Arbres, cartes et nombres de Hurwitz

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CNRS & École Polytechnique
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Plan de l’exposé

Revêtements ramifiés et cartes

Cartes et arbres

Énumération d’arbres et formule d’Hurwitz

Revêtements et cartes aléatoires
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Revêtements ramifiés et cartes

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Revêtements et cartes aléatoires
Ramified coverings of the sphere by itself

Let $B = \{ z \mid |z| < 1 \} \subset \mathbb{C}$ and let $\sim$ denote equivalence up to homeomorphisms.

A mapping $\phi : D \to I$ is a covering if, for all $x$ in $I$ there exists $n \geq 1$ and a neighborhood $V$ of $x$ such that $\phi^{-1}(V) \sim B \times \{1, \ldots, n\}$,

and the restriction of $\phi$ to each sheet $B_i$ (connected component of the preimage)
is an homeomorphism $\phi_{|B_i} : B_i \simto B$. 

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Example:
Let $A_r$ be the annulus $\{ z \mid r < |z| < 1 \} \subset \mathbb{C}$. Consider $\phi_k : A_r \to A_{r^k}$ with $\phi_k(z) = z^k$. 
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[Diagram showing ramified coverings of the sphere by itself]
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By continuity, the number $n = |\phi^{-1}(x)|$ of sheets of a covering $\phi$ does not depend on $x$: for instance $n = k$ for $\phi_k$. 
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The number \( n \) of sheets is called the **degree** of the covering.
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The number \( n \) of sheets is called the degree of the covering.

What is we try to extend from \( A_r \) to \( B \)?
Recall $\phi_k : A_r \to A_{r,k}$ with $\phi_k(z) = z^k$.

Extend from $A_r$ to $B$?

The mapping $\phi_k : B^* \to B^*$ is a covering, but not $\phi_k : B \to B$. 
Recall $\phi_k : A_r \to A_{r,k}$ with $\phi_k(z) = z^k$.

Extend from $A_r$ to $B$?

The mapping $\phi_k : B^* \to B^*$ is a covering, but not $\phi_k : B \to B$.

What happens at $x = 0$?
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What happens at $x = 0$?

The mapping $\phi_k : B \to B$ has a connected ramification of degree $k$ at $x = 0$. 
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A mapping $\phi$ is ramified at $x = 0$ if

- there is a neighborhood $V$ of the origin such that $\phi^{-1}(V) \sim B \times [1, \ldots, p]$ and,

- the restriction of $\phi$ to each component of $\phi^{-1}(V)$ is homeomorphic to $\phi_k$ for some $k$. 
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Regular (aka unramified) value = ramified with $\phi_1$ on each component.
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Regular (aka unramified) value $=$ ramified with $\phi_1$ on each component.
A mapping $\phi$ is a ramified covering of $\mathbb{S}$ by $\mathbb{S}$ if there exists a finite subset $X = \{x_1, \ldots, x_p\}$ such that:

- $\phi_{\mathbb{S}\setminus \phi^{-1}(X)}$ is a covering, and
- $\phi$ is ramified over each $x_i$
A mapping $\phi$ is a **ramified covering** of $S$ by $S$ if there exists a finite subset $X = \{x_1, \ldots, x_p\}$ such that:

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The \textbf{ramification type} over a critical value $x_i$ is the partition $\lambda^{(i)}$

The \textbf{passport} of a ramified covering is the list $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$

\[
\lambda^{(1)} = 1^5 \quad \lambda^{(2)} = 1, 2^2 \quad \lambda^{(2)} = 2, 3
\]

the passport $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ of a ramified covering
Ramified coverings of the sphere by itself (Cont’d)

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Ramified coverings of the sphere by itself (Cont’d)

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Ramified coverings of the sphere by itself (Cont’d)

To understand the "shape" of the covering, draw paths on $\mathcal{I}$ and study its preimages.

\[ \mathcal{D} = \mathbb{S} \]
\[ \mathcal{I} = \mathbb{S} \]

generically $n$ sheets

\[ \lambda(1) = 1^5 \]
\[ \lambda(2) = 1, 2^2 \]
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the passport $\Lambda = (\lambda(1), \ldots, \lambda(p))$ of a ramified covering

regular value  critical value  critical value

$\phi_3$

$\phi_2$

$\phi_2$

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Ramified coverings of the sphere by itself (Cont’d)

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Ramified coverings of the sphere by itself (Cont’d)

To understand the "shape" of the covering, draw paths on $\mathcal{I}$ and study its preimages.

- $n$ independant preimages as long as we stay away from critical points
- a contractible loop on $\mathcal{I}$

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Ramified coverings of the sphere by itself (Cont’d)

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- $n$ independent preimages as long as we stay away from critical points
- A contractible loop on $\mathcal{I}$ yields $n$ contractible loops on $\mathcal{D}$

$\mathcal{D} = S$

$\mathcal{I} = S$

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- $n$ independant preimages as long as we stay away from critical points
- A contractible loop on $\mathcal{I}$ yields $n$ contractible loops on $\mathcal{D}$ but if we wind around critical points

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  but if we wind around critical points
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To understand the ”shape” of the covering, draw paths on \( \mathcal{I} \) and study its preimages.

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  - but if we wind around critical points some sheets may get permuted
- visiting critical points create multiple values or ”vertices”

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regular value critical value critical value

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The partitions $\lambda^{(i)}$ are partitions of $n$, degree of the covering.

The passport $\Lambda = (\lambda^{(1)}, \ldots, \lambda^{(p)})$ of a ramified covering.
Monodromy, and permutations

Let us label \( \{1, \ldots, n\} \) the preimages of a regular point. Loop \( \Rightarrow \) permutation of sheet labels

Example: \((1, 2)(3, 4)(5)\) in cyclic notation
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\( \Rightarrow \) Equivalence classes of ramified coverings \( \equiv \) factorizations of permutations but geometric intuition is lost
coverings with 3 critical values and bipartite maps

\[ \mathcal{D} = S \]

\[ \mathcal{I} = S \]

3 critical values

\[ \lambda^\bullet = 2^31^2 \quad \lambda^o = 3^22 \quad \lambda^{\square} = 62 \]
coverings with 3 critical values and bipartite maps

$\mathcal{D} = \mathcal{S}$

$I = \mathcal{S}$

3 critical values $\lambda^\bullet = 2^3 1^2$ $\lambda^o = 3^2 2$ $\lambda^\square = 62$

1 regular value with labeled preimages
coverings with 3 critical values and bipartite maps

\[ D = S \]

\[ I = S \]

3 critical values
\[ \lambda^\bullet = 2^3 1^2 \quad \lambda^\circ = 3^2 2 \quad \lambda^\square = 62 \]

1 regular value with labeled preimages

On \( I \), draw an edge between \( \bullet \) and \( \circ \) via the basepoint
coverings with 3 critical values and bipartite maps

3 critical values  \( \lambda^\bullet = 2^3 1^2 \)  \( \lambda^\circ = 3^2 2 \)  \( \lambda^\square = 62 \)
1 regular value with labeled preimages

On \( \mathcal{I} \), draw an edge between \( \bullet \) and \( \circ \) via the basepoint.

We get a planar map:
that is, a graph embedded on the sphere with simply connected faces.

\[ \mathcal{D} = \mathbb{S} \]

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coverings with 3 critical values and bipartite maps

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On \( I \), draw an edge between \( \bullet \) and \( \circ \) via the basepoint.

We get a planar map:

that is, a graph embedded on the sphere with simply connected faces.

**Proof.** Faces are simply connected because a loop around the edge in \( I \) can be deformed to a loop around \( \square \).

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coverings with 3 critical values and bipartite maps

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coverings with 3 critical values and bipartite maps

On $\mathcal{I}$, draw an edge between $\bullet$ and $\circ$ via the basepoint

We get a planar map:

that is, a graph embedded on the sphere with simply connected faces

Proof. Faces are simply connected because a loop around the edge in $\mathcal{I}$ can be deformed to a loop around $\square$

Proposition. This is a bijection between bipartite planar maps and ramified coverings of $\mathbb{S}$ by $\mathbb{S}$ with 3 critical values.

$3$ critical values

$\lambda^\bullet = 2^31^2$  $\lambda^\circ = 3^22$  $\lambda^\square = 62$

$1$ regular value with labeled preimages
3 critical values, bipartite maps and permutations

\[ \mathcal{D} = \mathcal{S} \]

A loop around a critical value yields a permutation

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3 critical values
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1 regular value with labeled preimages
A loop around a critical value yields a permutation

\[ \sigma = (1, 3, 6)(2, 5, 4)(7, 8) \]
with cyclic type \( \lambda^\circ \)

3 critical values

\[ \lambda^\bullet = 2^3 1^2 \]

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\[ \lambda^\square = 62 \]
A loop around a critical value yields a permutation

\[ \sigma_\circ = (1, 3, 6)(2, 5, 4)(7, 8) \]
with cyclic type \( \lambda_\circ \)

\[ \sigma_\bullet = (1)(2, 6)(3, 5)(4, 7)(8) \]
with cyclic type \( \lambda_\bullet \)

Cycle types \( \Leftrightarrow \) degree distributions

3 critical values

1 regular value with labeled preimages
3 critical values, bipartite maps and permutations

A loop around a critical value yields a permutation

\[ \sigma = (1, 3, 6)(2, 5, 4)(7, 8) \]
with cyclic type \( \lambda^o \)

\[ \sigma = (1)(2, 6)(3, 5)(4, 7)(8) \]
with cyclic type \( \lambda^\bullet \)

Cycle types \( \leftrightarrow \) degree distributions

\[ \lambda^\bullet = 2^3 1^2 \quad \lambda^o = 3^2 2 \quad \lambda^{\square} = 62 \]

3 critical values
1 regular value with labeled preimages
A loop around a critical value yields a permutation

\[ \sigma \circ = (1, 3, 6)(2, 5, 4)(7, 8) \]

with cyclic type \( \lambda^\circ \)

\[ \sigma \bullet = (1)(2, 6)(3, 5)(4, 7)(8) \]

with cyclic type \( \lambda^\bullet \)

Cycle types \( \leftrightarrow \) degree distributions

What about \( \sigma \square \) and \( \lambda \square \)?

3 critical values

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3 critical values, bipartite maps and permutations

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$\sigma_\circ = (1, 3, 6)(2, 5, 4)(7, 8)$
with cyclic type $\lambda_\circ$

$\sigma_\bullet = (1)(2, 6)(3, 5)(4, 7)(8)$
with cyclic type $\lambda_\bullet$

Cycle types $\leftrightarrow$ degree distributions

What about $\sigma_\square$ and $\lambda_\square$?

$\sigma_\square = (2, 3)(1, 5, 7, 8, 4, 6)$
loops around $\square = \text{faces}$

3 critical values

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3 critical values, bipartite maps and permutations

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Cycle types \( \leftrightarrow \) degree distributions

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\[ \sigma \square = (2, 3)(1, 5, 7, 8, 4, 6) \]
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I = S

3 critical values

1 regular value with labeled preimages

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3 critical values, bipartite maps and permutations

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Cycle types \( \leftrightarrow \) degree distributions

What about \( \sigma_\square \) and \( \lambda_\square \)?

\[ \sigma_\square = (2, 3)(1, 5, 7, 8, 4, 6) \]
loops around \( \square = \text{faces} \)

But loop around \( \square = \text{concatenate loop around } \circ \text{ and } \bullet \)

**Proposition:** \( \sigma_\circ \sigma_\bullet = \sigma_\square \).
$m + 1$ critical values, $m$-constellations, permutations

$m + 1$ critical values

1 regular value with labeled preimages
The preimage of the \( m \)-star is called a star-constellation.

**Proposition.** Planar star-constellations with:
- \( n \) labelled \( m \)-stars,
- \( \lambda_j \) faces of degree \( j \),
- \( \lambda^{(i)}_j \) color \( i \) vertices of degree \( j \)
are in bijection with minimal transitive factorizations \( \sigma_1 \cdots \sigma_m = \sigma_\square \) with \( \sigma_i \) of cyclic type \( \lambda^{(i)} \).
Theorem. There is a bijection between

- Labelled ramified covering of $\mathcal{S}$ of type $\Lambda = (\lambda_0, \ldots, \lambda_m)$
- Factorizations $(\sigma_1 \cdots \sigma_m = \sigma_0)$ of type $\Lambda$
- labelled $m$-star-constellations of type $\Lambda$.

$\mathcal{D} = \mathcal{S} \iff$ minimality $\iff$ planarity.
Monodromy, permutations, constellations: a summary

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**Specializations.**

— $m = 2$: bipartite maps with $n$ edges

— $m = 2$, $\lambda_0 = 4^n$, all faces have degree 4: quadrangulations
  $\Rightarrow$ Jean-François Le Gall’s last year talk at this seminar
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— for all \( i \geq 1 \), \( \lambda^{(i)} = 21^{n-2} \): factorizations in transpositions.
  coverings with only **simple** branch points
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A ramified cover is simple if its $m$ ramifications have type $21^{n-2}$.

Then each face of degree 2 on the image has $n - 2$ preimages that are faces of degree 2, and 1 that is a quadrangle.
A ramified cover is simple if its $m$ ramifications have type $2^{n-2}$.

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Upon contracting multiple edges, only quadrangle remains.
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Then the faces of the preimage have distinct labels 1, ..., $m$ that are increasing in ccw direction around black vertices and in cw direction around white vertices.
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Such a map is called an increasing labelled quadrangulation.
Simple ramified covers, increasing quadrangulations

Theorem. Simple ramified covers of $S$ by itself with $m$ ramifications points are in bijection with increasing labelled quadrangulations with $m$ faces.

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Compter des classes d’équivalence de revêtements ramifiés ⇔ compter certaines plongements de graphes
Plan de l’exposé

Revêtements ramifiés et cartes

Cartes et arbres

Énumération d’arbres et formule d’Hurwitz

Revêtements et cartes aléatoires
Planar maps, spanning trees and duality

A planar map is a proper embedding of a connected graph on the sphere (considered up to homeomorphisms).

From now on, map means rooted planar map.
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A spanning tree is a subgraph which is a tree and visits every vertices. A tree-rooted map is a map with a spanning tree.
A **planar map** is a proper embedding of a connected graph on the sphere (considered up to homeomorphisms).

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The **dual** map of a map is the map of incidence between faces.
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**Euler's relation:**

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(#\text{vertices}-1) + (#\text{faces}-1) = #\text{edges}
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Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree
Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree
Rooted tree ≡ balanced parenthesis word

uduuduuddd
Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree
Rooted tree $\equiv$ balanced parenthesis word
\[ uduududuuddd \]
Non visited edges $\equiv$ balanced parenthesis word
Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree
Rooted tree $\equiv$ balanced parenthesis word
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Rooted tree $\equiv$ balanced parenthesis word
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Code of the tree-rooted map $= \text{tree decorated by a balanced parenthesis word}$
Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree
Rooted tree $\equiv$ balanced parenthesis word
$uduuddudd$

Non visited edges $\equiv$ balanced parenthesis word
$uuuduuddddud$

Writing the two codes during the walk:
$uuuududuuudududdddudd$

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Rooted tree $\equiv$ balanced parenthesis word
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Code of the tree-rooted map $=$ tree decorated by a balanced parenthesis word
$=$ shuffle of two balanced parenthesis words
Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree

Rooted tree $\equiv$ balanced parenthesis word

$ududududd$  

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Code of the tree-rooted map $=$ tree decorated by a balanced parenthesis word

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The number of tree rooted planar maps with $n$ edges is $\sum_{i=0}^{n} \binom{2n}{i} C_i C_{n-i}$ where

$C_n = \frac{1}{n+1} \binom{2n}{n}$

denotes Catalan numbers, counting balanced parenthesis words.
Encoding and counting tree-rooted maps

Starting at a root corner, turn around the tree
Rooted tree $\equiv$ balanced parenthesis word $ududdudddd$

Non visited edges $\equiv$ balanced parenthesis word $uuudduddduud$

Writing the two codes during the walk: $uuuddudduddudduddudddd$

Observe that closure edges turn clockwise around the tree.

Code of the tree-rooted map $=$ tree decorated by a balanced parenthesis word
$=$ shuffle of two balanced parenthesis words

The number of tree rooted planar maps with $n$ edges is $\sum_{i=0}^{n} \binom{2n}{i} C_i C_{n-i}$ where $C_n = \frac{1}{n+1} \binom{2n}{n}$ denotes Catalan numbers, counting balanced parenthesis words.
but we want rooted (not tree-rooted) maps

Let us recycle the idea used for tree-rooted maps, using a canonical spanning tree
but we want rooted (not tree-rooted) maps

Let us recycle the idea used for tree-rooted maps, using a canonical spanning tree.

Then write the code of the primal tree on the chosen canonical tree.
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The map is recovered from the code by closure.
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Let us recycle the idea used for tree-rooted maps, using a canonical spanning tree.

Then write the code of the primal tree on the chosen canonical tree.

The map is recovered from the code by closure.

Our code of the map will be a canonical decorated tree.

Question is How do we choose the canonical spanning tree so that the resulting decorated trees can be described and counted?
From tree-rooted maps to minimal accessible maps

Orient the tree edges away from the root
From tree-rooted maps to minimal accessible maps

Orient the tree edges away from the root
Orient the other edges counterclockwise around the tree
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The resulting orientation has no clockwise circuit.
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A oriented map is **accessible** if every vertex can be reach by an oriented path from the root.
From tree-rooted maps to minimal accessible maps

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The resulting orientation has no clockwise circuit.

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**Theorem** (Bernardi 2005) This is a bijection between tree-rooted maps with $n$ edges and minimum accessible maps with $n$ edges.
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The tree is recovered by reconstructing its contour.
Minimal orientations and canonical spanning trees

Idea:
Choose a minimal accessible orientation to get a spanning tree

Our pb becomes:
How to choose a canonical accessible minimal orientation?
Minimal orientations and canonical spanning trees

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A function $\alpha : V \to \mathbb{N}$ is feasible on a plane map $M$ if there exists an orientation of $M$ such that for each vertex $v$ the outdegree of $v$ is $f(v)$. 
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Theorem (Felsner 2004). Let \( \alpha \) be a feasible function on a plane map \( M \). Then the map \( M \) has a unique minimal \( \alpha \)-orientation.
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Our pb becomes: How to choose a canonical $\alpha$? (and check accessibility)

**Fact:** For many subclasses $\mathcal{F}$ of planar maps, there exists an $\alpha_\mathcal{F}$ s.t.:

A planar map is in $\mathcal{F}$ if and only if it admits an $\alpha_\mathcal{F}$-orientation.
Recall increasing quadrangulations are planar maps with faces of degree 4 such that:

- faces have labels in $\{1, \ldots, 2n - 2\}$
- around labeled vertices, face labels increase in ccw order
- around white vertices, face labels increase in cw order
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Orient each edge so that the minimum incident label is on the left
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Orient each edge so that the minimum incident label is on the left. This orientation is accessible, in fact strongly connected.
Recall increasing quadrangulations are planar maps with faces of degree 4 such that:

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Each black vertex has indegree \( \alpha_h(\text{black}) = m - 1 \), outdegree 1.
Each white vertex has indegree \( \alpha_h(\text{white}) = 1 \).
α-orientations for increasing quadrangulations

Recall increasing quadrangulations are planar maps with faces of degree 4 such that:

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Each white vertex has indegree \( \alpha_h(\text{white}) = 1 \).

This is our choice of canonical \( \alpha \) to decompose increasing quadrangulations.
opening of an increasing quadrangulation
opening of an increasing quadrangulation

endow with min $\alpha_c$-orient

(return cycles)
opening of an increasing quadrangulation

endow with min $\alpha_c$-orient

(return cycles)

find spanning tree
opening of an increasing quadrangulation
opening of an increasing quadrangulation

endow with min $\alpha_c$-orient
return cycles

find spanning tree

open
opening of an increasing quadrangulation

endow with min $\alpha_c$-orient

(return cycles)

find spanning tree

open

but forget half-edges
opening of an increasing quadrangulation

endow with min $\alpha_C$-orient (return cycles)

find spanning tree

open

but forget half-edges
give labels to edges
eliminate root black vertex
Proposition. The resulting simple Hurwitz trees has $n$ unlabelled vertices, $n - 1$ labeled vertices of degree 2, $2n - 2$ edges that increase ccw around labeled vertices.
From simple Hurwitz trees to increasing quadrangulations

A local rule to create increasing half edges

Cas 1:

Cas 2:

Two half-edges with same label $\Rightarrow$ edge and face of degree 4

Iterate the local rules as long as possible...
From simple Hurwitz trees to factorizations
From simple Hurwitz trees to factorizations

vertex label are useless for the bijection
From simple Hurwitz trees to factorizations

vertex label are useless for the bijection
From simple Hurwitz trees to factorizations

vertex label are useless for the bijection

adding buds
From simple Hurwitz trees to factorizations

- Vertex label are useless for the bijection
- Adding buds
- Parings and adding buds again
From simple Hurwitz trees to factorizations

vertex label are useless for the bijection

adding buds

Parings and adding buds again

gain
From simple Hurwitz trees to factorizations

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adding buds

Parings and adding buds again

again

again
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From simple Hurwitz trees to factorizations

Theorem [Duchi-Poulalhon-S. 2012] Closure is the reverse bijection between
- simple Hurwitz trees of size \( n \), and
- increasing quadrangulations, and
- simple ramified covers of \( \mathbb{S} \) by itself with \( m = 2n - 2 \) critical values.
From simple Hurwitz trees to factorizations

vertex label are useless for the bijection

adding buds

Parings and adding buds again

again

again

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Résumé des 2 premiers épisodes

Compter des classes d’équivalence de revêtements ramifiés

$\iff$

compter certaines plongements de graphes

$\iff$

compter certains arbres
Plan de l’exposé

Revêtements ramifiés et cartes

Cartes et arbres

Énumération d’arbres et formule d’Hurwitz

Revêtements et cartes aléatoires
Hurwitz formula for increasing quadrangulations

**Theorem** [Duchi-Poulalhon-S. 2012] Increasing quadrangulations (size $n$) are in bijection with simple Hurwitz trees having $n$ unlabelled vertices, $n - 1$ labeled vertices of degree 2, $2n - 2$ edges that increase ccw around labeled vertices.
Theorem [Duchi-Poulalhon-S. 2012] Increasing quadrangulations (size $n$) are in bijection with simple Hurwitz trees having $n$ unlabelled vertices, $n - 1$ labeled vertices of degree 2, $2n - 2$ edges that increase ccw around labeled vertices.
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The number of simple ramified cover of $\mathbb{S}$ by itself with $m = 2n - 2$ critical points is $n^{n-3}(2n - 2)!$.
Hurwitz formula for factorizations in transpositions

**Theorem.** Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition of $n$, and $\ell = \sum_i \ell_i$. The number of $m$-uples of transpositions $(\tau_1, \ldots, \tau_m)$ such that

- (product cycle type) $\tau_1 \cdots \tau_m = \sigma$ has cycle type $\lambda$
- (transitivity) the associated graph is connected
- (minimality) the number of factors is $m = n + \ell - 2$

is

$$n^{\ell-3} \cdot m! \cdot n! \cdot \prod_{i \geq 1} \frac{1}{\ell_i!} \left( \frac{i^i}{i!} \right)^{\ell_i}$$

**Proofs:**


(recurrences, Abel identities) (gfs and differential eqns) (geometry of LL mapping) (bijection + inclusion/exclusion)

$\lambda = n$, factorizations of $n$-cycles: $n^{n-2} \cdot (n-1)!$

$\lambda = 1^n$, factorizations of the identity: $n^{n-3} \cdot (2n-2)!$
Theorem. Let $\lambda = 1^{\ell_1}, \ldots, n^{\ell_n}$ be a partition of $n$, and $\ell = \sum_i \ell_i$. The number of $m$-uple of permutations $(\sigma_1, \ldots, \sigma_m)$ such that

- (factorization) $\sigma_1 \cdots \sigma_m = \sigma$ with cycle type $\lambda$
- (transitivity) $\langle \sigma_1, \ldots, \sigma_m \rangle$ acts transitively on $\{1, \ldots, n\}$
- (minimality) the total rank of factors is $\sum_i r(\sigma_i) = n + \ell - 2$

is

$$m \frac{((m - 1)n - 1)!}{(mn - (n + \ell - 2))!} \cdot n! \cdot \prod_i \frac{1}{\ell_i!} \left(\frac{mi - 1}{i}\right)^{\ell_i}$$

Proofs:

(Bousquet-Mélou–Schaeffer 2000) (Goulden–Serrano 2009)

(bijection + inclusion/exclusion)(gfs and differential eqns)

$\lambda = n$, factorizations of $n$-cycles: $\frac{1}{(mn+1)} \binom{mn+1}{n} \cdot (n - 1)!$

$\lambda = 1^n$, identity factorizations: $\frac{m}{(m-2)n+2} \frac{(m-1)^{n-1}}{(m-2)n+1} \binom{(m-1)n}{n} \cdot (n - 1)!$
Résumé des 3 premiers épisodes

Compter des classes d’équivalence de revêtements ramifiés

⇔

compter certaines plongements de graphes

⇔

compter certains arbres

les formules simples appellent des preuves constructives
Plan de l’exposé

Revêtements ramifiés et cartes

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Énumération d’arbres et formule d’Hurwitz

Revêtements et cartes aléatoires
Quadrangulations croissantes aléatoires uniformes

\( \bar{Q}_n = \{\text{quadrangulations croissantes à } n \text{ faces}\} \).

Quadrangulation croissante uniforme = variable aléatoire \( Q_n \) à valeur dans \( \bar{Q}_n \) avec

\[
\Pr(Q_n = q) = \frac{1}{|\bar{Q}_n|} = \frac{1}{n^{n-3}(2n-2)!}
\]

pour tout \( q \in \bar{Q}_n \).
Quadrangulations croissantes aléatoires uniformes

\[ \tilde{Q}_n = \{ \text{quadrangulations croissantes à n faces} \} . \]

Quadrangulation croissante uniforme = variable aléatoire \( Q_n \) à valeur dans \( \tilde{Q}_n \) avec

\[ \Pr(Q_n = q) = \frac{1}{\vert \tilde{Q}_n \vert} = \frac{1}{n^{n-3}(2n - 2)!} \quad \text{pour tout } q \in \tilde{Q}_n \]

- le choix de la distribution uniforme combinatoire est le plus immédiat

Parallèle naturel avec la distribution uniforme sur les quadrangulations enracinées:

\[ \Pr(\vec{Q}_n = q) = \frac{1}{\vert \vec{Q}_n \vert} = \frac{1}{2 \cdot 3^n (2n)!} \frac{(2n)!}{(n+2)!n!} \quad \text{pour tout } q \in \vec{Q}_n \]

Comment étudier \( Q_n \) ?
Propriétés des cartes aléatoires uniformes ?
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Triangulation uniforme aléatoire d’un disque

Delaunay de points aléatoires dans un disque

on est loin d’une discrétisation aléatoire d’une géométrie euclidienne

en physique on lie cela à la modélisation discrète de la gravité quantique
Quadrangulations uniformes comme surfaces aléatoires

L’allure d’une sphère aléatoire dépend un peu de qui dessine...

Objectif: Choisir une métrique intrinsèque et décrire les surfaces ainsi obtenues
Étudier les quadrangulations aléatoires uniformes

Distribution uniforme sur les quadrangulations à \( n \) faces, pour \( n \) grand

1ère approche: Étudier le comportement asymptotique de paramètres:

- degré d’un sommet aléatoire
- distance entre 2 sommets aléatoires
- loi 0-1 pour les propriétés locales
- longueur d’un plus petit cycle diviseur

⇒ espérance, moments, lois limites discrètes ou continues, qd \( n \to \infty \)

Bender, Canfield et al (90’s →) en combinatoire

Ambjørn, Watabiki et al (90’s →) en physique
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Exemple: $\Delta_n =$ distance entre 2 sommets aléatoires uniformes de $Q_n$

Théorème (Chassaing-S. 2004) $\mathbb{E}(\Delta_n) \sim c \cdot n^{1/4}$

$(n^{-1/4}\Delta_n) \xrightarrow{d} \max \text{ (serpent Brownien)}$
Étudier les quadrangulations aléatoires uniformes

Distribution uniforme sur les quadrangulations à \( n \) faces, pour \( n \) grand

2ème approche: Définir des surfaces aléatoires limites
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Distribution uniforme sur les quadrangulations à $n$ faces, pour $n$ grand

2ème approche: Définir des surfaces aléatoires limites

– convergence vers une limite d’échelle

⇒ la carte Brownienne

(Pb posé au séminaire Hypathie en 2002 à Lyon)

Marckert, Mokkadem, Le Gall, Miermont, …
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⇒ la carte Brownienne

Marckert, Mokkadem, Le Gall, Miermont, ...  
puis Weill, Curien, Benjamini,...

(Pb posé au séminaire Hypathie en 2002 à Lyon)

– convergence vers une limite infinie discrète

⇒ la quadrangulation infinie uniforme (UIPQ)

Angel, Schramm, ...

puis Durhus, Chassaing, Krikun, Bettinelli,...
Conclusions

– L’excursion Brownienne décrit la limite d’échelle de toute sorte d’excursions aléatoires discrètes plus ou moins complexes.

– L’arbre continu aléatoire est limite d’échelle de toute sorte d’arbres aléatoires discrets plus ou moins complexes.

⇒ On pense qu’il en est de même de la carte Brownienne.
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On dispose d’un cadre bijectif très général pour la construction de cartes par recollements d’arbres (Bernardi-Chapuy-Fusy 2011, Albenque-Poulalhon 2012)
On obtient ainsi en particulier un codage d’arbres pour les revêtements...
Il reste à utiliser ces constructions pour passer à la limite...