Binary trees, super-Catalan numbers

and

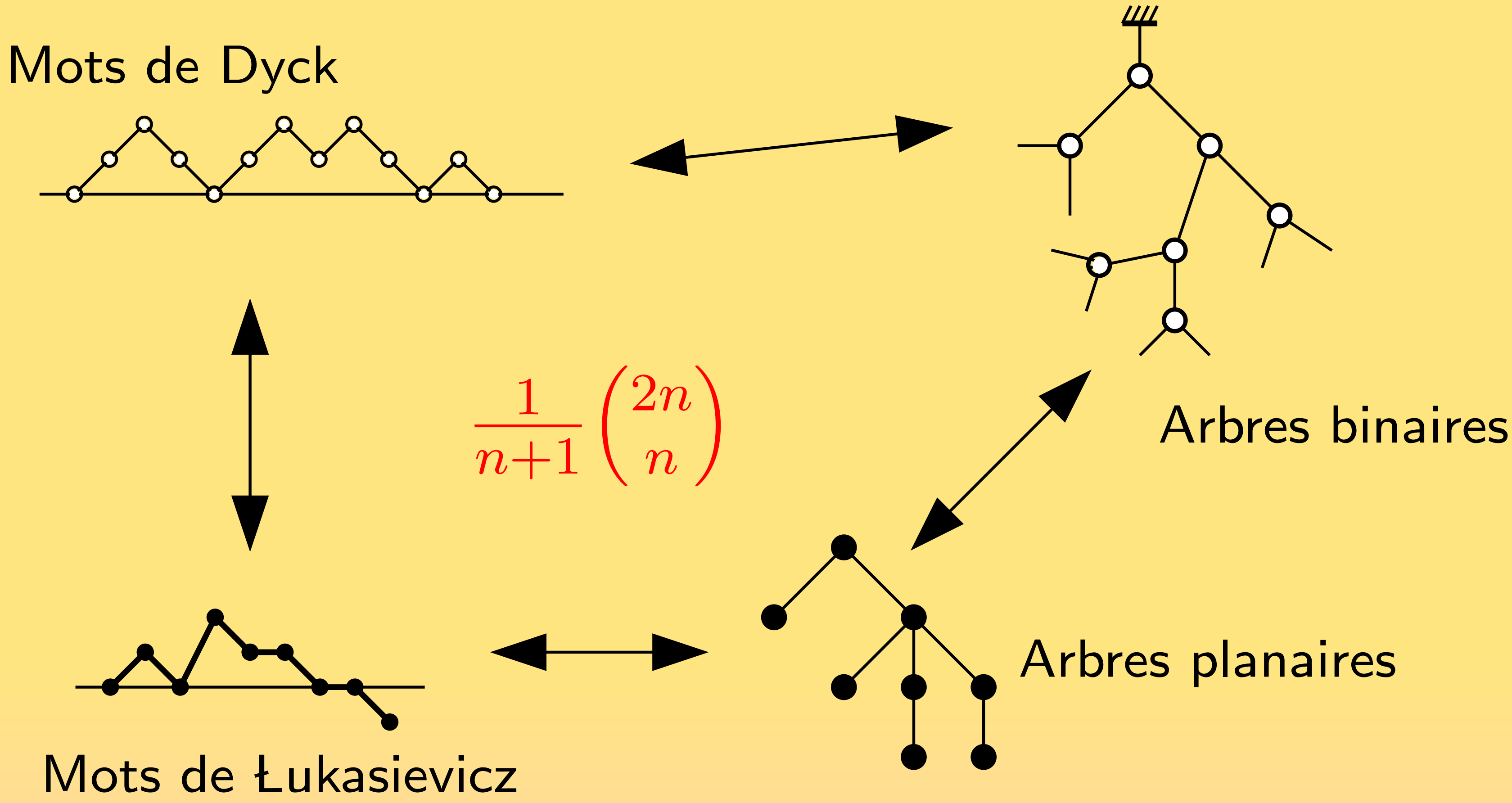
3-connected Planar Graphs

Gilles Schaeffer

LIX, CNRS/École Polytechnique

Based in part on a joint work with E. Fusy and D. Poulalhon

Mon premier souvenir d'un cours de combinatoire...



Mini-jardin de Catalan

(D'après photocopies de transparents de Viennot, 1993)

Today's subject: **Super Catalan numbers** (*Catalan*, *Gessel*)

$$\frac{1}{2}\frac{(2n)!(2m)!}{(n+m)!n!m!}$$

These numbers are integers for all positive m, n.

$\Rightarrow$ They deserve a combinatorial interpretation!

– For $m = 1$, Catalan numbers: $\frac{(2n)!}{n!(n+1)!} = \frac{1}{n+1}\binom{2n}{n}$.

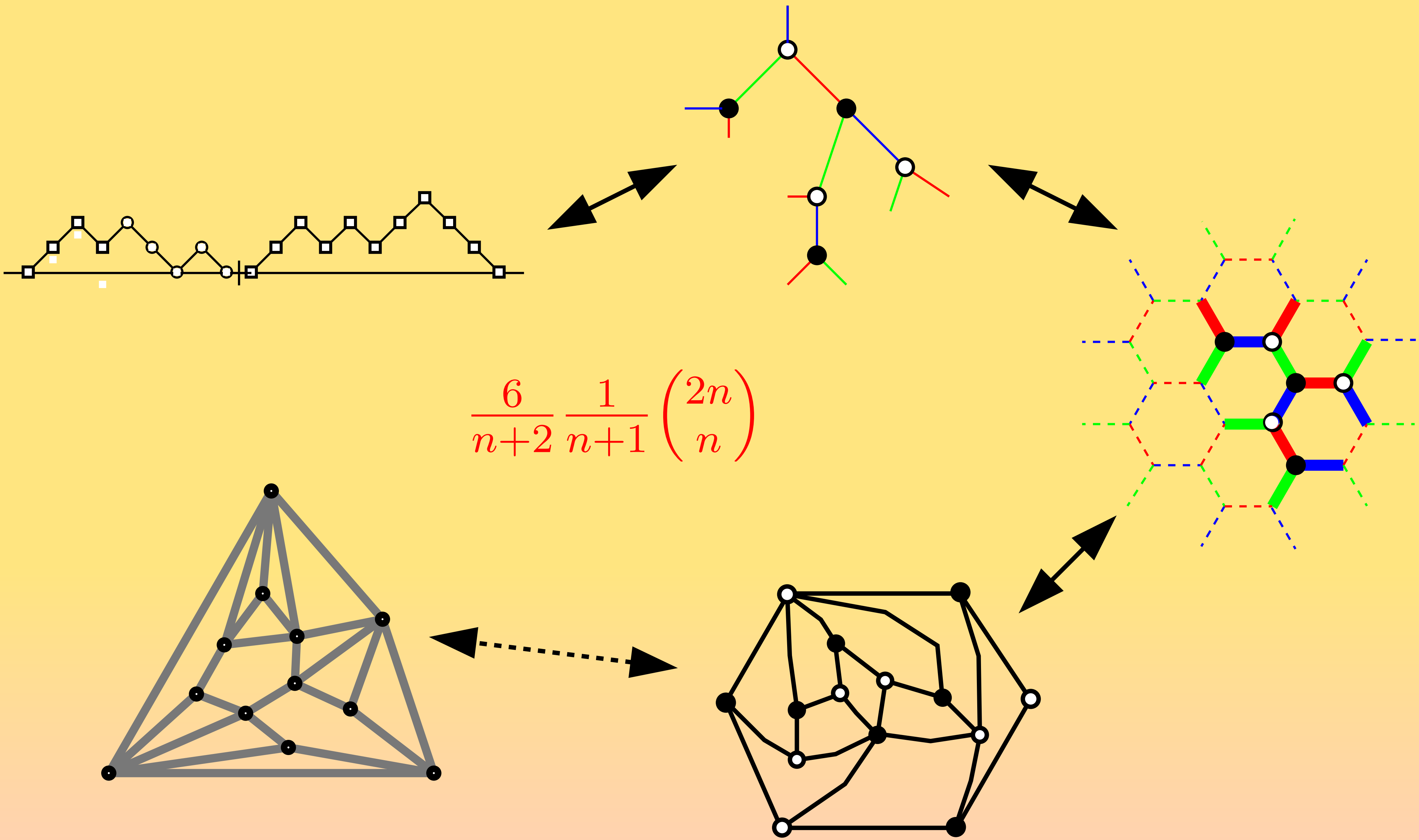
$$1, 2, 5, 14, 42, 132, 429, 1430 \ldots$$

– For $m = 2$, the numbers are: $\frac{6(2n)!}{(n+2)!n!} = \frac{6}{n+2}\frac{1}{n+1}\binom{2n}{n}$.

$$2, 3, 6, 14, 36, 99, 286, 858, \ldots$$

We shall discuss some interpretations for $m = 2$.

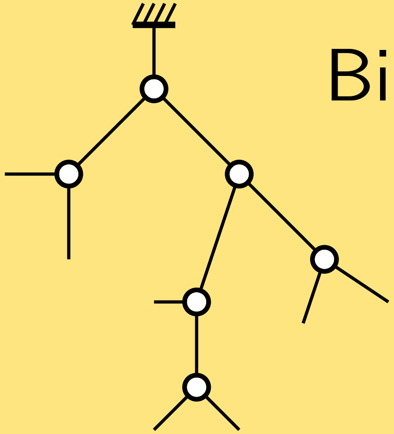
"More precisely", we aim at the following diagram:



$$\frac{6}{n+2}\,\frac{1}{n+1}\binom{2n}{n}$$

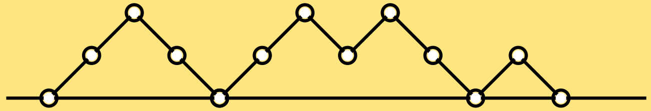
A one-page preliminary...

In the Catalan garden, I pluck the...



Binary trees
with n nodes

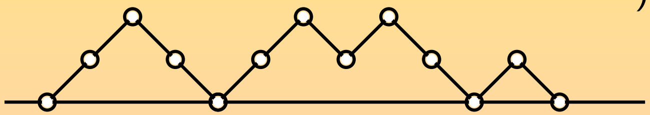
and

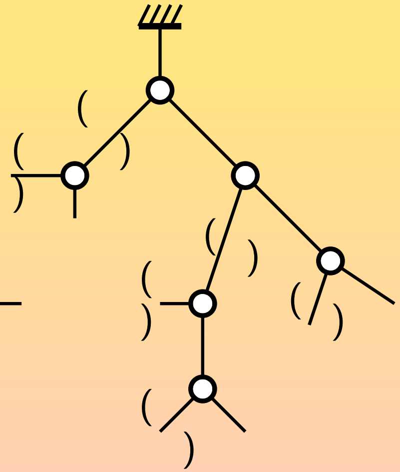


Dyck paths
with length $2n$.

They are counted by Catalan numbers: $\frac{1}{n+1}\binom{2n}{n}$

Here is a bijection:





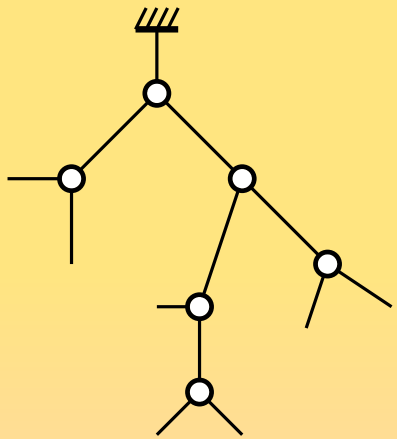
Turn around the tree,
write *up* or *down*
when *entering* or *exiting*
a left subtree.

First interpretations: unrooted binary trees

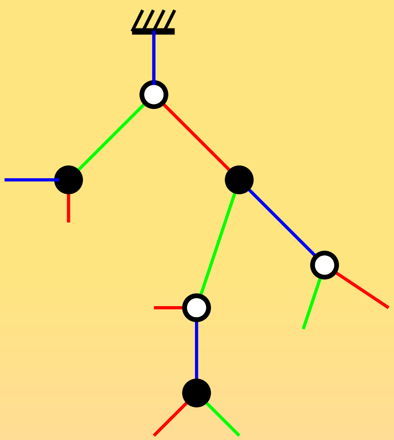
Colors make pictures more fun...

Edge-3-colored binary tree = a binary tree with colors on the edge and nodes such that there are two type of nodes:

Take a binary tree

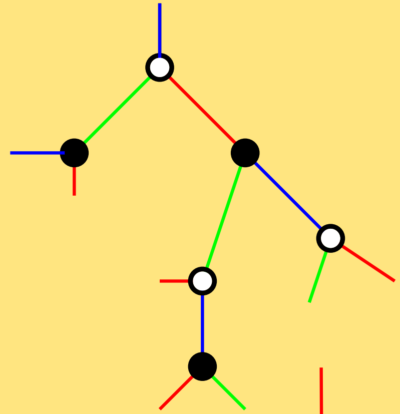


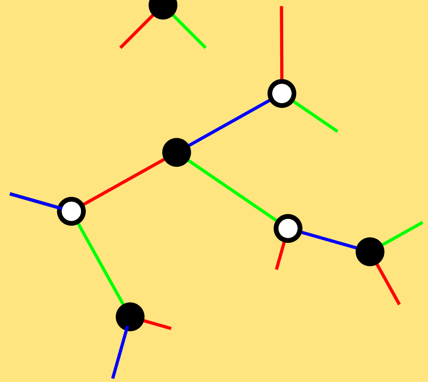
Choose colors for the root edge and the root vertex:



$$\Rightarrow \quad \#\{\text{colored tree with } n \text{ nodes}\} = 6 \cdot \frac{1}{n+1}\binom{2n}{n}.$$

Agriculture hors sol





unrooted 3-colored tree = like a 3-colored binary tree, but without the root...

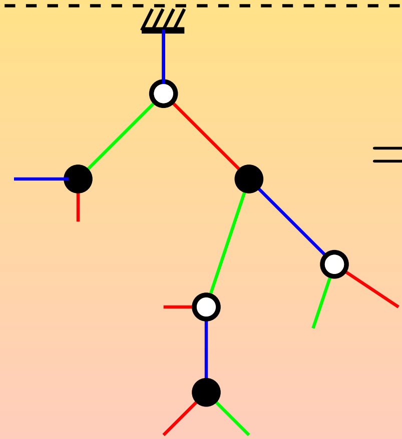
These trees have no symmetries: indeed symmetries of planar trees must leave the center invariant.

Here the center can be:

or:

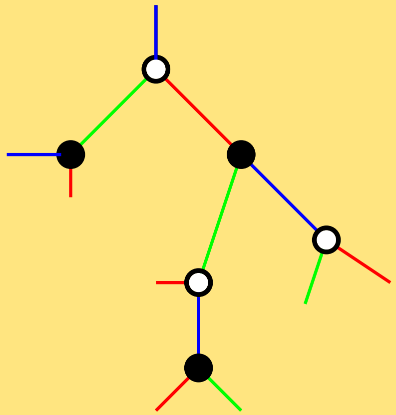
$\Rightarrow$ each tree has $n+2$ distinct rootings.

$$\#\{\text{unrooted 3-c trees with } n \text{ nodes}\} = \frac{6}{n+2} \cdot \frac{1}{n+1}\binom{2n}{n}$$

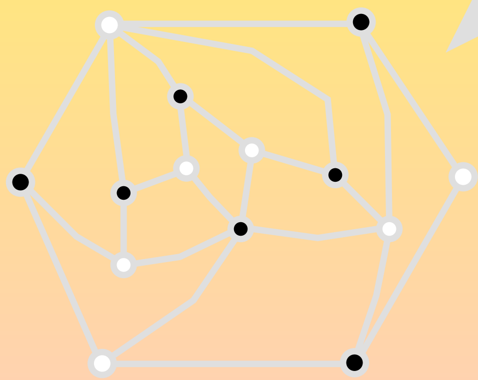


$$= \quad 6 \cdot \frac{1}{n+1}\binom{2n}{n}$$

Here is our first super-Cat-structure:

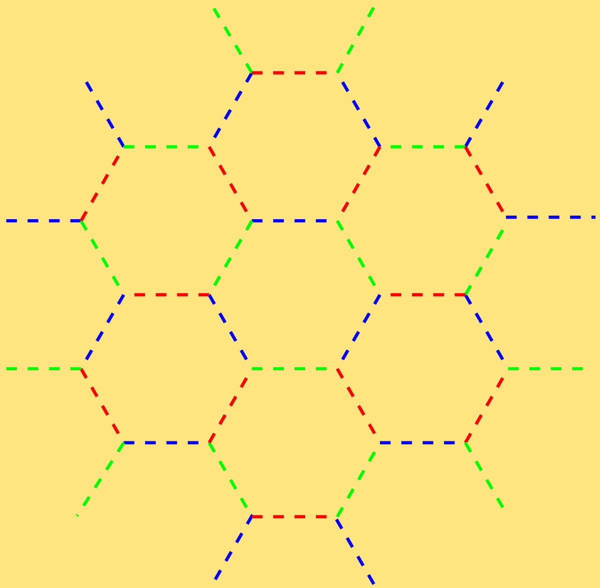


$$\frac{6}{n+2}\,\frac{1}{n+1}\binom{2n}{n}$$

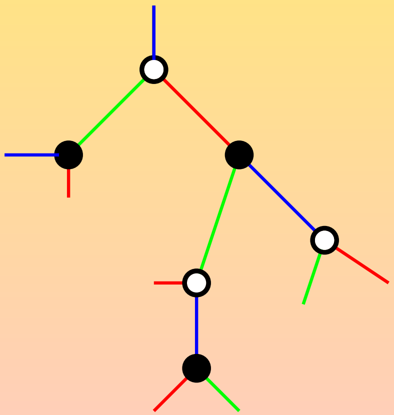


An elegant restatment:

Trees on the hexagonal lattice *(Pippenger & Schleich'03)*



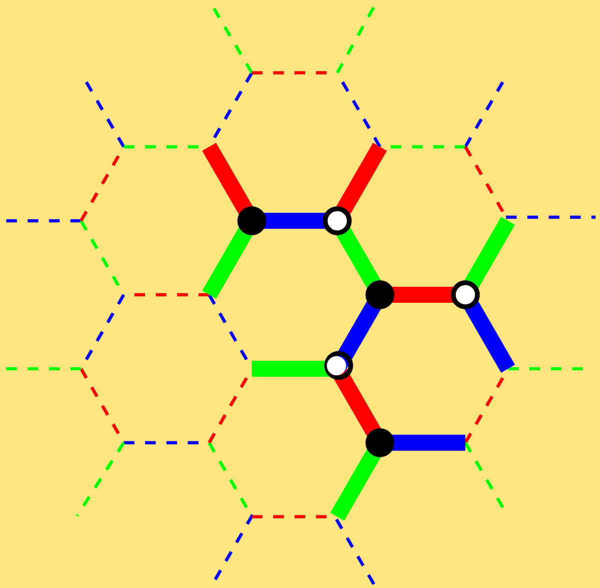
Up to translation and rotations, there is a unique way to embed an unrooted colored tree on the colored hexagonal lattice (possibly with overlaps).



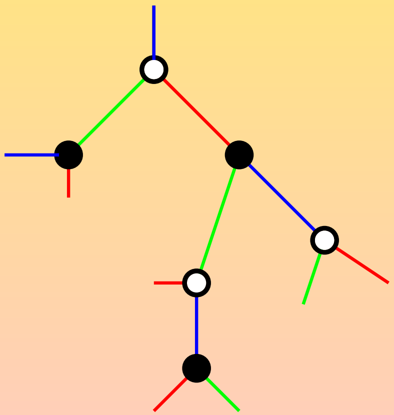
$$= \quad \frac{6}{n+2}\frac{1}{n+1}\binom{2n}{n}$$

An elegant restatment:

Trees on the hexagonal lattice *(Pippenger & Schleich'03)*



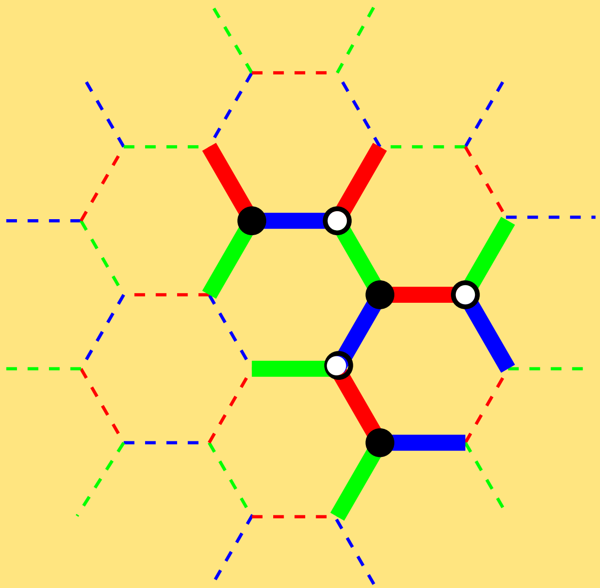
Up to translation and rotations, there is a unique way to embed an unrooted colored tree on the colored hexagonal lattice (possibly with overlaps).



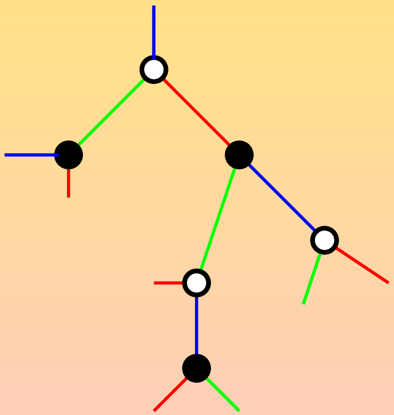
$$= \quad \frac{6}{n+2}\frac{1}{n+1}\binom{2n}{n}$$

An elegant restatment:

Trees on the hexagonal lattice *(Pippenger & Schleich'03)*



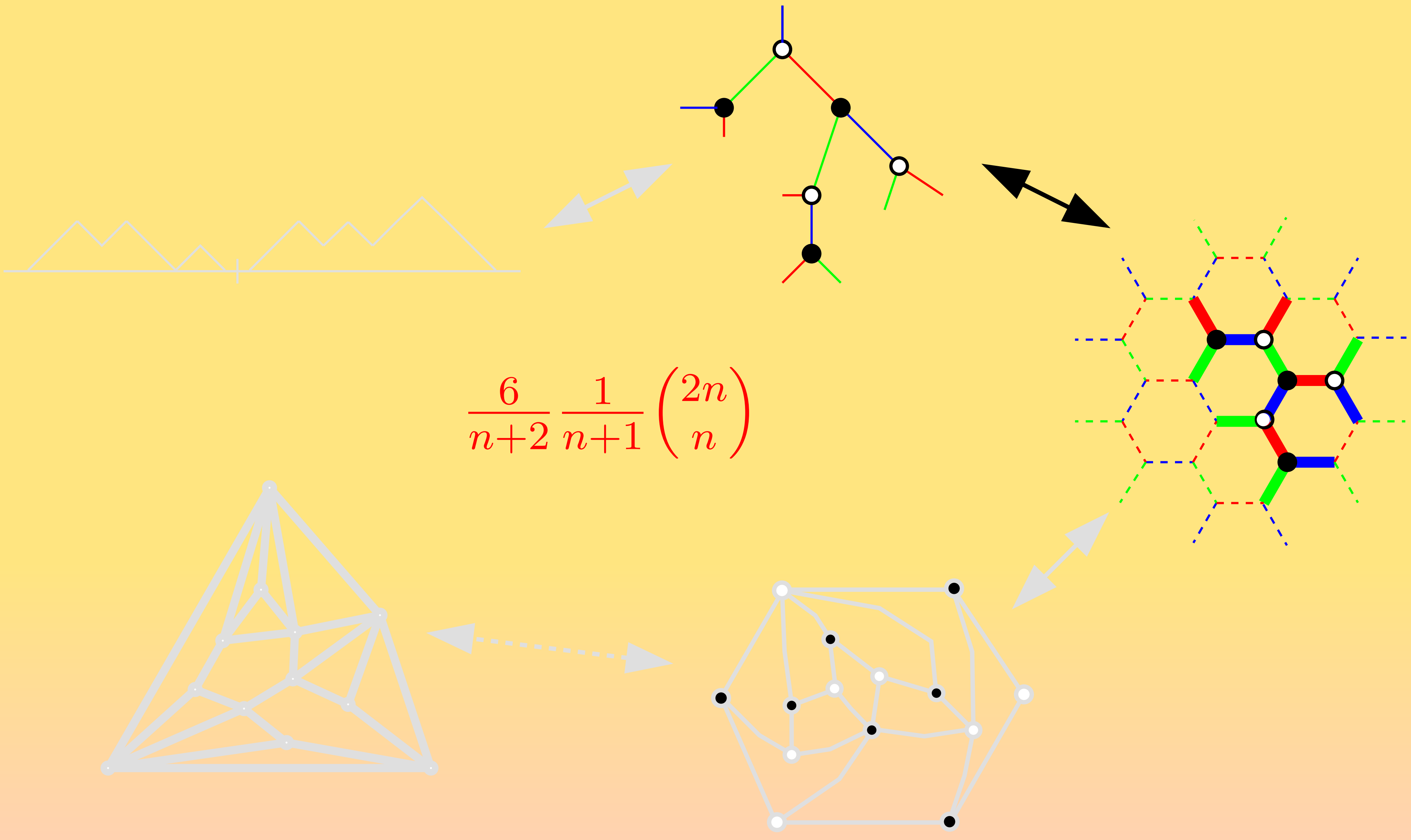
Up to translation and rotations, there is a unique way to embed an unrooted colored tree on the colored hexagonal lattice (possibly with overlaps).



$$= \quad \frac{6}{n+2}\frac{1}{n+1}\binom{2n}{n}$$

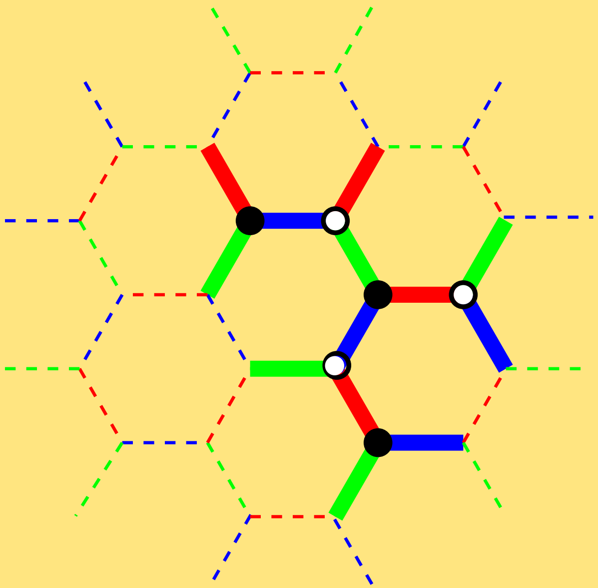
$$= \quad \#\{\text{hexagonal trees with } n \text{ nodes}\}.$$

We got an arrow !



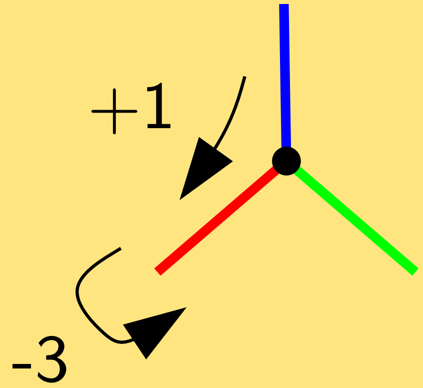
$$\frac{6}{n+2}\,\frac{1}{n+1}\binom{2n}{n}$$

Hexagonal trees are Mireille's embedded trees...

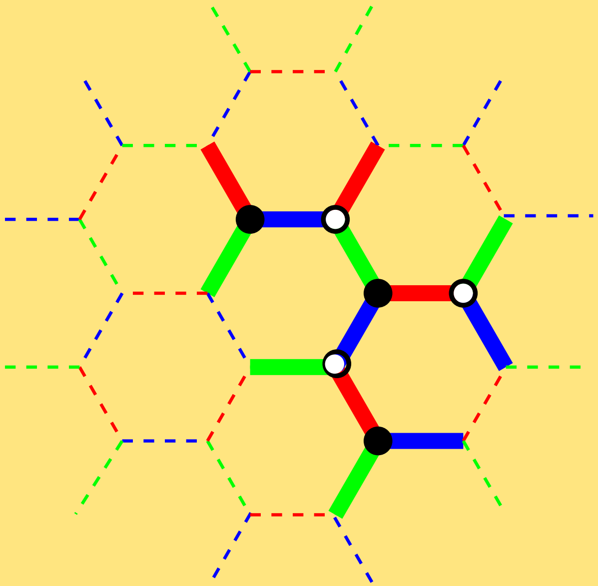


Turn counterclockwise around the tree, and label each side of edges:

- at each corner ℓ=ℓ+1
- at each leaf: ℓ=ℓ−3.

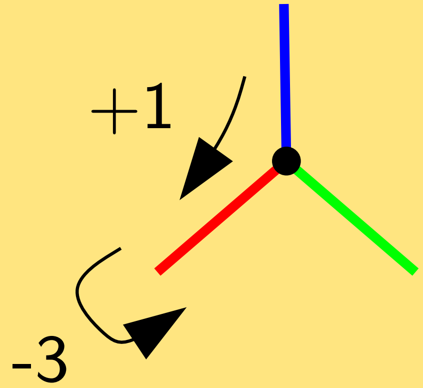


Hexagonal trees are Mireille's embedded trees...

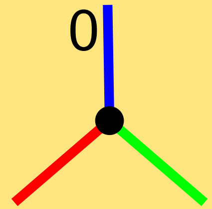


Turn counterclockwise around the tree, and label each side of edges:

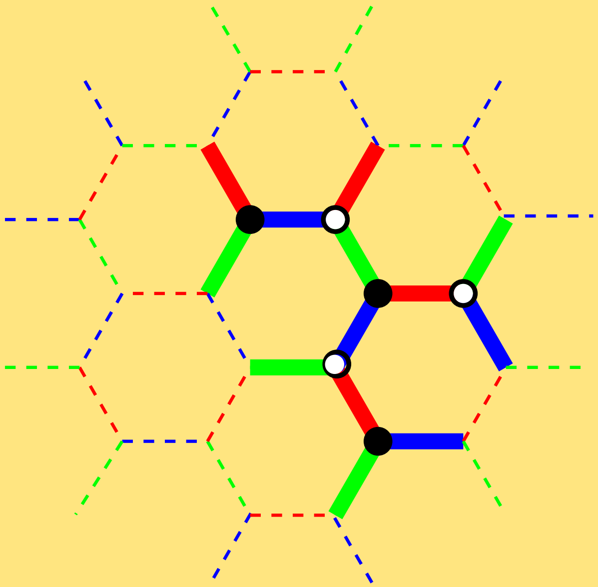
- at each corner $\ell = \ell + 1$
- at each leaf: $\ell = \ell - 3$.



Exemple:

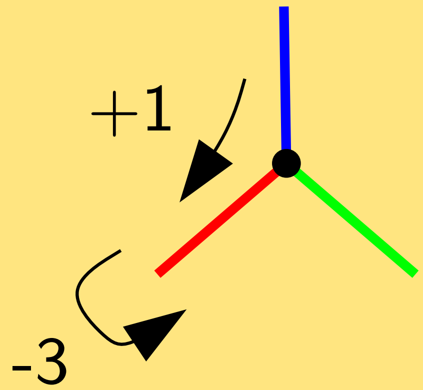


Hexagonal trees are Mireille's embedded trees...

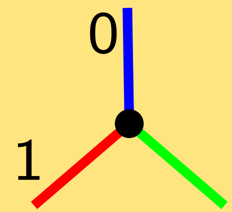


Turn counterclockwise around the tree, and label each side of edges:

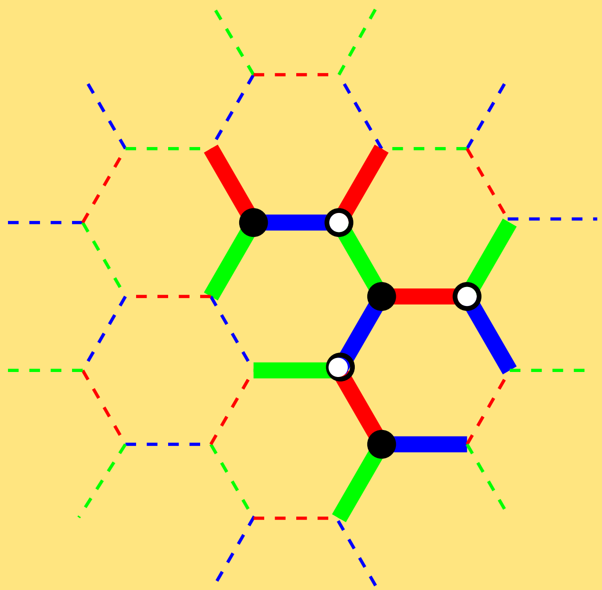
- at each corner $\ell = \ell + 1$
- at each leaf: $\ell = \ell - 3$.



Exemple:

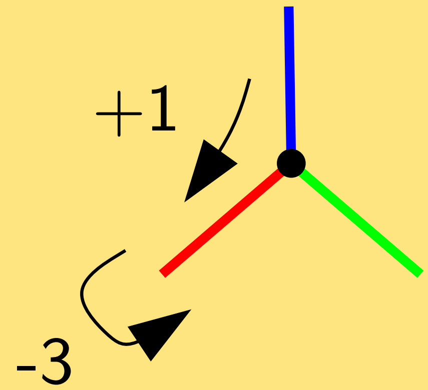


Hexagonal trees are Mireille's embedded trees...

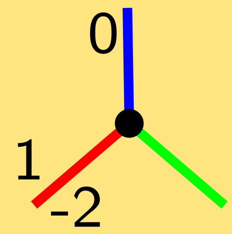


Turn counterclockwise around the tree, and label each side of edges:

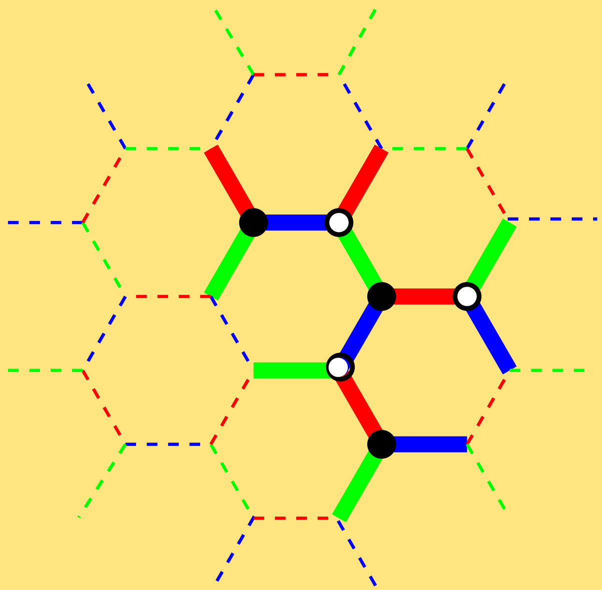
- at each corner $\ell = \ell + 1$
- at each leaf: $\ell = \ell - 3$.



Exemple:

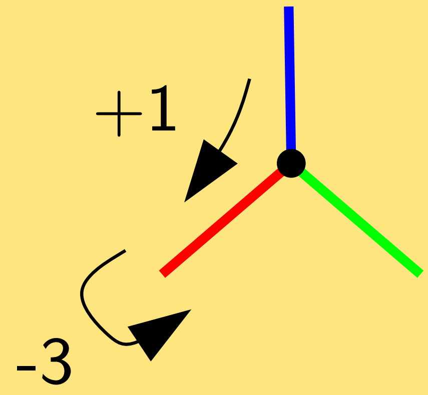


Hexagonal trees are Mireille's embedded trees...

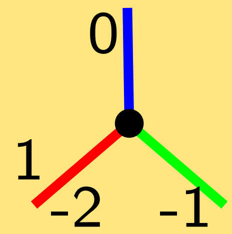


Turn counterclockwise around the tree, and label each side of edges:

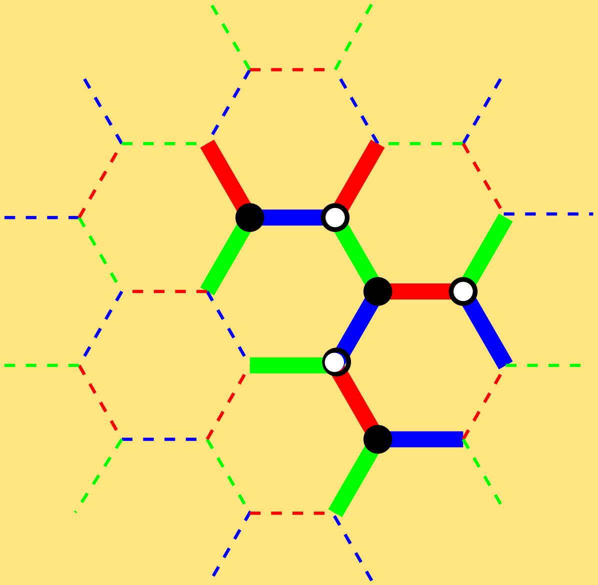
- at each corner $\ell{=}\ell{+}1$
- at each leaf: $\ell{=}\ell{-}3$.



Exemple:

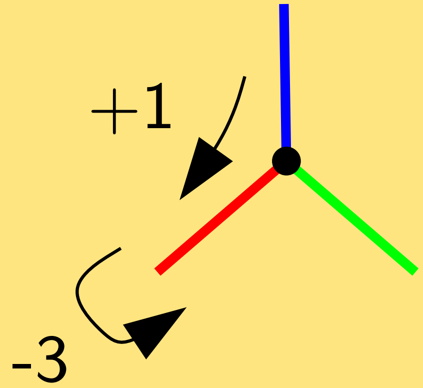


Hexagonal trees are Mireille's embedded trees...

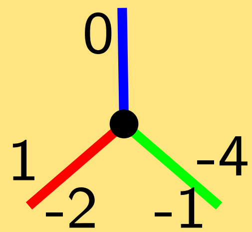


Turn counterclockwise around the tree, and label each side of edges:

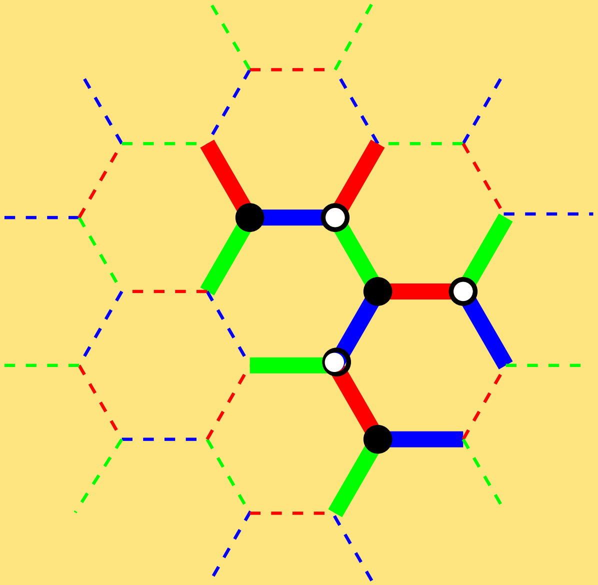
- at each corner $\ell = \ell + 1$
- at each leaf: $\ell = \ell - 3$.



Exemple:

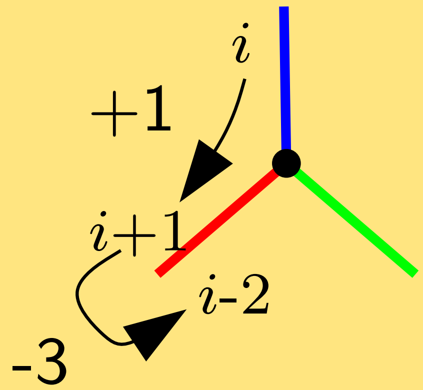


Hexagonal trees are Mireille's embedded trees...

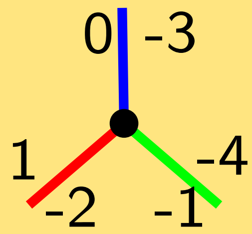


Turn counterclockwise around the tree, and label each side of edges:

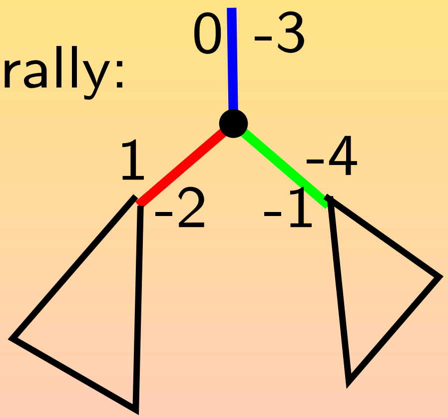
- at each corner $\ell = \ell + 1$
- at each leaf: $\ell = \ell - 3$.



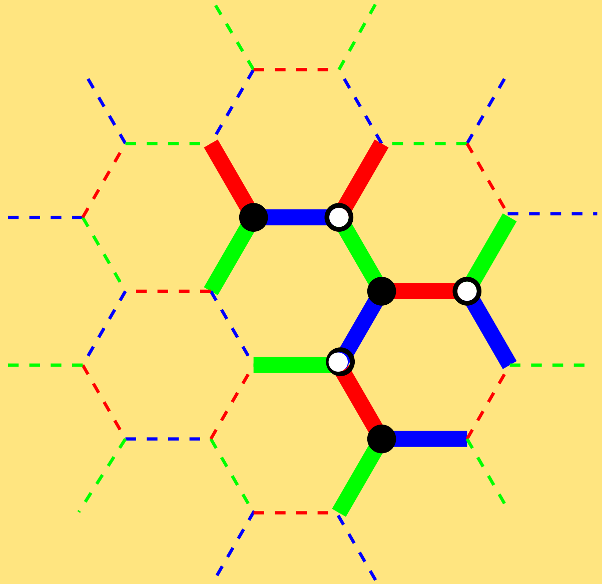
Exemple:



More generally:

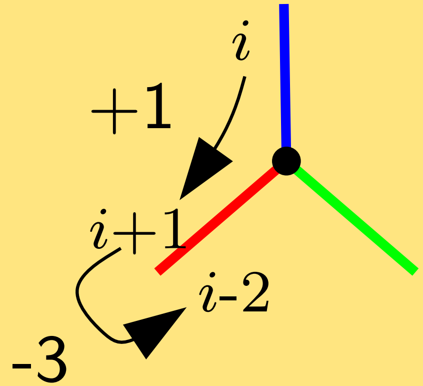


Hexagonal trees are Mireille's embedded trees...

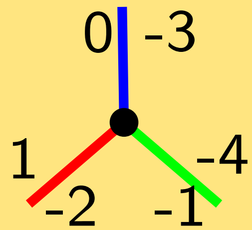


Turn counterclockwise around the tree, and label each side of edges:

- at each corner ℓ=ℓ+1
- at each leaf: ℓ=$\ell-3$.



Exemple:

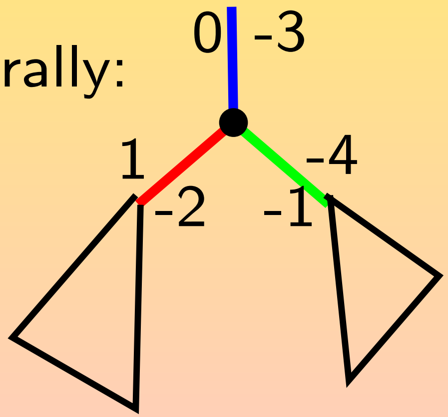


Read Mireille's labels on the left of inner edges.

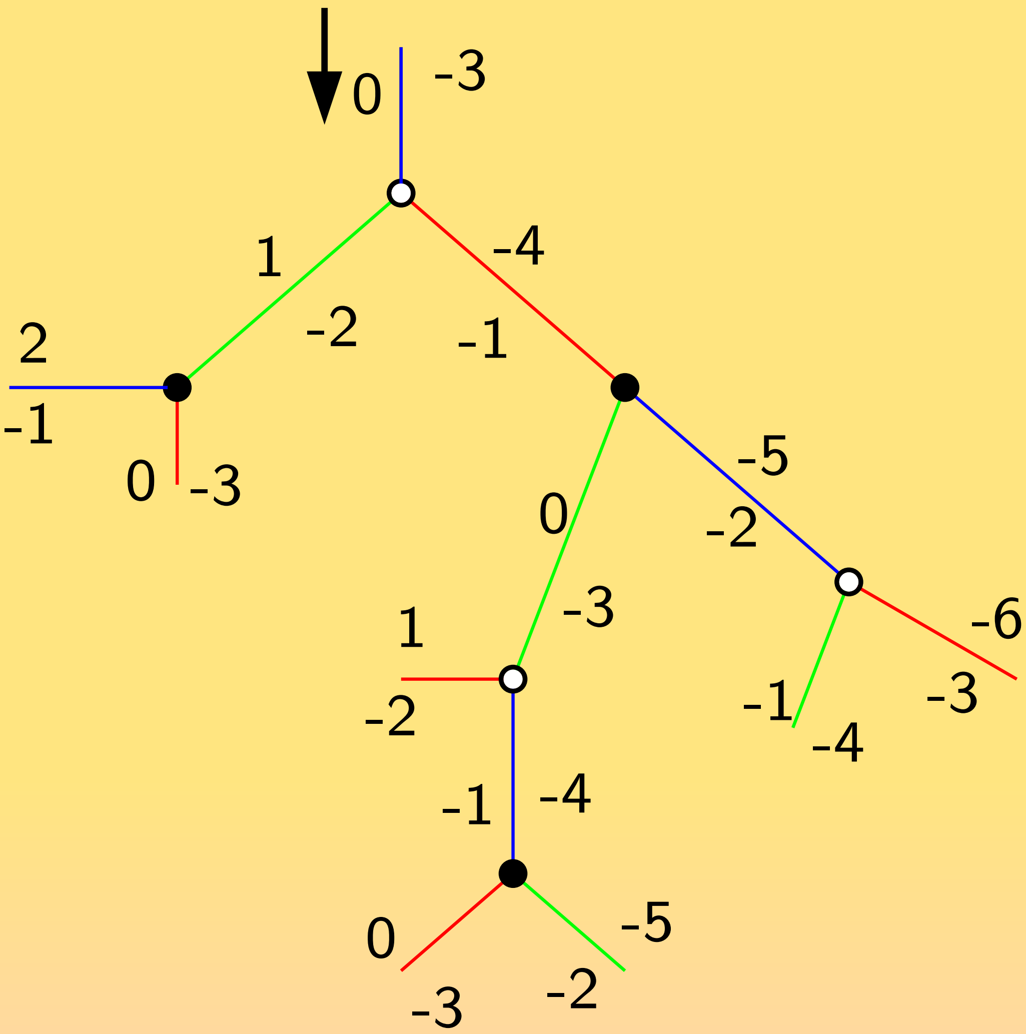
These labels should be viewed as angles (multiples of $\pi/3$).

After a full turn around the tree, the angle variation is $-2\pi = -6 \cdot (\pi/3)$.

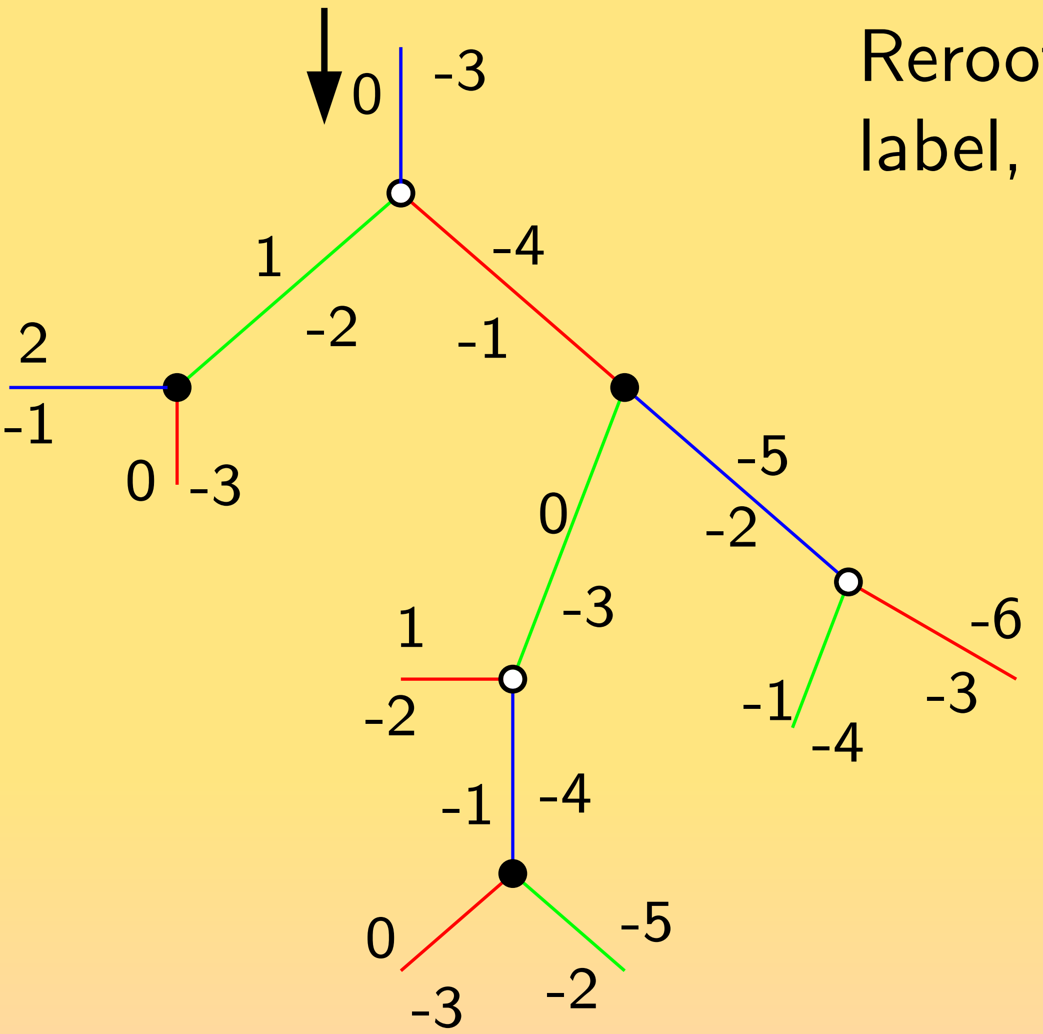
More generally:



A bigger example.

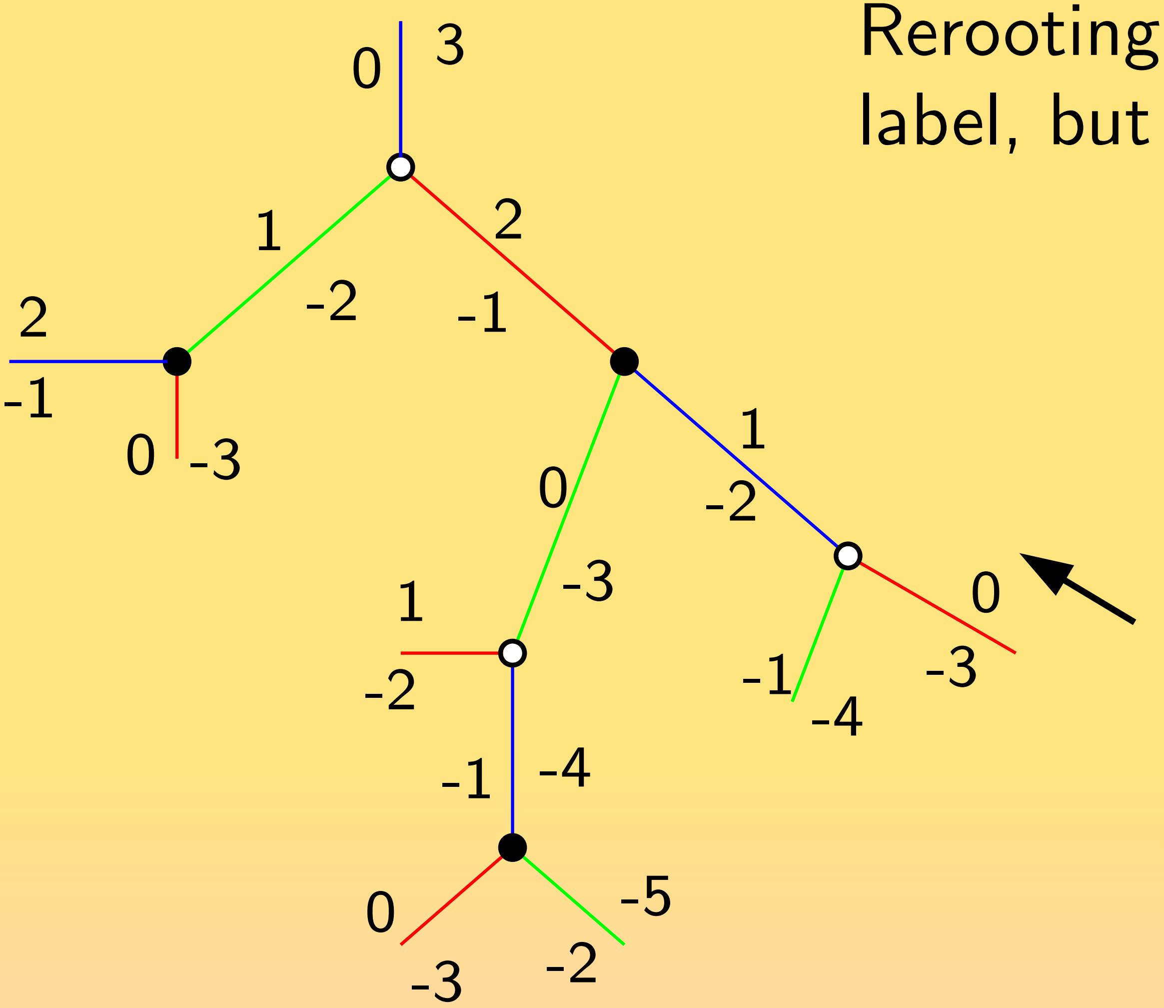


A bigger example.



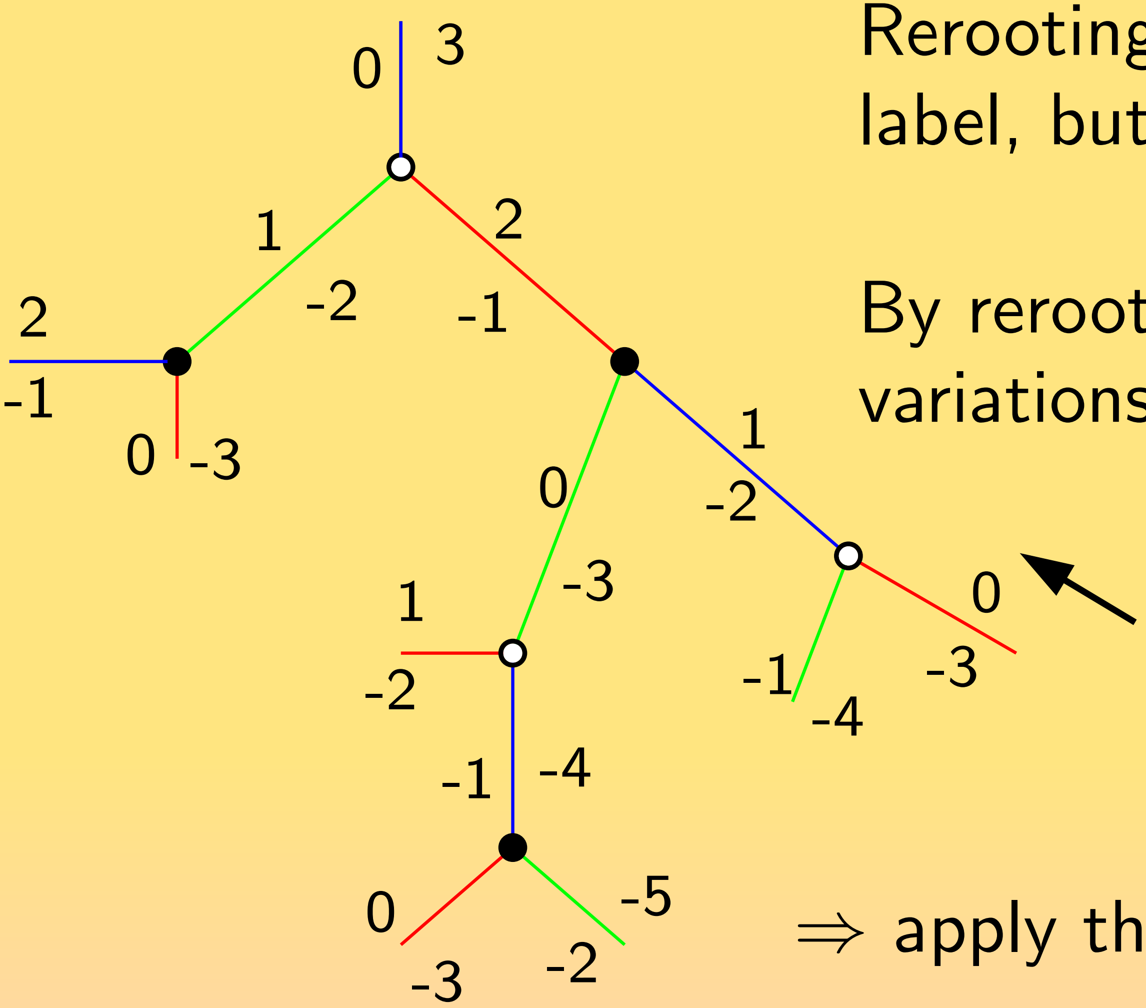
Rerooting changes the actual label, but not the variations!

A bigger example.



Rerooting changes the actual label, but not the variations!

A bigger example.



Rerooting changes the actual label, but not the variations!

By rerooting, the sequence of variations are cyclically permuted.

$\Rightarrow$ apply the cycle lemma.

$$\frac{6}{n+2}\,\frac{1}{n+1}\binom{2n}{n}$$

This should give Mireille's formula for positive binary trees: recall

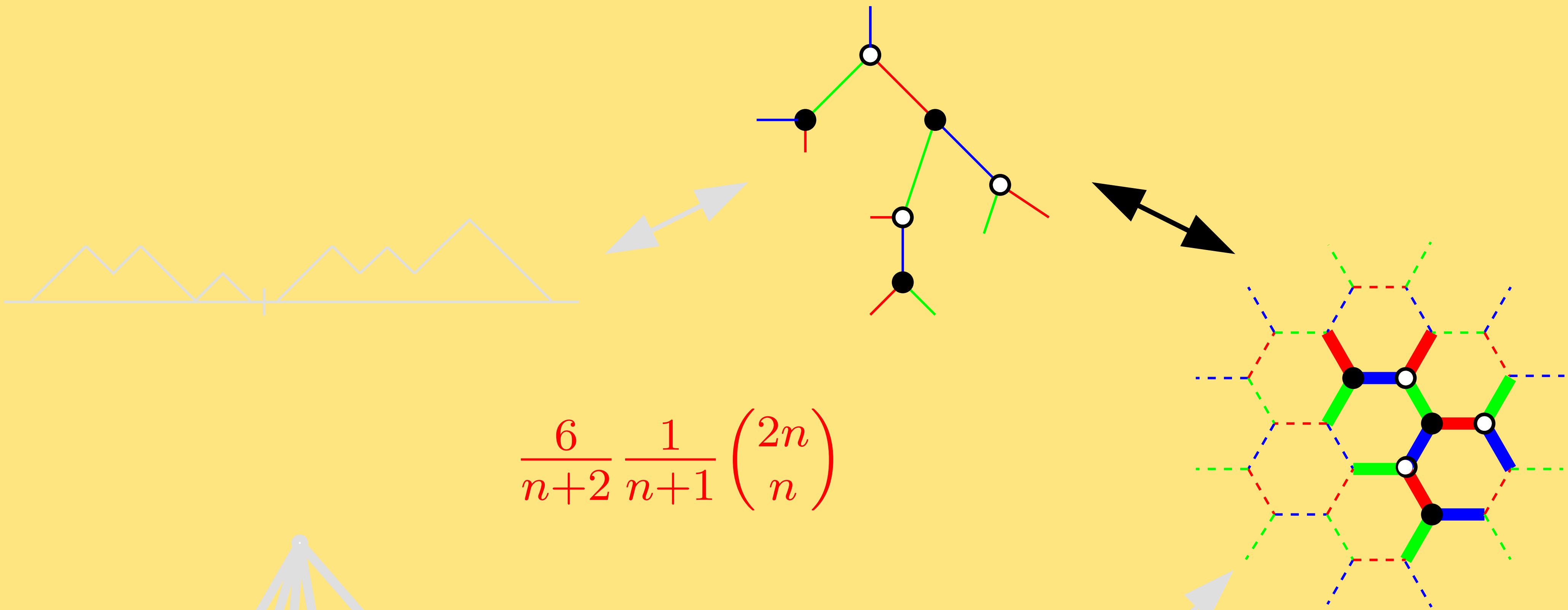
Theorem (part of her) *Let B_n be the number of rooted binary trees with n nodes with label ≥ 0. Then*

$$B_n^{\geq 0} + B_{n+1}^{\geq 0} = \frac{6(2n)!}{n!(n+2)!}.$$

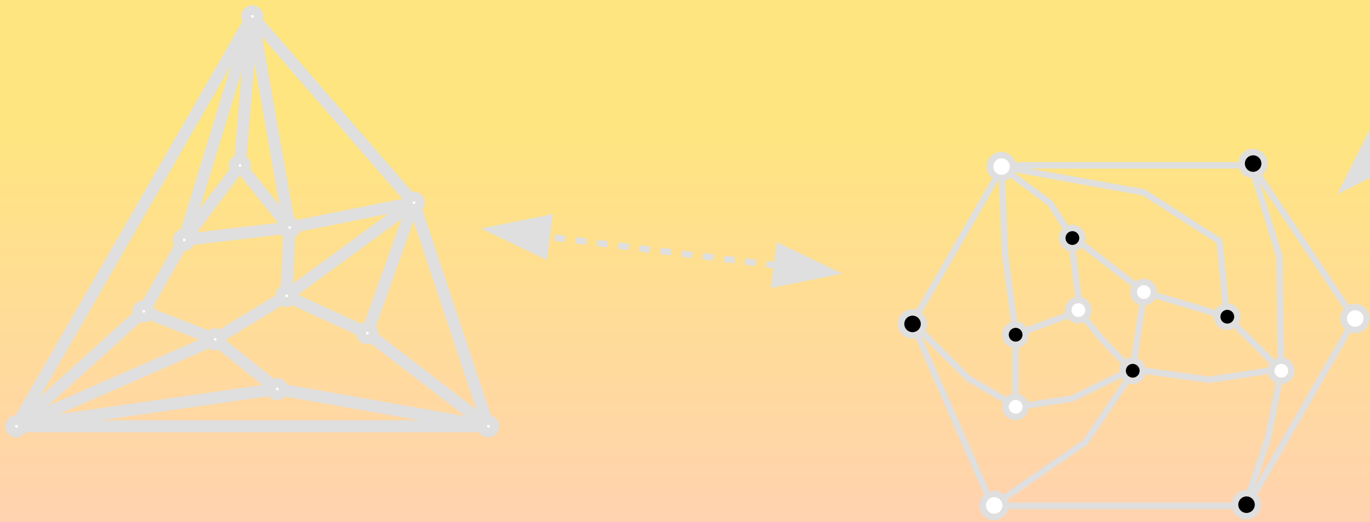
In other terms:

$$B_n^{\geq 0} + B_n^{\geq -1} = \frac{6(2n)!}{n!(n+2)!}.$$

We got an arrow !



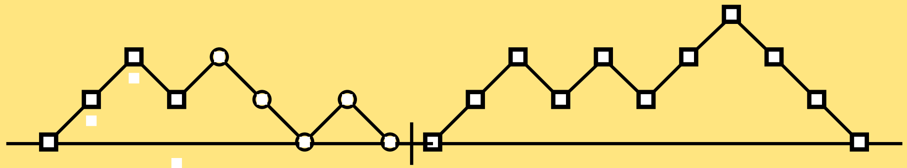
$$\frac{6}{n+2}\,\frac{1}{n+1}\binom{2n}{n}$$

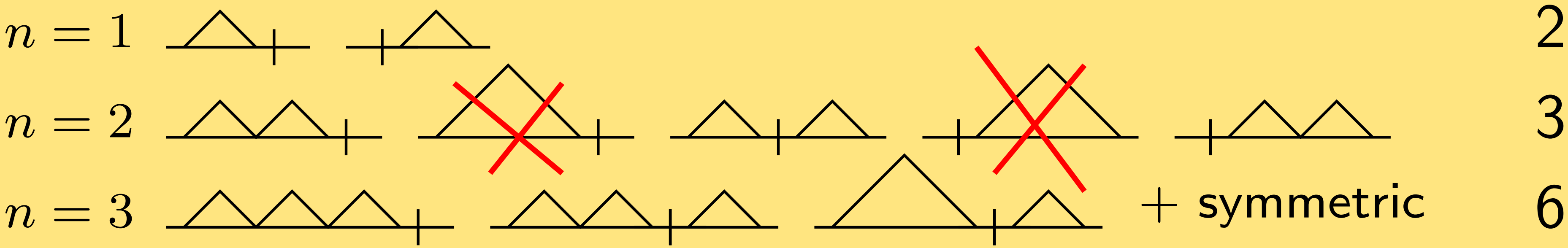


Second interpretation: Dyck paths

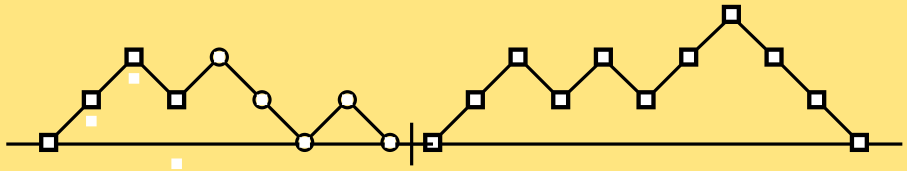
Let **a Gessel-Xin pair** be a pair of Dyck paths such that the height of the two paths differ at most by one.

Example:





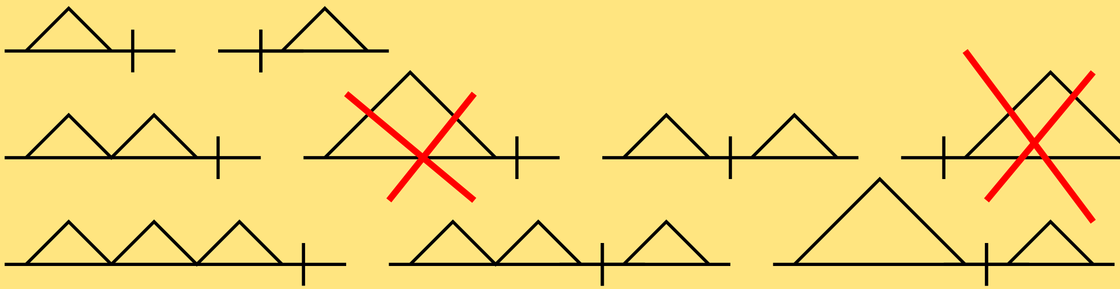
Let **a Gessel-Xin pair** be a pair of Dyck paths such that the height of the two paths differ at most by one.

Example: 

$n = 1$ … 2

$n = 2$ … 3

$n = 3$ … + symmetric … 6

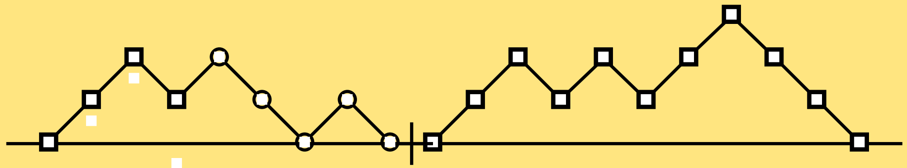


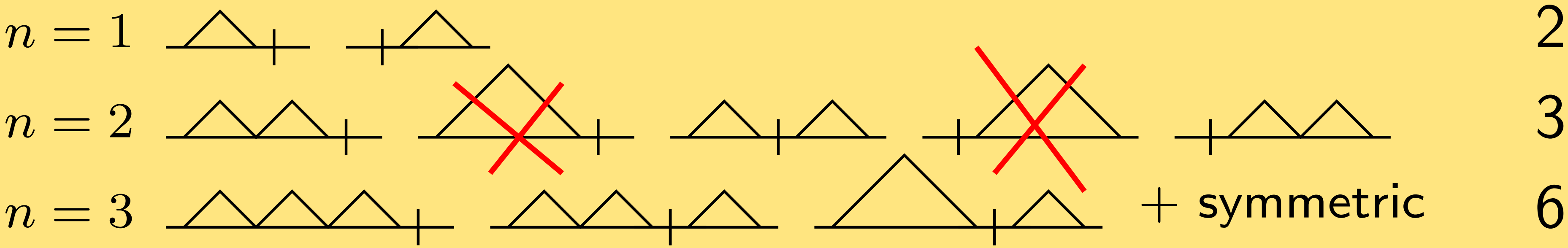
Theorem (Gessel & Xin). *The number of Gessel-Xin pairs with total length $2n$ is:*

$$4C_n - C_{n+1} \;=\; \tfrac{6(2n)!}{(n+2)!n!}.$$

Let **a Gessel-Xin pair** be a pair of Dyck paths such that the height of the two paths differ at most by one.

Example:



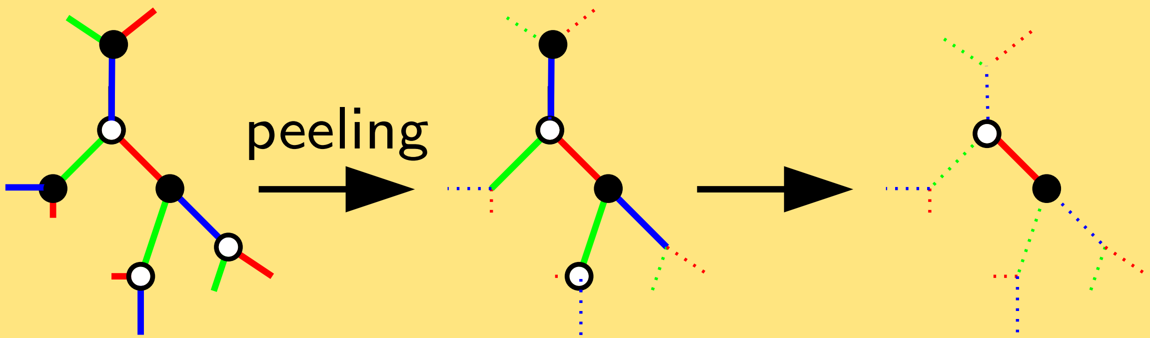


Theorem (Gessel & Xin). *The number of Gessel-Xin pairs with total length $2n$ is:*

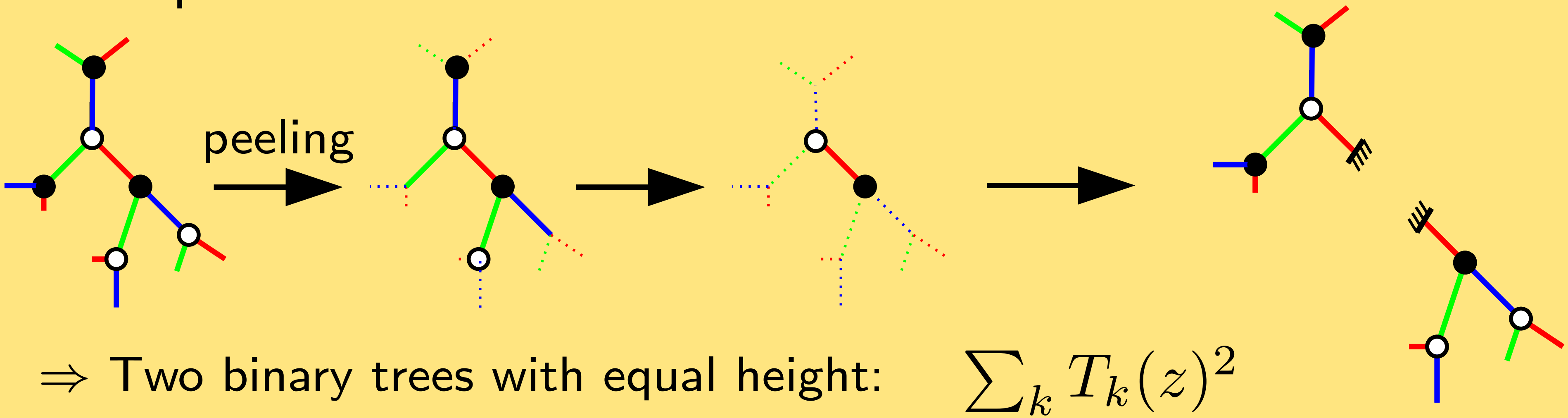
$$4C_n - C_{n+1} \;=\; \frac{6(2n)!}{(n+2)!n!}.$$

Can we relate this to the previous binary trees ?

Decomposition at the center of the tree

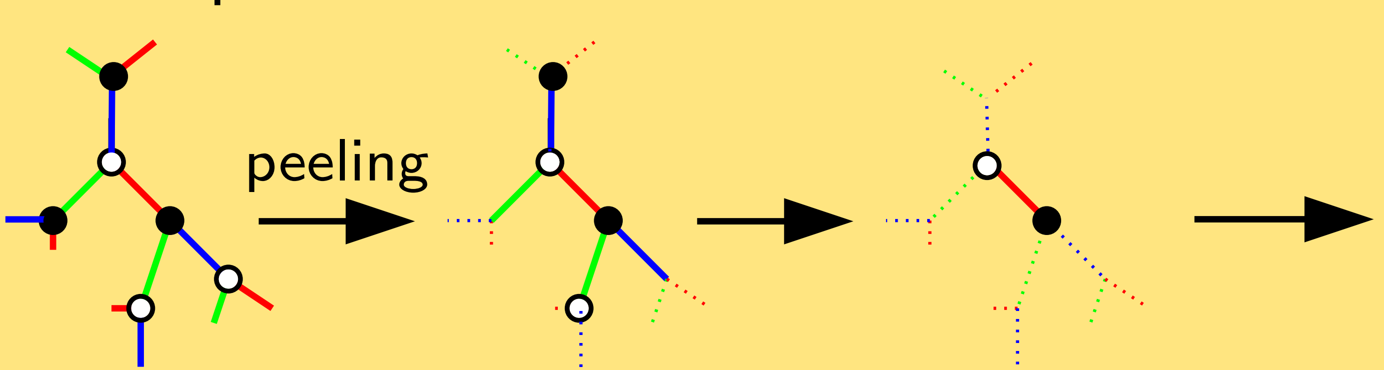


Decomposition at the center of the tree

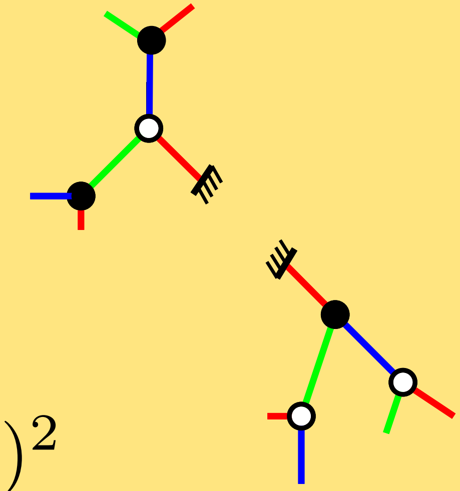


⇒ Two binary trees with equal height: $\sum_k T_k(z)^2$

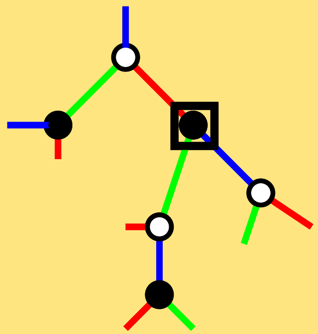
Decomposition at the center of the tree



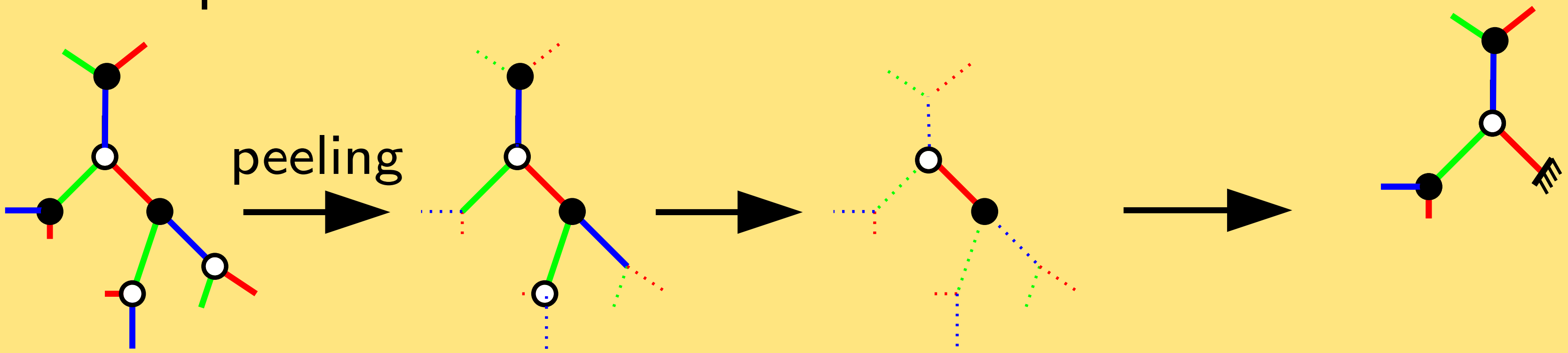
$\Rightarrow$ Two binary trees with equal height: $\sum_k T_k(z)^2$

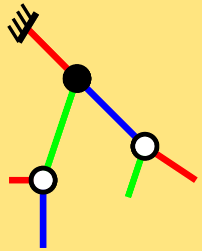


The center can also be a node:



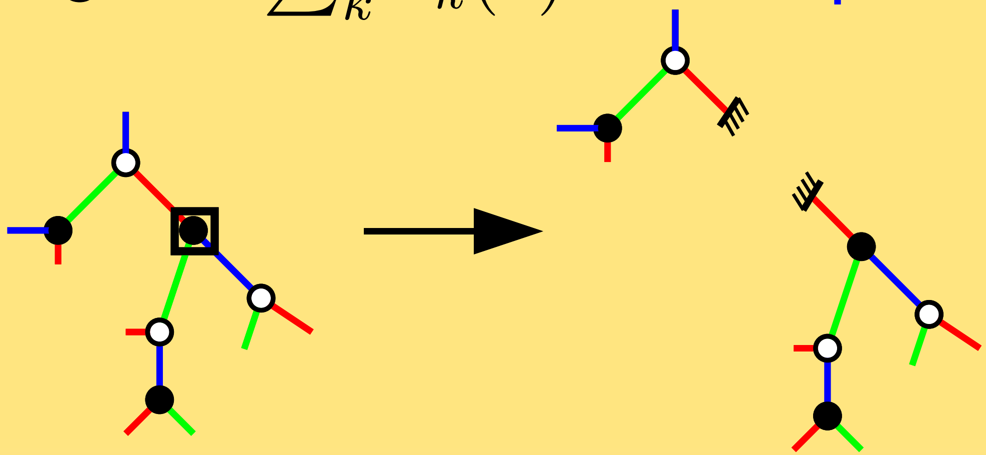
Decomposition at the center of the tree





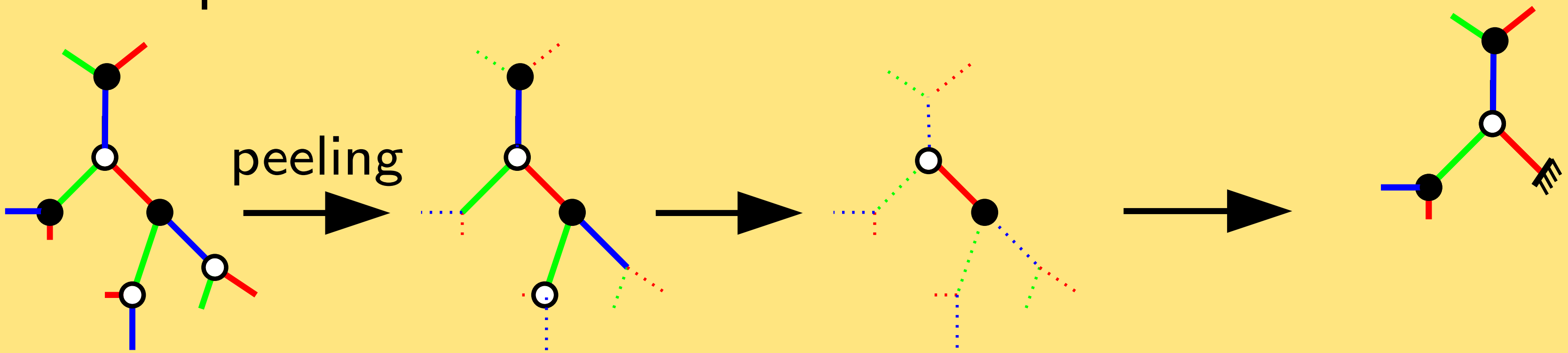
⇒ Two binary trees with equal height: $\sum_k T_k(z)^2$

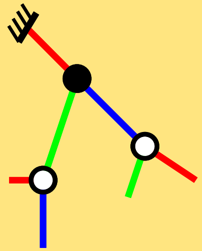
The center can also be a node:



⇒ Two binary trees with almost the same height: $\sum_k T_k(z)T_{k-1}(z)$

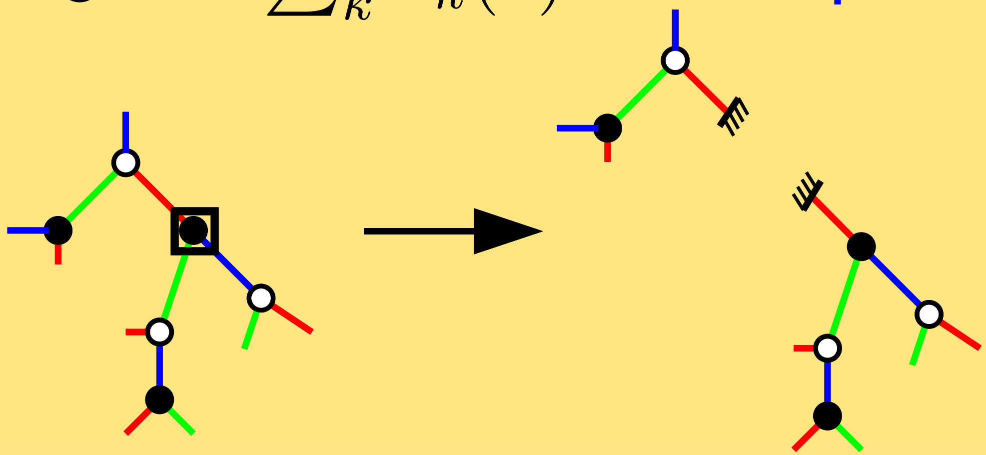
Decomposition at the center of the tree





$\Rightarrow$ Two binary trees with equal height: $\sum_k T_k(z)^2$

The center can also be a node:



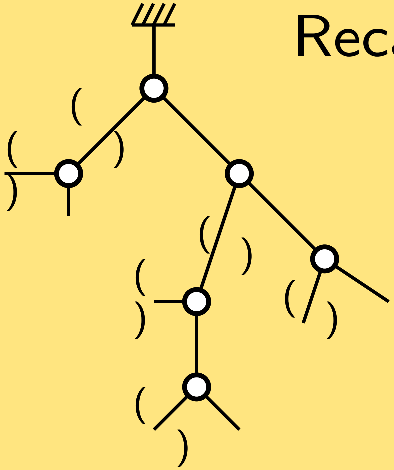
$\Rightarrow$ Two binary trees with almost the same height: $\sum_k T_k(z) T_{k-1}(z)$

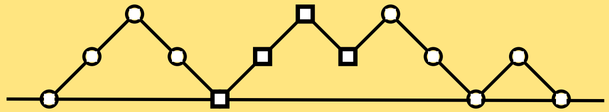
But this approach does not yield the relation to Dyck paths:

- Colors are not taken into account correctly...
- Not the right notion of height!

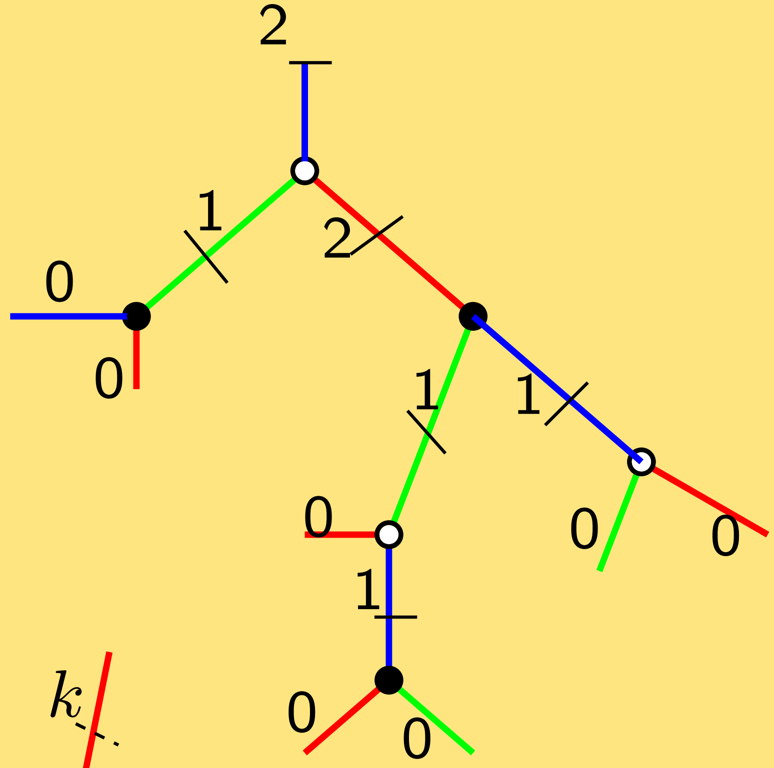
A notion of center inherited from Dyck paths.

Recall the bijection...



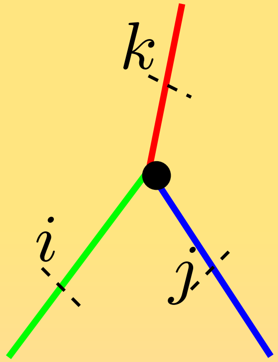


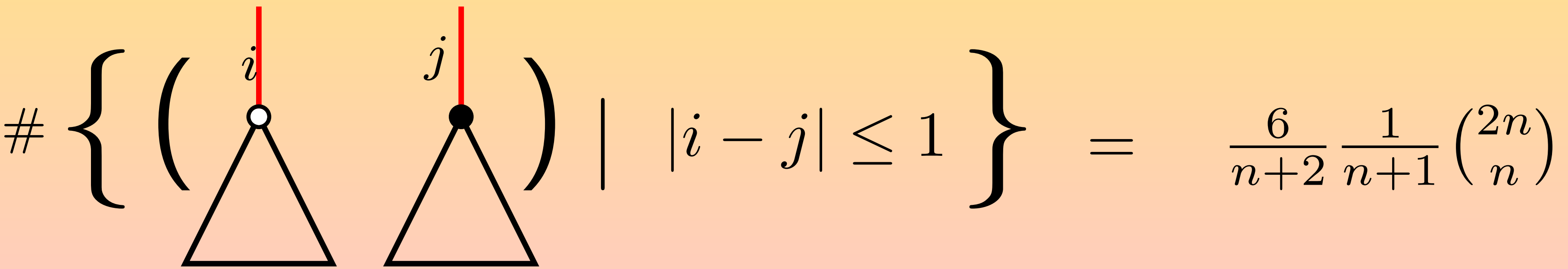
$\frac{1}{n+1}\binom{2n}{n}$



hence the rule for computing the height:

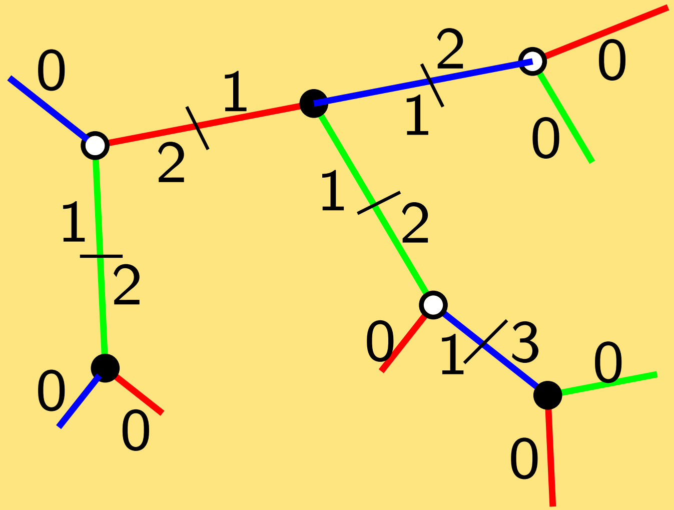
$$k = \max(i+1, j)$$





Depending on the position of the root, each edge can get two labels: there is a height labelling of an unrooted tree!

Exemple:

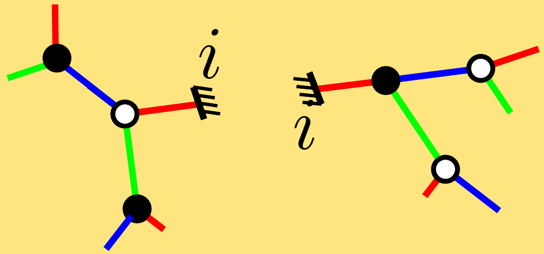


Theorem. *Exactly one of the following two cases occur:*

- *there is one edge with the 2 labels that are equal,*
- *or there is one vertex with the 3 incident labels that are equal.*

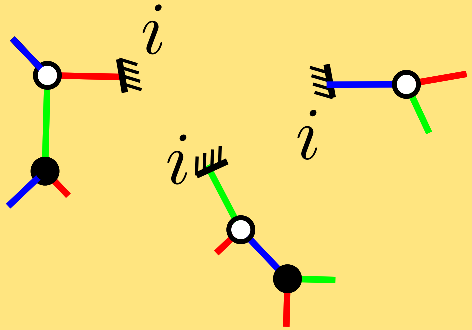
Decomposition at the center of the tree

The center is an edge:



$\Rightarrow$ Two binary trees with equal height: $\quad 3\sum_k D_k(z)^2$

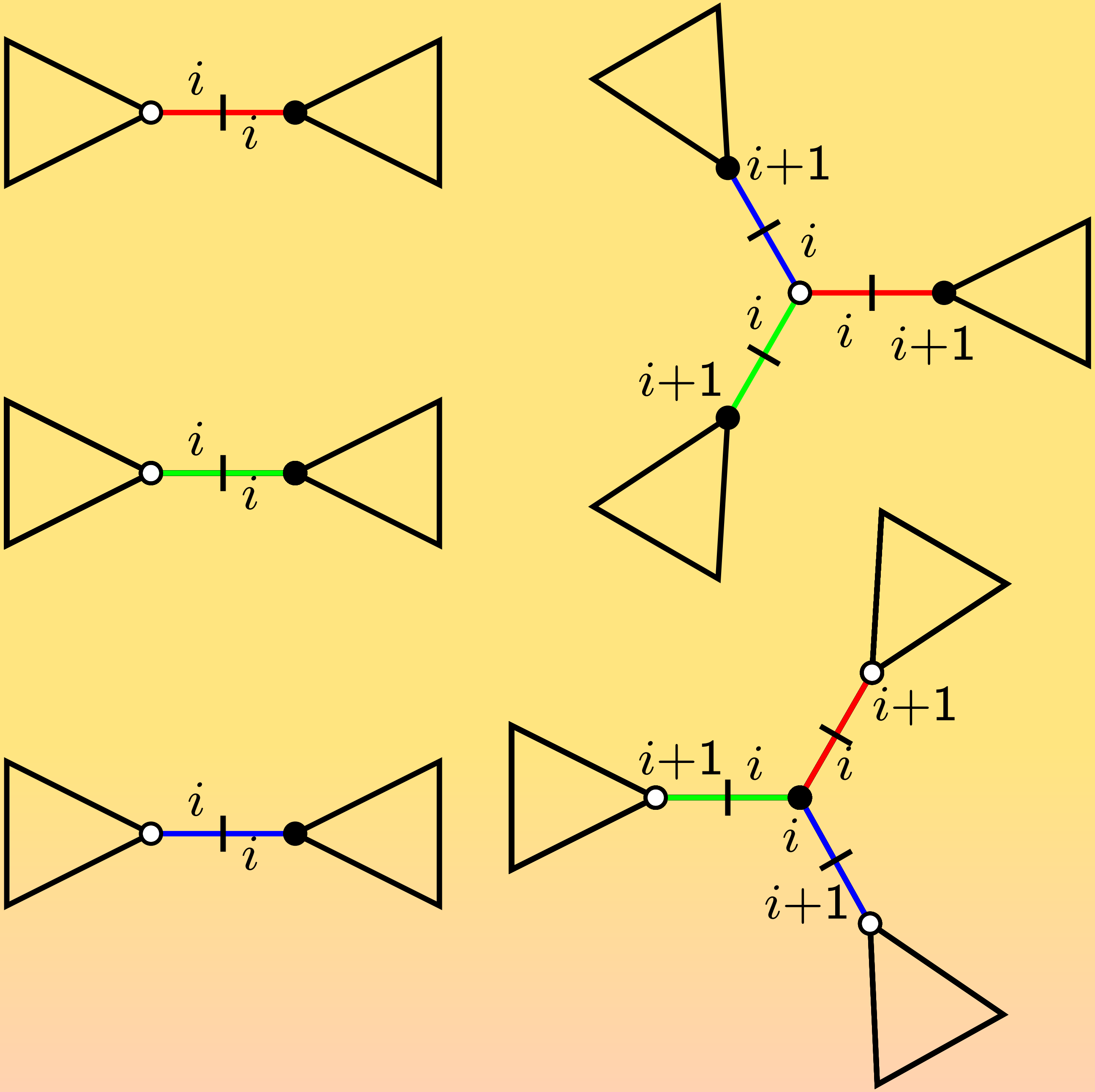
The center is a node:



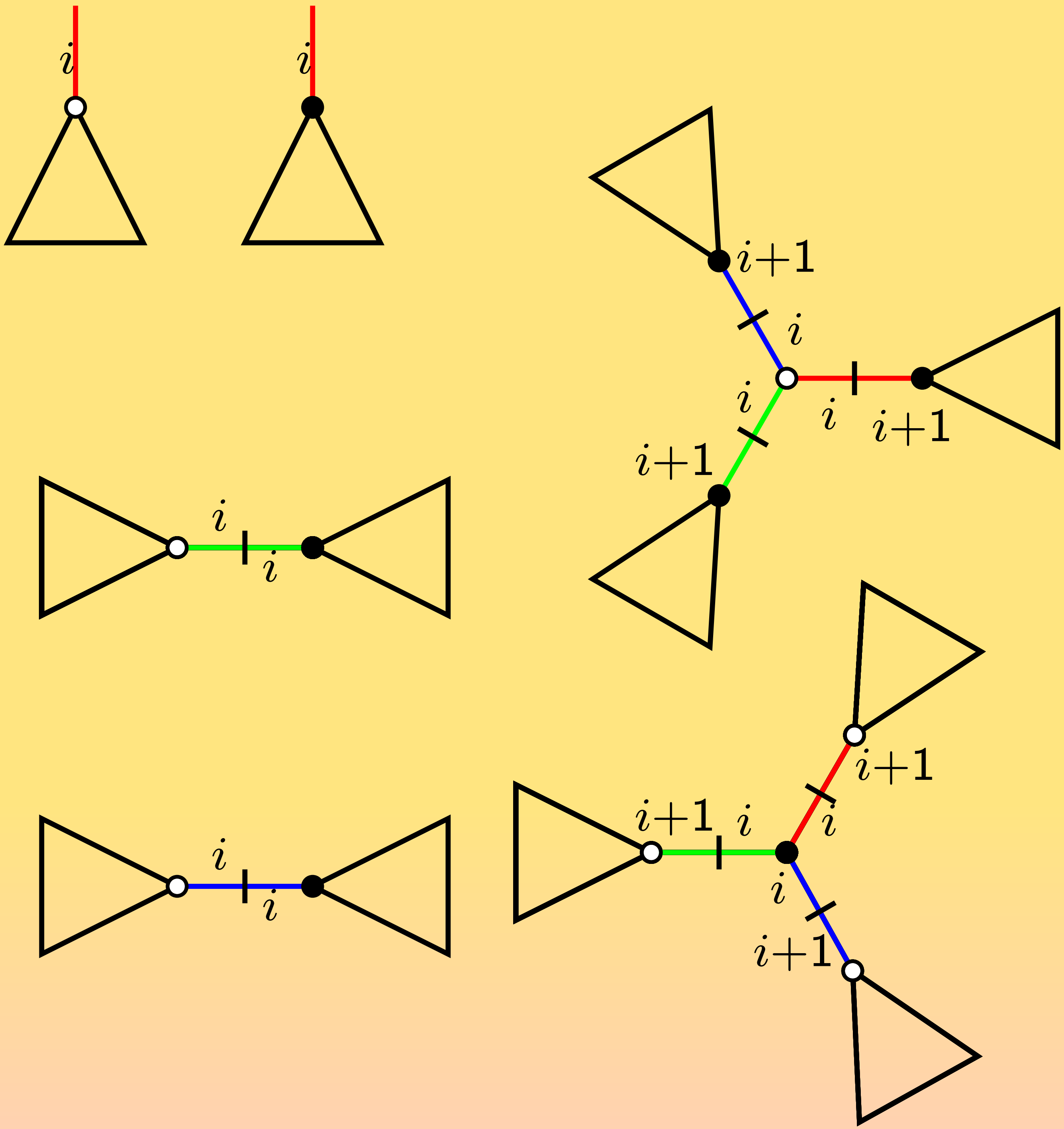
The center can also be a node:

$\Rightarrow$ Three binary trees with the same height: $\; 2\sum_k zD_k(z)^3$

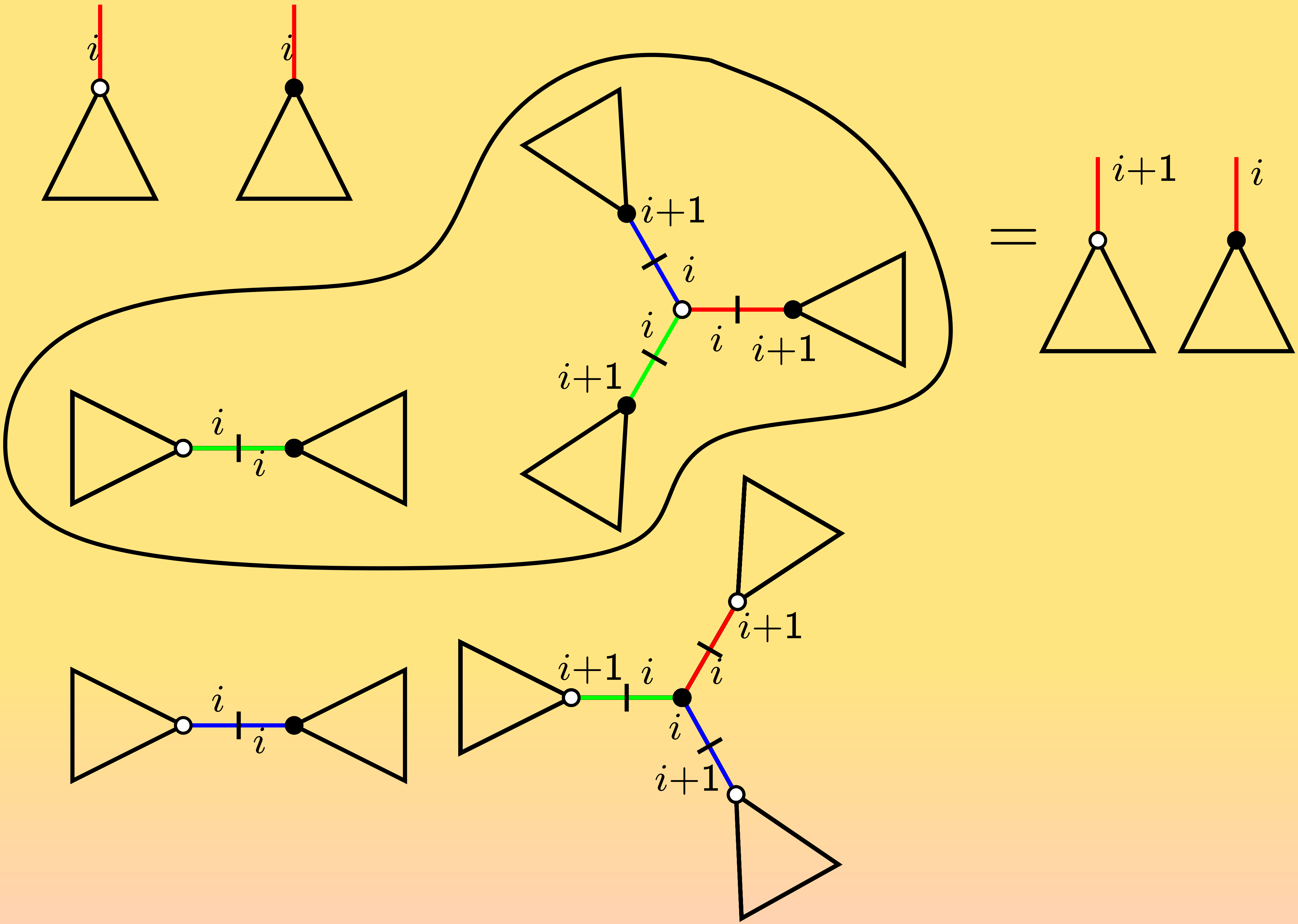
This is correct: $\sum_k 3D_k(z)^2 + 2zD_k(z)^3 = \sum \frac{6(2n)!}{n!(n+2)!}\, z^n$.
But what we want are pairs of Dyck paths with almost the same height.



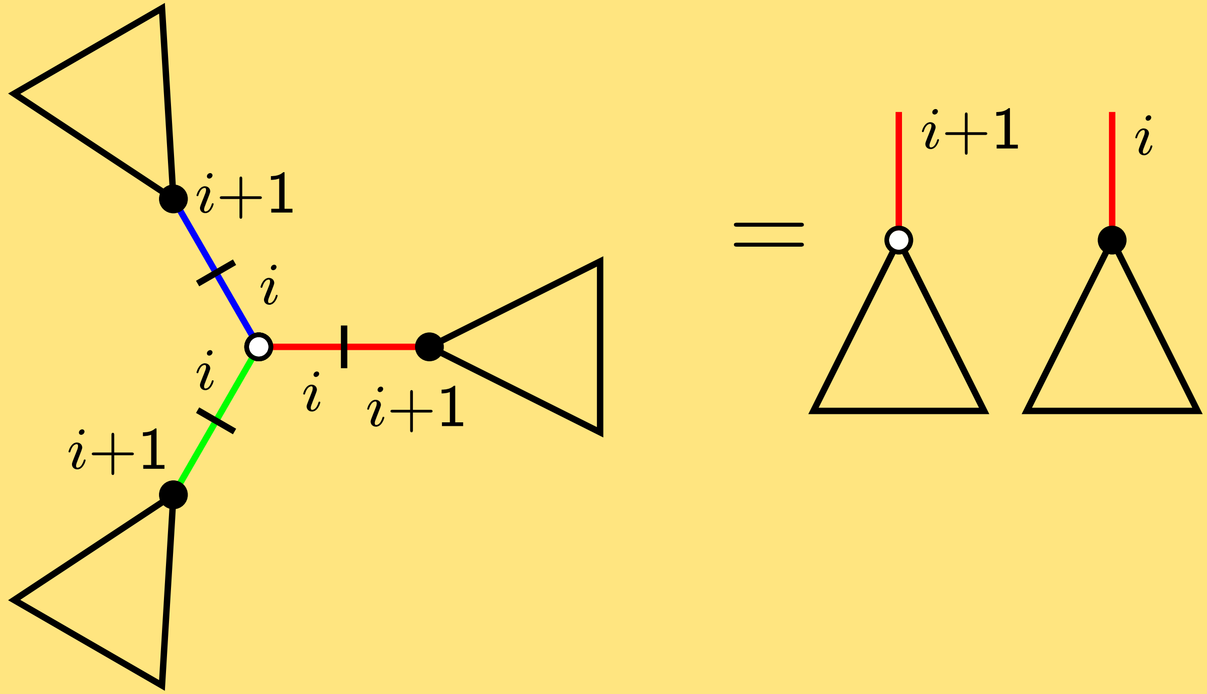
i
i
i
i
i
i
$i{+}1$
i
i
i
$i{+}1$
$i{+}1$
$i{+}1$
$i{+}1$
i
i
i
$i{+}1$

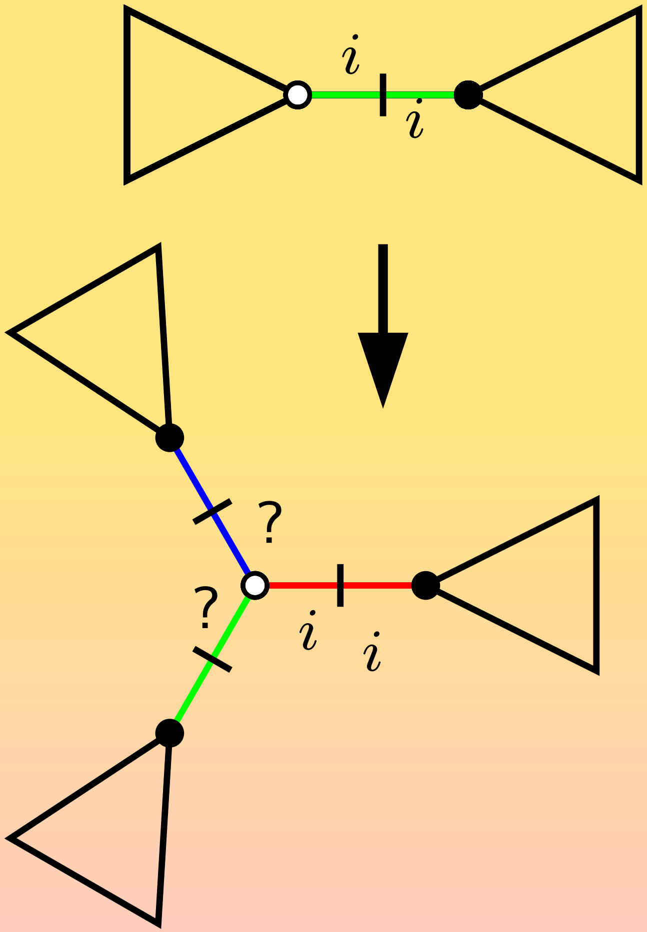


i
i
$i+1$
i
i
i
$i+1$
$i+1$
i
i
i
i
$i+1$
$i+1$
i
i
i
$i+1$



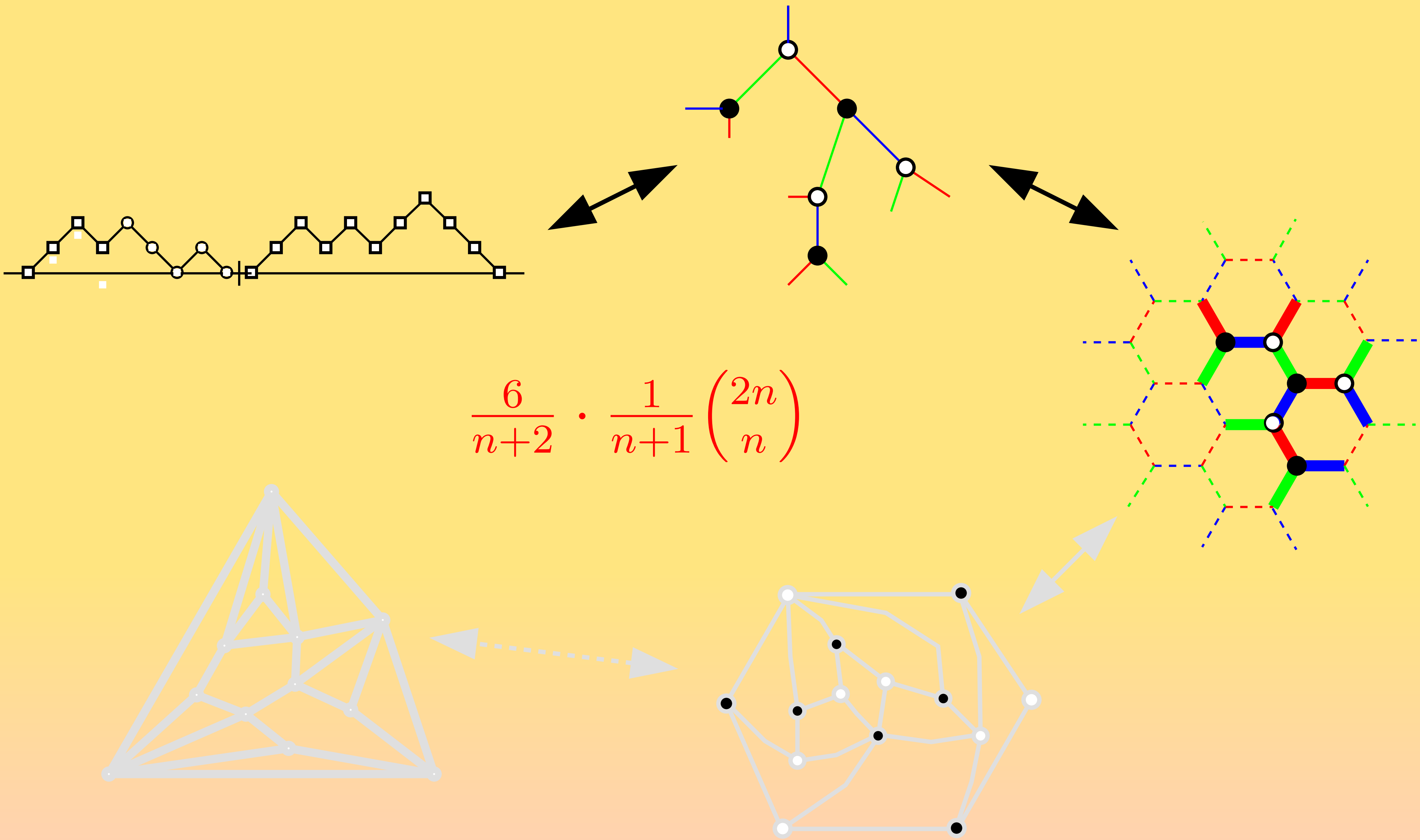
i
i
$i+1$
i
i
i
$i+1$
$=$
$i+1$
i





Looking at possible label and exchanging some subtrees, complete the missing terms!

Here is our diagram...

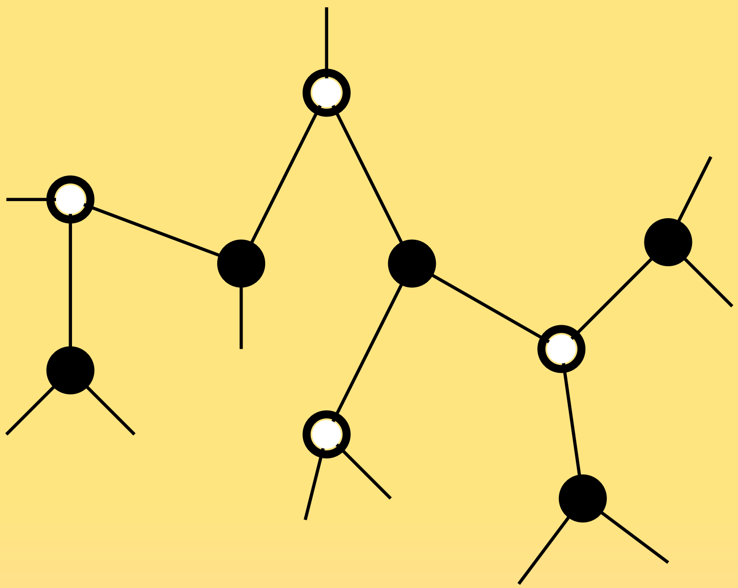


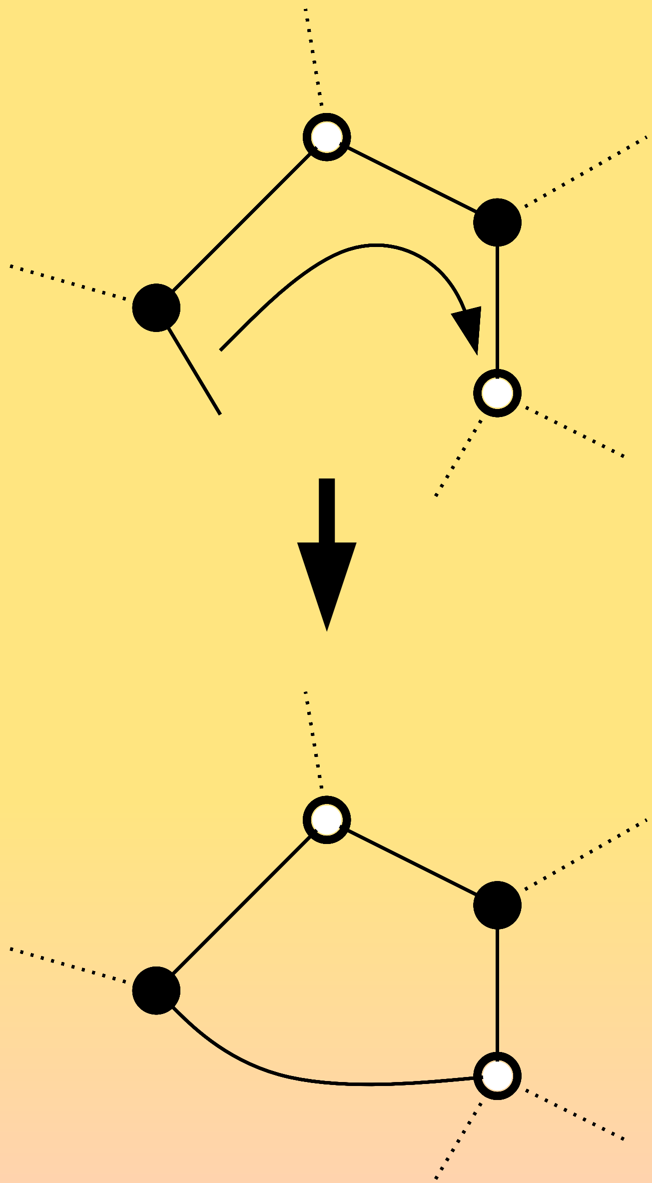
$$\frac{6}{n+2} \cdot \frac{1}{n+1}\binom{2n}{n}$$

Third interpretation: graphs...

A combinatorial operation: the local closure

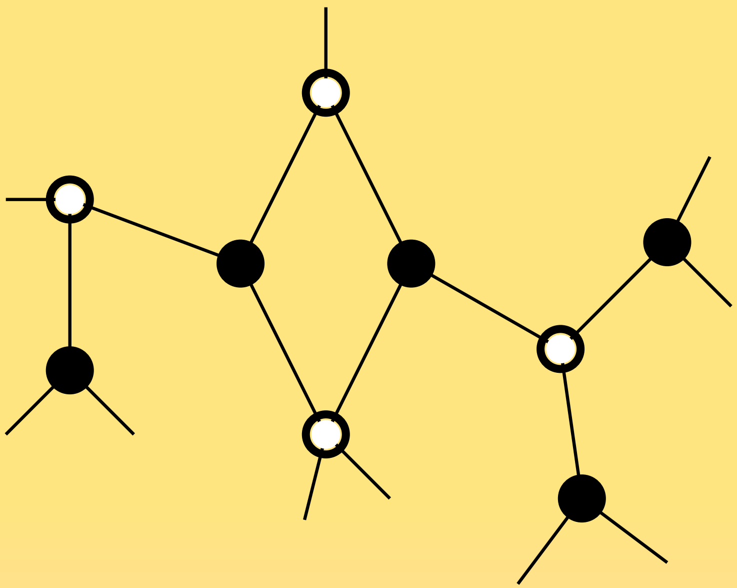
Start with a binary tree and apply greedily
the **local closure rule**

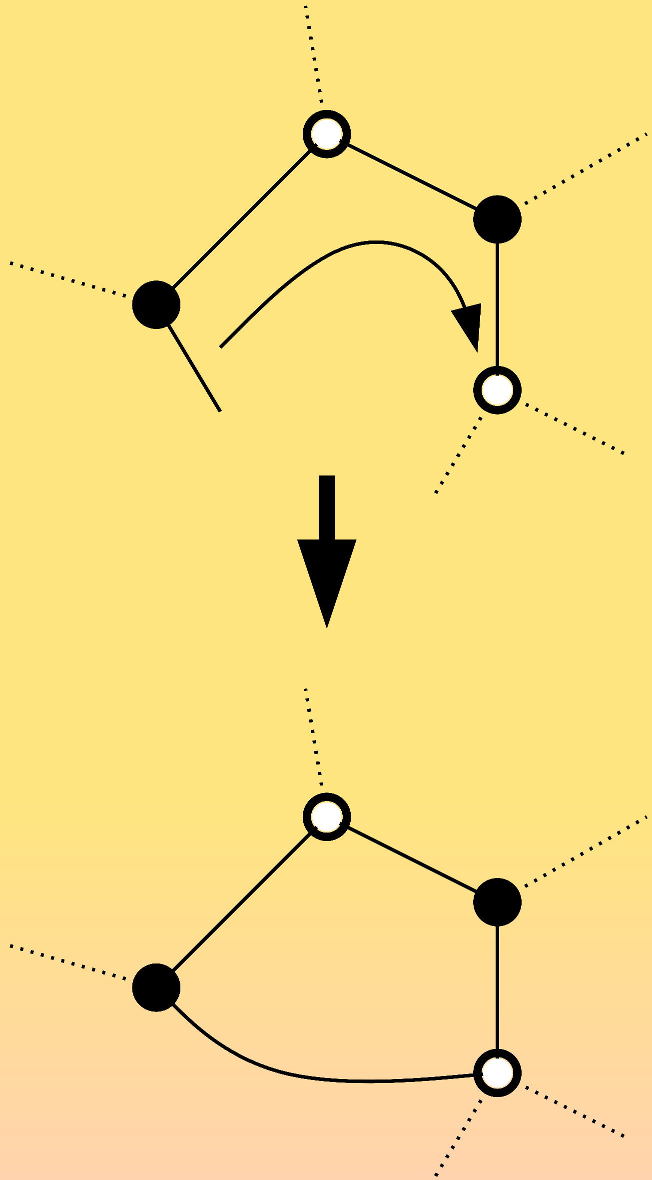




A combinatorial operation: the local closure

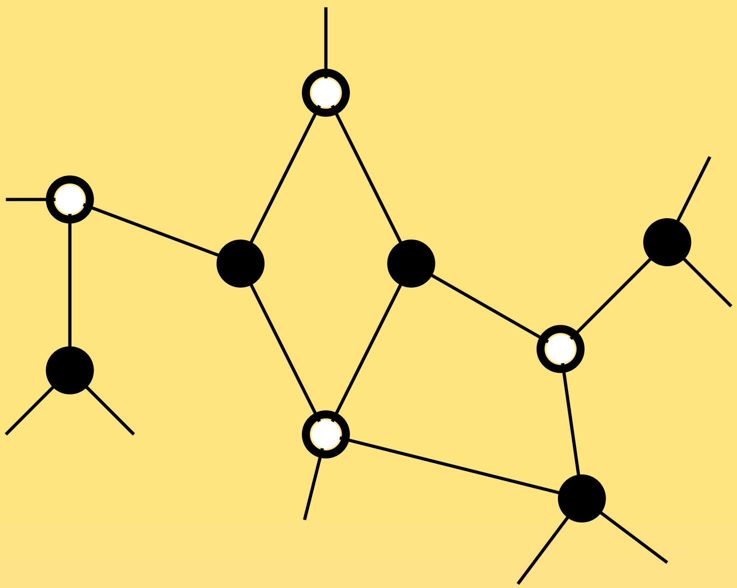
Start with a binary tree and apply greedily
the **local closure rule**

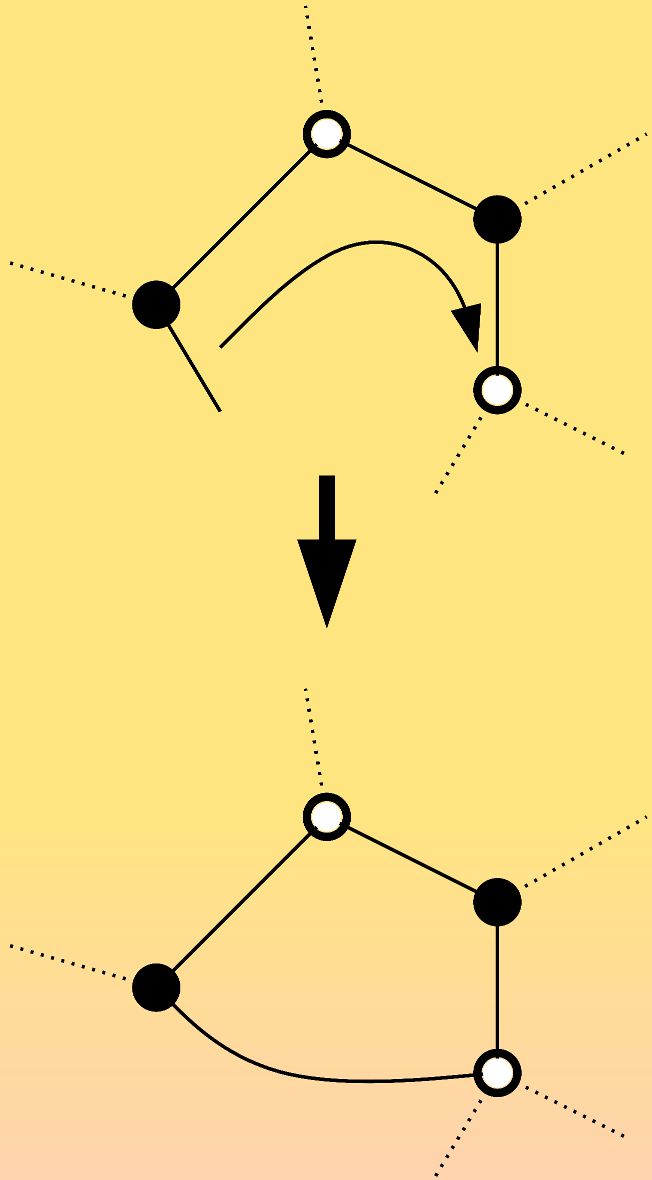




A combinatorial operation: the local closure

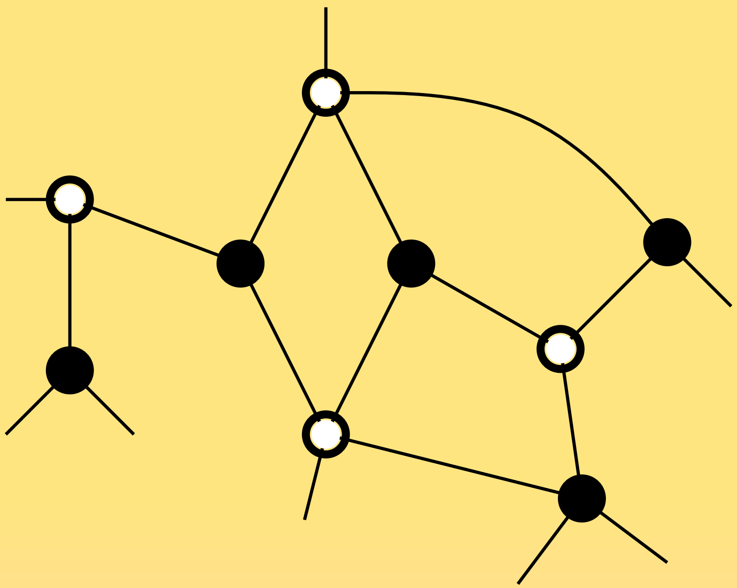
Start with a binary tree and apply greedily
the **local closure rule**

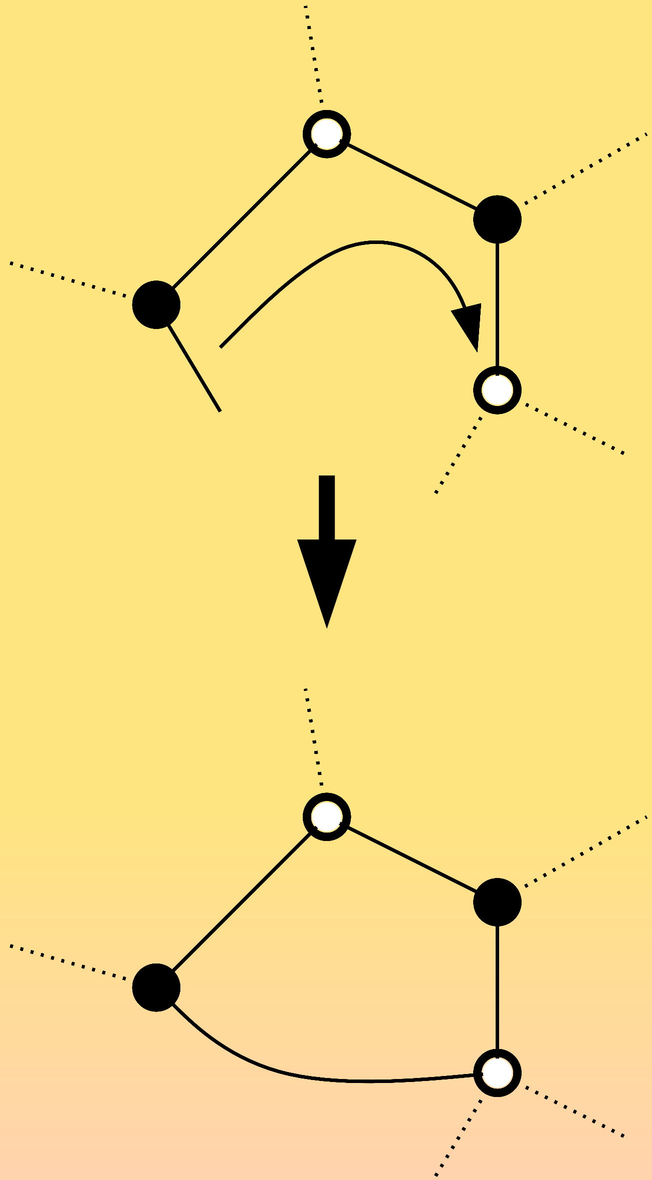




A combinatorial operation: the local closure

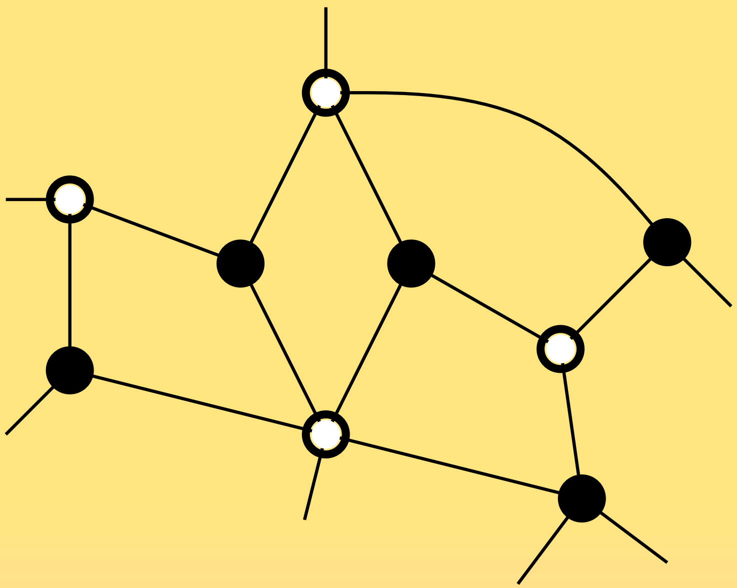
Start with a binary tree and apply greedily
the **local closure rule**

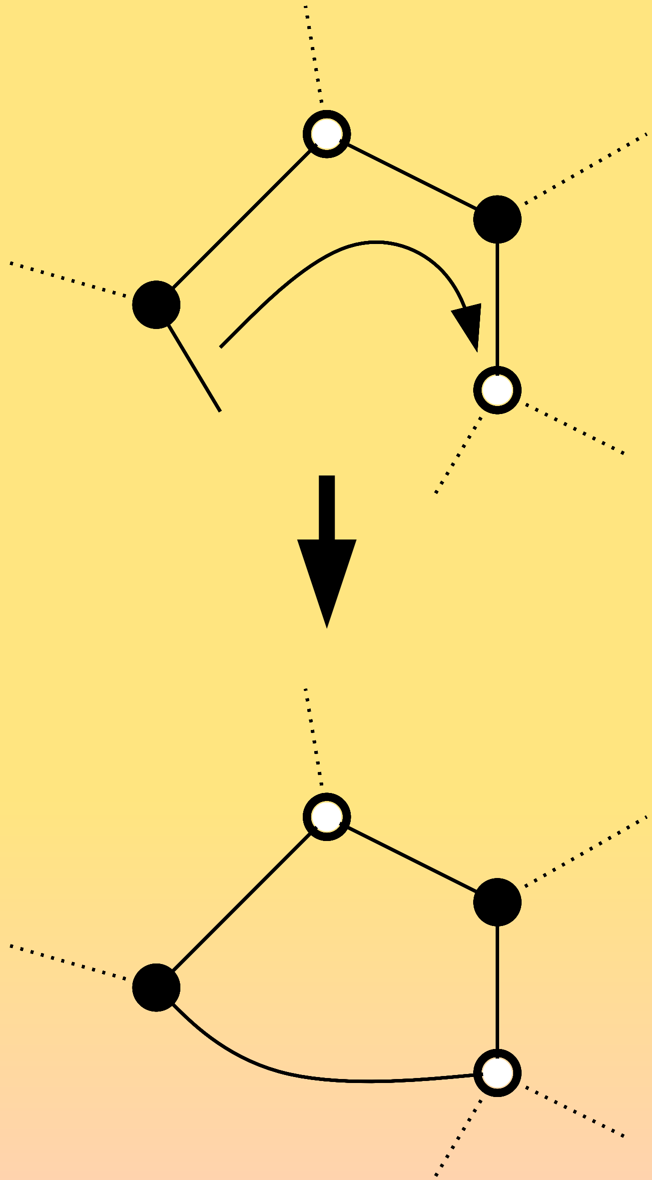




A combinatorial operation: the local closure

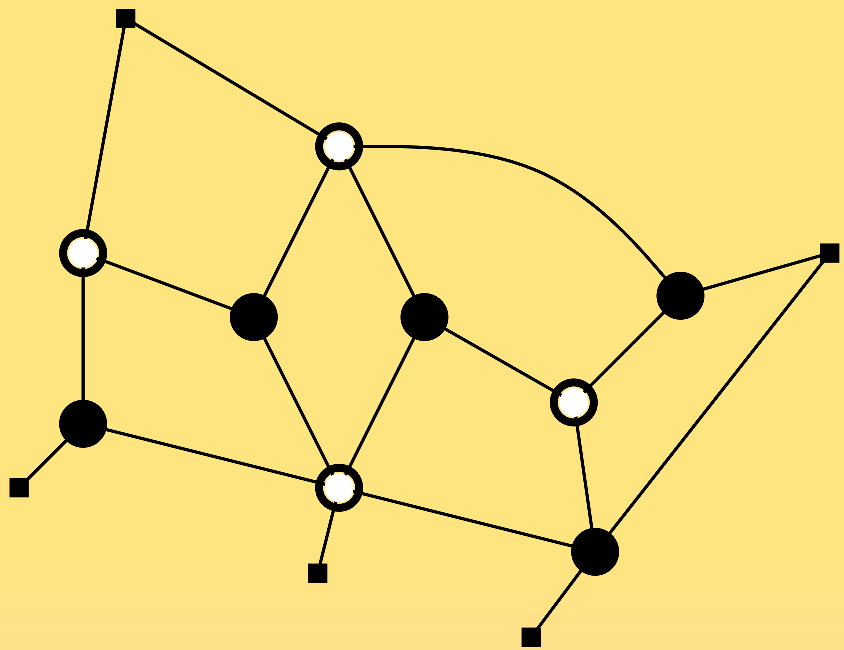
Start with a binary tree and apply greedily
the **local closure rule**

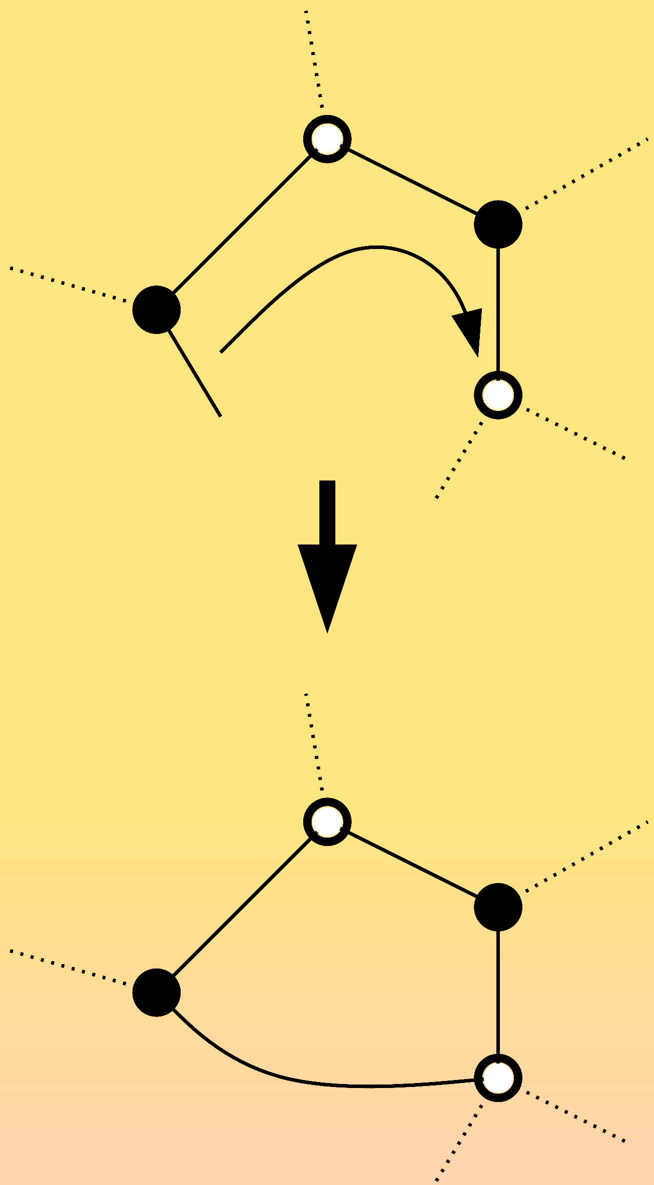




A combinatorial operation: the local closure

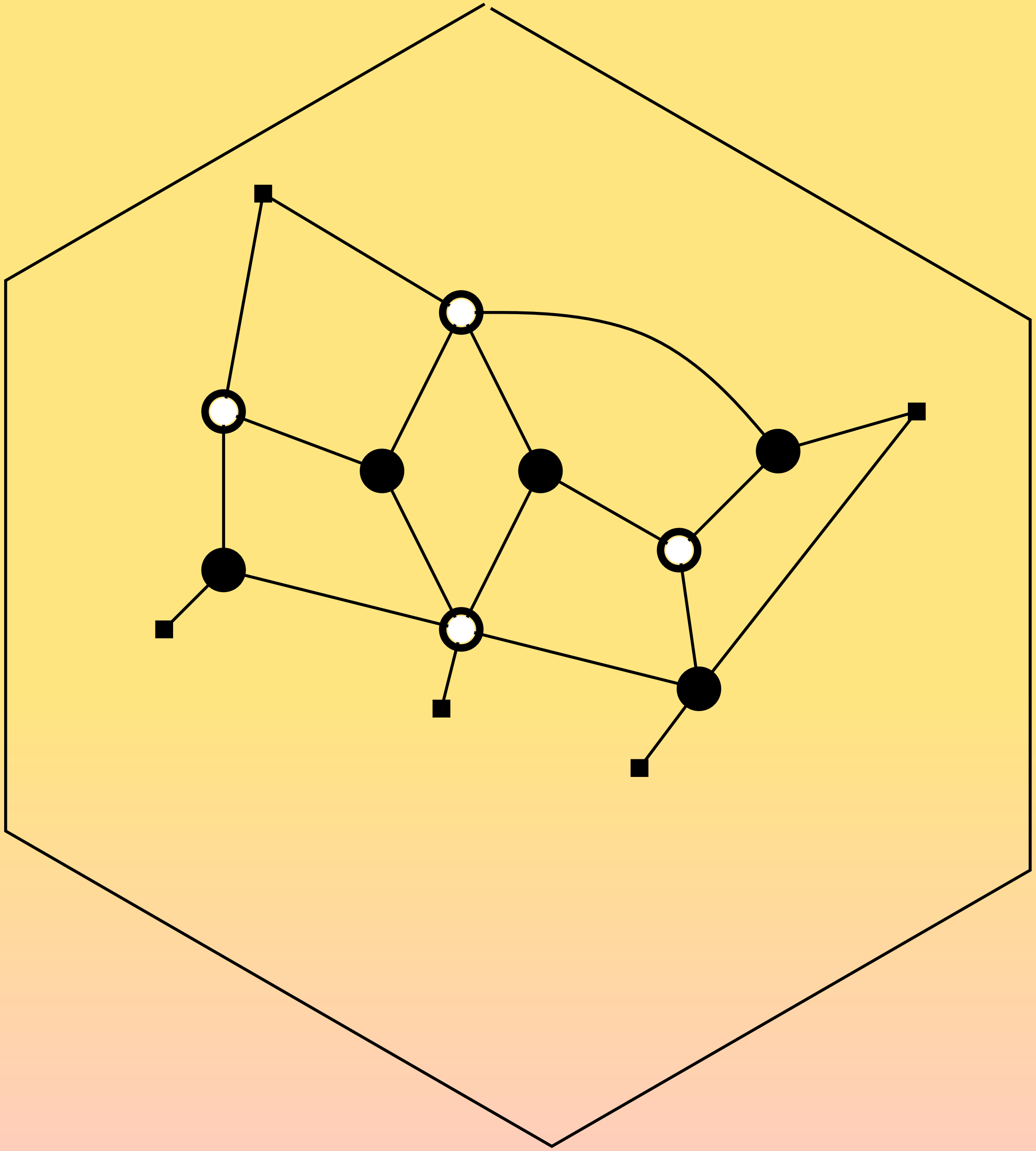
Start with a binary tree and apply greedily
the **local closure rule**





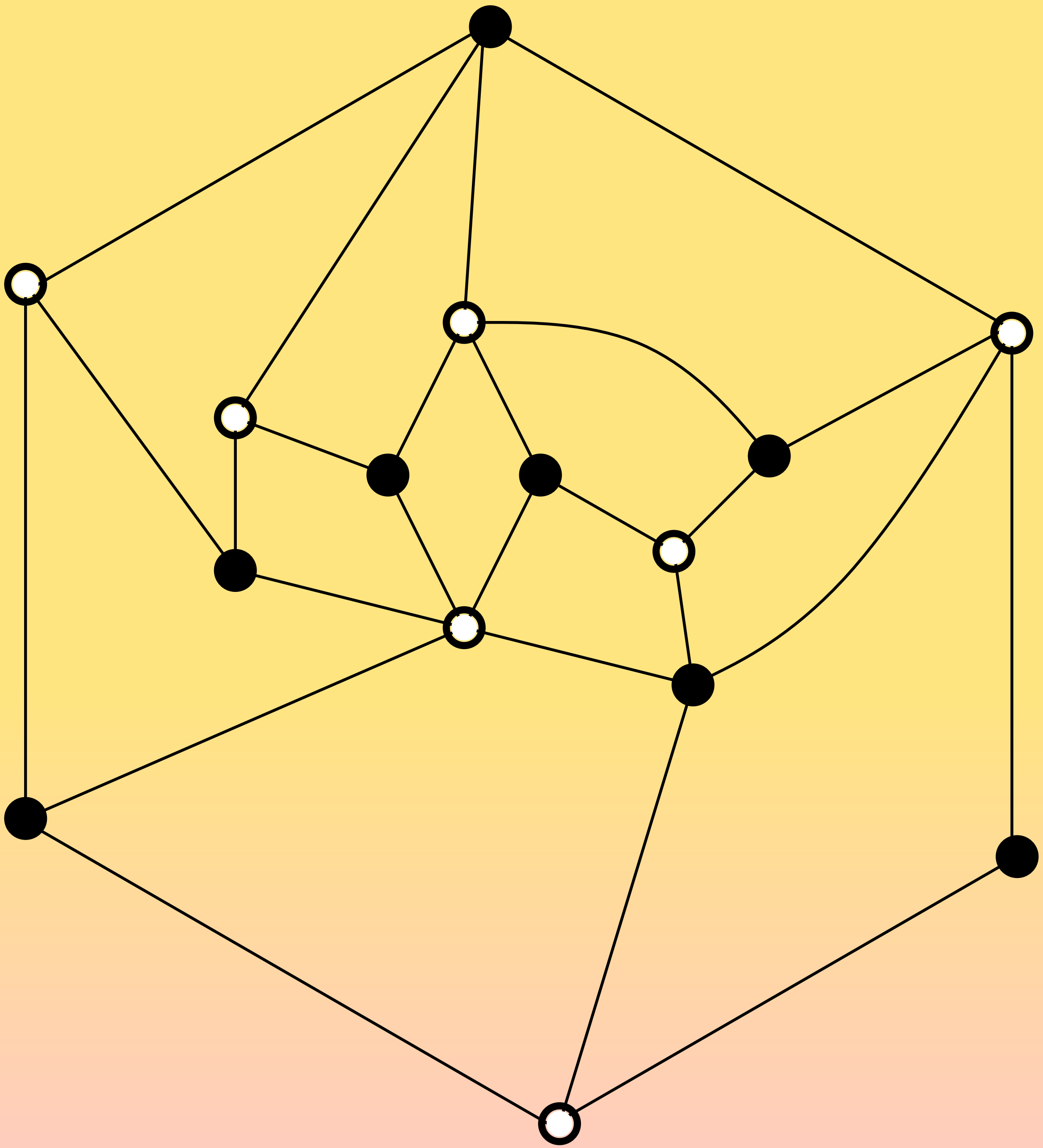
Exactly 6 new vertices are needed

A combinatorial operation: the complete closure



Add a hexagon around the picture

A combinatorial operation: the complete closure



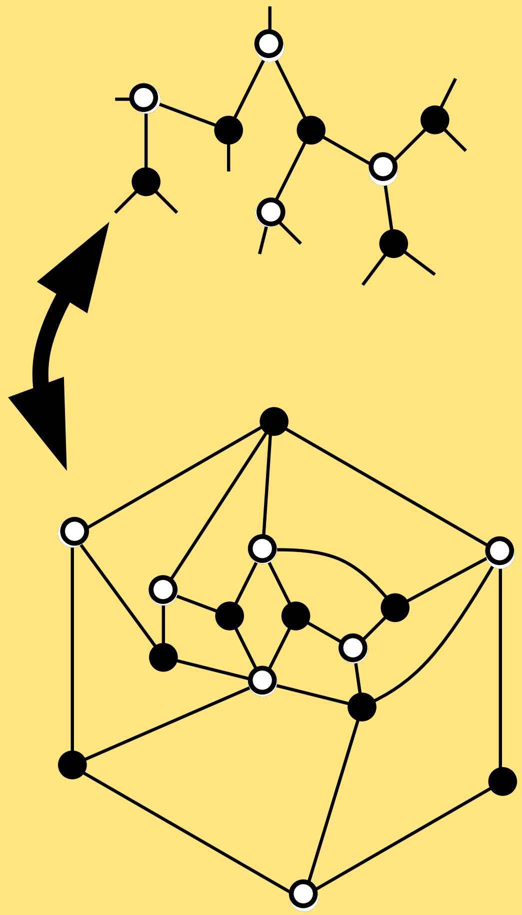
Add a hexagon around the picture

Form quadrangles...

This yields the quadrangulation of a hexagon.

Theorem (Fusy, Poulalhon, S. 05).
The closure is a bijection between

- *unrooted binary trees with n nodes,*
- *unrooted quadrangulations of a hexagon with n internal vertices.*

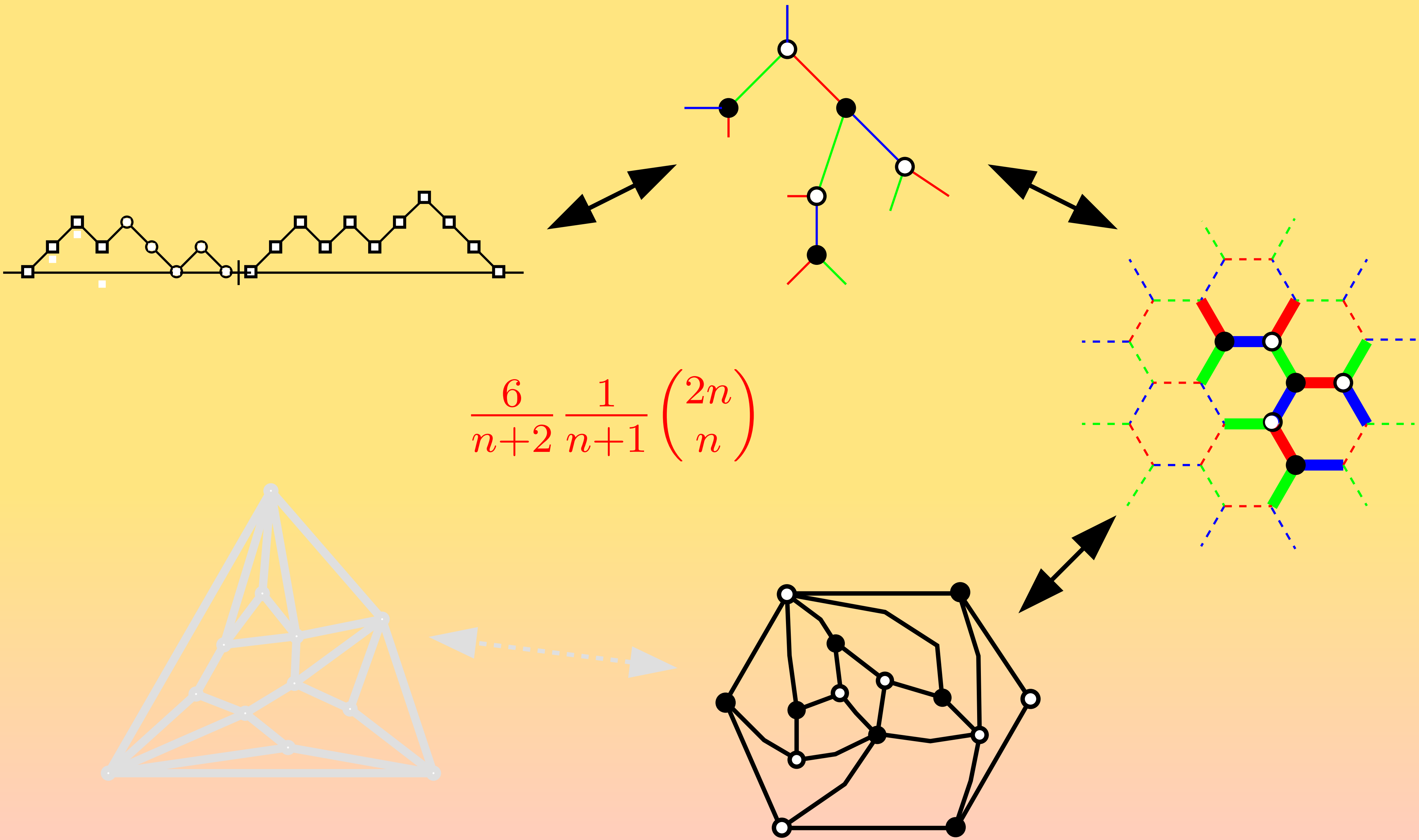


(I will not prove this theorem: it is *hard*...)

Corollary. (Mullin & Schellenberg 68)
The number of rooted quadrangulations of a hexagon is

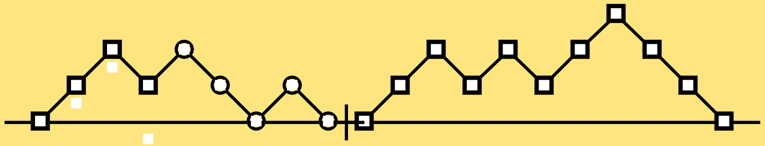
$$\frac{6}{n+2} \cdot \frac{1}{n+1} \binom{2n}{n}.$$

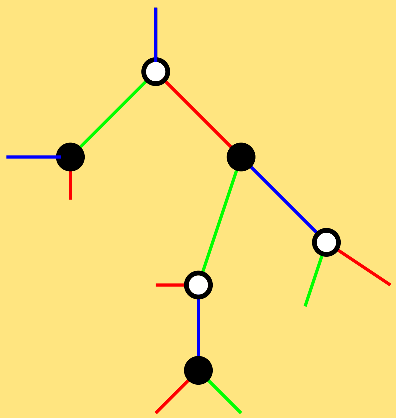
The diagram is almost complete, but we still miss the 3-connected planar graphs of the title of the talk.

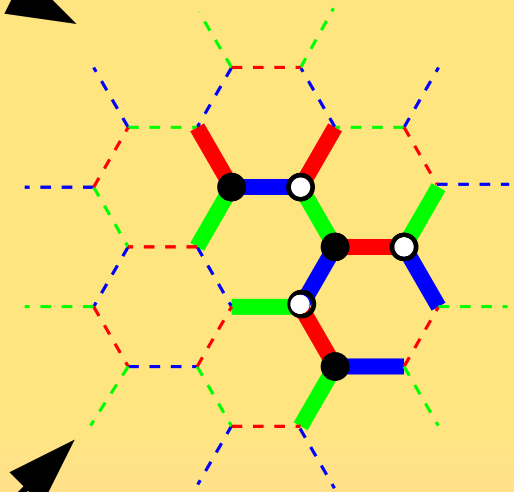


$$\frac{6}{n+2}\,\frac{1}{n+1}\binom{2n}{n}$$

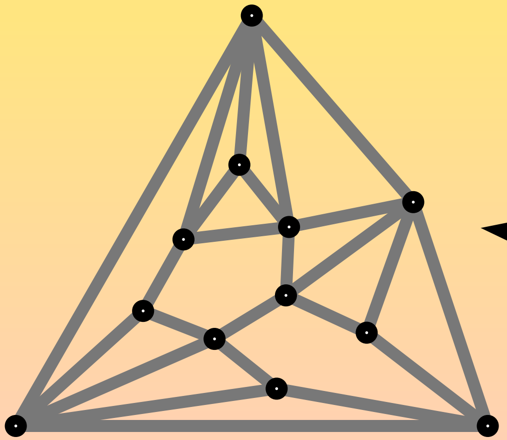
The diagram is almost complete, but we still miss the 3-connected planar graphs of the title of the talk.

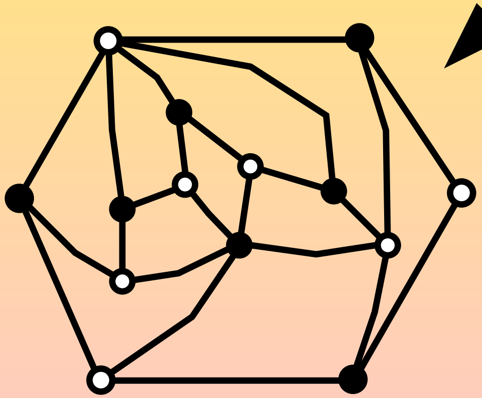






$$\frac{6}{n+2}\,\frac{1}{n+1}\binom{2n}{n}$$





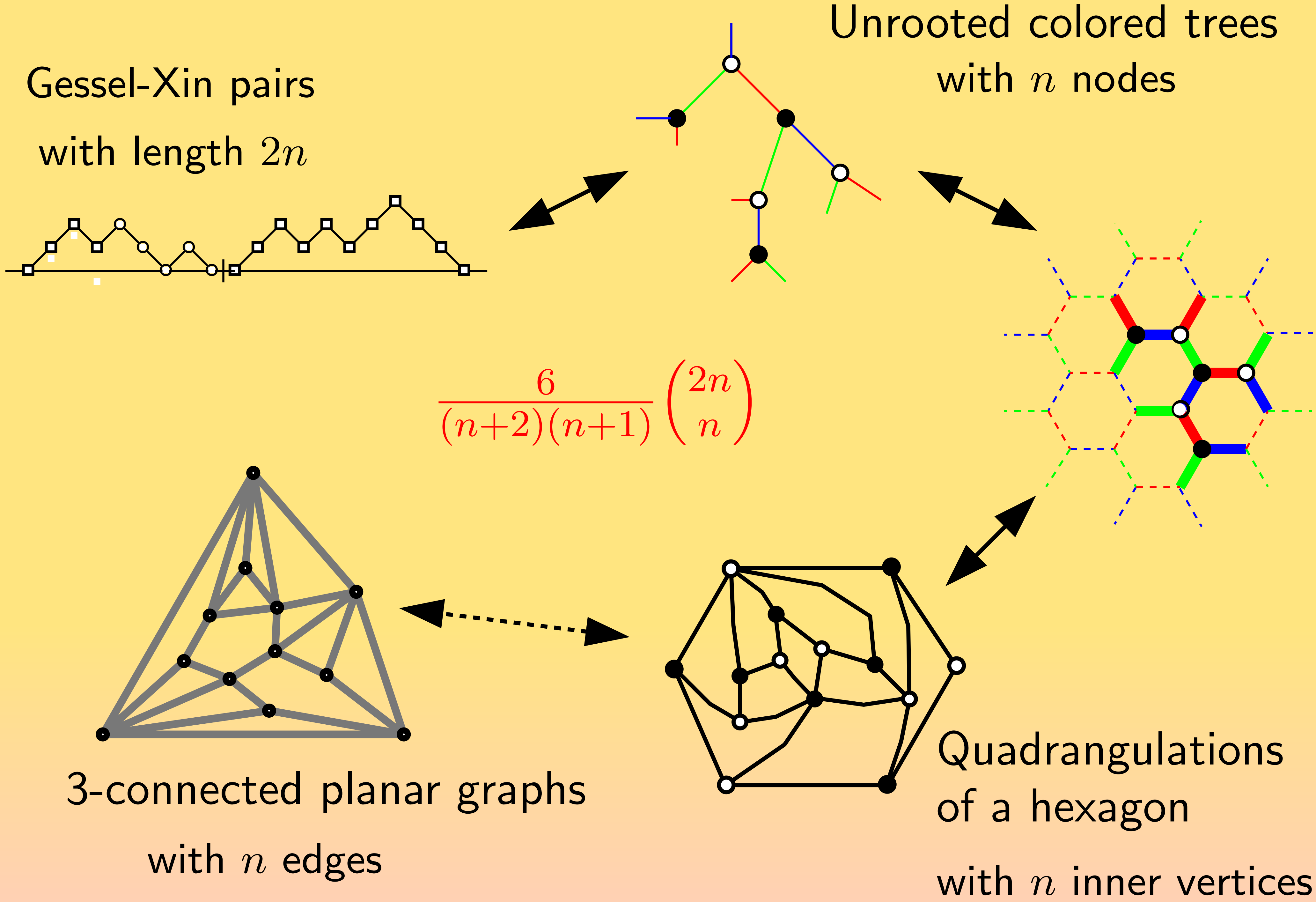
Quadrangulations of a hexagon are "almost" in bijection with 3-connected planar graphs.

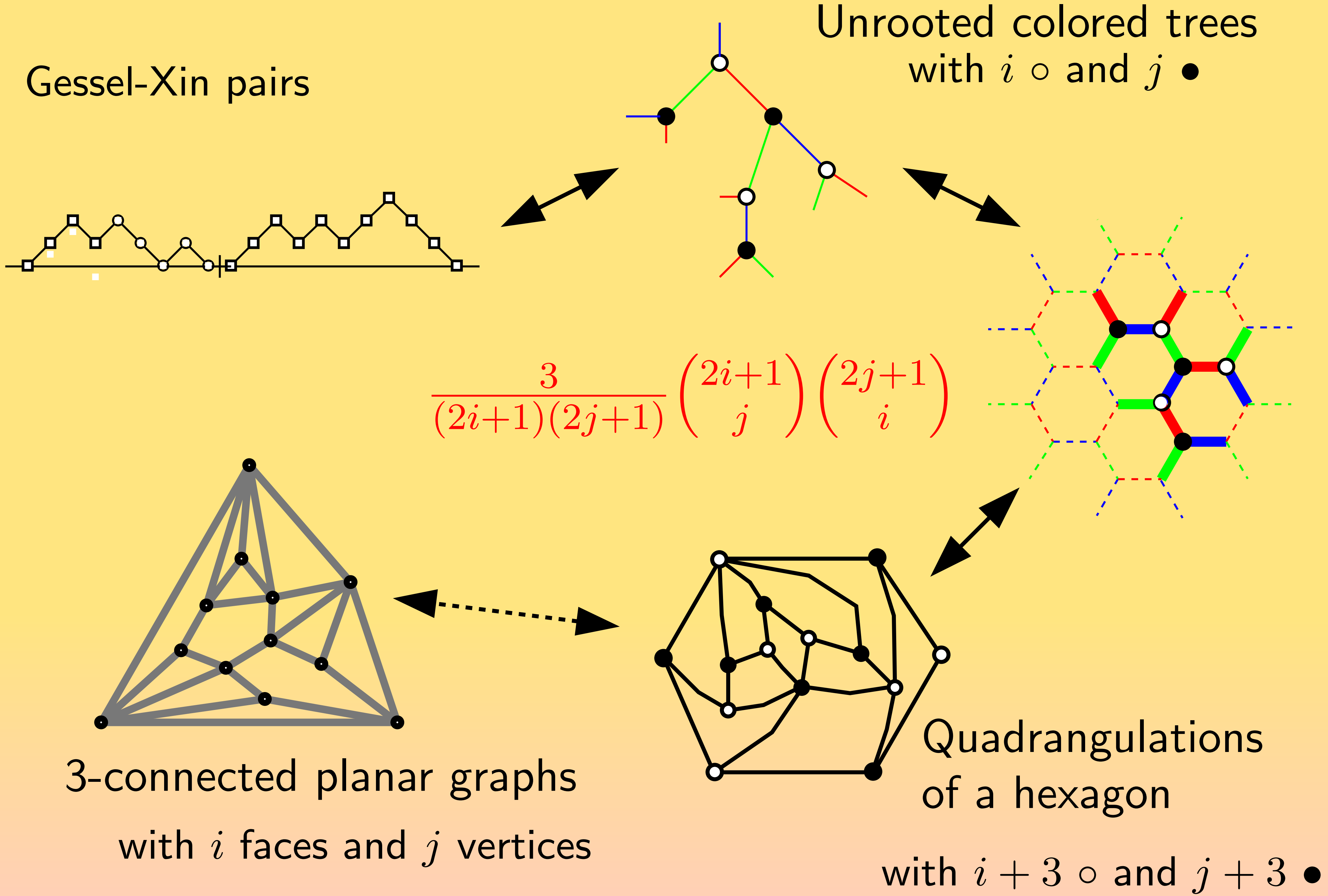
More precisely:

Theorem. (Tutte) *There is a simple bijection between*

- *3-connected planar maps with* n *edges,*
- *quadrangulations** *of a square with* n *faces.*

Theorem (Whitney). *3-connected planar graphs have essentially only one embedding in the plane.*


Unrooted colored trees
with n nodes
Gessel-Xin pairs
with length $2n$
$\frac{6}{(n+2)(n+1)}\binom{2n}{n}$
3-connected planar graphs
with n edges
Quadrangulations
of a hexagon
with n inner vertices


Unrooted colored trees
with i ○ and j ●
Gessel-Xin pairs
$\frac{3}{(2i+1)(2j+1)}\binom{2i+1}{j}\binom{2j+1}{i}$
3-connected planar graphs
with i faces and j vertices
Quadrangulations
of a hexagon
with $i+3$ ○ and $j+3$ ●

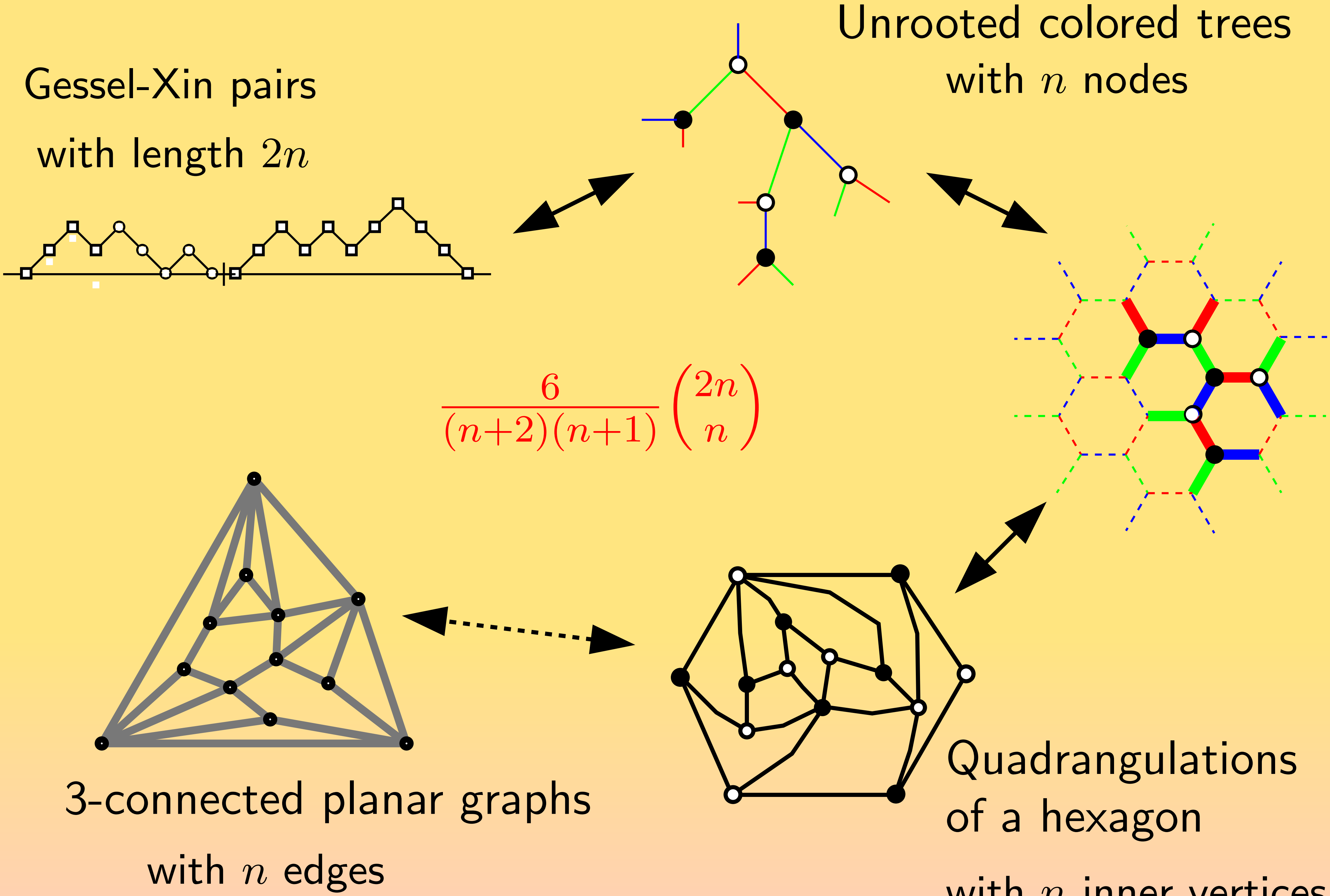
	univariate	bivariate
(order 1 super) Catalan	$\frac{(2n)!}{n!(n+1)!}$	$\frac{(2i+1)!(2j)!}{i!j!(2i+1-j)!(2j+1-i)!}$
order 2 super Catalan	$\frac{6(2n)!}{n!(n+2)!}$	$\frac{3(2i)!(2j)!}{i!j!(2i+1-j)!(2j+1-i)!}$
(m,n) super Catalan	$\frac{1}{2}\frac{(2n)!(2m)!}{n!m!(n+m)!}$	???

Maybe having a 2-variable version could help finding a combinatorial interpretation for all (m,n)...

	univariate	bivariate
(order 1 super) Catalan	$\frac{(2n)!}{n!(n+1)!}$	$\frac{(2i+1)!(2j)!}{i!j!(2i+1-j)!(2j+1-i)!}$
order 2 super Catalan	$\frac{6(2n)!}{n!(n+2)!}$	$\frac{3(2i)!(2j)!}{i!j!(2i+1-j)!(2j+1-i)!}$
(m,n) super Catalan	$\frac{1}{2}\frac{(2n)!(2m)!}{n!m!(n+m)!}$	$???$

Maybe having a 2-variable version could help finding a combinatorial interpretation for all (m,n)...

That's all. Merci de votre attention !


Gessel-Xin pairs
with length $2n$
Unrooted colored trees
with n nodes
$\frac{6}{(n+2)(n+1)}\binom{2n}{n}$
3-connected planar graphs
with n edges
Quadrangulations
of a hexagon
with n inner vertices