Obvious and hidden tree structures in 2d-triangulations

from algebraic generating functions
to random surfaces

Gilles Schaeffer  CNRS & École Polytechnique
Supported by ERC RStG 208471 ”ExploreMaps”

Quantum gravity in Orsay, march 2013
random triangulations as random surfaces

from the combinatorial literature (equivalent pict in physics literature)
Counting maps and triangulations
Trees, independence, algebraic generating series
Stack triangulations
General 2d triangulations
Realizers
Plane graphs and planar maps

Plane graph = \{\text{Embedding of a connected graph in the plane}\}
Plane graphs and planar maps

Plane graph = \{ Embedding of a connected graph in the plane \}

vertex
edge
face

loop
multiple edges
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plane quadrangulation
Plane graphs and planar maps

Plane graph = \{ Embedding of a connected graph in the plane or on the sphere \}

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plane quadrangulation

sphere triangulation

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there are finitely many maps with given number of edges/vertices/faces
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planar maps are discrete (combinatorial) structures
there are finitely many maps with given number of edges/vertices/faces
my main activity is to count these things... also on more general surfaces
but not much to say about 3d or higher until now...
Counting / enumerative combinatorics

A set $\mathcal{A}$ of combinatorial structures, endowed with a size: $\mathcal{A} \rightarrow \mathbb{N}$, $a \mapsto |a|$. We assume $\mathcal{A}_n = \{a \in \mathcal{A} \mid |a| = n\}$ finite for all $n$.

The counting problem is to compute $a_n = |\mathcal{A}_n|$, $n \geq 0$.

The generating function (gf) of the family $\mathcal{A}$ according to the size is

$$A(t) = \sum_{n \geq 0} a_n t^n = \sum_{a \in \mathcal{A}} t^{|a|}.$$
Counting maps

Why do people care about counting maps?

- Tutte et al. (1962→ 2013, decompositions and functionnal equations)
  first with the idea to prove the 4 color theorem
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Theorem (Tutte, 1963): Let \( Q_n = \{\text{rooted quadrangulations with } n \text{ faces}\} \)
and let \( Q(t) = \sum_{q \in Q_n} t^{|q|} \) be the gf where \(|q| = \#\text{faces of } q\).
\[
Q(t) = 1 + 2t + 9t^2 + \ldots
\]

Then \( Q(t) \) is solution of the system \[
\begin{cases}
Q(t) = R(t) - tR(t)^3 \\
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so that \( Q(t) = \frac{(1-12t)^{3/2}-1+18t}{54t^2} \) and \(|Q_n| = \frac{2}{n+2} \frac{3^n}{n+1} \binom{2n}{n} \).
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*\text{a posteriori} justification: a beautiful formula!*
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A lot of analogous results for other families of maps $\mathcal{F}$:

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- many more results of algebraicness of gfs
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- a dozen of nice counting formulas for the $|\mathcal{F}_n|$
- many more results of algebraicness of gfs

- Cori, Vauquelin *et al.* (70/80’s → 2012, bijections with trees) to *explain* the nice formulas and algebraicness
Counting maps

Why do people care about counting maps?

- Brezin, Itzykson, Parisi, Zuber, et al. (1978→ 2013, matrix integrals)

  Key remark (t’Hooft): Perturbative expansion of hermician matrix integrals lead to map generating functions...

⇒ powerful tools and wide extension of Tutte’s counting results

→ Eynard’s talk

here is how I explain to my colleagues that physicists are interested by this: counting maps is a first step in studying the uniform distribution on maps of size $n$, which happens to be an interesting model of random surface.

Ising model on square lattice = toy model of matter
Ising model on random maps ”=” 2d quantum geometry coupled with matters

→ Loll, Budds, Bouttier’s talks
Counting maps

Why do people care about counting maps?

- Goulden, Jackson et al. (80’s → 2012, characters of the symmetric group) for Hurwitz problem: counting ramified covers of the sphere by itself

\[ D = S \]

\[ I = S \]

Hurwitz formula (1894):

\[ h_n(\lambda) = n^{\ell-3} \cdot (n + \ell - 2)! \cdot \prod_{i \geq 1} \frac{1}{\ell_i!} \left( \frac{i^i}{i!} \right)^{\ell_i} \]

\# \{ \text{covers with } n + \ell - 2 \text{ simple ramifications and 1 of type } \lambda = 1^{\ell_1} \ldots n^{\ell_n} \}
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This is what I have been doing for the last 3 years!

An alternative (but equivalent in the limit) model of 2d quantum geometry.
Counting maps and triangulations

Trees, independence, algebraic generating series

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General 2d triangulations

Realizers
Rational and algebraic series in combinatorics

A formal power series $A(t)$ is algebraic (on $\mathbb{Q}(t)$) if it satisfies a (nontrivial) polynomial equation: $P(t, A(t)) = 0$.

It is rational if it can be written as $A(t) = \frac{P(t)}{Q(t)}$ with $P(t)$ and $Q(t)$ polynomials.

A family of combinatorial structures is algebraic or rational if its gf is.
Why do we care about rational and algebraic series

Good closure properties (+, ×, /, derivative, composition) and efficient computational tools (partial fraction decomposition, Puisieux expansion, elimination, resultant, Gröbner,...)

Coefficients can be computed in linear time from the equation.

Algebraicness of a series can be guessed from first coefficients of its expansion (for instance using the tools gfun and Maple)

The asymptotic expansion of coefficients can be determine almost automatically

\[ a_n \sim \frac{\kappa}{\Gamma(d+1)} \rho^{-n} n^d \]

with \( \kappa \) and \( \rho \) some algebraic constants on \( \mathbb{Q} \) and \( d \in \mathbb{Q} \setminus \{-1, -2, \ldots\} \)
## Algebraic series and combinatorics

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**A typical example:** ordered trees

![Diagram of ordered trees](image-url)
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A typical example: ordered trees

Let $A(t)$ be the generating function of ordered trees according to the number of edges

\[
\begin{align*}
\tau & = \tau + \tau' + \ldots \\
\Rightarrow \quad A(t) & = 1 + tA(t)^2 
\end{align*}
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As a result: 
\[
A(t) = \frac{1-\sqrt{1-4t}}{2t} = \sum_{n \geq 0} \frac{1}{n+1} \binom{2n}{n} t^n,
\]

i.e. ordered trees are counted by Catalan numbers.

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another example:

2-leaf trees
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Let $C_2(t)$ be the gf of plane trees with 2 leaves per inner vertex, according to the number of inner edges.

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| $|\alpha| = |(\beta, \gamma)| = |\beta| + |\gamma|$ |

Another example: 2-leaf trees

Let $C_2(t)$ be the gf of plane trees with 2 leaves per inner vertex, according to the number of inner edges.

$$C_2(z) = C_1(z) + zC_2(z)^2$$

$$C_1(z) = C_0(z) + zC_1(z)C_2(z)$$

$$C_0(z) = 1 + zC_0(z)C_2(z)$$
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$C_0 = \frac{1}{1-zC_2}$, $C_1 = \frac{1}{(1-zC_2)^2}$, $C_2 = \frac{1}{(1-zC_2)^3}$

As a result: $C_2(z) = \sum_{n \geq 0} \frac{1}{3n+1} \binom{4n}{n} z^n$. 

another example: 2-leaf trees
Combinatorial interpretation: \( \mathbb{N} \)-algebraic structures

\( \mathbb{N} \)-algebraic structures = families that can be defined by algebraic specification (aka context-free grammars)
Combinatorial interpretation: $\mathbb{N}$-algebraic structures

$\mathbb{N}$-algebraic structures = families that can be defined by algebraic specification (aka context-free grammars)

$N(t) = 1 + tN(t) + t^2B(t)N(t) + t^3N(t)R(t)^2$

$B(t) = 1$

$R(t) = 1 + t^2R(t)N(t) + t^3N(t)B(t)^2 + t^3B(t)R(t)N(t)$

$\mathbb{N}$-algebraic structures clearly have algebraic generating functions
Combinatorial interpretation: $\mathbb{N}$-algebraic structures

$\mathbb{N}$-algebraic structures $=$ families that can be defined by algebraic specification
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$\mathbb{N}$-algebraic structures clearly have algebraic generating functions

Intuition: $\mathbb{N}$-algebraic structures are tree-like, with independant subtrees (conditionally to the root colors)
Combinatorial interpretation: \( \mathbb{N} \)-rational structures

When all equations are linear, the structures are called \( \mathbb{N} \)-rational, and they have rational generating series.

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**Intuition:** \( \mathbb{N} \)-rational structures have a linear structure and their growth is controled by a finite number of states (finite state machine)
Validity of the combinatorial intuition

The intuition is "always correct" for rational structures

\[ \text{combinatorial structure + rational gf } \implies \text{ } \mathbb{N}\text{-rational structure} \]

(empirical implication, one can create ad-hoc counterexamples)

On the contrary there are many examples of combinatorial structures that are algebraic but display no natural tree-like structure

\[ \text{combinatorial structure + algebraic gf } \not\implies \text{ } \mathbb{N}\text{-algebraic structure} \]

\[ \implies \text{ the bijective problem} \]

give combinatorial explanations for algebraic gf i.e. prove \( \mathbb{N}\)-algebraicness to understand better algebraic structures

in other terms, when the gf is algebraic one would like to make explicit a tree-like structure
Counting maps and triangulations
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Stack-triangulations as 3d triangulations

A ”trivial” model of 3d-triangulations:
a set of tetrahedra, pairs of identified faces
each face belong to:
one identification (internal face) or none (boundary face)

\[ n \text{ tetrahedra} \rightarrow 4n \text{ faces} \]

if connected, at least \( n - 1 \) indentifications, at most \( 2n + 2 \) boundary faces

A stack-triangulation is a connected 3d-triangulation with \( n \) tetrahedra and \( 2n + 2 \) free faces (aka a maximal boundary 3d-triangulation)

Equivalently a 3d-triangulation is stack if it has a tree-like structure.
The boundary of a stack-triangulation is topologically a sphere
Stack-triangulations as 3d triangulations

A "trivial" model of 3d-triangulations:

- A set of tetrahedra, pairs of identified faces
- Each face belongs to:
  - One identification (internal face)
  - Or none (boundary face)

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If connected, at least \( n - 1 \) indentifications, at most \( 2n + 2 \) boundary faces

A stack-triangulation is a connected 3d-triangulation with \( n \) tetrahedra and \( 2n + 2 \) free faces (aka a maximal boundary 3d-triangulation)

Equivalently a 3d-triangulation is stack if it has a tree-like structure.

The boundary of a stack-triangulation is topologically a sphere
Stack-triangulations as 3d triangulations

A "trivial" model of 3d-triangulations:

- A set of tetrahedra, pairs of identified faces
- Each face belongs to:
  - One identification (internal face)
  - Or none (boundary face)

$n$ tetrahedra $\rightarrow 4n$ faces

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Take a boundary triangle and root it.
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This correspondence is invertible.

This yields a bijective decomposition:

\[ T \equiv \Delta \cdot ( \Delta + T)^3 \]
Counting stack-triangulations

Stack-triangulations are specified by the eq: $T \equiv \triangle \cdot (\Delta + T)^3$

which translate into the algebraic equation $T(z) = z(1 + T(z))^3$

Hence the number of rooted stack-triangulations with $n$ tetrahedra is:

$$2i\pi [z^n]T(z) = \int \frac{T(z)}{z^{n+1}} dz = \int \frac{t}{z(t)^{n+1}} z'(t) dt = \int \frac{(1-2t)(1+t)^{3n-1}}{t^n} dt$$

$$[z^n]T(z) = [t^{n-1}](1+t)^{3n-1} - 2[t^{n-2}](1+t)^{3n-1} = \binom{3n-1}{n-1} - 2\binom{3n-1}{n-2} = \frac{1}{2n+1} \binom{3n}{n}$$

These numbers are the number of ternary trees. In this case there is no real surprise.
Large random stack-triangulations

The uniform distribution on stack-triangulations of size $n$ yields the uniform distribution on ternary trees with $n$ nodes.

Ternary trees (as all ”simple” families of trees) converge upon rescaling by a factor $\sqrt{n}$ to the continuum random tree (aka Brownian tree).

The geometry is of gluing tetrahedra is trivial however because vertices get multiply identified. Yet Albenque and Marckert (2005) have proved that the convergence hold in the sense of Gromov Hausdorff (cf. Legall’s talk).
Stack-triangulations as 2d triangulations

Stack-triangulations can be projected on their boundary, and in the plane upon putting a point of a boundary face at infinity.

A 2d-triangulation is (the projection of) a stack triangulation if
- either it is a tetrahedra
- or it contains a vertex of degree 3 and the removal of this vertex and the incident edges is again a stack-triangulation.

Conversely all stack-triangulations can be constructed from an original outer triangle by iteratively subdividing triangles in 3.
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The $\mathbb{N}$-algebraic structure arises from a decomposition in 3 independent regions.
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Not all 2d-triangulations are stack-triangulations!

No vertex of degree 3 here...

Are general random triangulations of size $n$ much different from random stack triangulations of size $n$?
Counting maps and triangulations
Trees, independence, algebraic generating series
Stack triangulations
General 2d triangulations
Realizers
Counting general triangulations

Tutte has found by computations that

\[ T_n = \{ \text{rooted triangulations with } 2n \text{ faces} \} = \frac{2(4n-3)!}{n!(3n-1)!} \]

Equivalently the generating function is algebraic

\[ T(z) = A(z) - 2A(z)^2 \quad \text{where} \quad A(z) = \frac{z}{(1-A(z))^3} = C_2(z). \]

This formula can also be obtained from BIPZ matrix integral approach (although not directly: the direct computation yields triangulations with multiple edges; one needs to perform a simple renormalization to eliminate these multiple edges).
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• find "\( \mathbb{N} \)-algebraic" canonical trees inside the structures

• give a direct \( \mathbb{N} \)-algebraic decomposition (Bouttier-Guitter 2013)
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**Theorem** Closure is a one-to-one correspondence between 2-leaf trees with $n$ nodes and marked triangulations with $n + 2$ vertices.
From trees to triangulations

Theorem (Poulalhon-S. 2004) Closure is a one-to-one correspondence between 2-leaf trees with $n$ nodes and marked triangulations with $n + 2$ vertices.

Corollary The number of triangulations with $n + 2$ vertices is $\frac{2(4n-3)!}{n!(3n-1)!}$
Some hints about the machinery
Some hints about the machinery
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Lemma. Closure endows the triangulation with an orientation without clockwise cycles.

Indeed all faces are closed by counterclockwise edges.
Some hints about the machinery

Lemma (Poulalhon-Schaeffer 2004/ Bernardi 2005.)
A planar map endowed with an accessible orientation without clockwise cycles admits a unique spanning tree such that external edges are all counterclockwise.

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Corollary The closure is a bijection between 2-leaf trees with \( n \) nodes and triangulations with a minimal accessible 3-orientation.
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Theorem (Schnyder, 1992) Every planar triangulation admits a 3-orientation, and all 3-orientations are accessible.

Define a partial order on the 3-orientations of a triangulation by circuit reversal.
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**Corollary** Every planar triangulation has a unique accessible 3-orientation without clockwise cycle.

**Corollary** The previous closure is a bijection between 2-leaf trees with $n$ nodes and planar triangulations with $n + 2$ vertices.
**Summary**

**Theorem** Closure is a one-to-one correspondence between 2-leaf trees with $n$ nodes and marked triangulations with $n + 2$ vertices.

**Bonus** The closure edges form almost geodesics toward the root. This allows to study the number of vertices at distance $r$ from the root.
Triangulations converge to the Brownian map

Theorem (Albenque, Adderio-Berry 2013) Upon rescaling edge length by a factor $n^{1/4}$, uniform random triangulations of size $n$ converge to the Brownian map of Legall in the sense of Gromov-Hausdorff.

The meaning of this statement will be explained in Le Gall’s talk who will discuss the proof of his earlier analog result for $2g$-angulations and for triangulations with loops and multiple edges.
Partial conclusion

"Many” exact enumeration results on planar (or higher genus maps).

Quite a number of them involve algebraic gf. As shown by Eynard, this corresponds to a peculiar (non fundamental he would say) property of their spectral curve.

Yet it corresponds to cases which we can solve ”even more explicitely” by combinatorial tools: by revealing hidden tree structure, we prove that these models are in some sense $\mathbb{N}$-algebraic.

The machinery based on orientation without clockwise cycle is very general.

For some not completely clear reasons, the revealed tree structures also allows to study geodesics in the corresponding maps, and to prove convergence of the rescaled surfaces to the Brownian map.
Counting maps and triangulations
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Realizers as a toy model on triangulations

Let $T$ be a triangulation with boundary $\{x_1, x_2, x_3\}$.

$I = \{\text{internal vertices}\}$.

(Here $|I| = 6$)
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Let $T$ be a triangulation with boundary $\{x_1, x_2, x_3\}$.

A Schnyder wood is a partition $T_1, T_2, T_3$ of the internal edges of $T$ such that:

i) $T_i$ is a spanning tree of $I \cup \{x_i\}$,

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Upon orienting edges of each tree toward its root, each vertex has 1 outgoing edge of each color, and

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ii) Colors must satisfy the local Schnyder rule:

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ii) Colors must satisfy the local Schnyder rule:

A realizer is a triangulation endowed with a Schnyder wood.

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**Corollary** Every triangulation admits a Schnyder wood.
Realizers can be counted exactly:

**Theorem (Bonichon 2003)** Realizers of size $n + 3$ vertices are in one-to-one correspondence with pairs of Dyck paths of length $2n$.

**Corollary** The number of realizers of size $n + 3$ is

$$C_{n+2}C_n - C_{n+1}^2 = \frac{6(2n)!(2n+2)!}{n!(n+1)!(n+2)!(n+3)!}$$

Realizers can thus be viewed as a curious exactly solvable model on triangulations.
We give arbitrary positions to the 3 points $x_1$, $x_2$ and $x_3$, and we want barycentric coordinates for internal points.
Schnyder’s drawing algorithms

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If $v$ has coordinates $(v_1, v_2, v_3)$, it is placed in position $v_1x_1 + v_2x_2 + v_3x_3$ (Assuming $v_i \in (0, 1)$ and $\sum v_i = 1$.)
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and Coordinate $v_i$ corresponds to the area of Triangle $t_i$.

We shall use a combinatorial analog to the areas of the 3 triangles to make canonical drawings of triangulations.
Assume we have a Schnyder forest:

- Let $P_i(v)$ the path from $v$ to $x_i$ in $T_i$.
- Let $R_i(v)$ the region bounded by $P_{i+1}(v)$, $P_{i+2}(v)$ and $(x_{i+1}, x_{i+2})$.
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The combinatorial analog of the triangle areas is given by the number of faces included in each regions:

$$V_i(v) = \frac{|R_i(v)|}{|T|}$$
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The combinatorial analog of the triangle areas is given by the number of faces included in each region:

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**Theorem (Schnyder)** The drawing of $T$ with straight lines with each vertice $v$ at its barycentric coordinate $(V_1(v), V_2(v), V_3(v))$ is planar, whatever the original placement of $x_1, x_2, x_3$ (non aligned).
Schnyder’s drawing algorithms

A sketch of proof

we only need to show that branches are geometrically oriented as expected.

**Lemma.** In the neighborhood of a vertex, the parallels to the 3 sides of the triangle \((x_1, x_2, x_3)\) separate the 6 type of edges.
Schnyder’s drawing algorithms

A sketch of proof

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**Lemma.** In the neighborhood of a vertex, the parallels to the 3 sides of the triangle \((x_1, x_2, x_3)\) separate the 6 type of edges.

This allows to prove that each region \(R_i(v)\) of the new drawing contains the same vertices than the \(R_i(v)\) of the initial drawing.

If an edge crosses one of the 3 edges emanating from \(v\), it intersects 2 different regions in the new drawing and thus also in the old one. This contradicts planarity of the original picture.

□
Some questions to conclude

About realizers:
In which class of universality do realizers fall? (eg what is the central charge of the underlying toy model?)

What is the Hausdorff dimension of realizers?
Has the Schnyder drawing of a realizer any physical relevance?

About a variant called transversal structures:

Definition: A transversal structure of a triangulation of a square is a partition of edges such that locally:

Lemma: A triangulation of the square admits a transversal structure if and only if it contains no separating 3-cycle.

Is this related with the local causal structure introduced by Loll?
Merci de votre attention