Graphes colorés réguliers aléatoires

Aspects combinatoires d'un modèle de la gravité quantique en dimension $D \geq 3$

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Regular colored graphs, why?

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Two "discrete → continuum" approaches for $D = 3$ (I know of):

— Lorenzian geometries, $D = 2 + 1$: layers of triangulations

Experimental results with random sampling, no exact results

— Euclidean geometries, $D = 3$: arbitrary pure simplicial complexes?

Partial results following the Tensor Track (survey©Rivasseau)

To learn more: workshop Quantum gravity in Paris-Orsay in march.

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we concentrate on Regular colored bipartite graphs

(there are a few other examples)

Regular colored graphs, why?

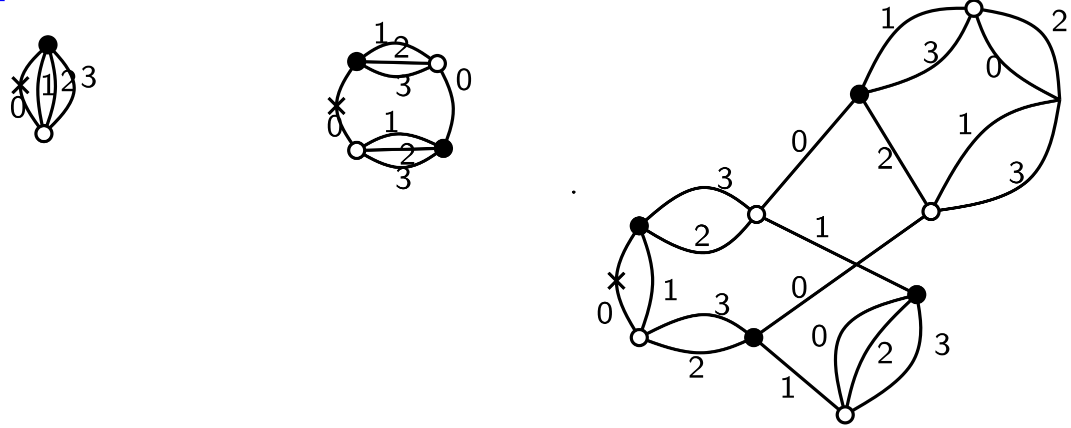
Definition: $(D+1)$-regular edge colored bipartite graphs:

— k white vertices, k black vertices

— $(D+1)k$ edges, k of which have color c, for all $0 \leq c \leq D$.

— each vertex is incident to one edge of each color

Examples:



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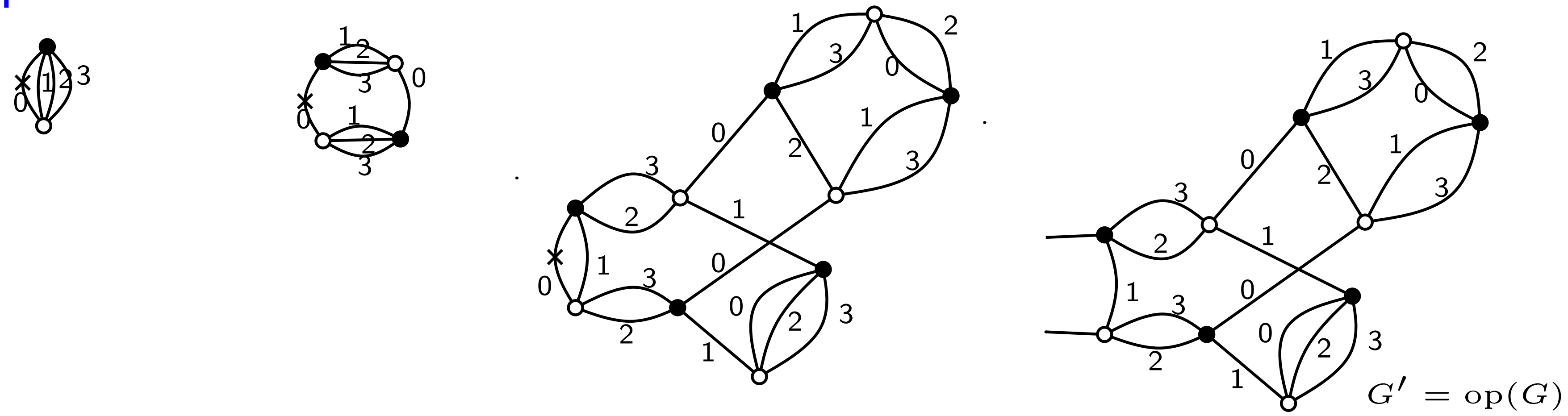

As usual a graph is rooted if one edge is marked.

Equivalently, a graph is open, if one edge is broken into two half edges.

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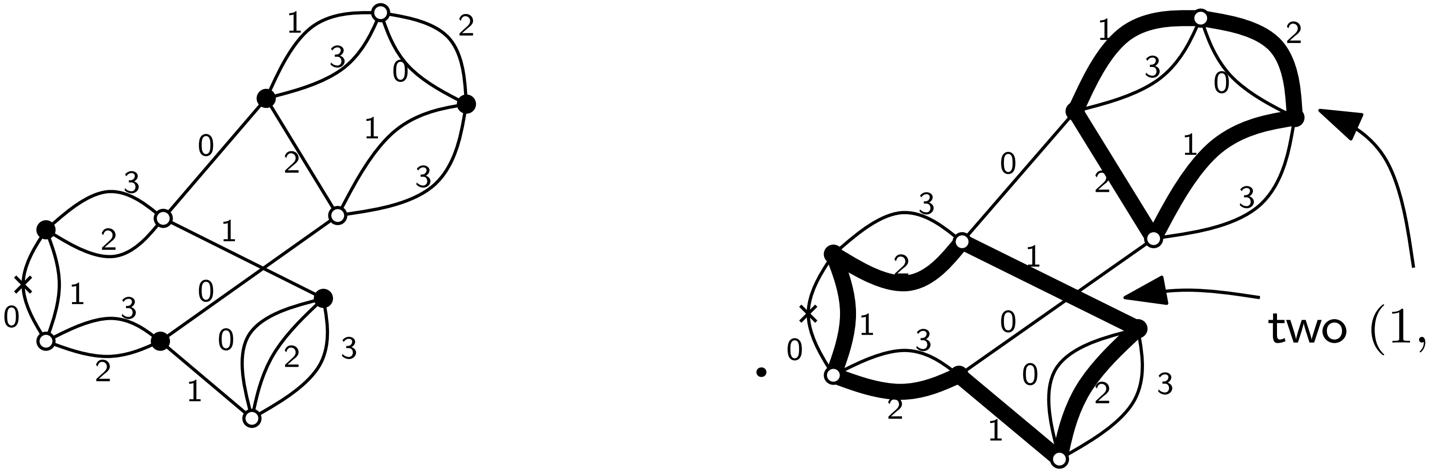

two $(1, 2)$-faces

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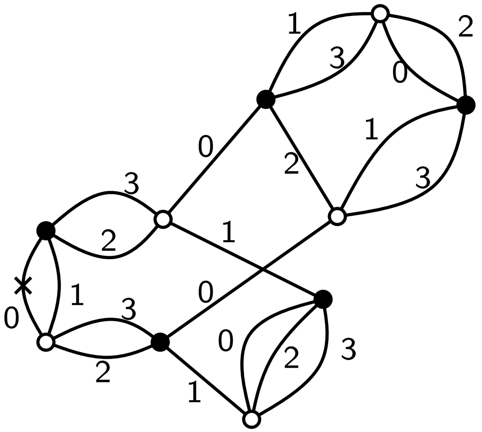

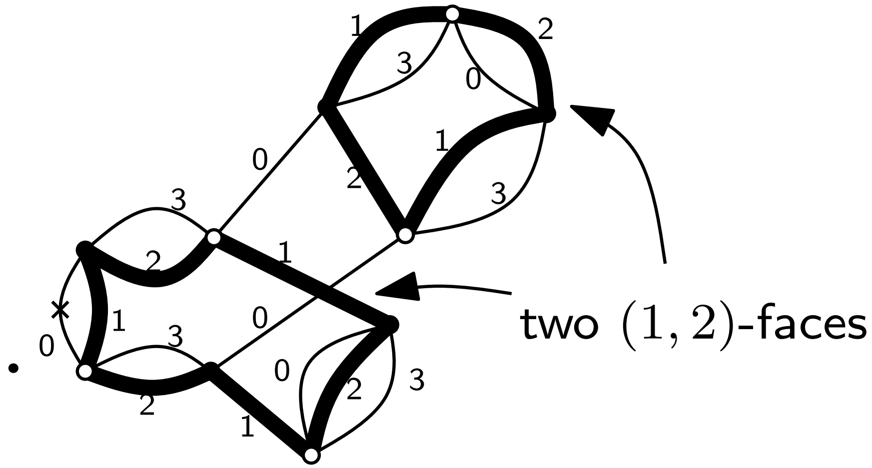

two $(1,2)$-faces

Let $F_p^{c,c'}$ count faces of color $\{c, c'\}$ and degree $2p$; $F_p = \sum_{\{c,c'\}} F_p^{\{c,c'\}}$

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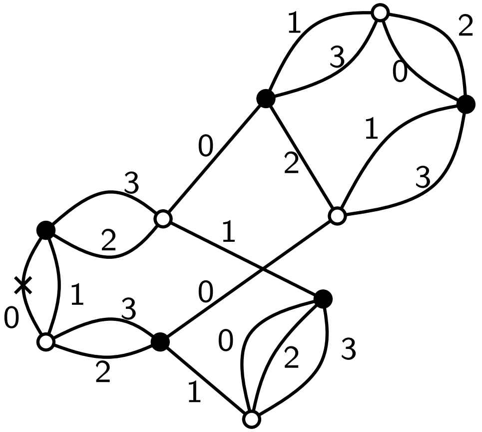

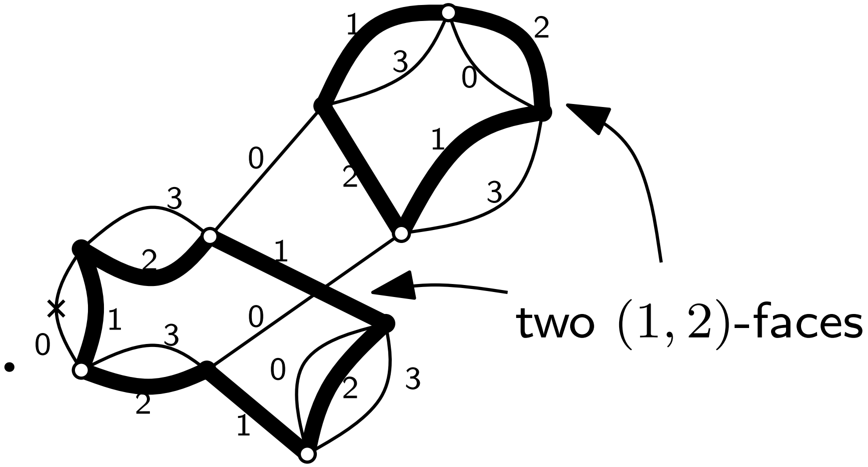

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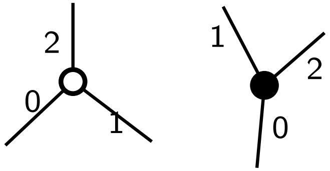
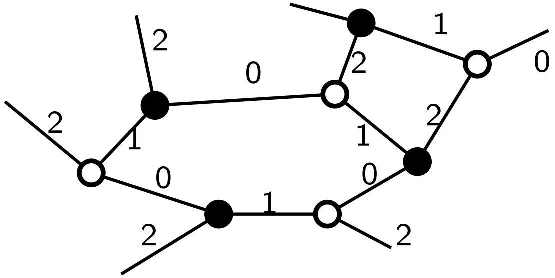
In the case $D = 2$, there are 3 colors, and the faces are the faces of a canonical embedding of the graph as a map.





Regular colored graphs, why?

Lemma. The reduced degree $\delta = \binom{D}{2}k + D - F$ is a non-negative integer.

Sketch of proof. Show that δ is the average genus among all possible canonical embedding (*jackets*) obtained by fixing the cyclic arrangement of colors around vertices.

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First observations:

For $D = 2$, coefficient of F_2 negative

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For $D = 2$, coefficient of F_2 negative

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For $D \geq 4$, coefficient of F_2 positive

$\Rightarrow$ finitely many graphs if δ and F_1 are fixed.

Same hold for $D = 3$ but non trivial.

Summary of the first episode

Matrix integral expansions

$\rightsquigarrow$

3-regular colored maps

k black vertices, F faces

$$2g = k - F + 2$$

(colored triangulations)

D-tensor integral expansions

$\rightsquigarrow$

D-regular colored graphs

k black vertices, F "faces"

$$\delta = \binom{D}{2}k - F + D$$

(D-dimensional pure colored complexes)

Classification by degree:

degree is not a topological invariant of underlying D-manifold:
it depends on the colored complex used to triangulate it

but it governs the expansion of the integral

Why this precise integral / family of graph?

More representative than simpler models: the barycentric sub-division of any manifold complex is a regular colored graph.

There are richer models for $D = 3$, but this model works for any D.

What's next?

Matrix integral expansions

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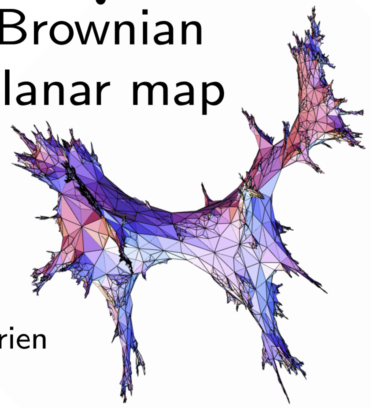
Define a r.v. with uniform distribution on objects of size k, and look for continuum limit of rescaled objects when $k \to \infty$

$g = 0$

Brownian planar map



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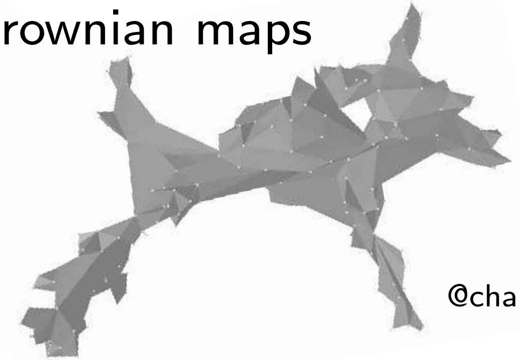
$g > 0$

Higher genus Brownian maps



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$\delta = 0$

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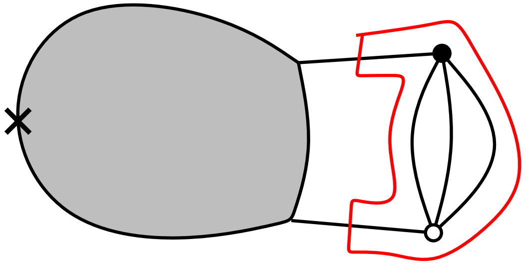
today's topic

The case $\delta = 0$

The case $\delta = 0$: Melonic graphs (Gurau, Rivasseau et al.)

Lemma. If G has degree 0 then it contains a non-root melon.



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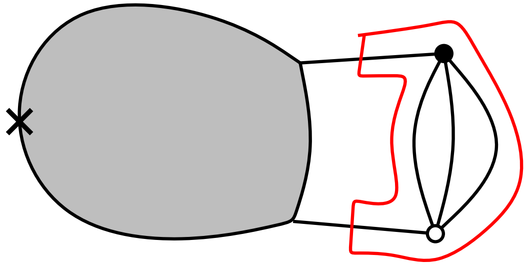

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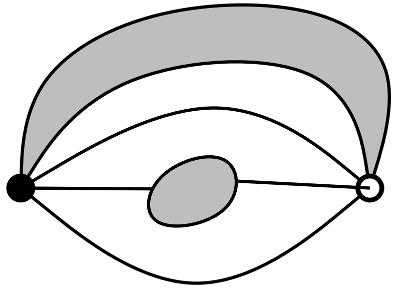
Proof. In view of counting lemmas, there exists a face of length 2.

Since δ is average genus, all "jackets" are planar.

If possible choose a jacket such that the 2-cycle isolates a non-trivial part attached by 2 edges, and iterate.



If this is not possible, we have a melon.

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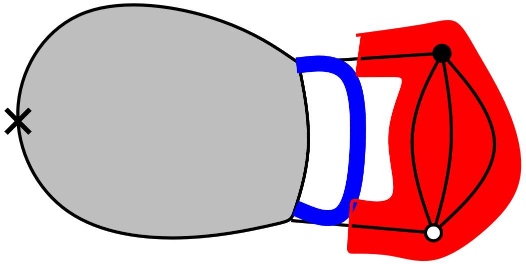

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$$k \to k - 1$$
$$F \to F - \binom{D}{2}$$

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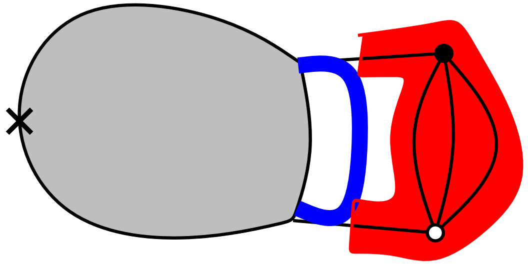

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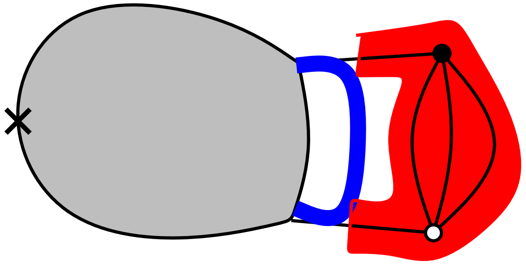

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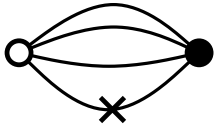
A melonic graph is a colored regular graphs that can be obtained by a series of insertion of melons in



Thm[Gurau *et al*] Colored regular graphs of degree 0 $\Leftrightarrow$ melonic graphs.

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Trees should be decomposed from the root, not from leaves...

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Inductive definition of rooted melonic graphs:

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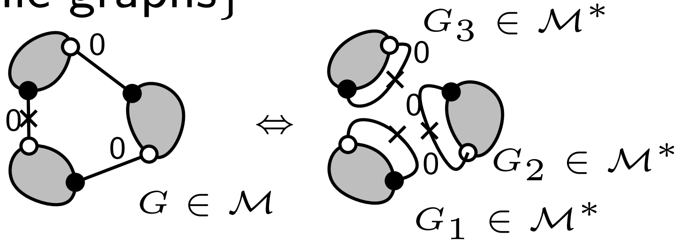

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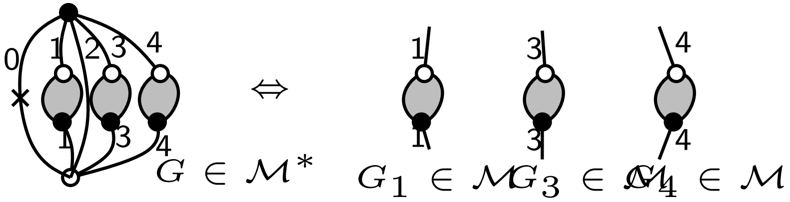

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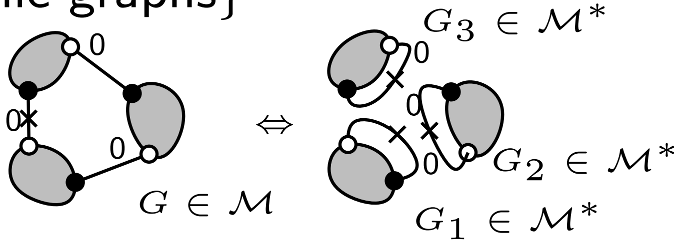

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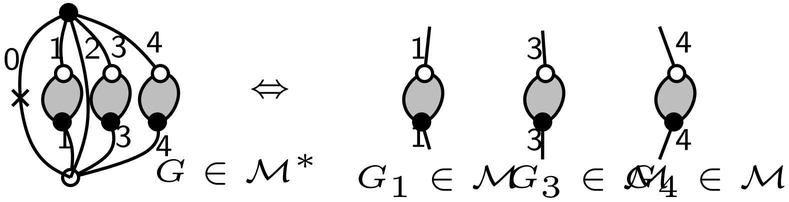

Melonic graphs "are" multitype Galton-Watson trees

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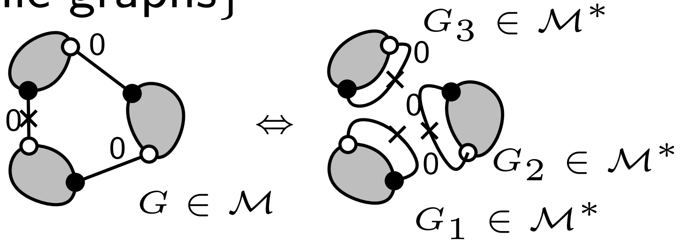

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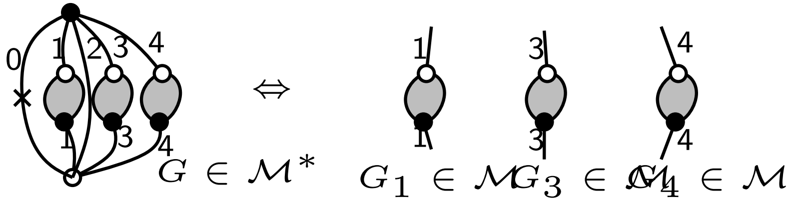

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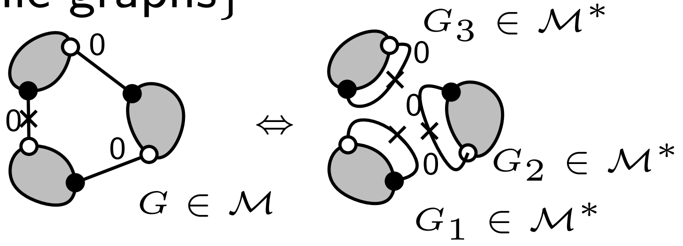

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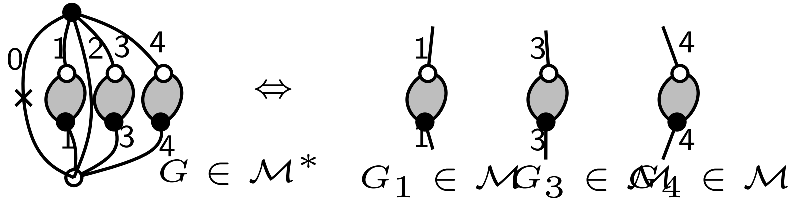

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The gf of rooted melonic graphs has a square root dominant singularity.

$$T(z) = a - b\sqrt{1 - z/z_0} + O(1 - z/z_0) \qquad \text{where } z_0 = \frac{D^D}{(D+1)^(D+1)}$$

The number of melonic graphs of size k grows like $cte \cdot z_0^{-k} k^{-3/2}$

The global picture

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(colored triangulations)

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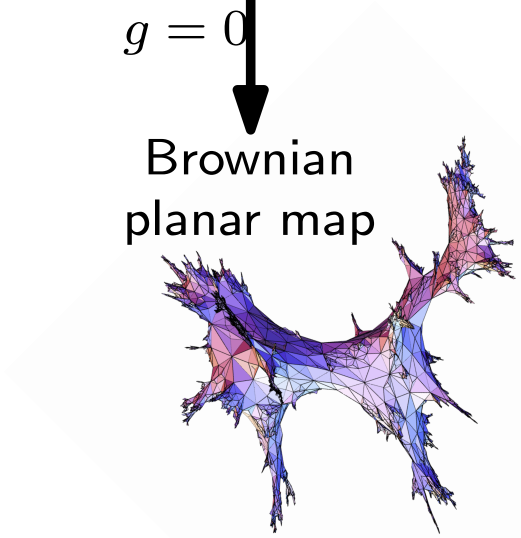
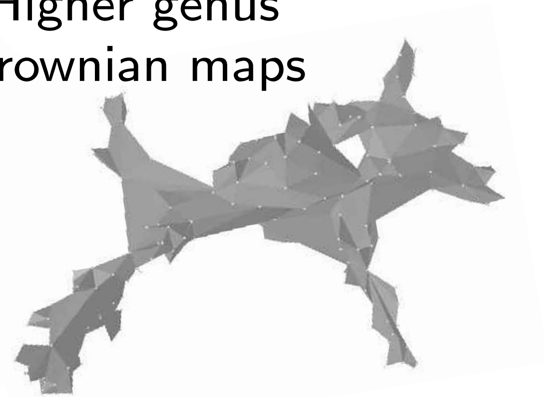


$g > 0$

Higher genus Brownian maps



$\delta = 0$

the CRT

(Gurau-Ryan'13)

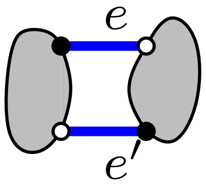
$\delta > 0$

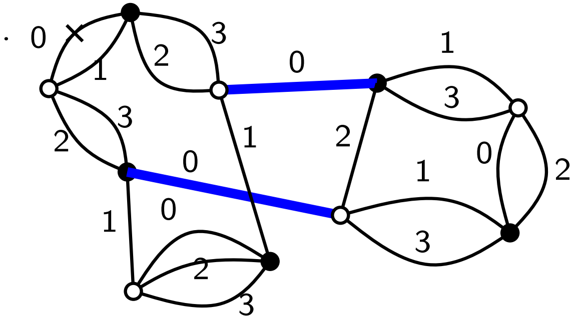
?

The case $\delta > 0$

Melons and the melon-free core

Plan: Study regular colored graphs via structural analysis of 2-edge-cuts





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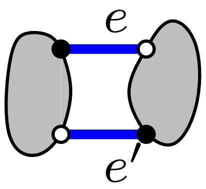
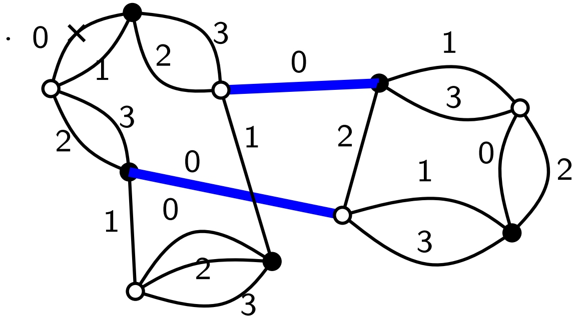
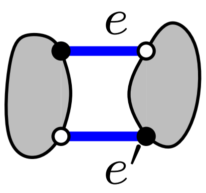


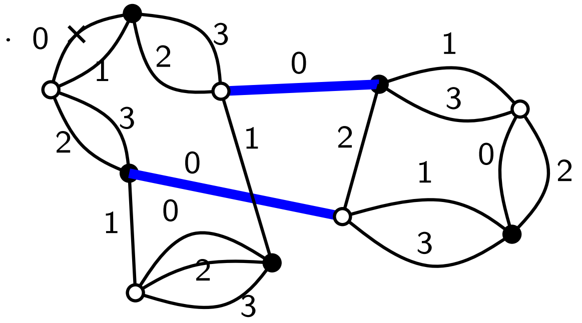


Lemma. $\{e, e'\}$ is 2-edge-cut iff any simple cycle visiting e visits e'.

Melons and the melon-free core

Plan: Study regular colored graphs via structural analysis of 2-edge-cuts

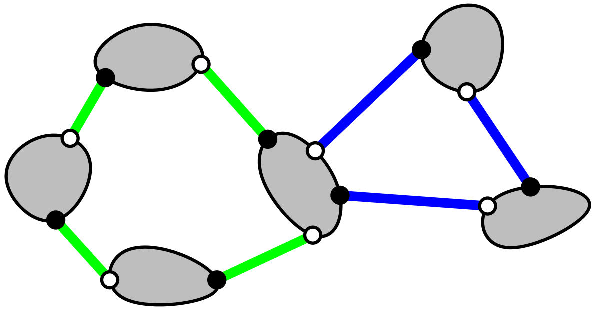




Lemma.$\{e, e'\}$ is 2-edge-cut iff any simple cycle visiting e visits e'.

Lemma. 2-edge-cuts form disjoint cut-cycles

where each cut-cycle is a maximal set of pairwise 2-cuts.



Melons and the melon-free core

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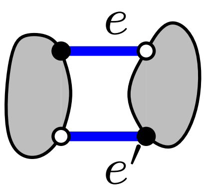

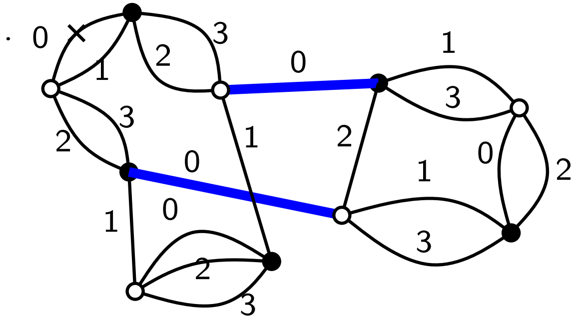

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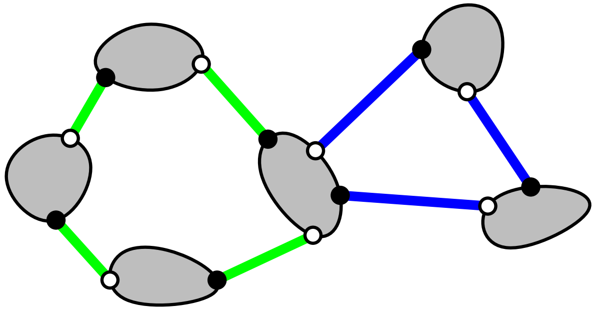
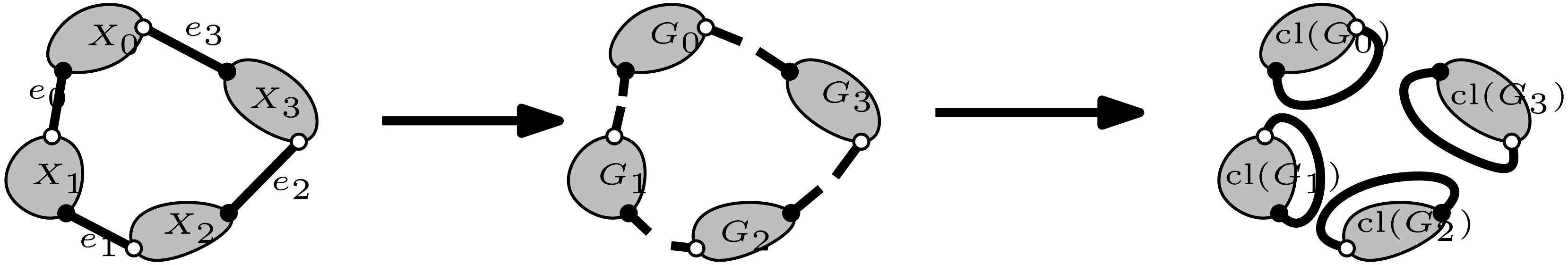
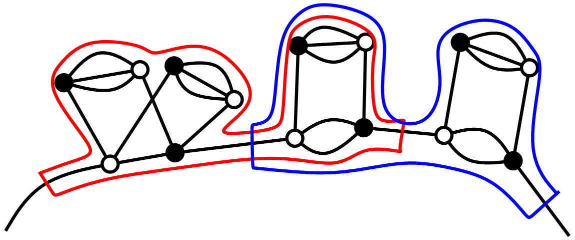


Decomposition along a cut-cycle:



Melons and the melon-free core

Lemma. The union of two non-disjoint open melonic subgraphs of an open regular colored graph is a melonic subgraph.



Proof: In view of the degree constraint, the boundary of an open melonic subgraph consists of its two open edges.

Therefore the open edges of the two components belong to a same open cut-cycle of the union, which is melonic by induction.

Melons and the melon-free core

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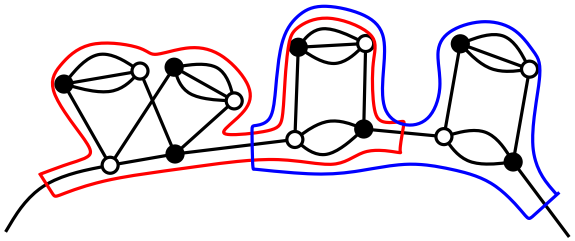
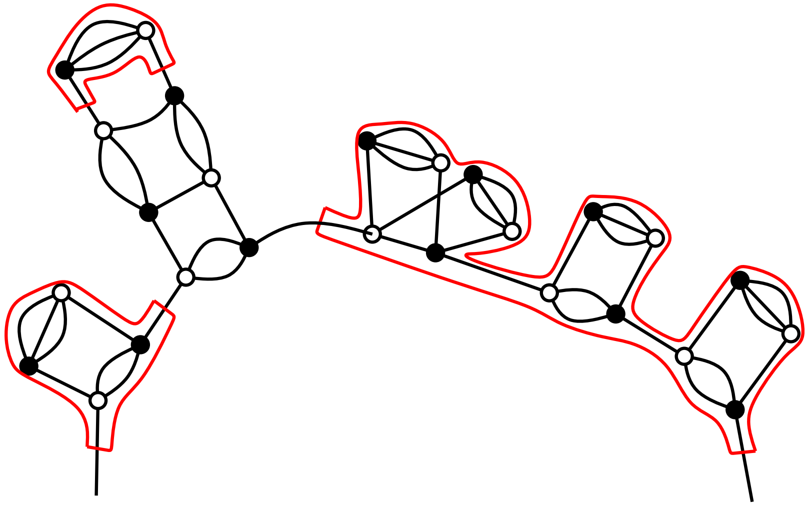


Corollary Maximal open melonic subgraphs are disjoint.



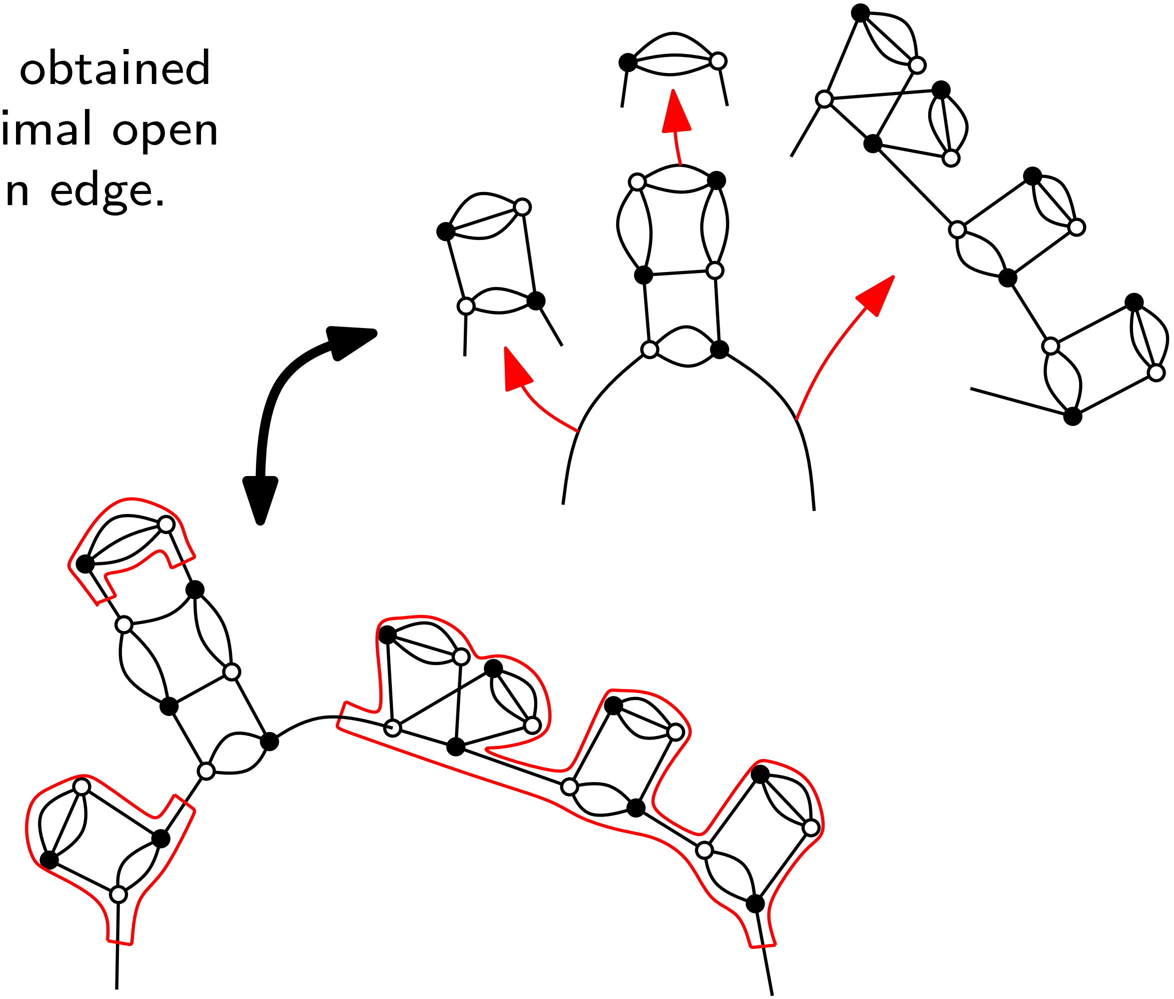
Melons and the melon-free core

Proposition. Core decomposition is a size preserving bijection between

— pairs $(C;(M_0,\dots,M_{(D+1)p}))$ with C a rooted melon-free graphs with $(D+1)p$ edges and $M_0,\dots,M_{(D+1)p}$ melonic graphs,

— and rooted regular colored graphs.

The melon-free core is obtained by replacing each maximal open melonic subgraph by an edge.



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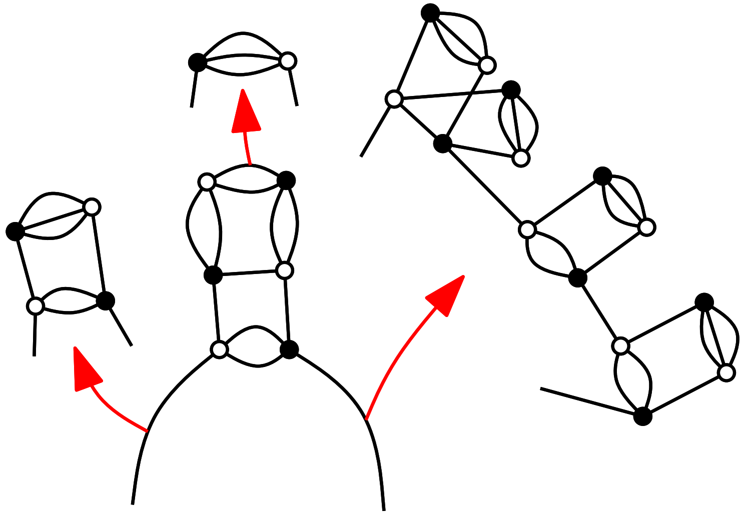
Proposition. The degree of a graph equals the degree of its core.



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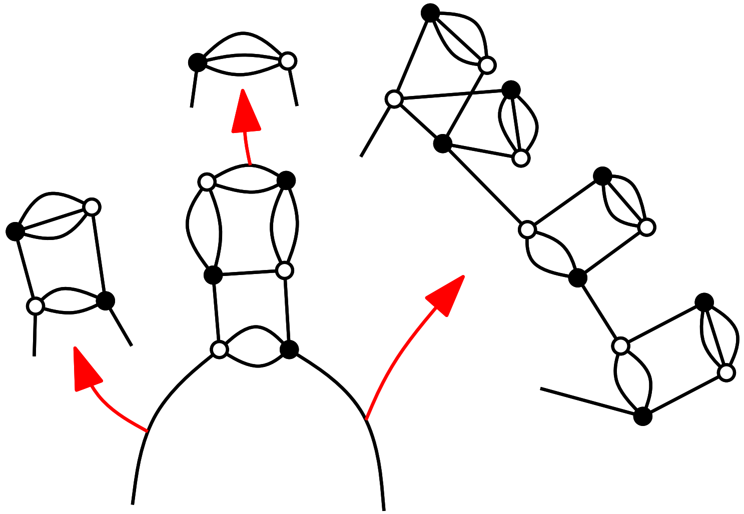

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$$F_C(z) = z^p T(z)^{(D+1)p+1}$$

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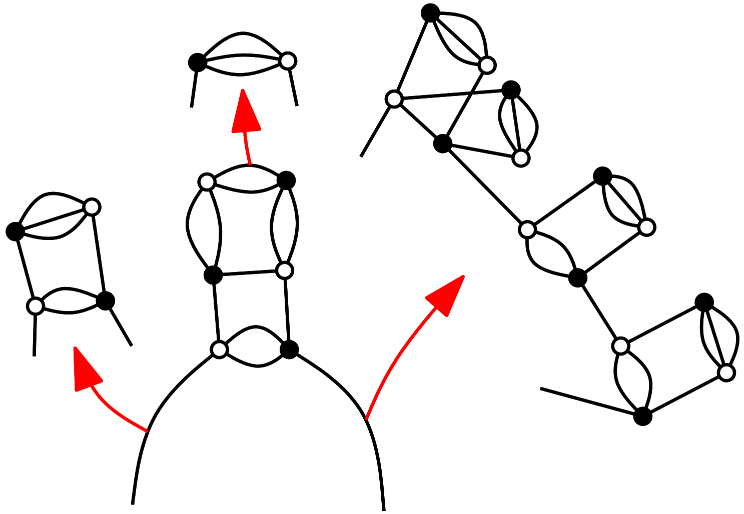

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$\Rightarrow$ The gf of rooted regular colored graphs of degree δ can be written as

$$F_\delta(z) = T(z) \sum_{C \in \mathcal{C}_\delta} (zT(z)^{(D+1)})^{|C|}.$$

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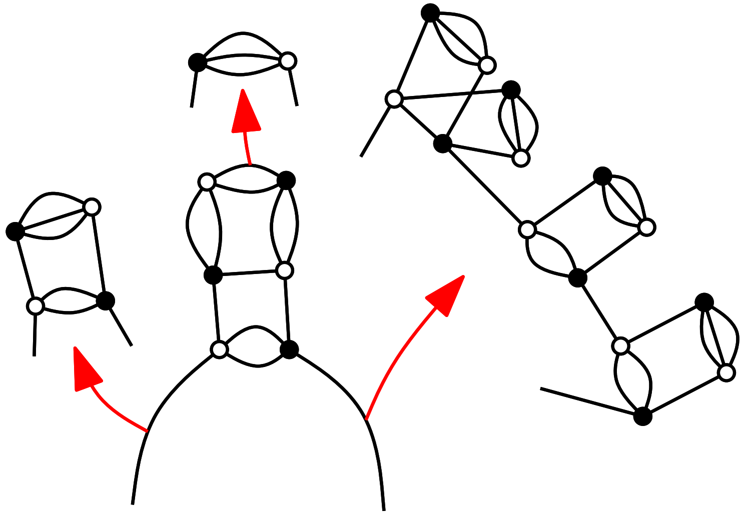

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Problem. For each $\delta > 0$, there exists an infinite number of melon-free graphs of degree δ: the above expression is not very useful...

Summary of the first two episodes

Colored regular graphs

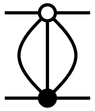
$\Updownarrow$

Melon-free cores $+$ Melons

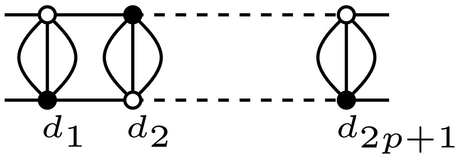
The scheme

Problem. For each $\delta > 0$, there exists an infinite number of melon-free graphs of degree δ.

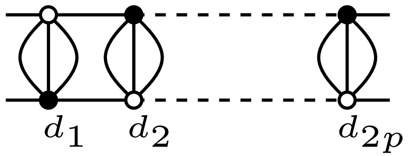
Some configurations can be repeated without increasing δ.
In particular, chains of $(D-1)$-dipoles:



$(D-1)$-dipole



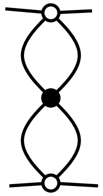
odd chain



even chain

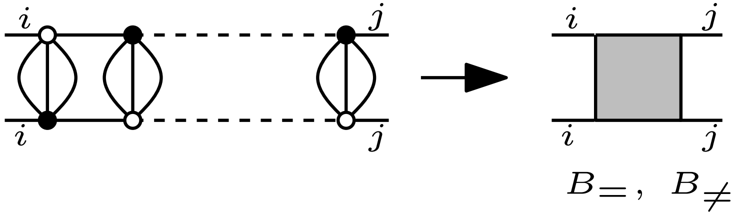
A chain is proper if it contains at least two $(D-1)$-dipoles.

Lemma. Maximal proper sub-chains are disjoints.

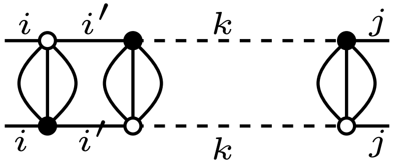


The scheme

Maximal chain replacement: chain-vertices



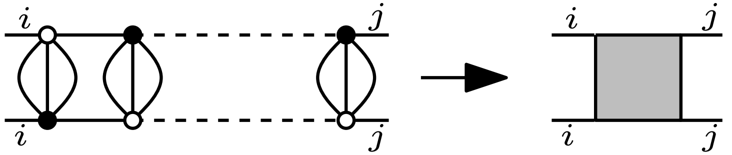
But not all chains are equivalent for the cycle structure:



parallel edges in chain have same labels

The scheme

Maximal chain replacement: chain-vertices



$B_{=}, \ B_{\neq}$

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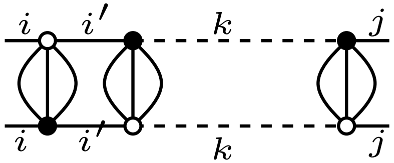
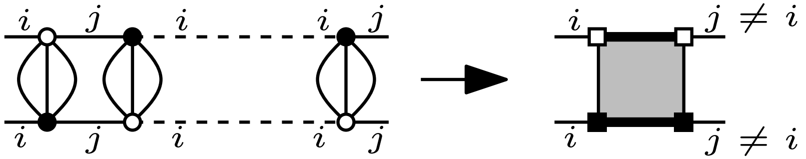
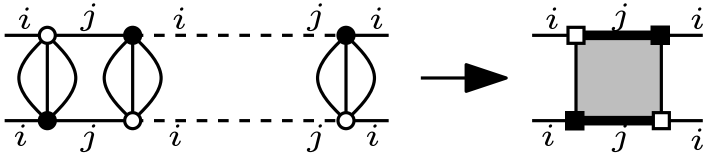


parallel edges in chain have same labels

At most one type of cycle can traverse the whole chain:

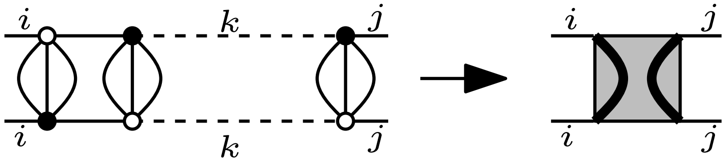


odd connected chains



even connected chains

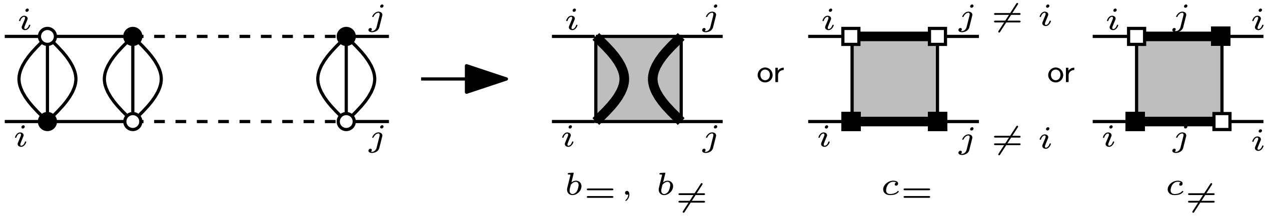
broken chains



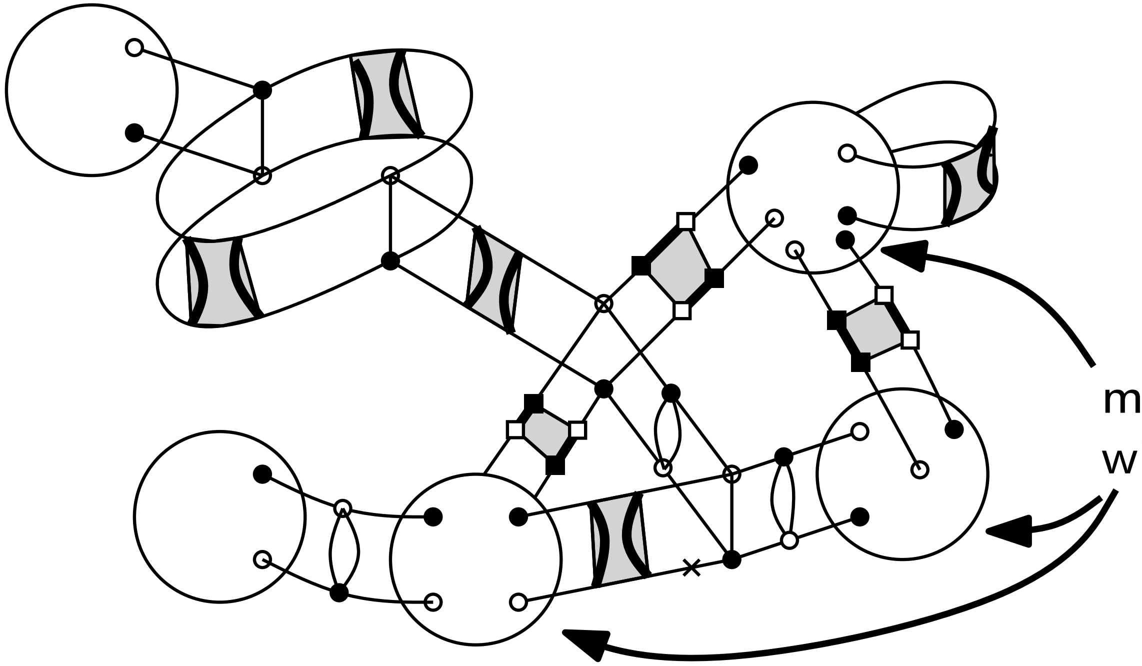
$k \neq i, j$

The scheme

Maximal chain replacement: chain-vertices



The scheme of a melon-free graph: do all replacements.



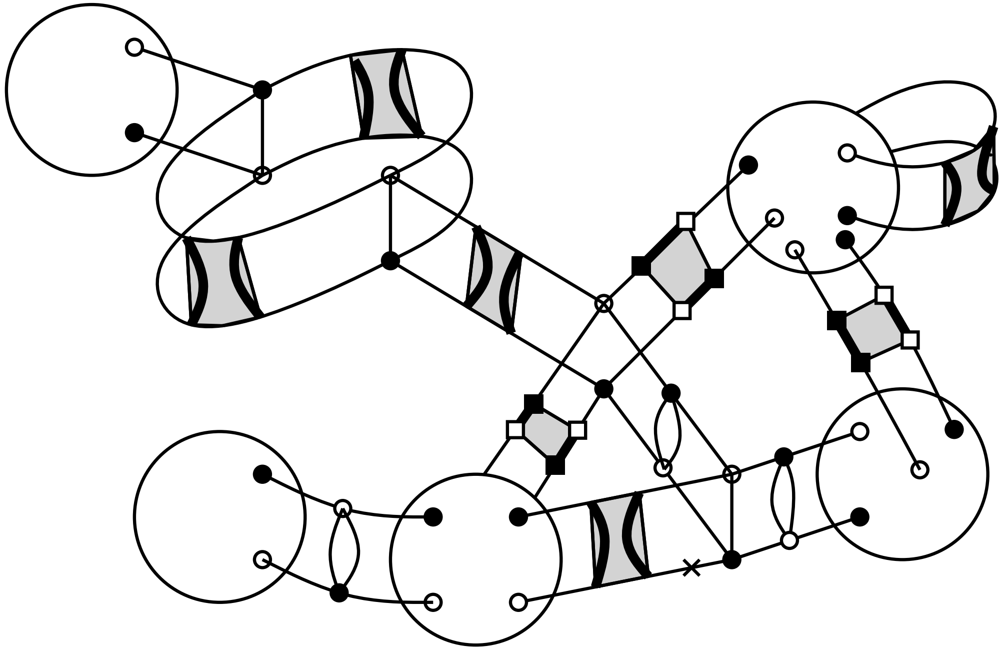
melon-free graphs without chain

By construction, 2 graphs with same scheme have the same degree.

$\Rightarrow$ this common degree is the degree of the scheme.

The scheme

Proposition. The scheme decomposition is a size and degree preserving bijection between pairs $(S;(C_0,\dots,C_n))$ where S is a scheme with n chain-vertices and $C_0,\dots,C_n$ are chains, and melon-free graphs.



The scheme

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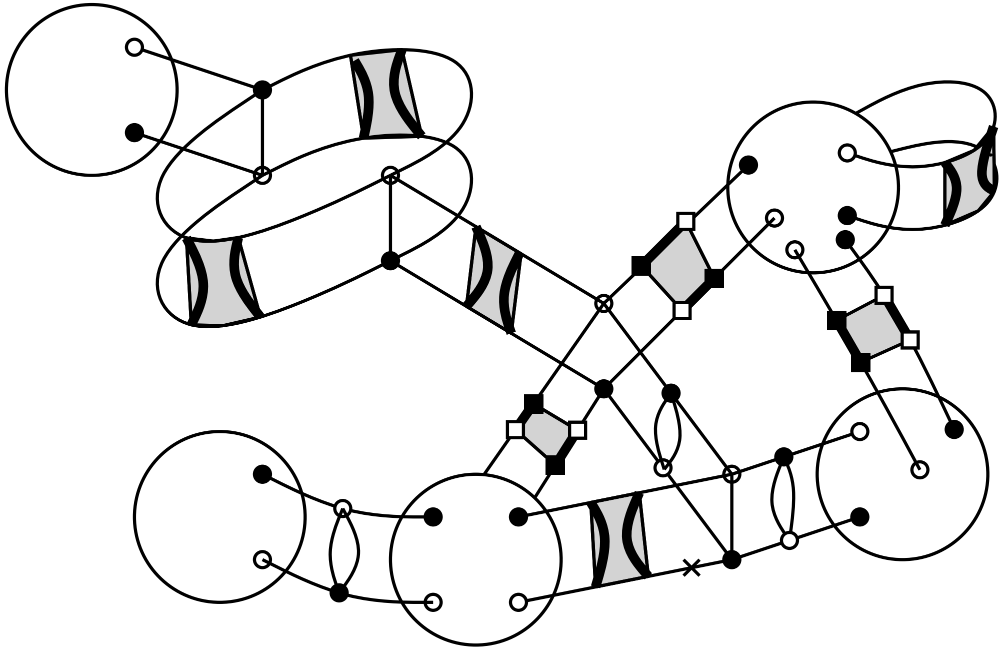
Proposition. Let S be a scheme with $b_{\neq}, b_{=}, c_{\neq}, c_{=}$ chain-vertices of each type. The gf of melon-free graphs with scheme S is

$$G_S(u) = \frac{u^p D^{b_=} (D-1)^b u^{b_= + c_{\neq} + 2b + 2c}}{(1-Du)^b (1-u^2)^{b+c}} \qquad \begin{aligned} b &= b_= + b_{\neq} \\ c &= c_= + c_{\neq} \end{aligned}$$



The scheme

Theorem. The number of schemes with degree δ is finite.

Lemma. The number of chain-vertices, $(D-1)$-dipoles and, for $D \geq 4$, $(D-2)$-dipoles in a scheme of degree δ is bounded by 5δ.

> **Idea:** The deletion of a dipole in a melon-free graph has in general the effect of decreasing the genus or disconnecting the graph in parts that all have positive genus. Actual proof is a bit technical.

Lemma. For $D = 3$ the number of graphs with a fixed number of 2-dipoles is finite. For $D \geq 4$, the number of graphs with fixed numbers of $(D-1)$-dipoles and $(D-2)$-dipoles is finite.

> **Idea:** For $D = 3$, ad-hoc argument.
> For $D \geq 4$, refine the counting argument of earlier slides.

Summary of the first three episodes

Colored regular graphs

$\Updownarrow$

Melon-free cores $+$ Melons

$\Updownarrow$

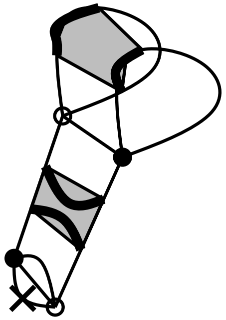
Schemes $+$ Chains $+$ Melons

Exact formulas

Theorem. Let $\delta \geq 1$. The gf of rooted colored graphs of degree δ w.r.t. black vertices is

$$F_\delta(z) = T(z) \sum_{s \in S_\delta} G_S(zT(z)^{D+1}) \quad \text{where } G_s(u) = \frac{u^p D^{b_=} (D-1)^b u^{b_= + c_{\neq} + 2b + 2c}}{(1-Du)^b (1-u^2)^{b+c}}$$

and $T(z) = 1 + zT(z)^D$



Corollary (Kaminski, Oriti, Ryan). For $\delta = D - 2$,

$$F_{D-2}(z) = \binom{D}{2} \frac{z^2 T(z)^{2D+3}}{1-z^2T(z)^{2D+2}} \frac{1}{1-DzT(z)^{D+1}}$$

Explicit next term, for $\delta = D$, is already a mess...

Asymptotic formulas and dominant terms

Theorem. Let $\delta \geq 1$. The gf of rooted colored graphs of degree δ w.r.t. black vertices has the asymptotic development

$$F_\delta(z) = \sum_{s \in S_\delta} f_{p,b,D}^{c_{\neq},c} (1 - z/z_0)^{-b/2} + O(1 - z/z_0)$$

where $f_{p,b}^{c_{\neq},c}(D)$ is a simple rational fraction in D: $f_{p,b,D}^{c_{\neq},c} = \frac{D^{3b/2-p-c_{\neq}-1}}{2^{b/2}(D-1)^c(D+1)^{c+b/2}}$

In this finite sum the dominant terms are the one that maximize b, the number of broken chains in the scheme.

Asymptotic formulas and dominant terms

Proposition. The maximum number of broken chains in a scheme of degree δ is the maximum of the following linear program:

$$b_{\max} = \max\left(2x + 3y - 1 \mid (D-2)x + Dy = \delta;\ x, y \in \mathbb{N}\right)$$

Moreover the corresponding dominant schemes consists of:

— b_{max} broken chain-vertices ($2x + y - 1$ spanning, $2y$ surplus).

— x connected chain-vertices each forming a loop at a $(D-2)$-dipole,

— $x + y - 1$ connecting $(D-2)$-dipoles, and one root-melon.

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For $3 \leq D \leq 5$. The maximum is obtained for $y = 0$: $\delta = (D-2) \cdot x$.
$\Rightarrow$ "binary trees" with $2x - 1$ chains, $x + 1$ end-dipoles
(the root and x wheels), $x - 1$ inner dipoles .

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For $D \geq 7$. The maximum is obtained for $x = 0$: $\delta = D \cdot y$
$\Rightarrow$ "ternary graphs" with $3y - 1$ chains, x inner dipoles, one root melon.

Conclusions

Fixed degree regular colored graphs

$= \text{scheme} \circ \text{chains} \circ \text{melons}$

finite number

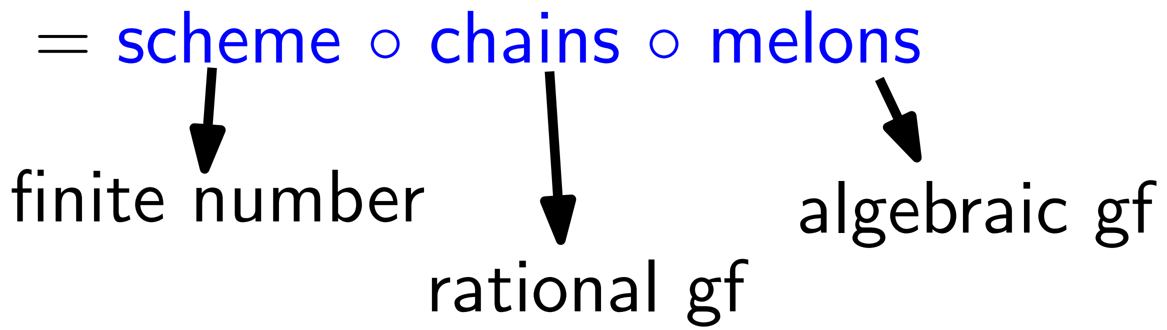
rational gf

algebraic gf

$\Rightarrow$ Exact counting

Conclusions

Fixed degree regular colored graphs



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Dominant schemes:

for $3 \leq D \leq 5$: for $\delta = d \cdot (D-2)$, rooted binary trees with d leaves

for $D \geq 7$: for $\delta = d \cdot D$, rooted 3-regular graphs with $3d-1$ vertices

Conclusions

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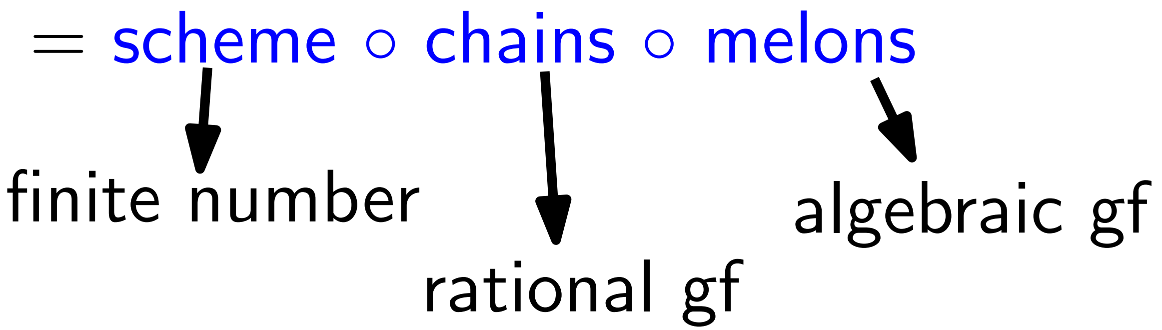

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Similar results were obtained by Dartois, Gurau and Rivasseau
for a simpler model, they obtain the same rich asymptotic behavior.

Conclusions

Scaling limits: δ fixed, size n going to infinity

Melonic graphs rescaled by $n^{-1/2}$ cv to CRT (Gurau-Ryan)

For $\delta \geq 1$, normalization is still $n^{-1/2}$ and we expect something similar to Addario-Berry, Broutin, Goldschmidt's critical random graphs (work in progress with Albenque)

Double scaling limits: compute $\sum_\delta N^{-\delta} domin(F_\delta(z))$

Upon sending $N \to \infty$ with $N(1 - z/z_0) = cte$, limit exists for $D \leq 5$

— resum lower order terms and look for a triple scaling limit?

— for $D \geq 6$, is it possible to say something about the divergent series?

These computations should probabibly be done first for the simpler model of Dartois, Gurau, Rivasseau.