polynomial system in $F(u_0)$, u_0, f: algebraic solutions !

Rooted planar maps. The solution

- We obtain an algebraic generating function $f(z)$:

$$f(z) = \sum_M z^{|M|} = 1 - \frac{1 - 18z - (1 - 12z)^{3/2}}{54z^2}$$

(remark that $F(z, u)$ is also algebraic $\rightarrow$ face degree)

- Transfert theorems (*e.g.*) yield an asymptotic expansion:

$$\#\{\text{rooted maps with } n \text{ edges}\}\ c \cdot n^{-5/2} \cdot 12^n.$$

The exponent 5/2 is characteristic of planar map emunerations (compare to 3/2 for various simple trees).

A first summary.

- Polynomial equations with one catalytic variable should have algebraic solutions (cf. Mireille Bousquet-Mélou):

$$P(F(z,u),\ z,\ u,\ f_1(z),\ldots,f_k(z)) = 0$$

(if you know examples, we are interested in collecting them !)

- Root deletion applies to many families of maps and yields "universal" asymptotic behavior:

$$\#\{\text{rooted } \mathcal{F}\text{-maps of size } n\} = c\rho^{-n} n^{-5/2}$$

where c and ρ depend on the family $\mathcal{F}$.

- In some cases the explicit formulas are nice.

Nice formulas, random maps and why planar maps are almost Galton-Watson trees

Tutte's formulas for rooted planar maps. (60's)

The root deletion method provides surprisingly nice formulas in several cases, among which:

$$\#\{\text{triangulations with } 2n \text{ faces}\} = \frac{2}{2n+2}\frac{2^n}{2n+1}\binom{3n}{n} \sim \frac{c_1}{n^{5/2}}\,(27/2)^n$$

$$\#\{\text{4-regular maps with } n \text{ vert.}\} = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n} \sim \frac{c_2}{n^{5/2}}\;12^n$$

All families should behave the same

$\Rightarrow$ concentrate on those simpler models !

(like binary trees in tree enumeration, bernoulli walks, ...)

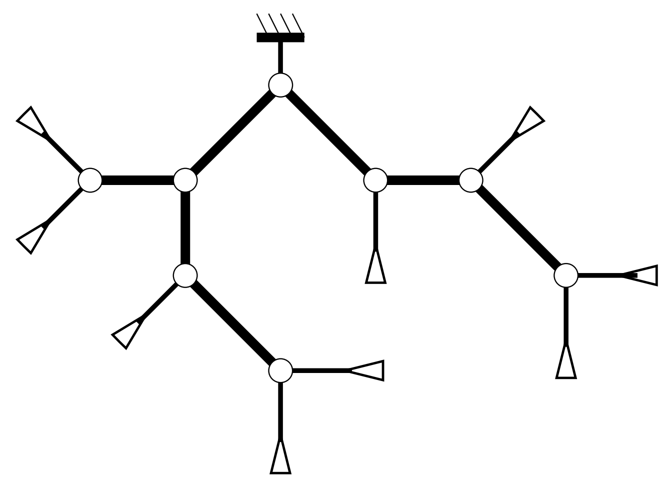
Tutte's formulae. A bijective proof (i).

$\#\{$ 4-regular maps with n vertices $\}$ is $\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}$.

There are

$$\frac{1}{n+1}\binom{2n}{n}$$

binary trees with n nodes.



Such trees have n (internal) nodes and $n+2$ leaves (root included).

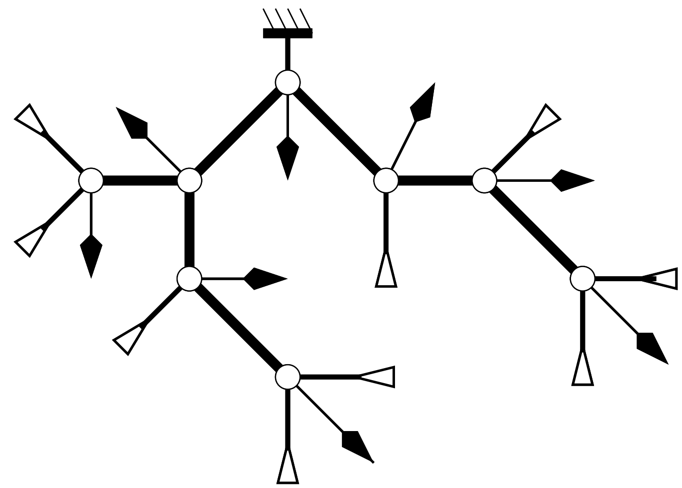
Tutte's formulae. A bijective proof (ii).

$\#\{$ 4-regular maps with n vertices $\}$ is $\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}$.

On each node, a bud can be added in three ways, giving rise to

$$\frac{3^n}{n+1}\binom{2n}{n}$$

blossom trees with n nodes.



Blossom trees have n buds and $n+2$ leaves around the tree.
Upon matching them counterclockwise, two leaves remain *unmatched*.

Tutte's formulae. A bijective proof (ii).

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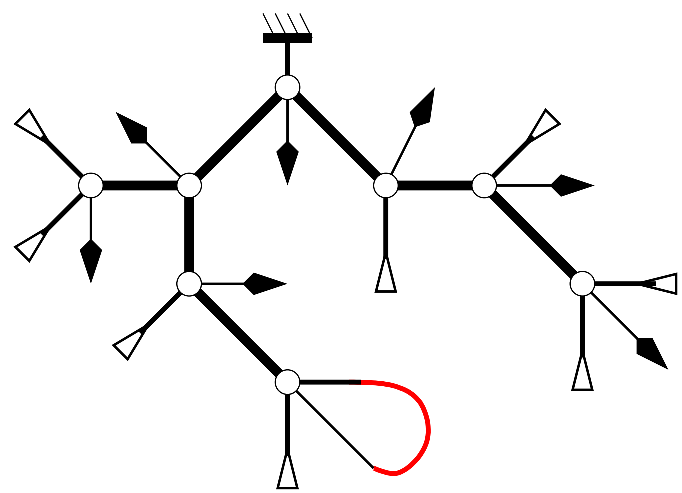

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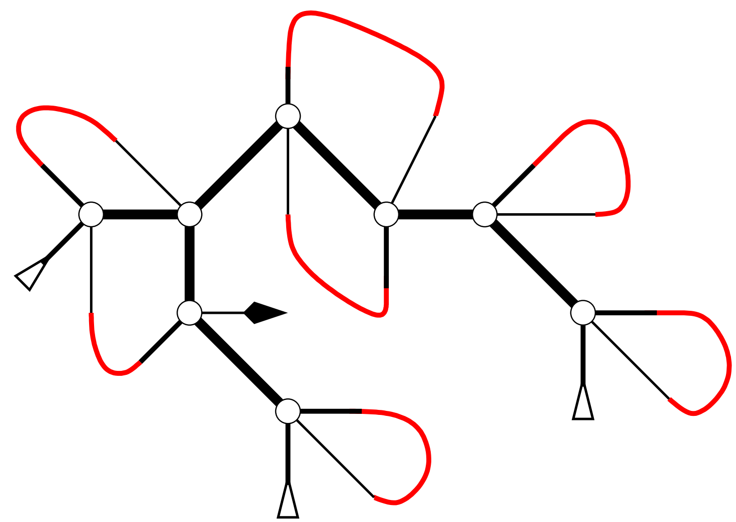

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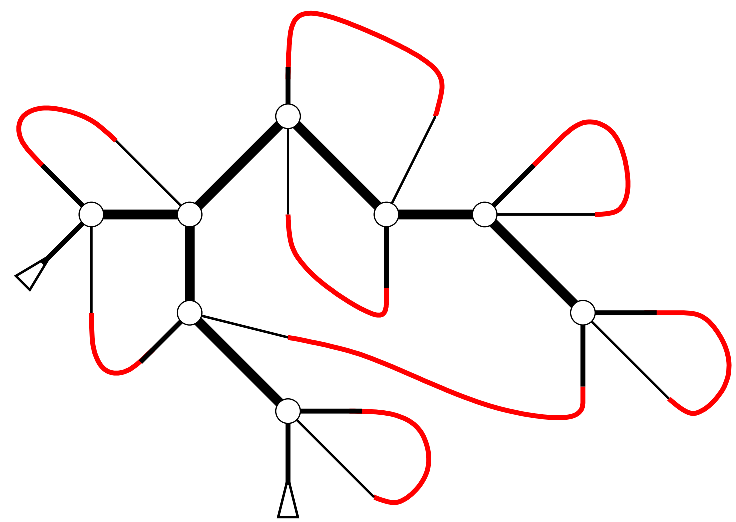
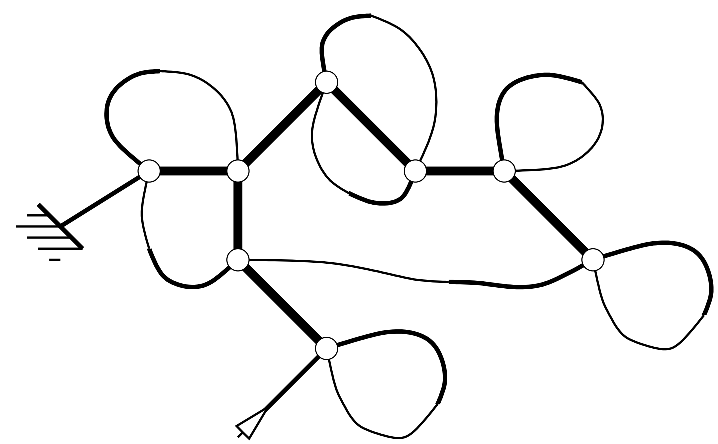


Blossom trees have n buds and $n+2$ leaves around the tree.
Upon matching them counterclockwise, two leaves remain *unmatched*.

Tutte's formulae. A bijective proof (iii).

#{ 4-regular maps with n vertices } is $\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}$.

The matching procedure does not depend on which leaf is the root. A blossom tree is *balanced* if its root remains unmatched.



Each *conjugacy class of trees* contains $n+2$ blossom trees, 2 of which are balanced: the number of balanced blossom tree is thus

$$\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}.$$

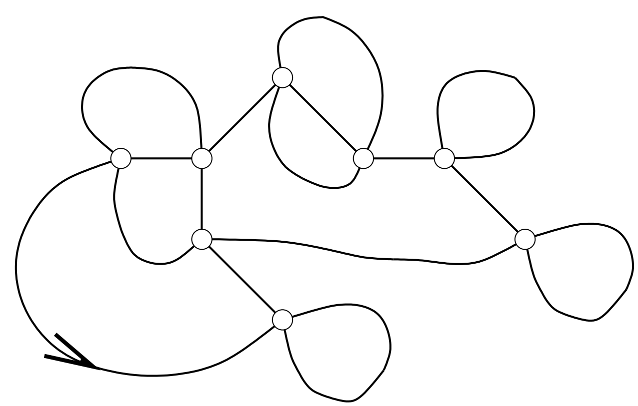
Tutte's formulae. A bijective proof (iv).

#{ 4-regular maps with n vertices } is $\frac{2}{n+2} \cdot \frac{3^n}{n+1}\binom{2n}{n}$.

Theorem (S. 1998):

Closure is one-to-one between

- balanced blossom trees with n nodes
- and 4-regular maps with n vertices.



The converse bijection is based on a bfs traversal of the dual graph.

Random planar maps

The random planar map.

Random planar maps are defined by:

the uniform distribution on rooted planar maps with n edges.

But we can as well use a subfamily:

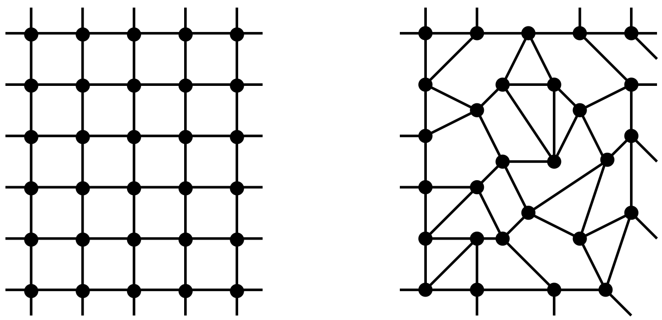
- uniform on 4-regular maps with n vertices
- uniform on balanced blossom trees with n nodes
- uniform on blossom trees with n nodes
- G.W. trees with 3 types of offspring 2, conditioned to have n nodes

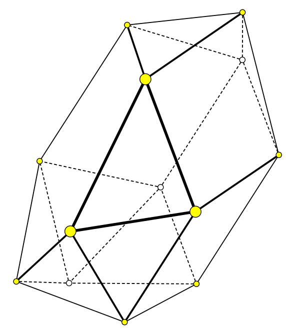
$\Rightarrow$ map parameters lead to fancy parameters on trees.

Random planar maps as random lattices

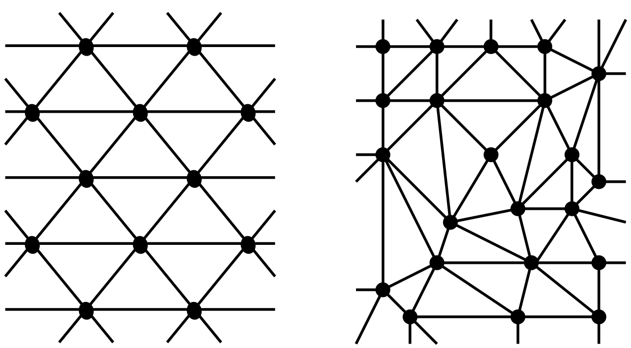
In physics papers, they would rather take:

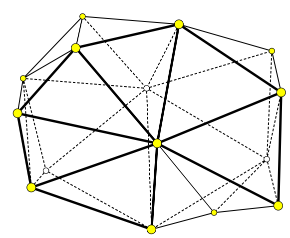
random 4-regular maps (ϕ^4 lattice model).





or random triangulations (dual ϕ^3 model).





Why random maps in physics ? (a naive point of view)

Consider a 2d universe...

- Conventional gravity: the universe is flat.

  $\Rightarrow$ discretised by a regular grid.

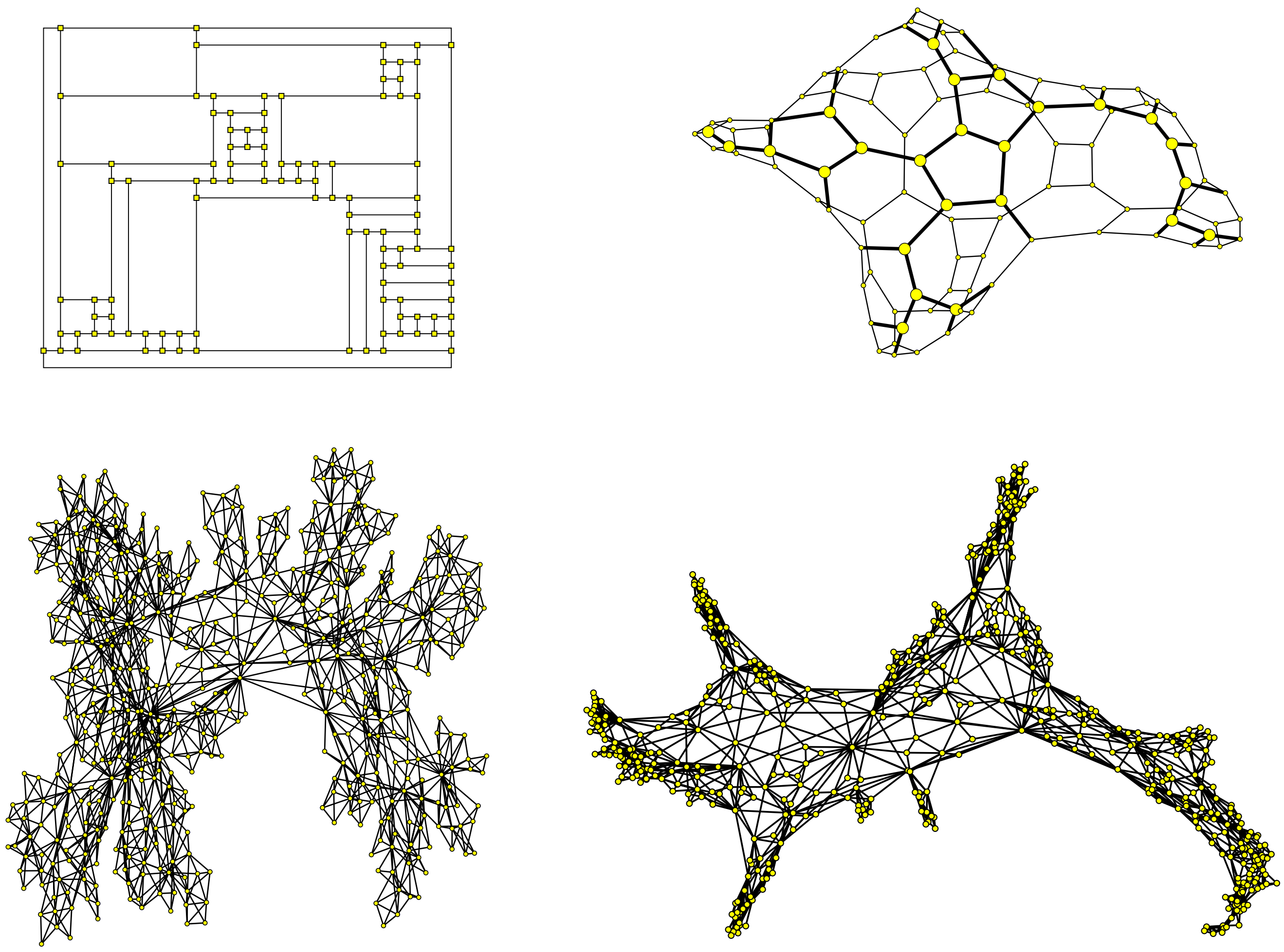
- Quantum gravity: a distribution of proba on possible universes.

  $\Rightarrow$ discretised by a random map.

- planar case is easier $\Rightarrow$ assume spherical topology to start with.

This lead some physicists to rediscover many formulas of Tutte using "perturbative expansion of matrix integrals".

A gallery of random maps

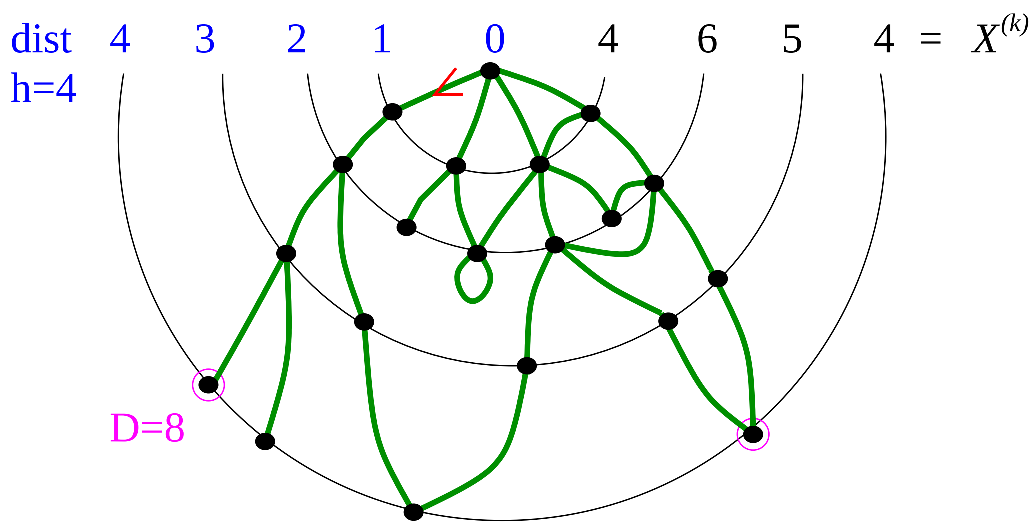


What is the typical geometry of a random map ?

(or triangulations or 4-regular maps, . . .)

The random planar map. Profile and diameter (i).

- $X_n^{(k)}$ is the number of vertices at distance k of the root
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \dots, X_n^{(k)}, \dots)$



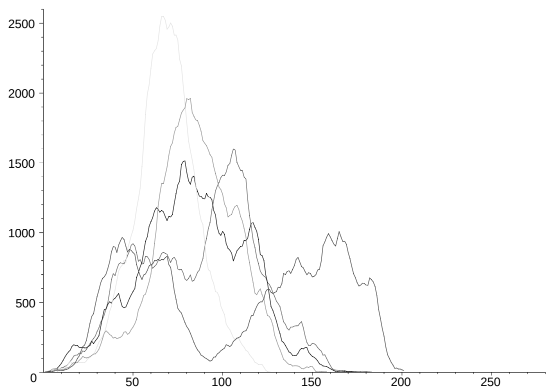
- h_n is the height (maximal distance from the root)
- D_n is the diameter of a random n-triangulation

In particular $h_n \leq D_n \leq 2h_n$.

The random planar map. Distances and diameter (ii).

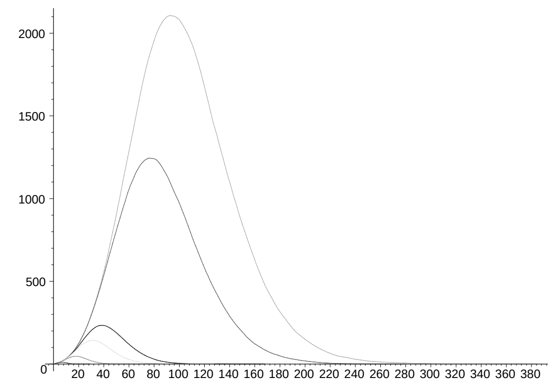
Experimentation using random sampling algorithms:

Six random profiles:



All for maps of size $n = 100,000$.

Averaged profiles:



For various n (100 to $100,000$).

$\Rightarrow$Conjecture (S. 1998) The correct scaling is $k = tn^{1/4}$.

For this scaling I expect normalised $X_n^{(k)}$ to converge to a random process $X(t)$ supported on $\mathbb{R}^+$.

The random planar map. Distances and diameter (iii).

In particular this "should" imply

- Two beautyful heuristic calculations by physicists Watabiki, Ambjørn *et al.* (1994:) *The Hausdorff dimension is* $1/4$ *:*

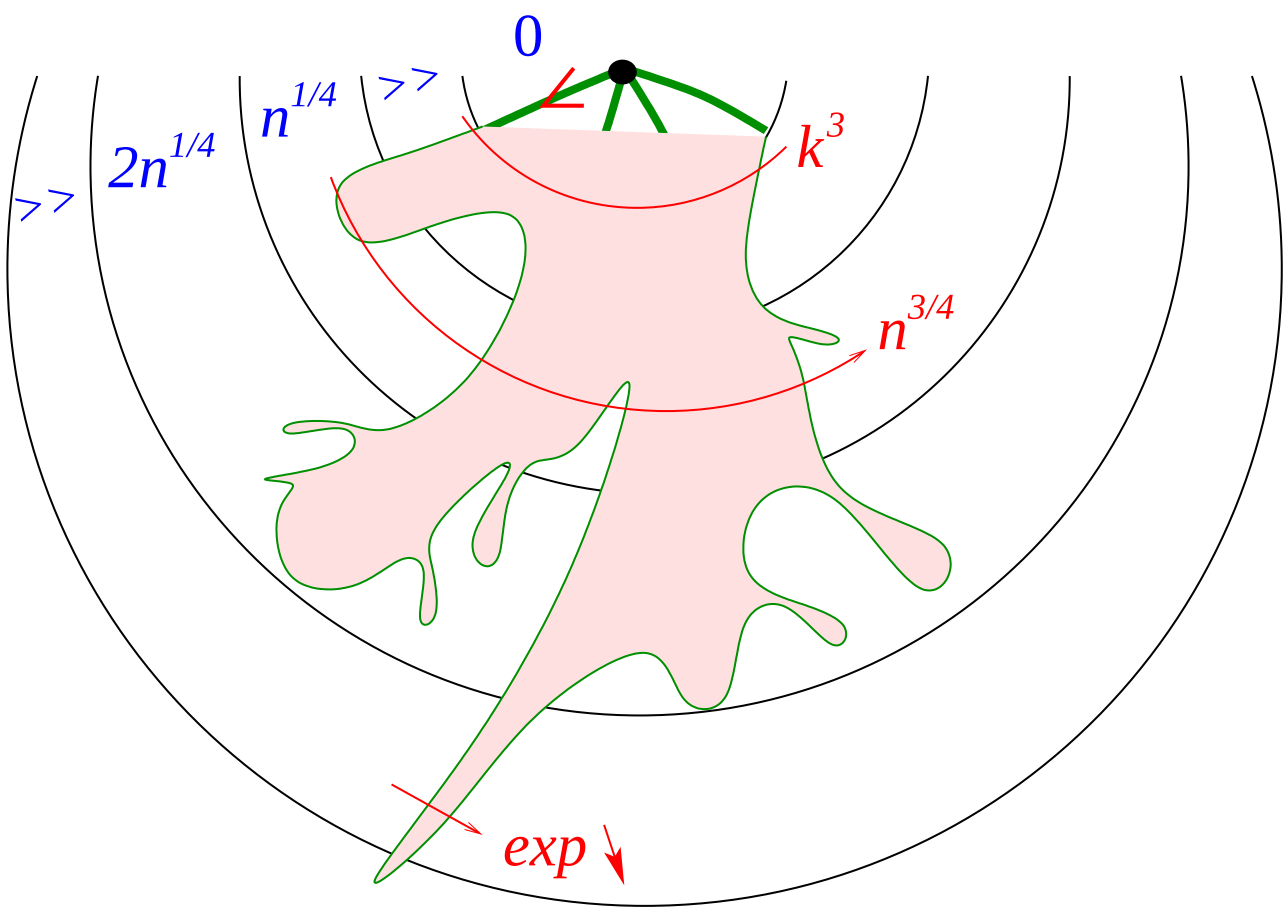
$$\text{meaning} \quad \text{for } k \ll n^{1/4}, \qquad \mathbb{E}(\textstyle\int_0^k X_n^{(i)}) \sim k^4,$$
$$\text{for } k \gg n^{1/4}, \qquad \mathbb{E}(X_n^{(k)}) \text{ is exp. decreasing}$$

- Conjecture (S. 2001):

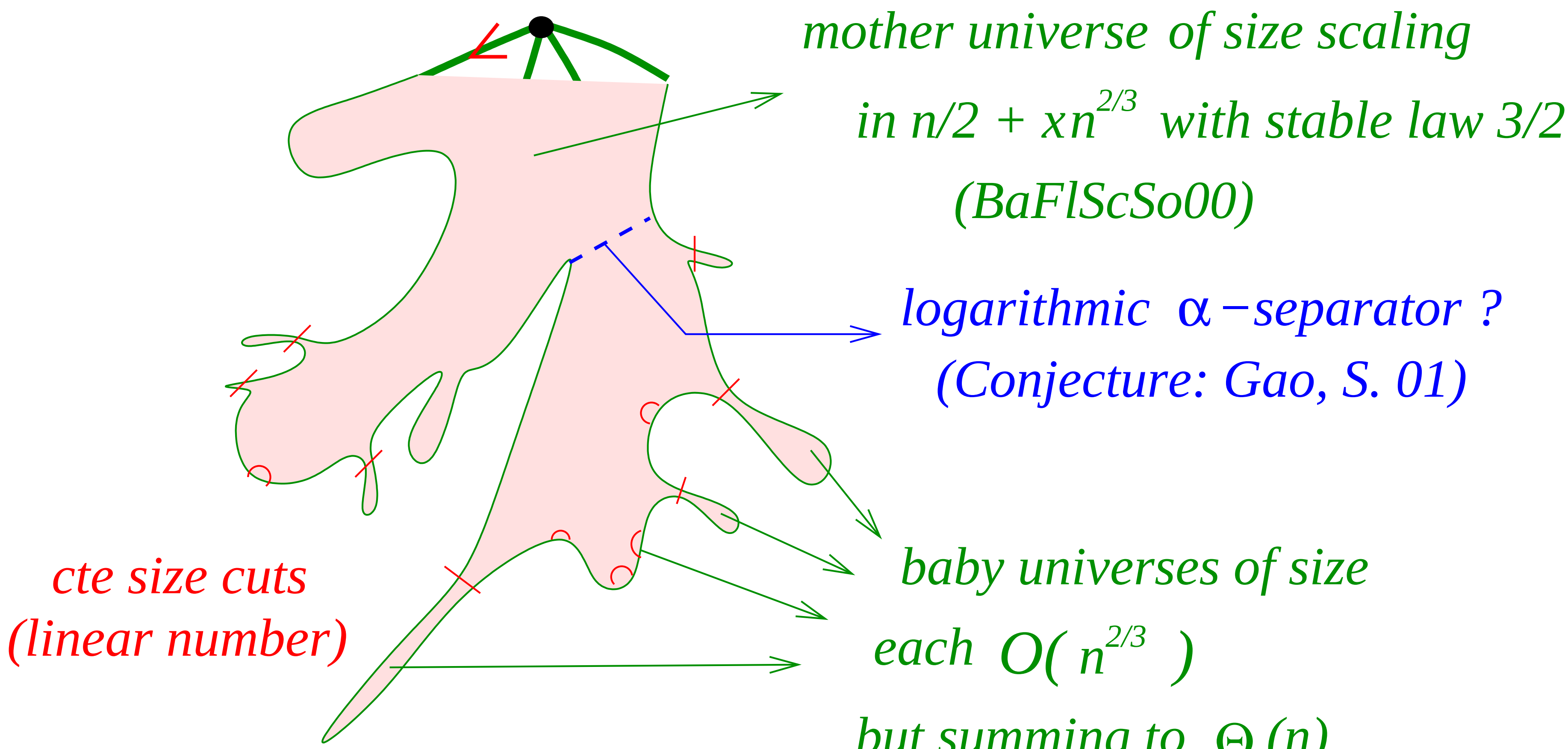
$$\mathbb{E}(h_n) \sim n^{1/4}\, e^{\alpha - \alpha (e/n)^{1/4}} \text{ where } \alpha = \sqrt{2 + \tfrac{13}{6}\sqrt{3}}.$$

(constant α given here for loopless cubic maps).

The random planar map. A tentative picture of distances.



The random planar map. A tentative picture of cuts.



A second summary

- The random planar maps model has many variants (triangulations, bipartite maps, convex polyhedra, . . .)

- Parameters of interests have similar flavor as for simple trees (profile, height, maximal degree, 0-1 laws, . . .)

- All knows results satisfy the expected “universality”: critical exponents agree for different families.

An application to knot theory:

the asymptotic number of prime alternating links

join work with Sébastien Kunz-Jacques.

Knots and links.

- The unknot is the simplest knot . . .
  A knot is made of one lace, a link may have more.

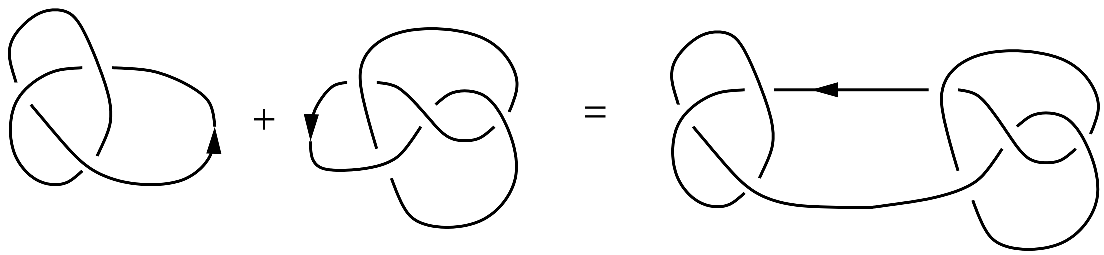
- A planar diagram of a link: a generic projection.

- The size of a link is its minimal number of crossings in a planar diagram.

- The 3 Reidemeister moves connect all its diagrams.

Knots and links. Prime factor decomposition

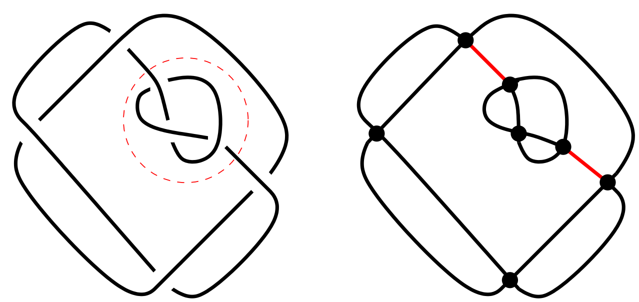
The **product** of two knots.



Knots and links have a **unique** decomposition in **prime** factors.

Prime links cannot be decomposed:
no 2-cut in their minimal diagrams

Example of defect of primality $\longrightarrow$



Knots and links. Enumerative knot theory ?

Count prime knots and links w.r.t. number of crossings !

or equivalently

Count equivalence classes of diagrams under Reidemeister moves.

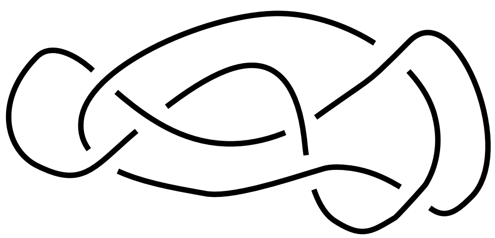
This seems to be a very difficult problem...

We shall restrict our attention to easier subclasses.

Knots and links. Alternating links

Alternating diagrams: each edge is undercrossing at one end,

and overcrossing at the other.



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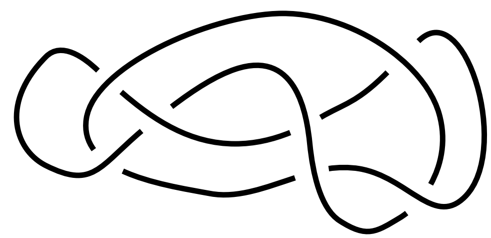


II

Find which one is alternating !

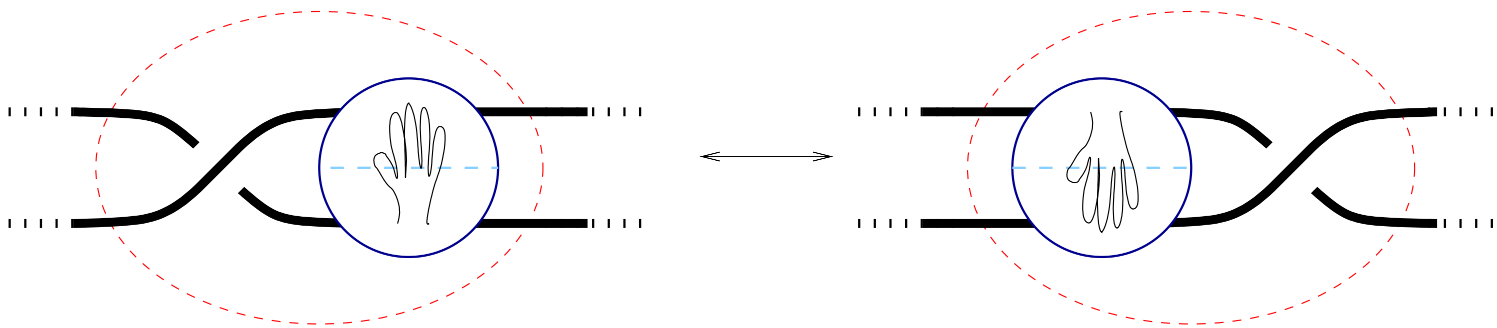
An alternating link is one that admits an alternating diagram.

Not all knots are alternating,

but these have nicer properties than general knots...

Knots and links. Flype and Tait's conjecture

A **flype** transforms one diagram into another:



Theorem (Menasco and Thistlethwaite, 1993)
Any two prime alternating diagrams of a prime alternating link are connected by a sequence of flypes.

Corollary. *All prime alternating diagrams of a prime alternating link have the same size (= number of crossings).*

The number of prime alternating links

A simpler problem ? Count prime alternating links

or equivalently

Count equivalence classes of diagrams under the action of flypes.

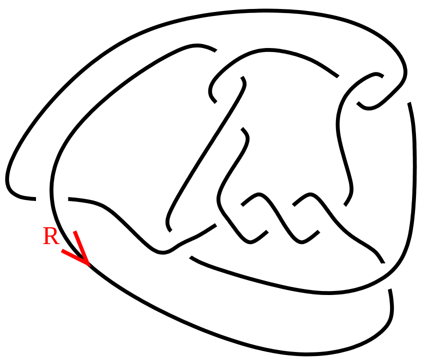
Theorem (Sundberg and Thistlethwaite, 1998)
The number A_n of prime alternating links of size n satisfies

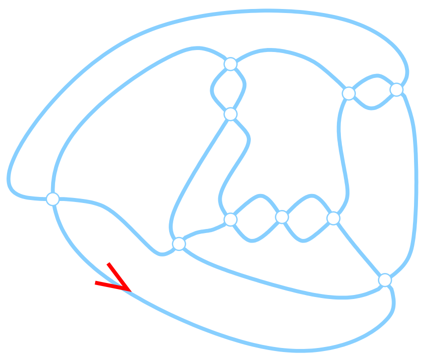
$$c_1 \lambda^n n^{-7/2} \leq A_n \leq c_2 \lambda^n n^{-5/2}.$$

Our aim: the exact asymptotic behavior.

Our means: analytic combinatorics of random planar maps.

Diagrams and planar maps.





Proposition. *There is a one-to-one correspondence between*

- *rooted (prime) alternating diagrams with n nodes,*
- *rooted 4-regular planar maps (without 2-cut) with n vertices.*

Idea: The over-undercrossing structure of the root vertex can be consistently propagated to all others.

Rooted diagrams. Enumeration.

We have seen that

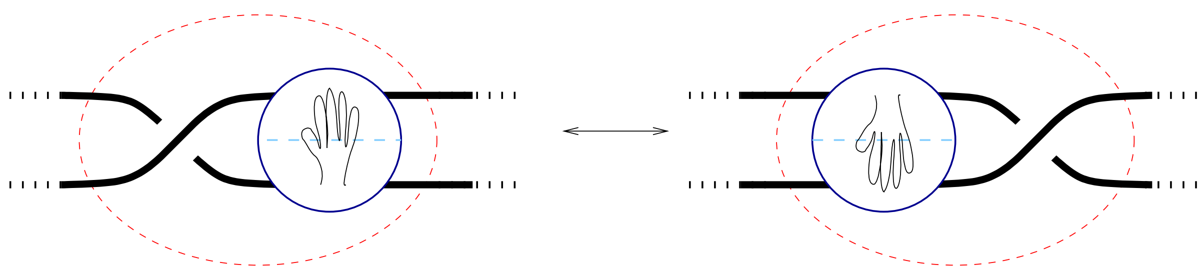
$$\#\{\text{rooted diagrams of size } n\} = \frac{2}{n+2}\frac{3^n}{n+2}\binom{2n}{n}.$$

Similarly

$$\#\{\text{rooted prime diagrams of size } n\} = \frac{4}{2n+2}\frac{1}{2n+1}\binom{3n}{n}.$$

(proof by root deletion or bijection with *ternary* blossom trees).

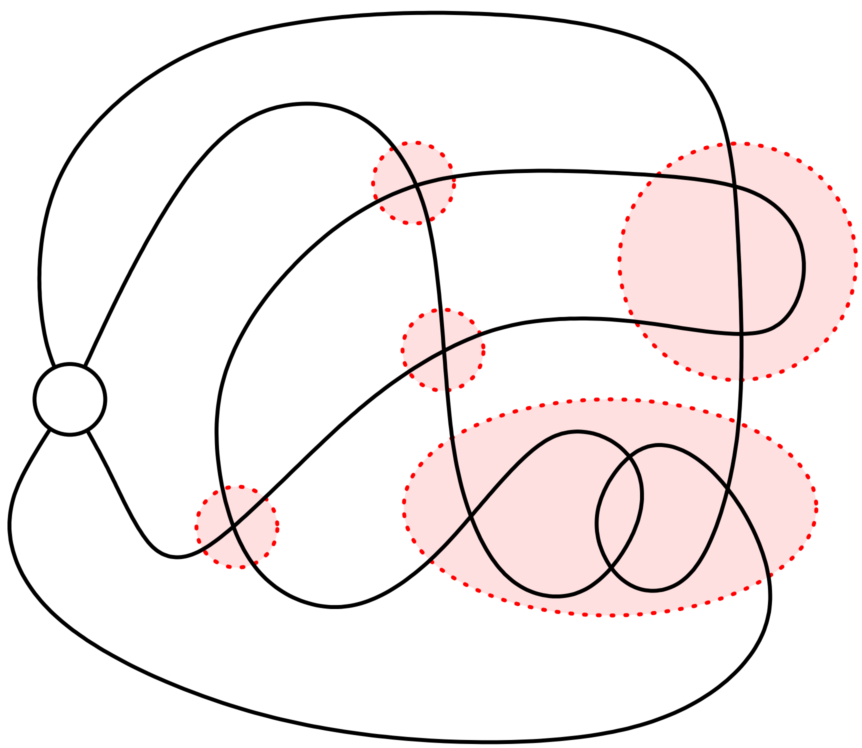
But we need to take flypes into account.



Flypes act inside "Conway circles" *i.e.* 4-cuts.

Rooted diagrams. Conway circle decomposition.

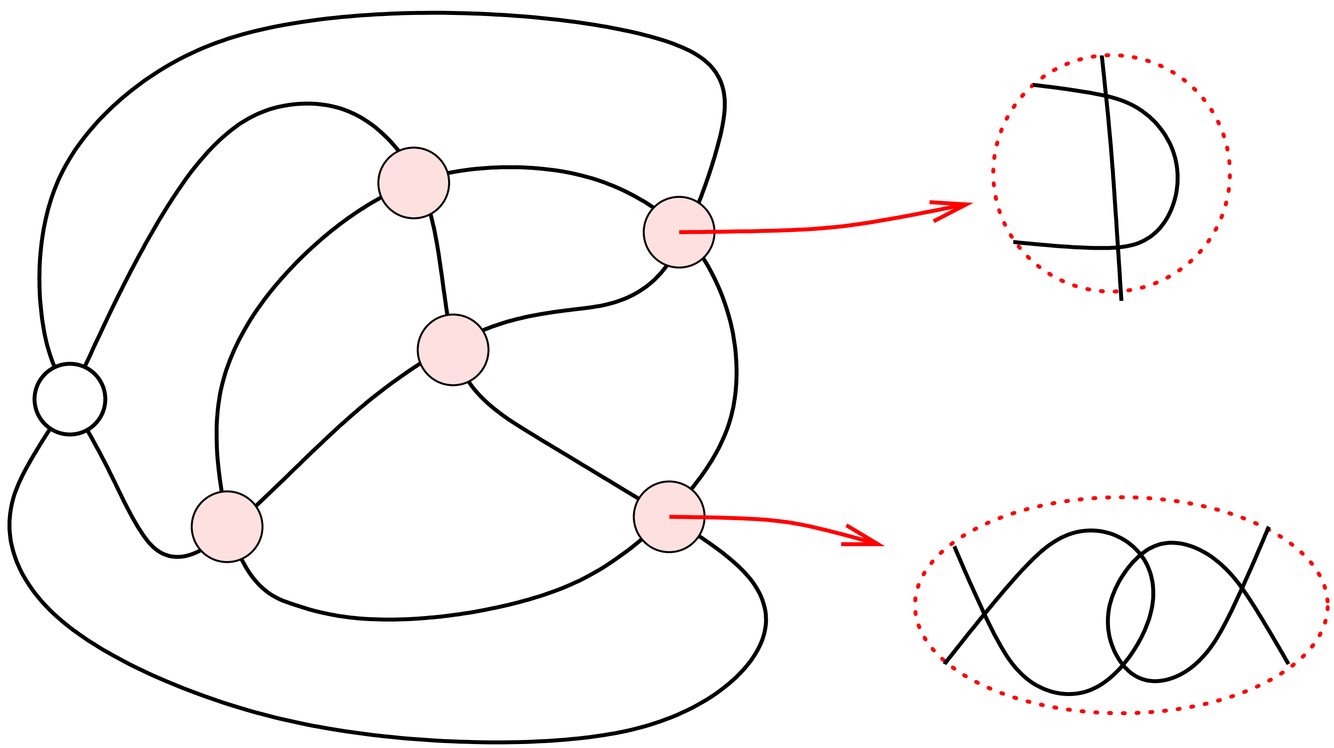
Look for *maximal* Conway circles.



$\Rightarrow$ they define a tree like decomposition.

Rooted diagrams. Conway circle decomposition.

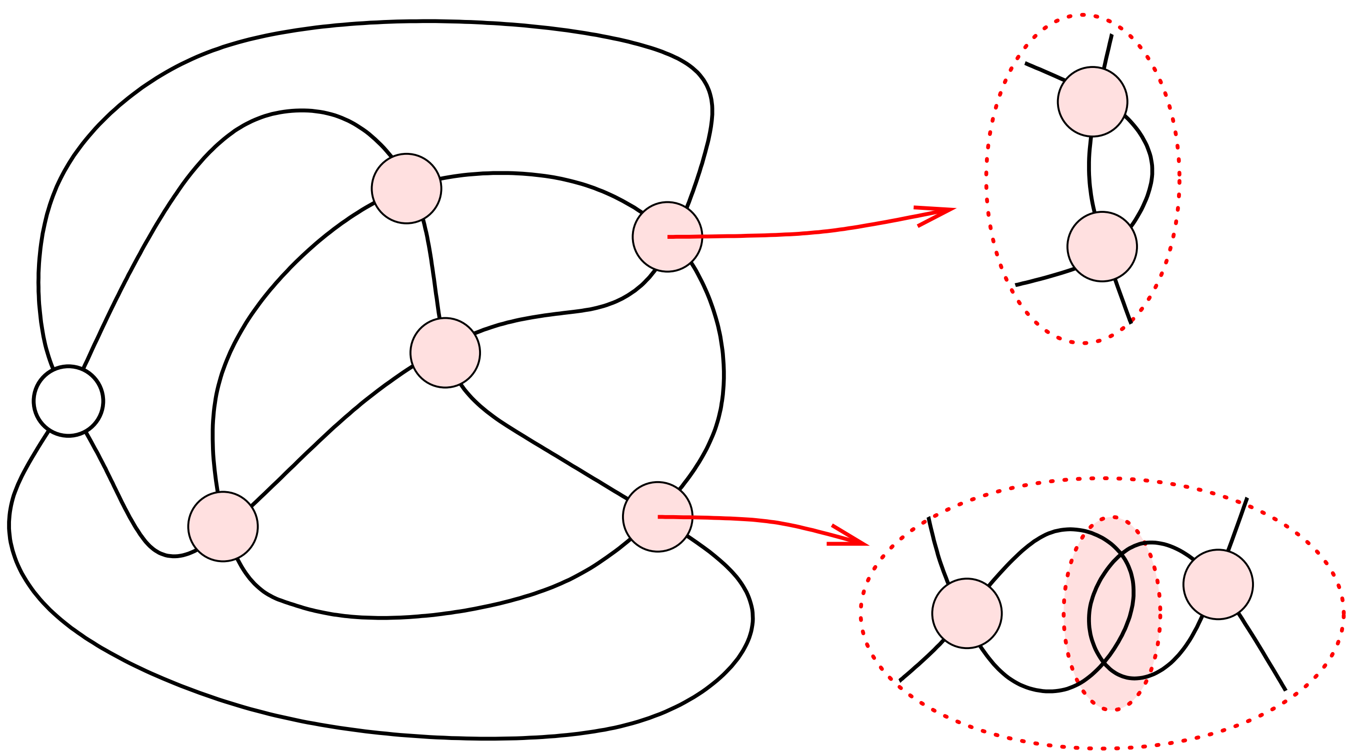
Look for *maximal* Conway circles.



⇒ they define a tree like decomposition.

Rooted diagrams. Conway circle decomposition.

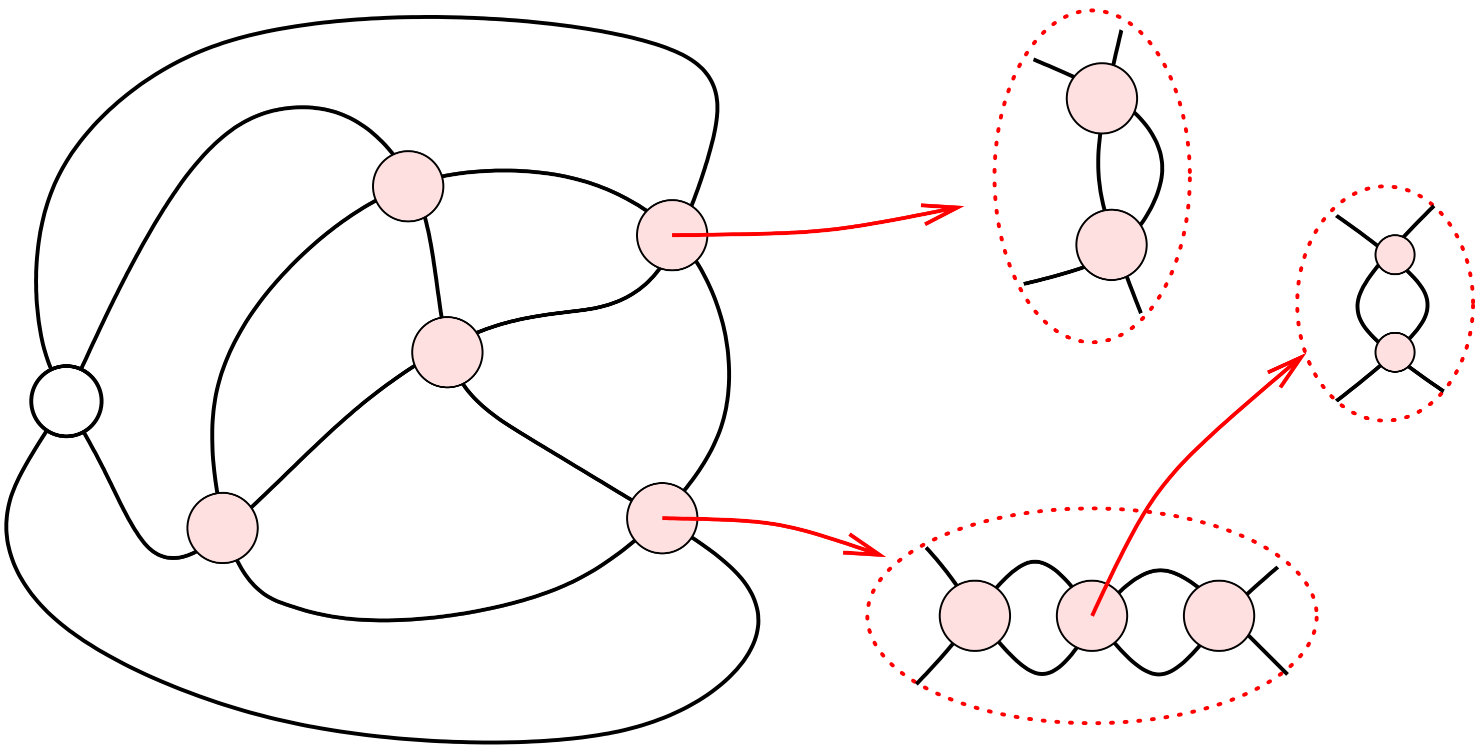
Look for *maximal* Conway circles.



$\Rightarrow$ they define a tree like decomposition.

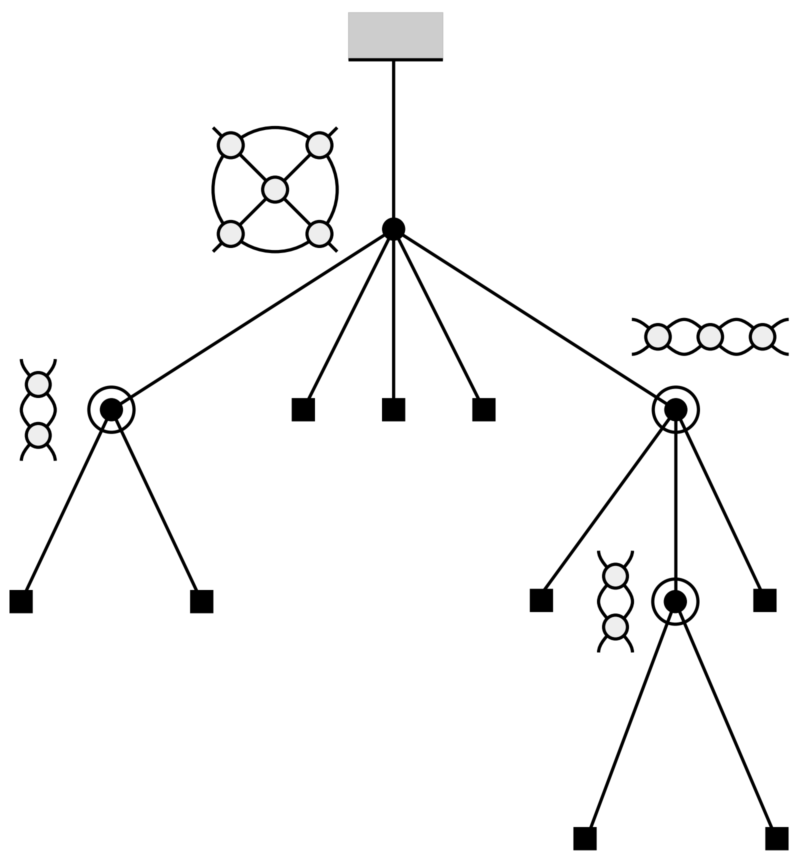
Rooted diagrams. Conway circle decomposition.

Look for *maximal* Conway circles.



⇒ they define a tree like decomposition.

Rooted diagrams. Tree-like structure.



"Nodes" of this tree are 4-regular maps of two type:

- indecomposable (4-cut free),
- vertical or horizontal sums.

Node degree is number of circles.

Leaves are the original crossings.

Rooted diagrams. Equations for rooted diagrams.

Let $I(z)$, $V(z)$, $H(z)$ be GF of "trees" according to the root node (indecomposable, v- or h-sum). Then

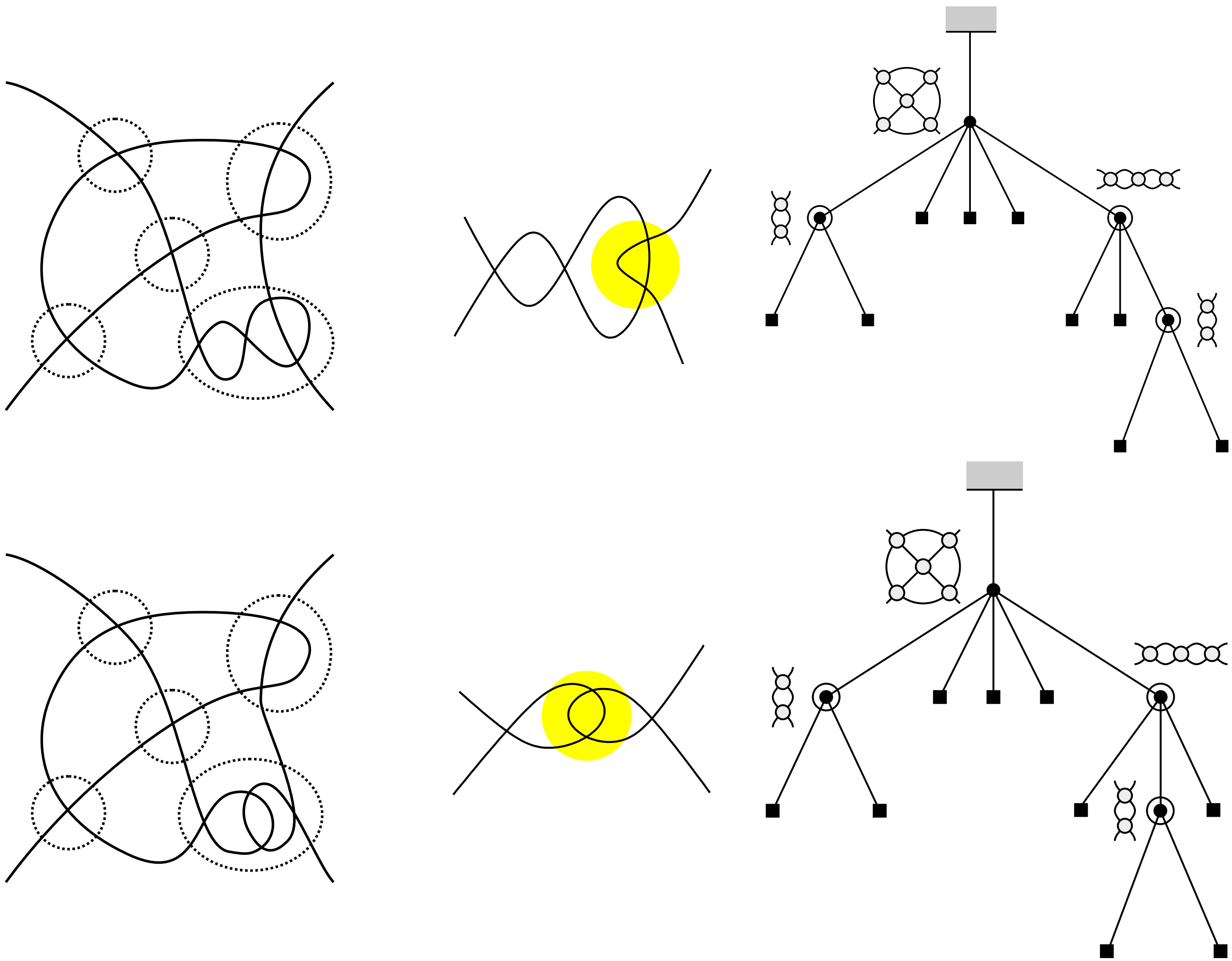
$$\begin{aligned} D(z) &= z + I(z) + V(z) + H(z) \\ I(z) &= \sum_{k\geq 3} p_k D(z)^k = P(D(z)) \\ V(z) &= H(z) = \frac{(z + I(z) + V(z))^2}{1 - (z + I(z) + V(z))}, \end{aligned}$$

$P(z) = \sum p_k z^k$ is GF of indecomposable nodes (4-cut free maps).

But $D(z)$ is GF of rooted diagrams (known and algebraic)

$\Rightarrow$ all series (including P) are algebraic.

Flypes act on the decomposition.



Articulated trees

Theorem (adapted from Sundberg & Thistlethwaite):

There is a one-to-one correspondence between

- *rooted prime alternating links of size n, and*
- *articulated trees with n leaves.*

To count articulated tree, put them in normal form !

Articulated trees. Equations

Let $\hat{I}(z)$, $\hat{V}(z)$, $\hat{H}(z)$ be GF of articulated trees.

Then

$$\begin{aligned}
\hat{D}(z) &= z + \hat{I}(z) + \hat{V}(z) + \hat{H}(z) \\
\hat{I}(z) &= P(\hat{D}(z)) \\
\hat{V}(z) &= \hat{H}(z) \;=\; \tfrac{1}{1-z}\tfrac{1}{1-(\hat{I}+\hat{V})} - 1 - z - (\hat{I}+\hat{V}).
\end{aligned}$$

The series P is still the same

$\Rightarrow$ all series (including P) are algebraic.

Articulated trees. Asymptotic number.

Theorem (Sundberg & Thistlethwaite, 1998):

The asymptotic number of rooted prime alternating links satisfies

$$a_n \sim c_0 \lambda^n n^{-5/2}$$

with

$$\lambda = \frac{101 + \sqrt{21001}}{40} \approx 6.15$$

and c_0 another known algebraic constant.

Prime alternating links. Unrooting

From the point of view of knot theory, rooting is not natural:

we really want the number of unrooted links

Tight bounds on the number of rootings for links give:

Theorem (Sundberg and Thistlethwaite, 1998)
The number A_n of prime alternating links of size n satisfies

$$C_1 \lambda^n n^{-7/2} \leq A_n \leq C_2 \lambda^n n^{-5/2}.$$

Can we do better by estimating the number of rootings of a random link ?

Prime alternating links. Unrooting

For random planar maps, unrooting is trivial:

Theorem (Wormald, 1994)
A random planar map with n edges has almost surely $8n$ rootings (with exponential bounds).

In other terms, symmetric maps are exponentially negligible among large random maps.

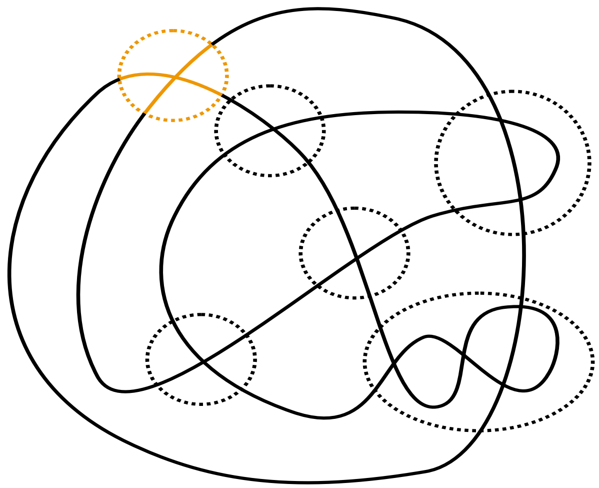
As now usual, a "universal" result.

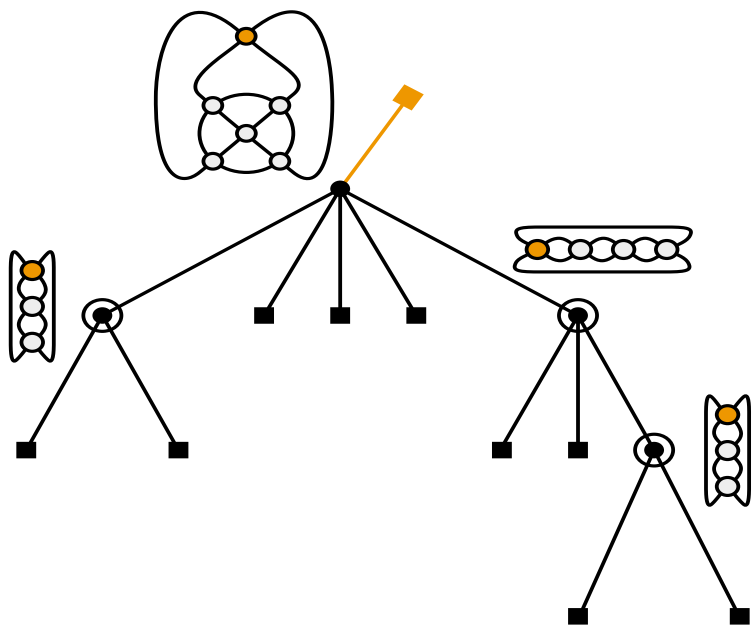
But for links the situation is more complicated.

Prime alternating links. Unrooting

There is an interference between:

possible rootings and *flype equivalence*





Prime alternating links. Unrooting

The following steps allow to circumvent this difficulty:

- The parameter *number of rootings* is compatible with the tree-like decomposition.

  $\Rightarrow$ marking in GF + singularity analysis

Theorem (S., Kunz-Jacques 2000)
The expected number of rootings is cn with concentration

- Global symmetries can be proved exponentially neglegible.

Prime alternating links. Final result

Theorem(S., Kunz-Jacques 2000)

The number of prime alternating links of size n satisfies for n going to infinity:

$$A_n \sim \frac{a_n}{8cn} \sim c' \lambda^n n^{-7/2}$$

where

$$\lambda = \frac{101 + \sqrt{21001}}{40} \approx 6.15 \qquad \text{and} \qquad c = \frac{1}{2}\left(\frac{371}{\sqrt{21001}} - 1\right) \approx 0.78.$$

and c' is a known algebraic constant.

Corollaries: parameters of random links.