Random triangulations,

planar maps,

and a Brownian snake

PART II

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An overview of the talk

A combinatorial model

Planar maps and triangulations

Random planar maps

as a discrete model of random geometries

Encoding the distance

From quadrangulations to embedded trees

Quadrangulations and Brownian snakes

Toward a continuum random map ?

A summary of the first part.

- The random planar maps model has many variants (triangulations, bipartite maps, convex polyhedra,...)

- Various parameters of interests can be analytically studied (maximal degree, baby universes, separators,...)

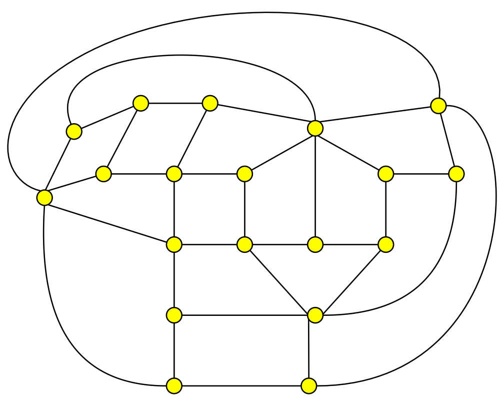
- All knows results satisfy the expected "universality": critical exponents agree for different families.

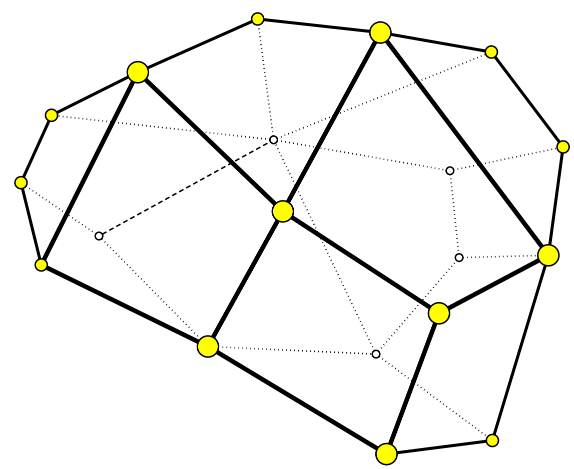
Seek a limit model encoding more than just one parameter...

$\Rightarrow$ concentrate on a simple variant.

Random quadrangulations.

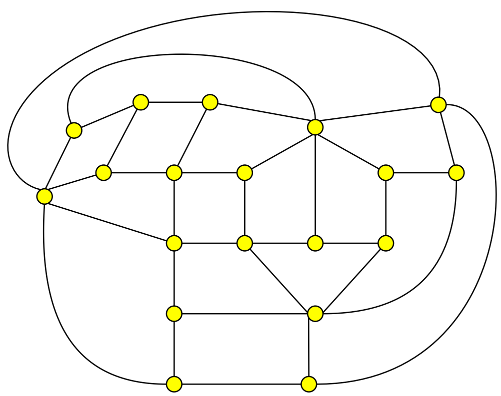
The simple family we choose is that of random quadrangulations.

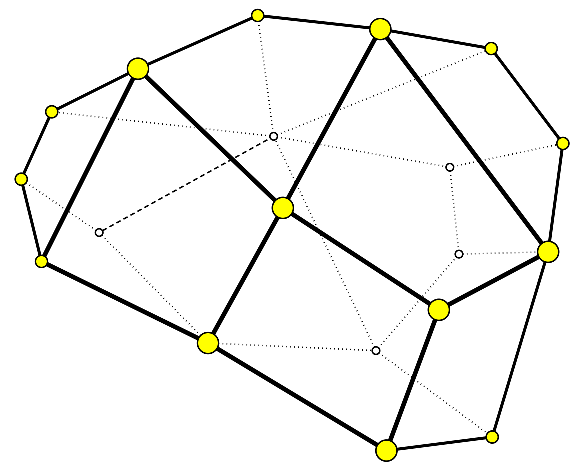




Why ?

Random quadrangulations. Enumeration.





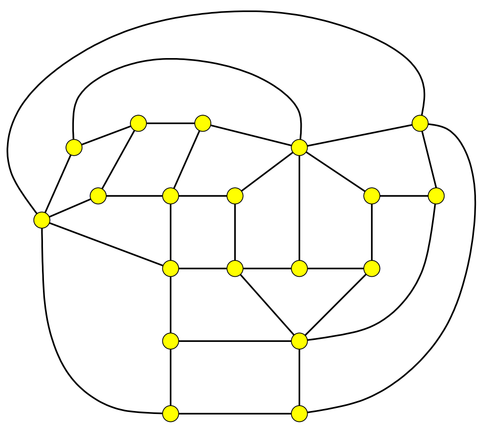
Theorem (Tutte 62). The number of rooted quadrangulations with n faces is

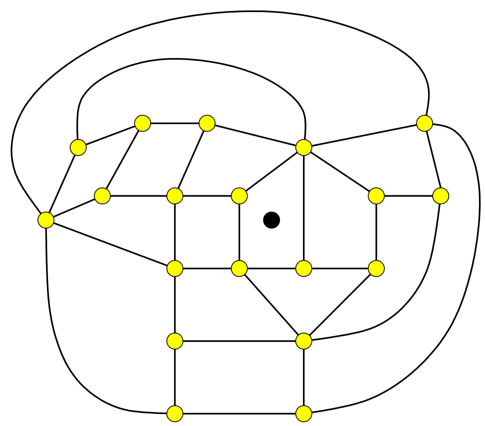
$$\frac{2}{n+2}\,\frac{3^n}{n+1}\binom{2n}{n}.$$

this might remind you the formula for 4-regular maps.

Random quadrangulations. As dual 4-regular maps

No surprise, this is just duality on planar graphs:





Add a vertex in each face, and dual edges.

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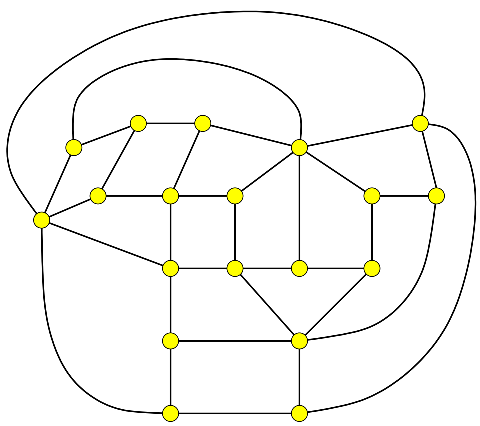

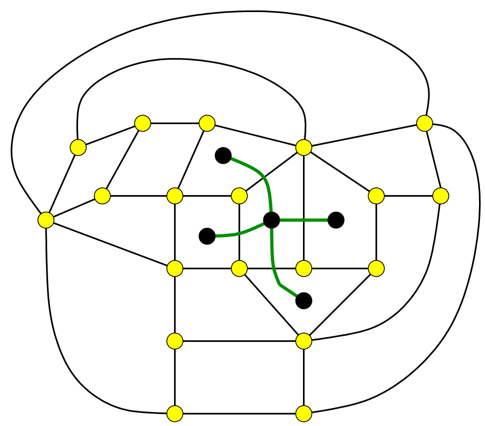

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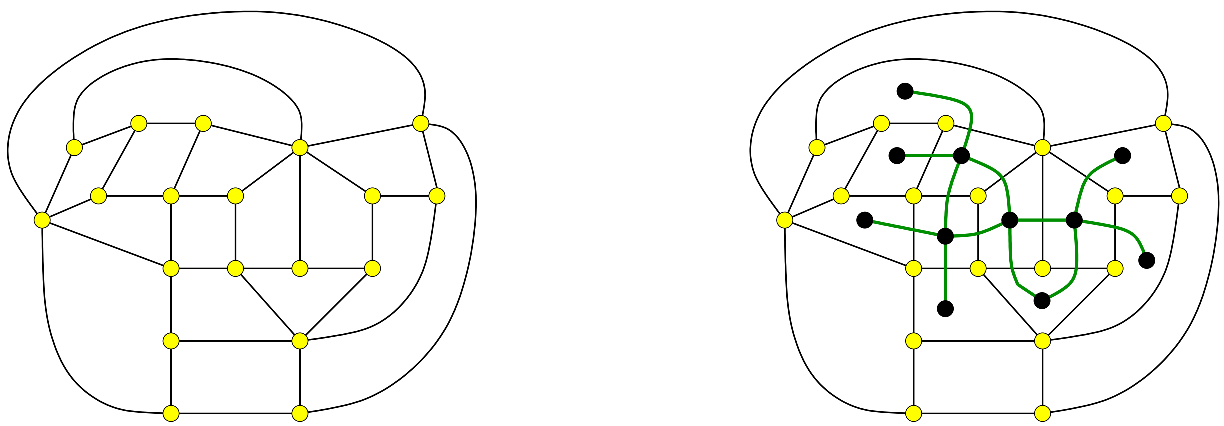

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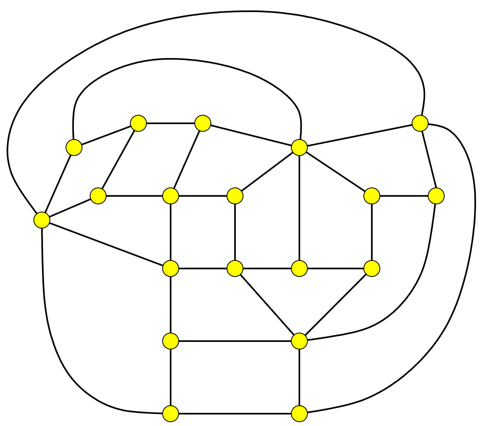

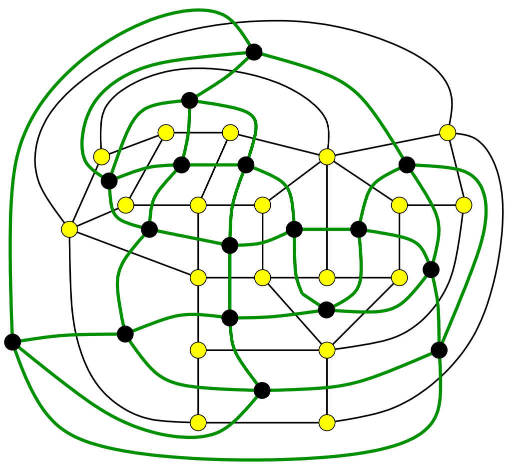

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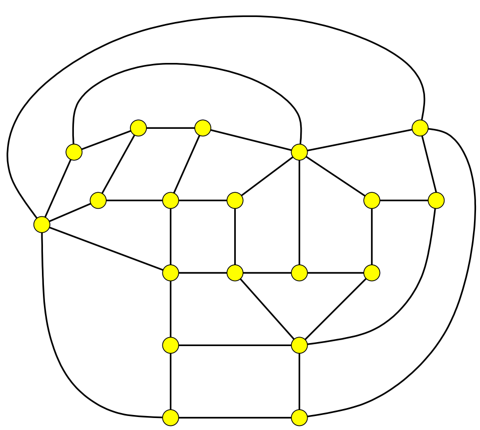
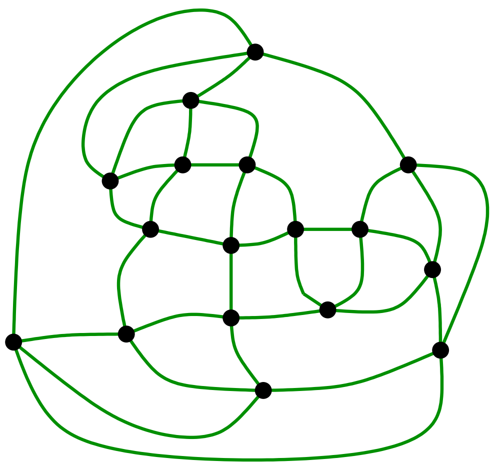
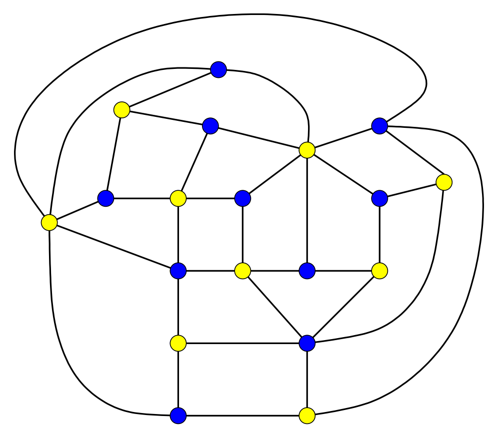
 

Add a vertex in each face, and dual edges.

This is one-to-one between quadrangulations and 4-regular maps.

Random quadrangulations are bipartite graphs.

The vertices of a planar quadrangulation can be colored in two colors so that all edges join different colors.



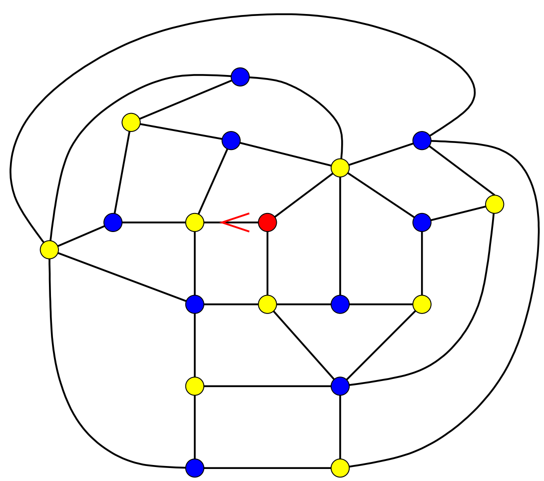
Indeed all faces have even length, and so have all cycles.

Distances in quadrangulations

Yet another parameter ?

Profile and radius of a quadrangulation with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of the (red) root
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



- r_n is the radius (maximal distance from the root)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius of a quadrangulation with n faces.

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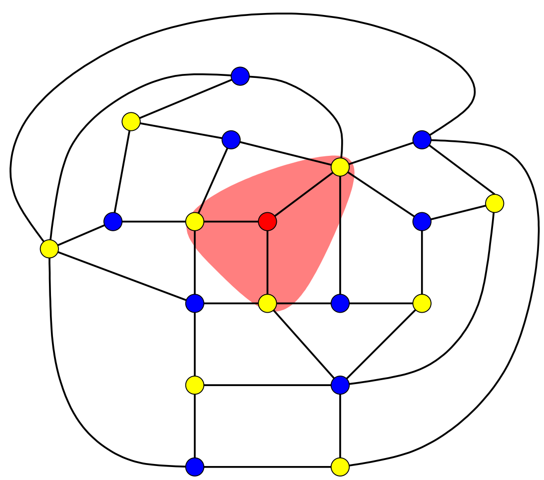

$X_n^{(1)} = 3$

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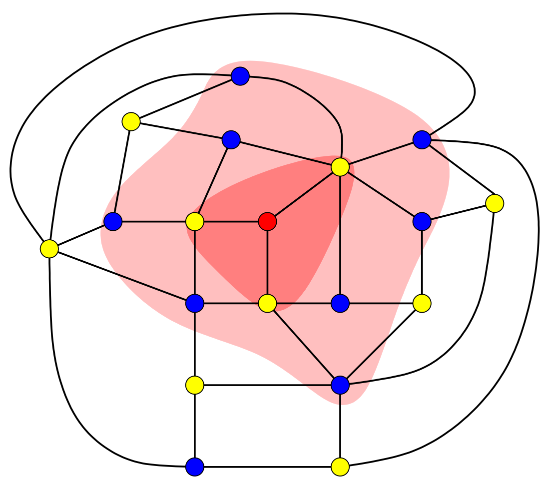

$X_n^{(1)} = 3$
$X_n^{(2)} = 8$

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Profile and radius of a quadrangulation with n faces.

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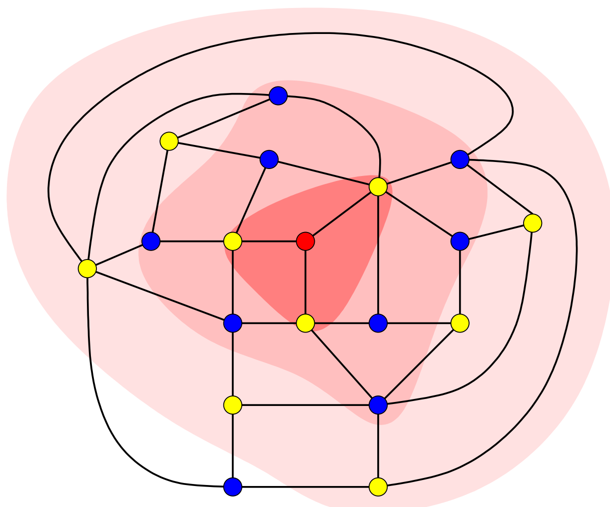

$X_n^{(1)} = 3$

$X_n^{(2)} = 8$

$X_n^{(3)} = 6$

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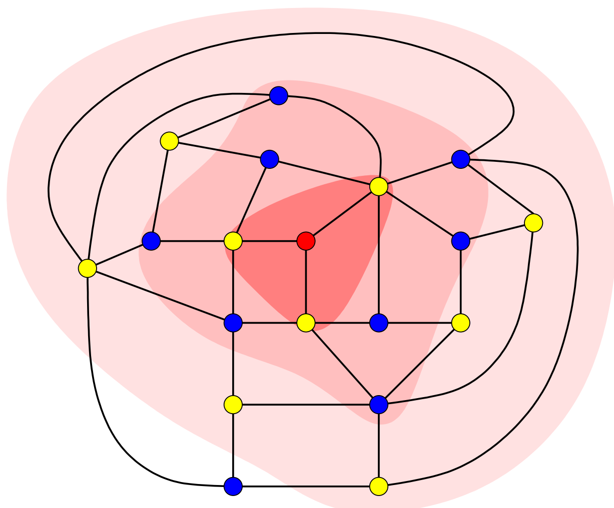


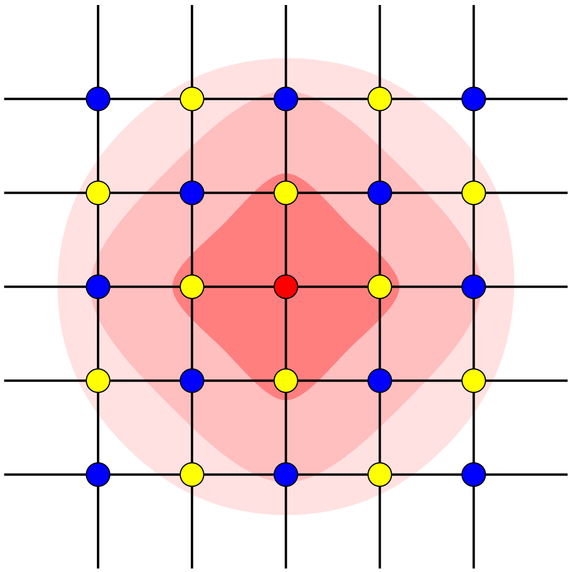
$X_n^{(1)} = 3$
$X_n^{(2)} = 8$
$X_n^{(3)} = 6$
$X_n^{(4)} = 1$
$r_n = 4.$

- r_n is the radius (maximal distance from the root)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius. On the grid ?

On a grid with n faces ($\sqrt{n} \times \sqrt{n}$), the behaviour is clear:



In particular,

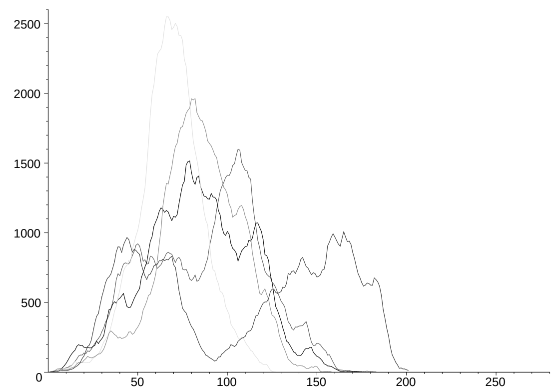
$$X_n^{(k)} = \Theta(k) \text{ for } k < n^{1/2}, \quad \text{and } r_n \text{ grows like } n^{1/2}.$$

How do these parameters behave on random quadrangulations ?

Profile and radius. Experimentaly.

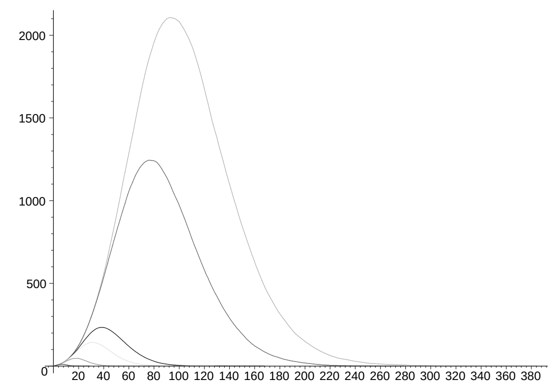
Experimental datas from random sampling:

Six random profiles:



All for maps of size $n = 100,000$.

Averaged profiles:



For various n (100 to $100,000$).

Conjecture (S. 1998). The correct scaling is $k = tn^{1/4}$, and

– $n^{-3/4} X_n^{(tn^{1/4})} \xrightarrow{\text{law}} X(t)$, a process supported on $\mathbb{R}^+$,

– the radius satisfies $\mathbb{E}(r_n) \underset{n\to\infty}{\sim} cte \cdot n^{1/4}$.

Profile and radius. Heuristic results.

These conjectures agree with previous results from physics.

For random triangulations:

- **Two beautyful heuristic calculations** by physicists Watabiki, Ambjørn *et al.* (1994). *The Hausdorff dimension is* 4 *:*

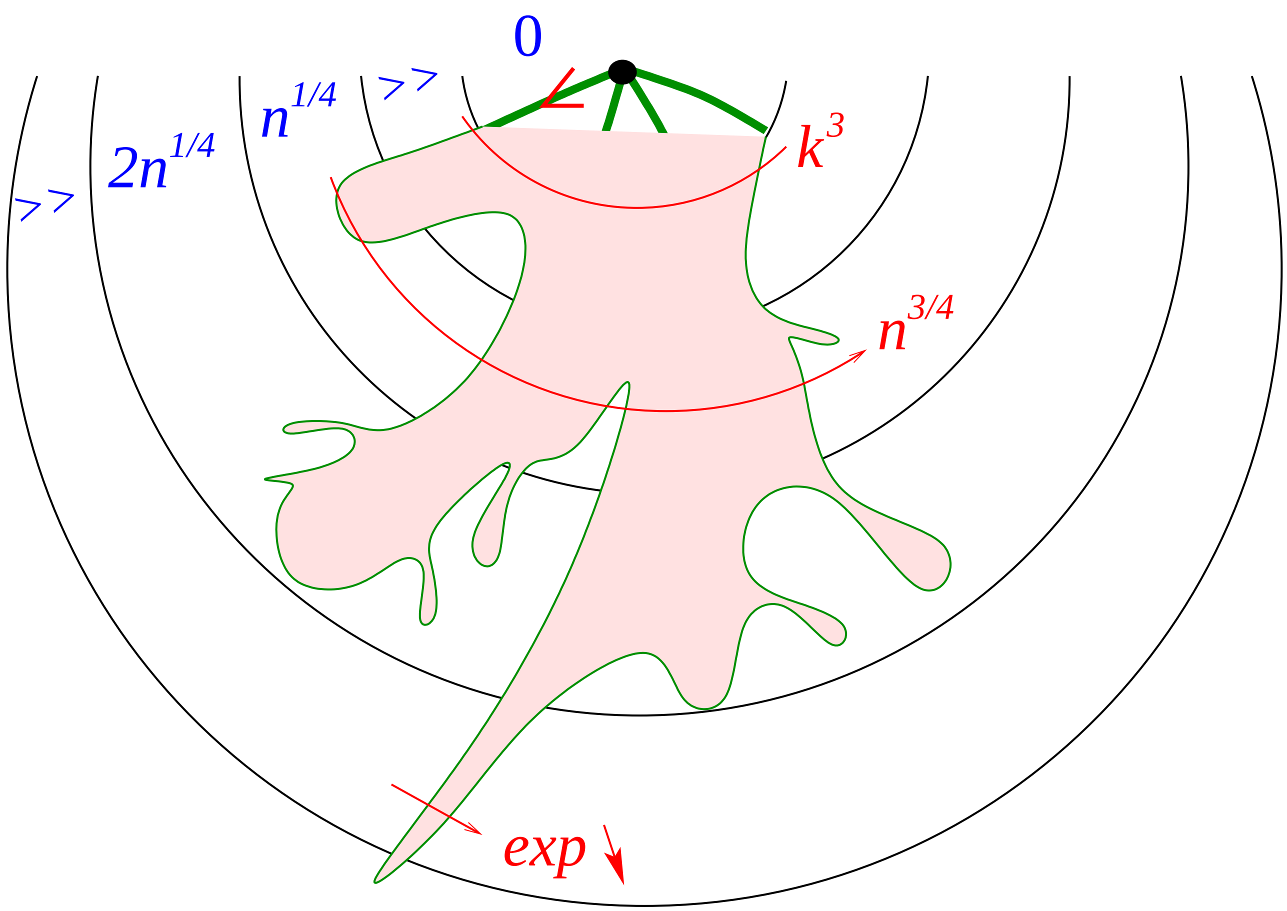
  meaning for $k \ll n^{1/4}$, $\mathbb{E}(\int_0^k X_n^{(i)}) \sim k^4$,

  for $k \gg n^{1/4}$, $\mathbb{E}(X_n^{(k)})$ is exp. decreasing

  They prove that only possible scaling is indeed $k = tn^{1/4}$.

However their result does not give the radius or the limit process.

Random quadrangulations. A tentative picture of distances.



Encoding the distances in a tree

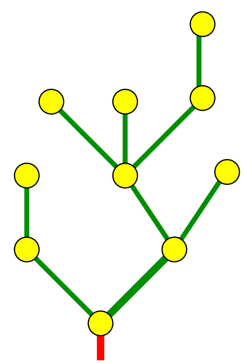
Another encoding by trees

Rooted planar trees. Catalan again.

A *rooted plane tree* is made of a root vertex attached to a sequence of rooted plane trees.

The number of rooted plane trees with n edges is

$$\frac{1}{n+1}\binom{2n}{n}$$



Proof. The contour walk of a rooted plane tree is a Dyck path.

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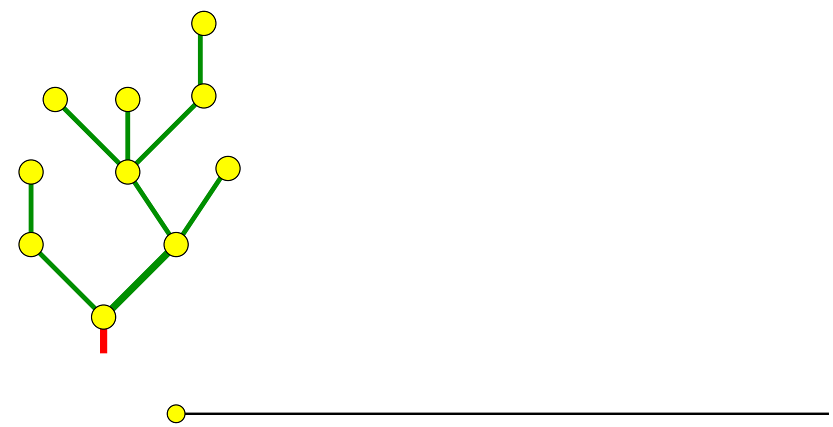

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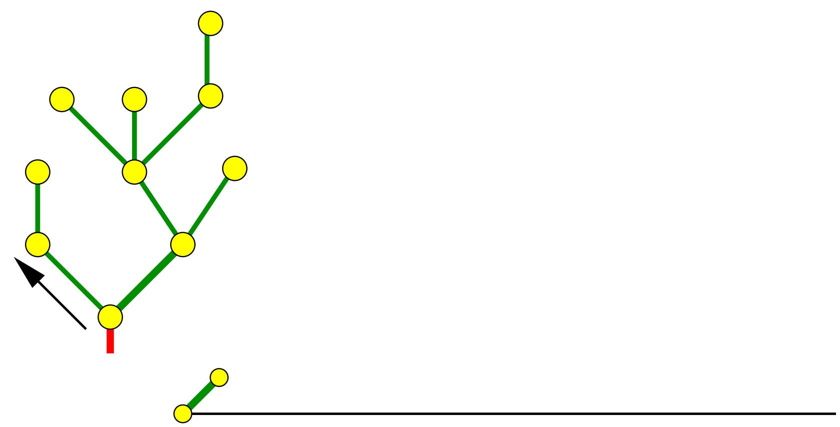

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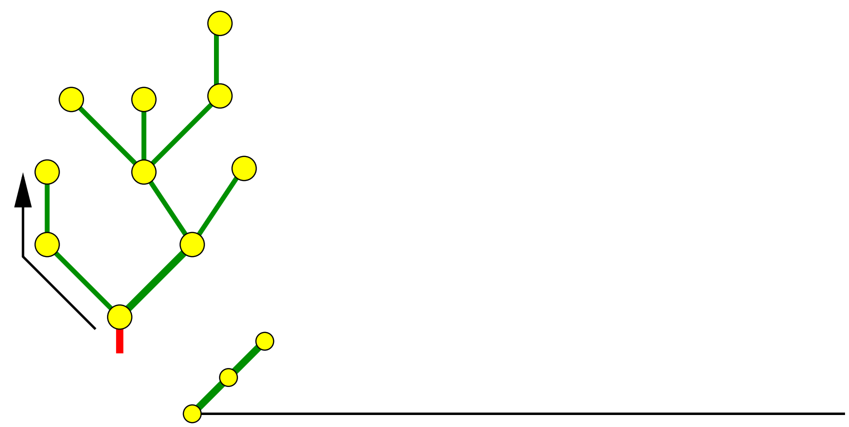

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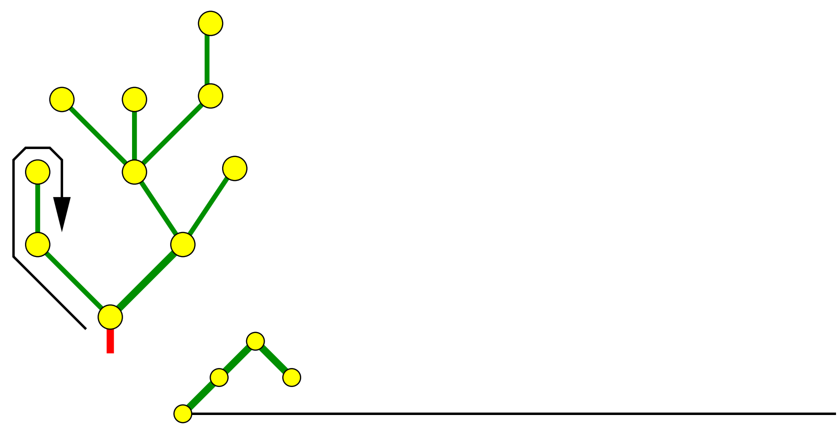

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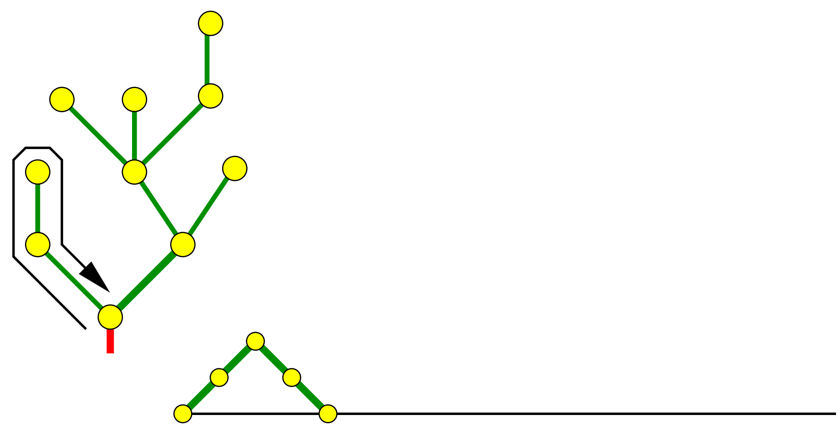

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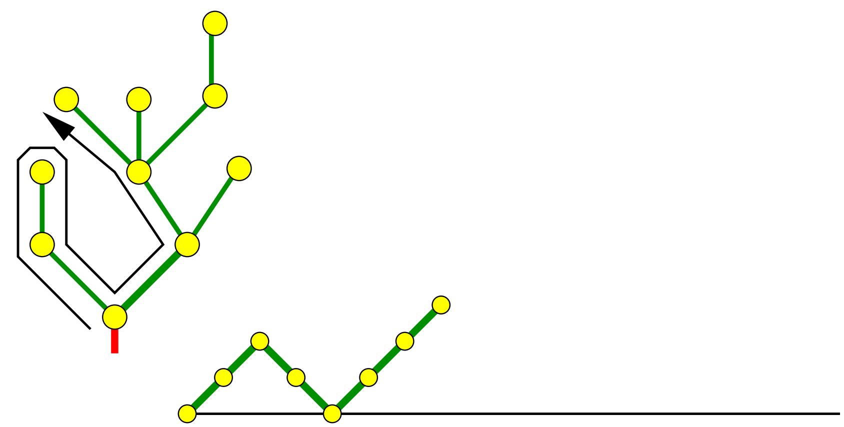

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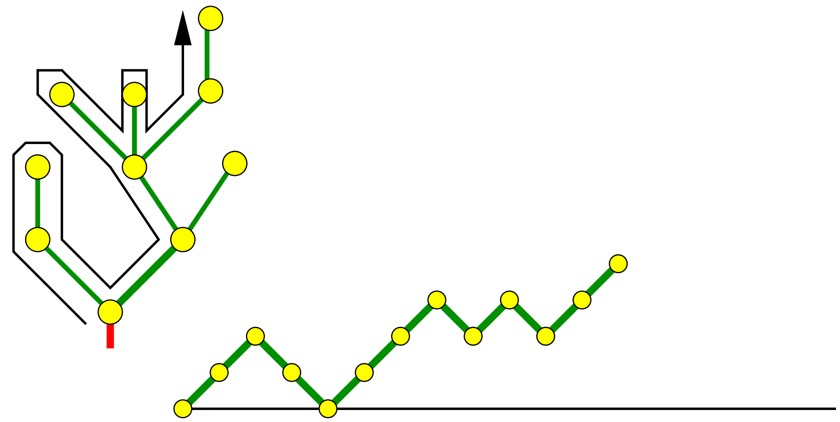

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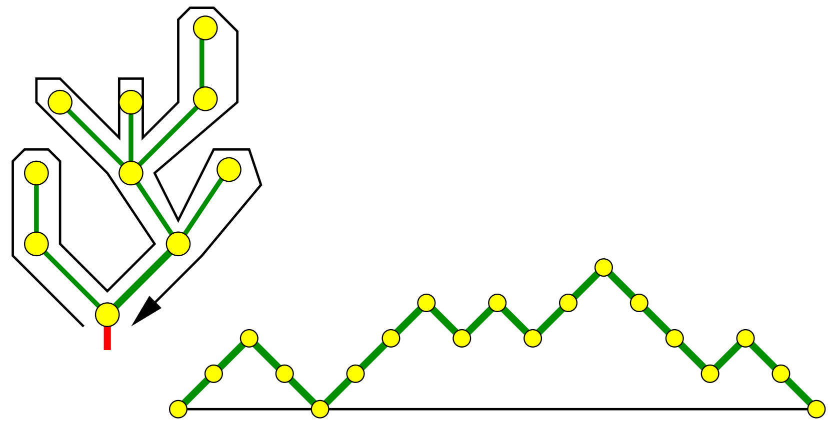

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$\Rightarrow$ cf. Part I: Catalan numbers.

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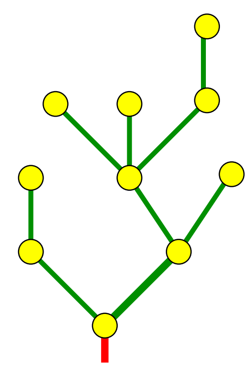


Let $\mathcal{T}_n$ be the set of rooted plane trees with n edges.
From now on U_n denote a r.v. uniform on $\mathcal{T}_n$.

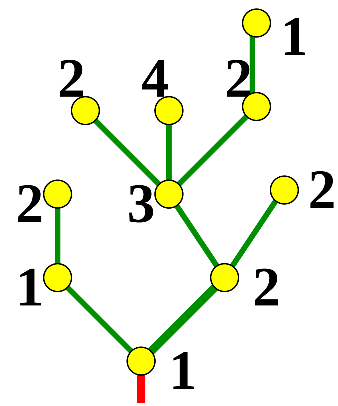
Well labelled trees. Cori and Vauquelin '84.

Definition. A *well labelled tree* (T, ϕ) is a rooted plane tree T with integer labels $\phi(v)$ such that:

(i) the root has label one: $\phi(r) = 1$.

(ii) labels differ at most by one along each edge (v, w): $|\phi(v) - \phi(w)| \leq 1$

(iii) all labels are positive: $\phi(v) > 0$.



Let (T_n, ϕ_n) be a uniform random well labelled tree with n edges.

Observe that T_n is not uniform on $\mathcal{T}_n$.

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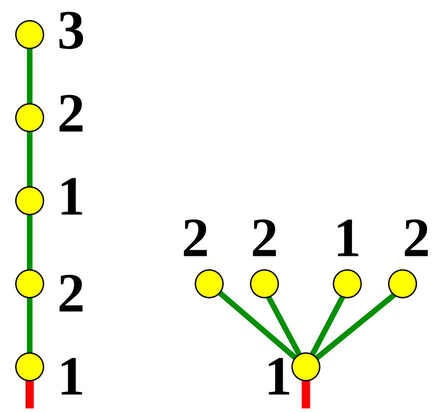

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Observe that T_n is not uniform on $\mathcal{T}_n$.

A distance preserving encoding. Statement

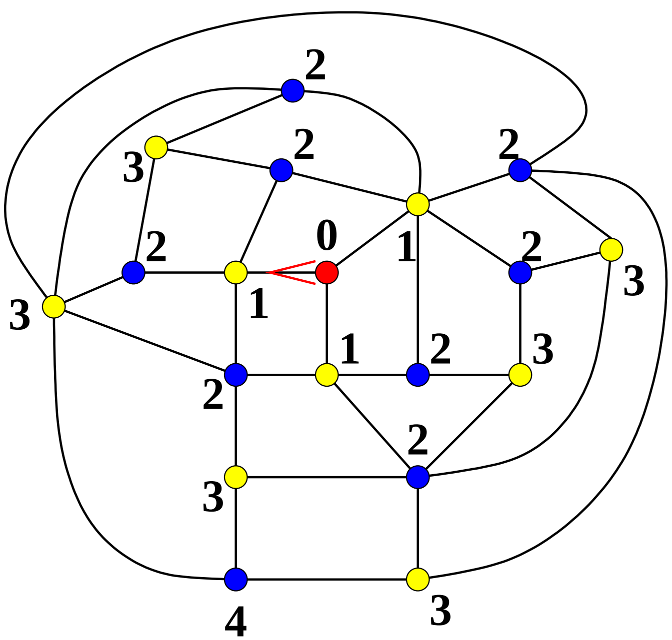
Theorem (S. 1998).

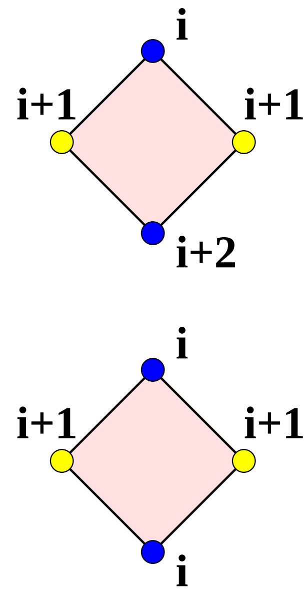
There is a one-to-one correspondence between
– rooted quadrangulations with n faces, and
– well labelled trees with n edges,
that maps the profile onto the label distribution.

Cori and Vauquelin (1984) gave another bijection that proves the theorem *without* the last requirement.

A distance preserving encoding. Proof

Let us label vertices by distances.

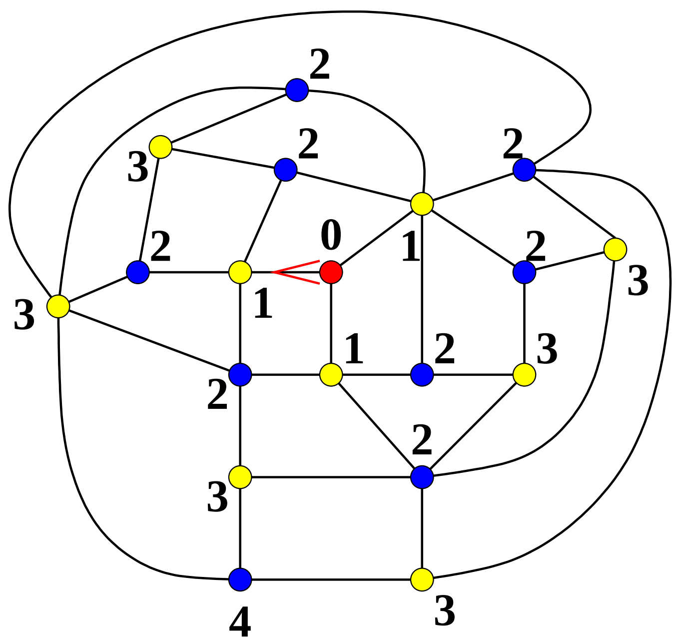


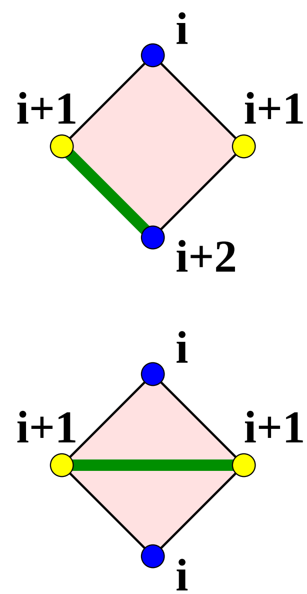


There are only two possible configurations around a face (up to colors).

A distance preserving encoding. Local rules

Consider the following two local rules.





Apply these rules to all faces.

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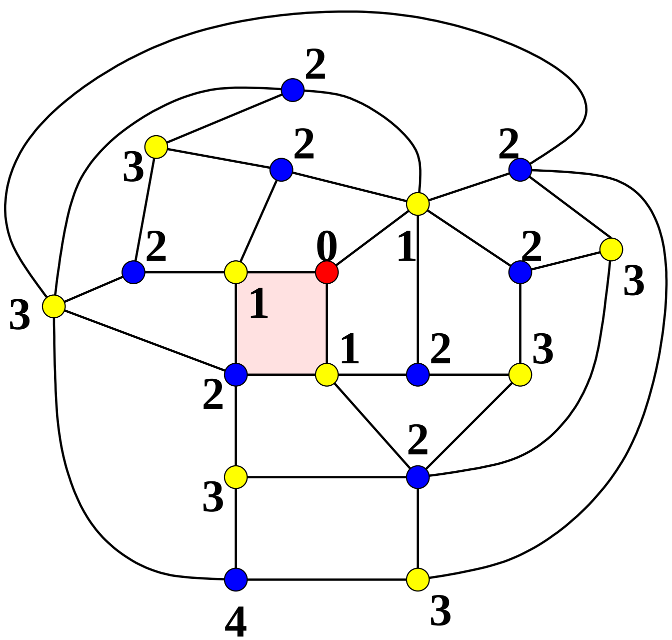

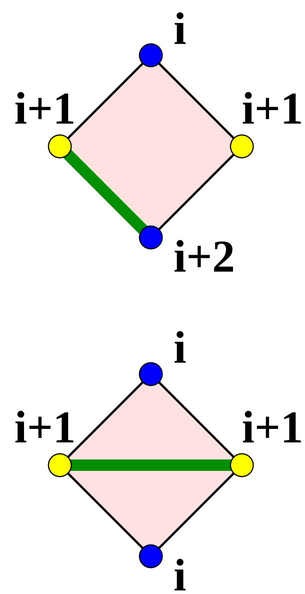

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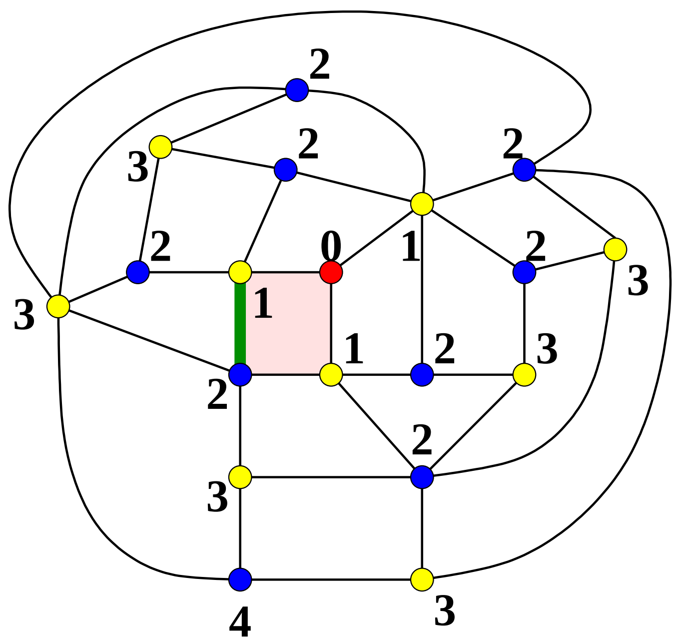

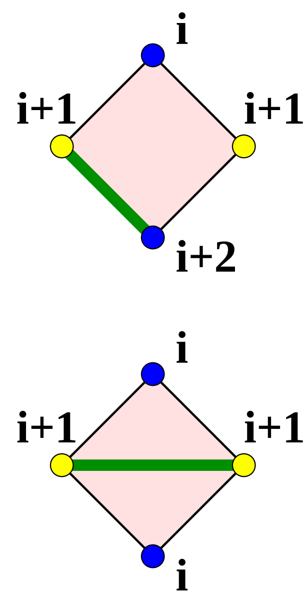

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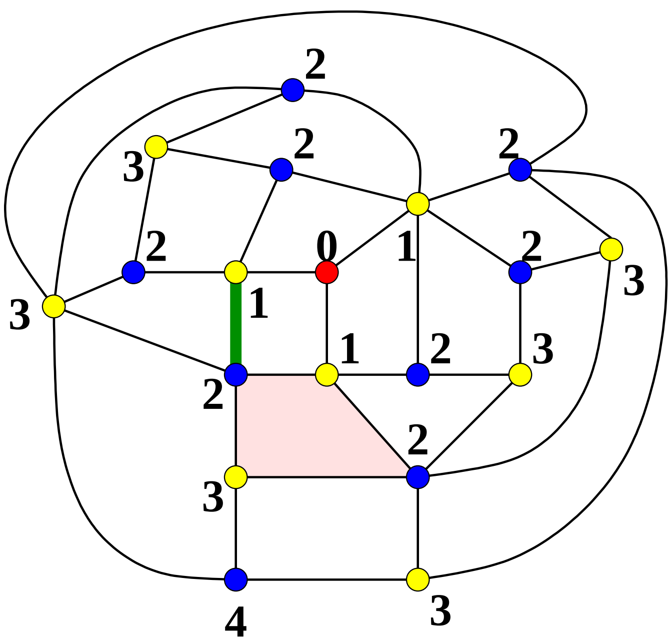

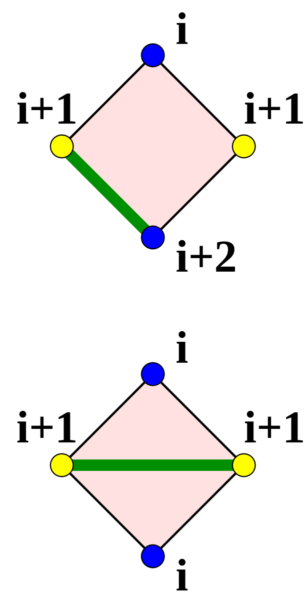

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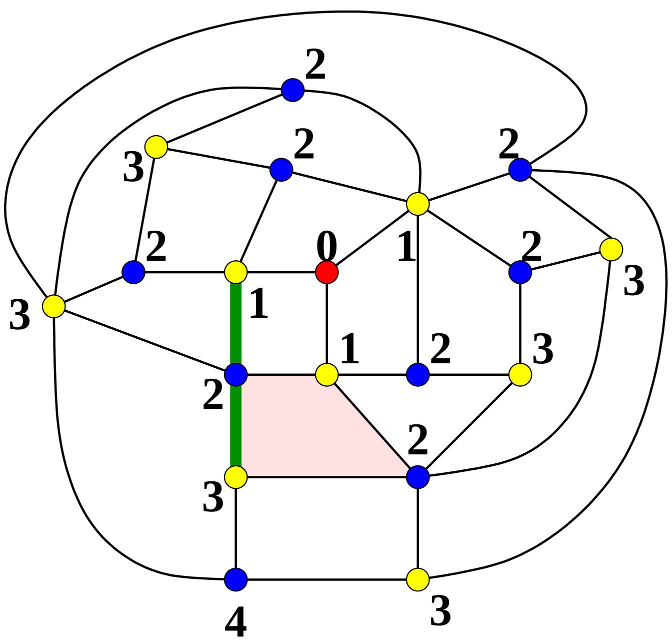

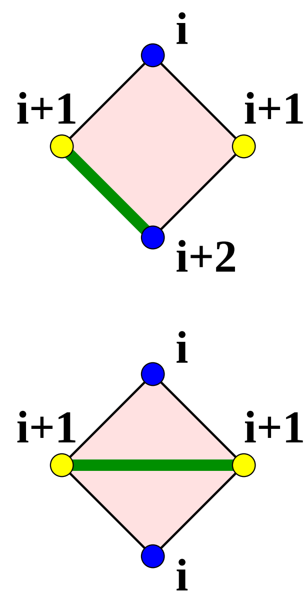

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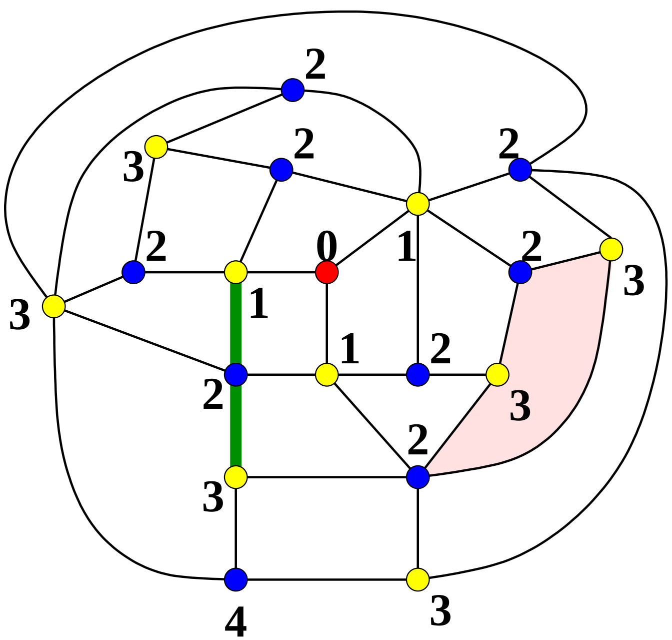

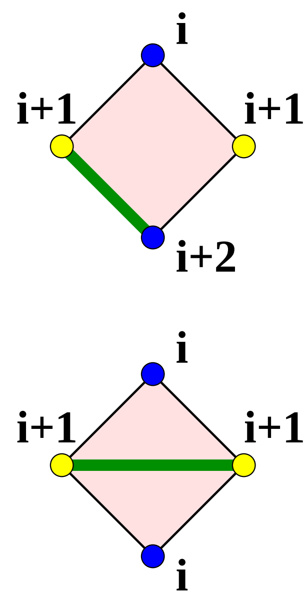

Apply these rules to all faces.

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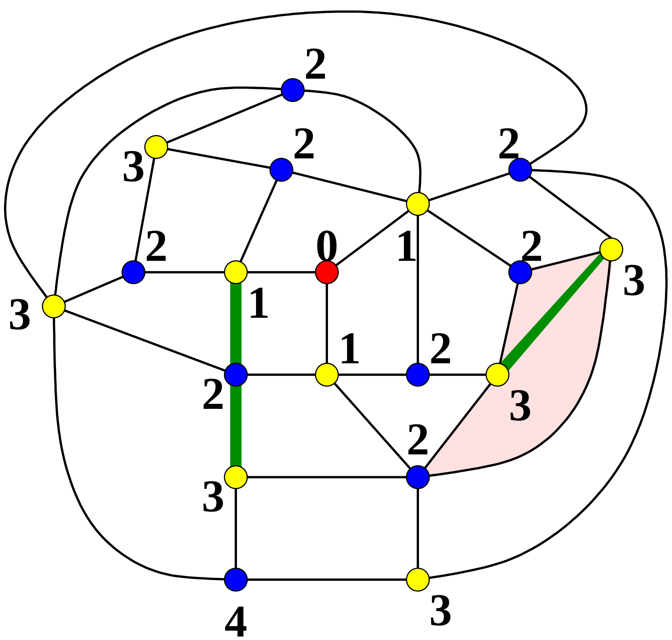

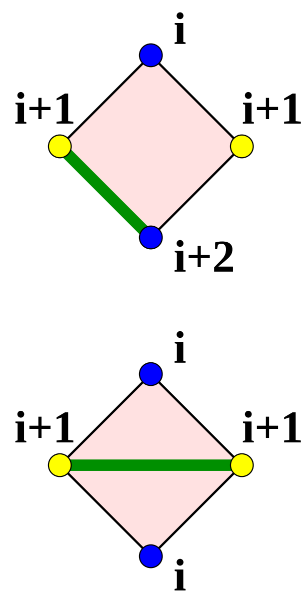

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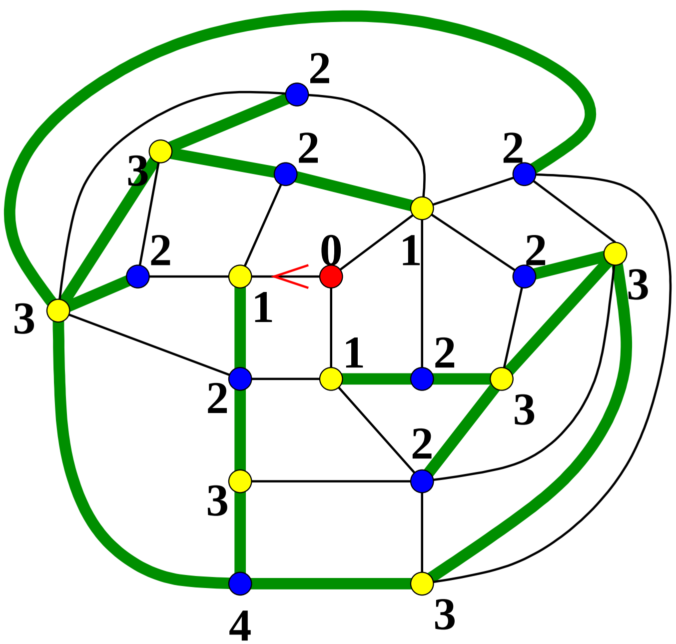

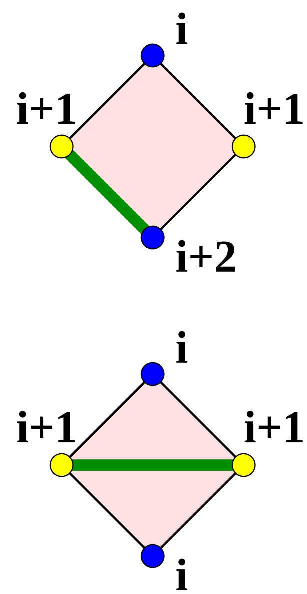

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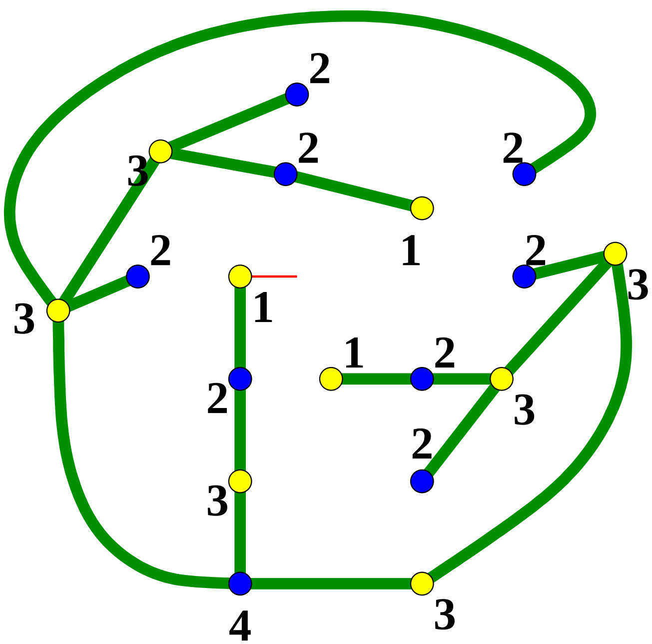

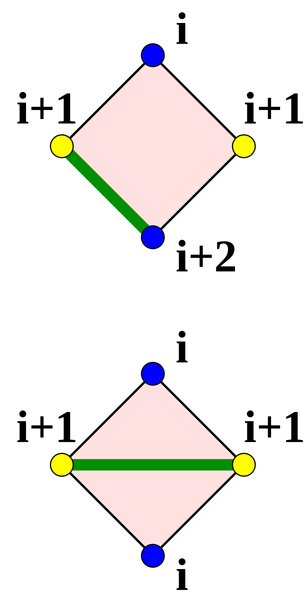

Proposition: the edges produced by local rules form a tree.

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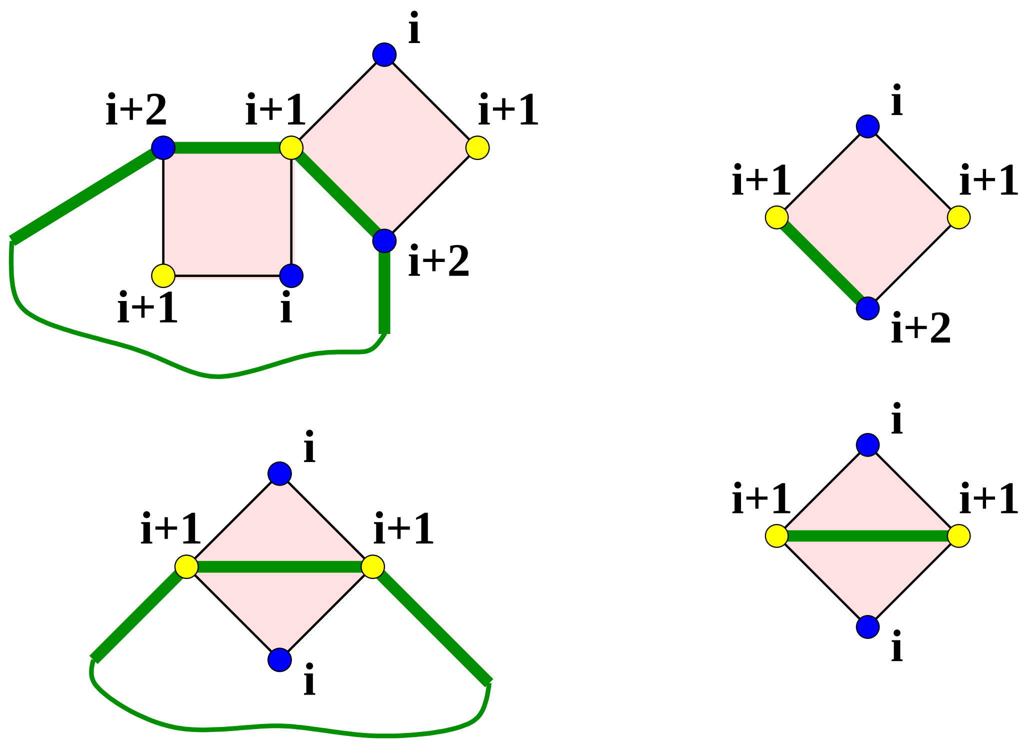
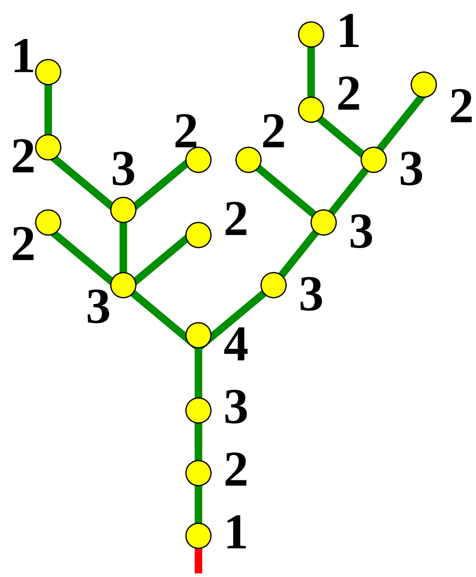
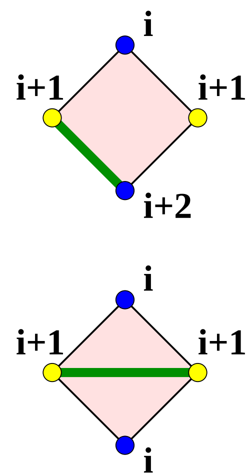


The root can be only in one of the two regions delimited by a cycle.
Taking $i+1$ minimal on the cycle, a contradiction is obtained
between rules and labelling by distance.

A distance preserving encoding. “End” of proof.

By construction, labels in the tree differ at most by one along edges.





The resulting tree is thus a well labelled tree.

Missing edges are recovered by a greedy $(i \to i-1)$ matching around the tree.

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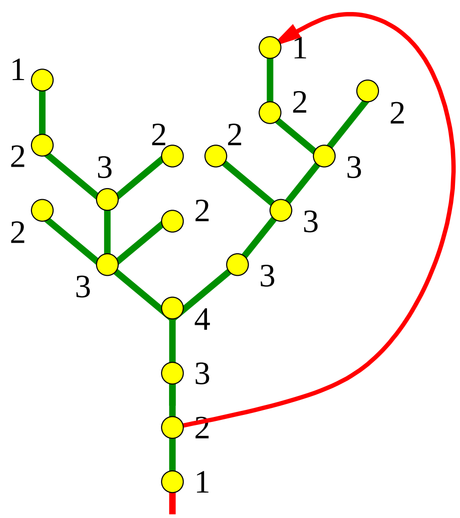

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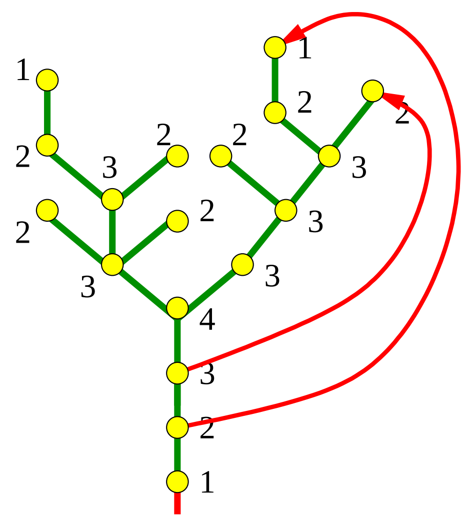

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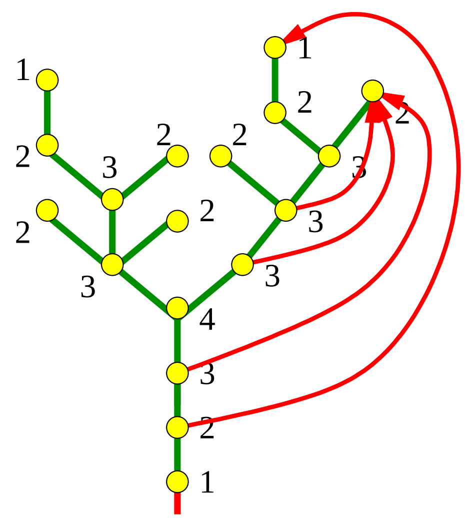

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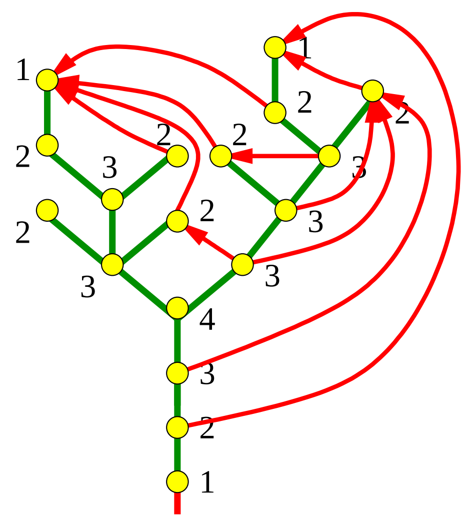

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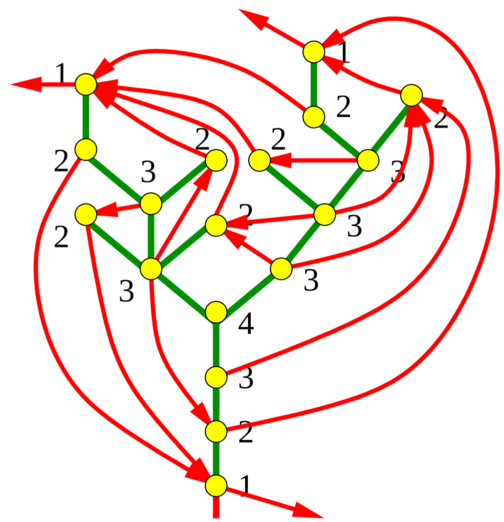

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A first summary

Uniform distribution on quadrangulations with n faces

$$=$$

Uniform distribution on well labelled trees with n edges

The profile $(X_n^{(k)})_{k\geq 1}$ is the label distribution $(L_n^{(k)})_{k\geq 1}$.

The radius $r_n = \max(k \mid X_n^{(k)} > 0)$ is the largest label of (T_n, ϕ_n).

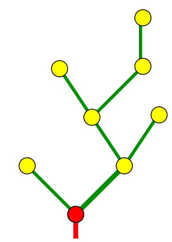
These are *identities in law*, not just asymptotic results.

Embedded trees, ISE and Brownian snakes

Aldous model of random mass distribution

Embedded trees. Aldous '93

Aldous ('93) introduced a model of random trees embedded in the lattice $\mathbb{Z}^d$.



– Start with U_n (uniform on $\mathcal{T}_n$).

– Give length one to all edges.

– Embed U_n in $\mathbb{Z}^d$:

  – put the root of U_n at the origin,

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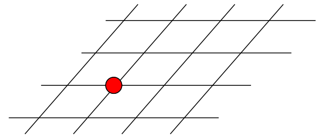

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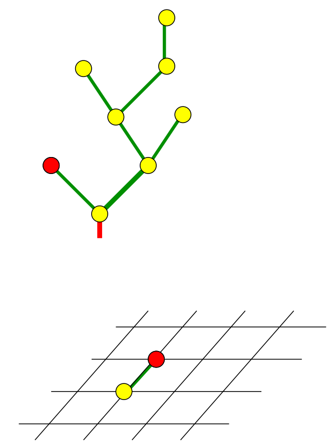

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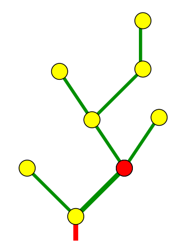

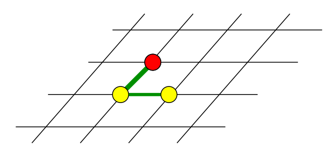

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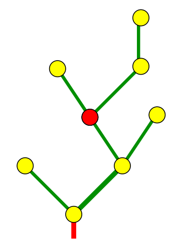

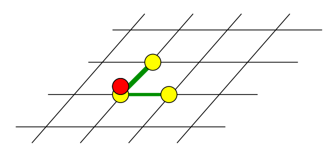

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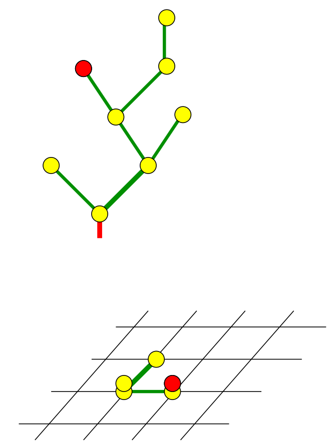

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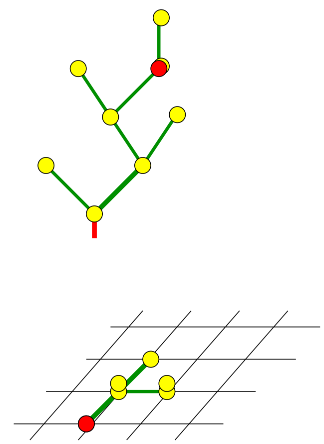

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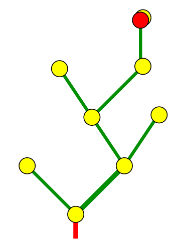

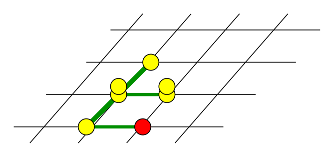

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Embedded trees. Mass distribution and ISE

Putting masses on vertices yield a random mesure on $\mathbb{Z}^n$,

$$\mathcal{J}_n = \frac{1}{n} \sum_{v \in U_n} \delta_{\psi_n(v)}.$$

Theorem. (Aldous '93, Borgs *et al.* '99)

> There is a random mesure $\mathcal{J}$ on $\mathbb{R}^d$, called *Integrated SuperBrownian Excursion* such that, upon scaling the lattice to $n^{-1/4}\mathbb{Z}^d$, $\mathcal{J}_n$ weakly converges to $\mathcal{J}$.

Intuition: Branches of U_n have typically length or order $\sqrt{n}$.
The embedding ϕ of a branch of length ℓ is a random walk.
$\Rightarrow$ most vertices are embedded at distance $\sqrt{\ell} = n^{1/4}$ from origin.

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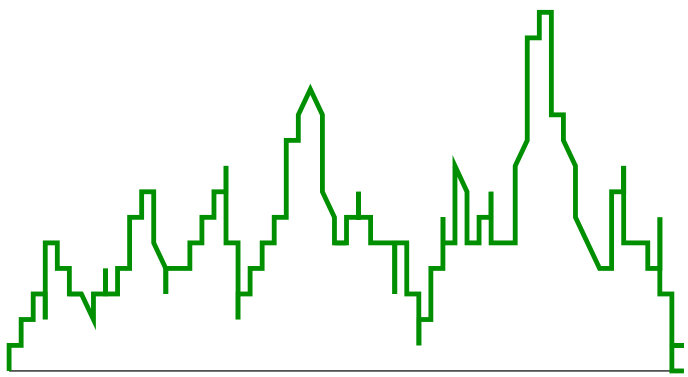
Derbez & Slade '98: ISE as continuum limit of lattice trees for $d > 8$.

Hara & Slade '98: ISE as continuum limit of incipient infinite cluster in percolation for $d > 6$.

Embedded trees. ISE and Brownian snakes

The mesure ISE admit an alternative description in terms of a *Brownian snake* (cf. Le Gall's book '99).

Let us give an informal description:



An excursion e, describing the vertical extension of the snake as time evolves.

At $t = 0$, a Brownian motion of length $e(0) = 0$.

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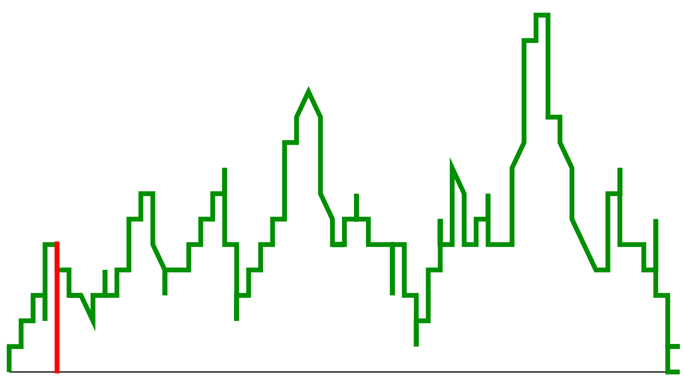

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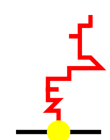

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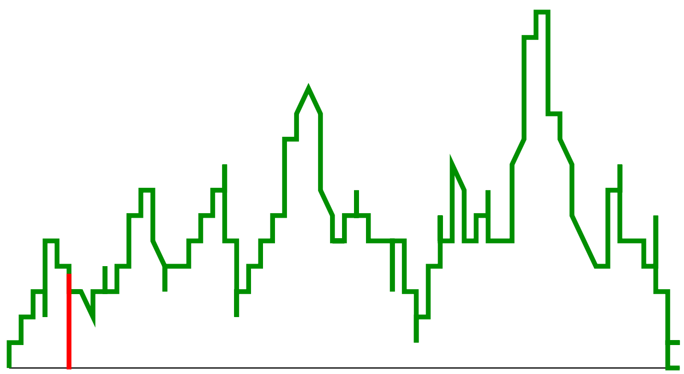

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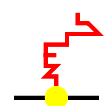

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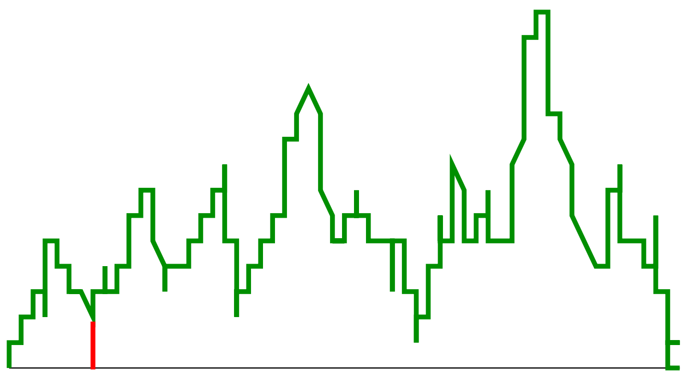

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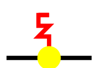

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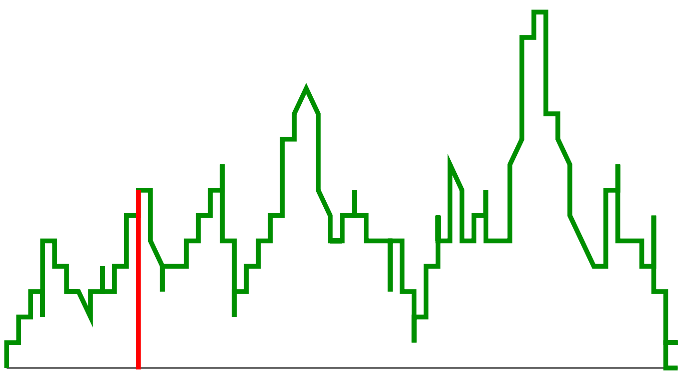

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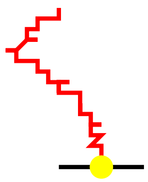

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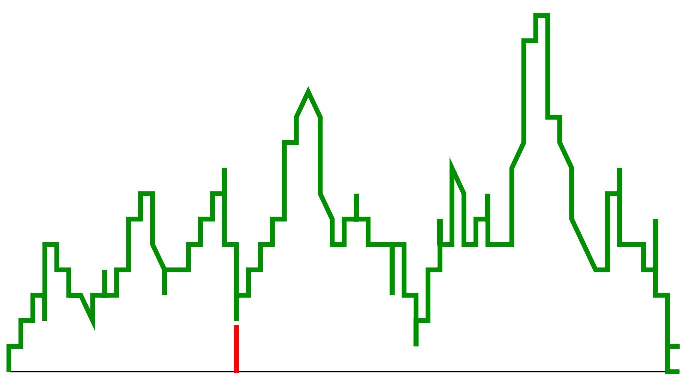

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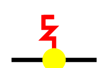

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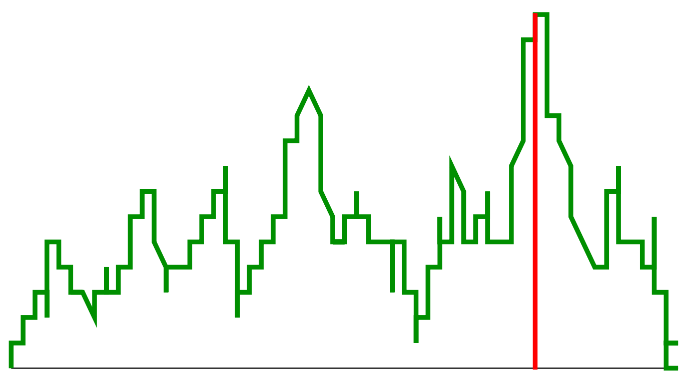

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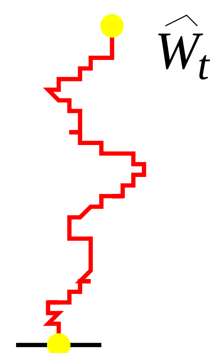

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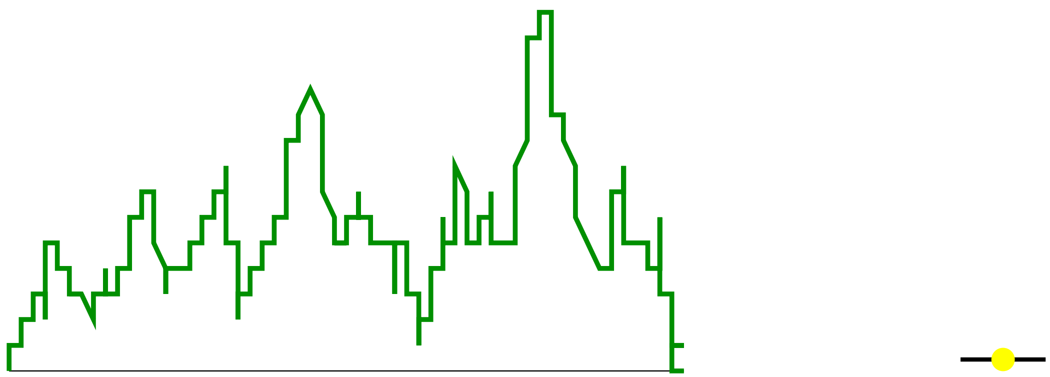

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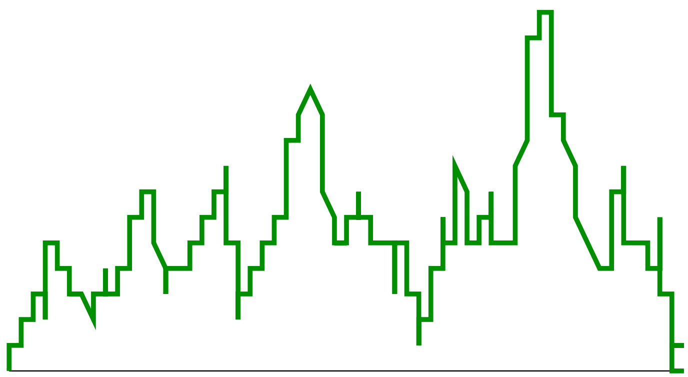

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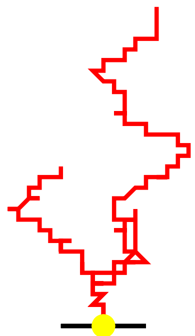

The total trace of the snake (branching r.w.).

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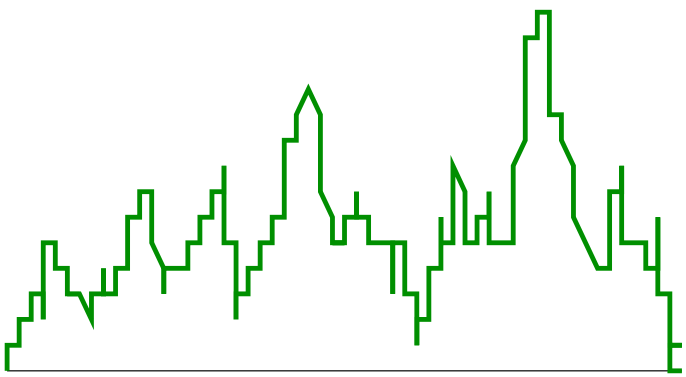

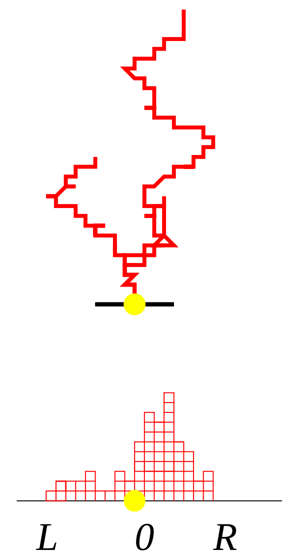

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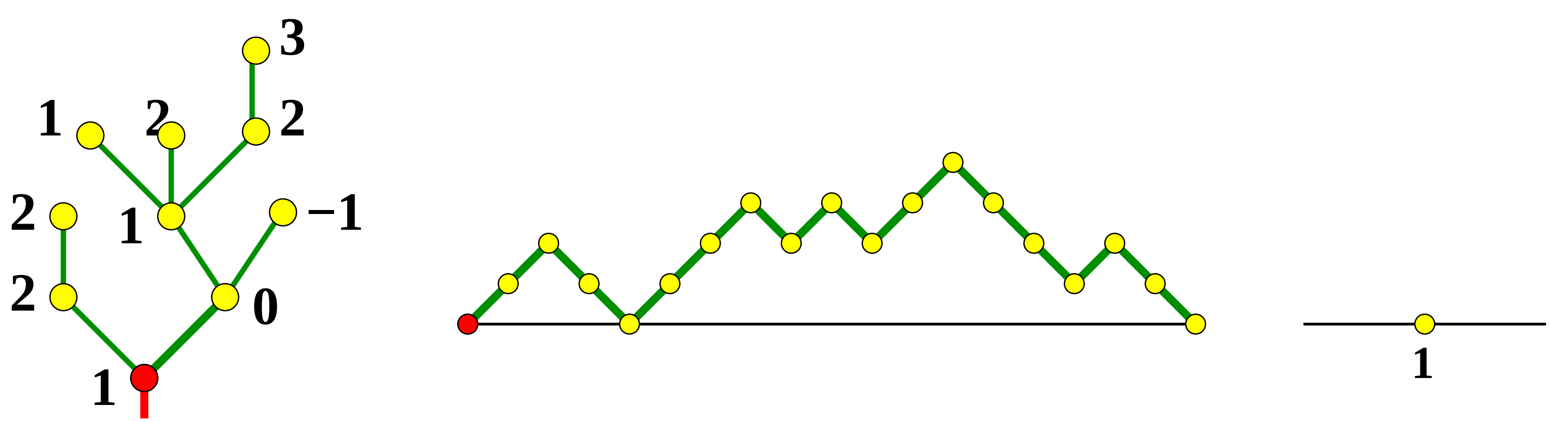
ISE is recovered as: $\forall g$ test function, $\int g \, d\mathcal{J} = \int_0^1 g(\hat{W}_s) \, ds$.

Embedded trees. Contour walks and convergences

The contour walks associated to the embedded tree (U_n, ψ_n) yields a discrete analog (E_n, V_n) of the Brownian snake.



In the case $d = 1$, the embedding ψ_n can be represented by labels.

Let $E_n(t)$ be the Dyck path and $V_n(t)$ describe the head of the snake.

If $v(t)$ is the vertex traversed at time t, then $\psi_n(v(t)) = V_n(t)$.

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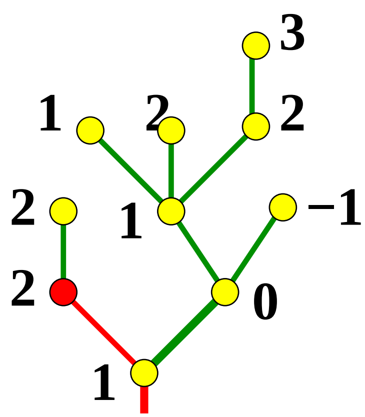

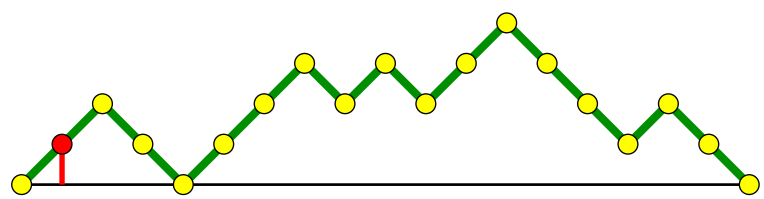

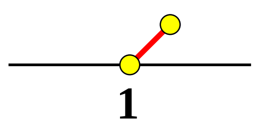

In the case $d = 1$, the embedding ψ_n can be represented by labels.

Let $E_n(t)$ be the Dyck path and $V_n(t)$ describe the head of the snake.

If $v(t)$ is the vertex traversed at time t, then $\psi_n(v(t)) = V_n(t)$.

Embedded trees. Contour walks and convergences

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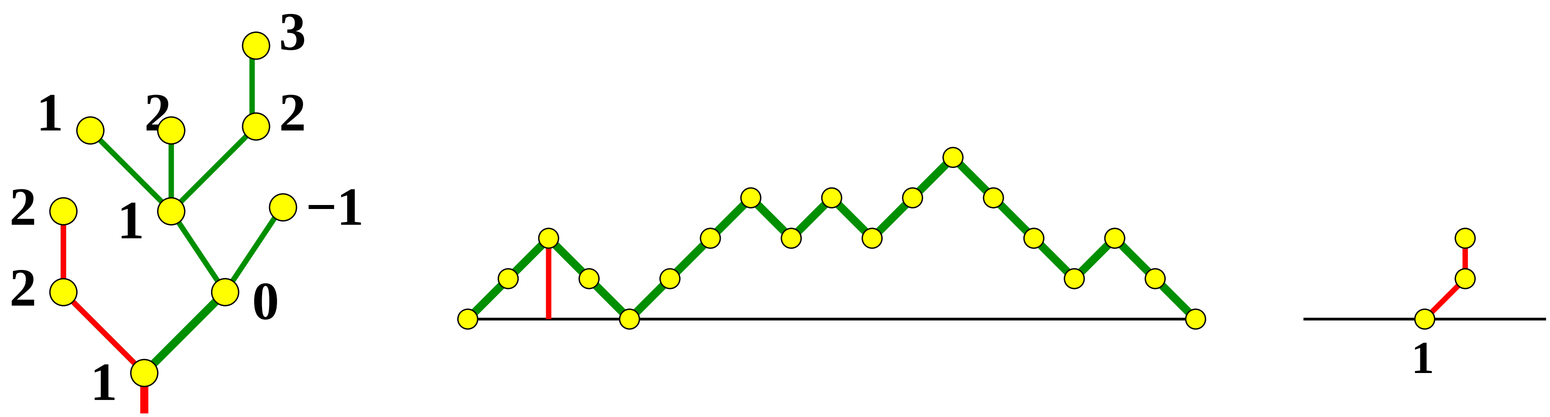

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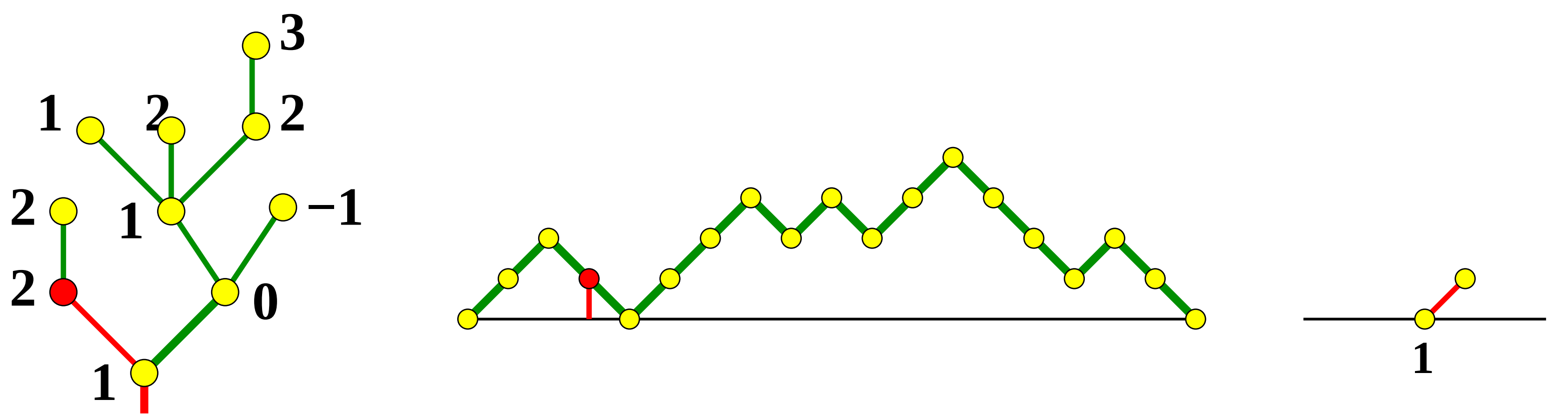

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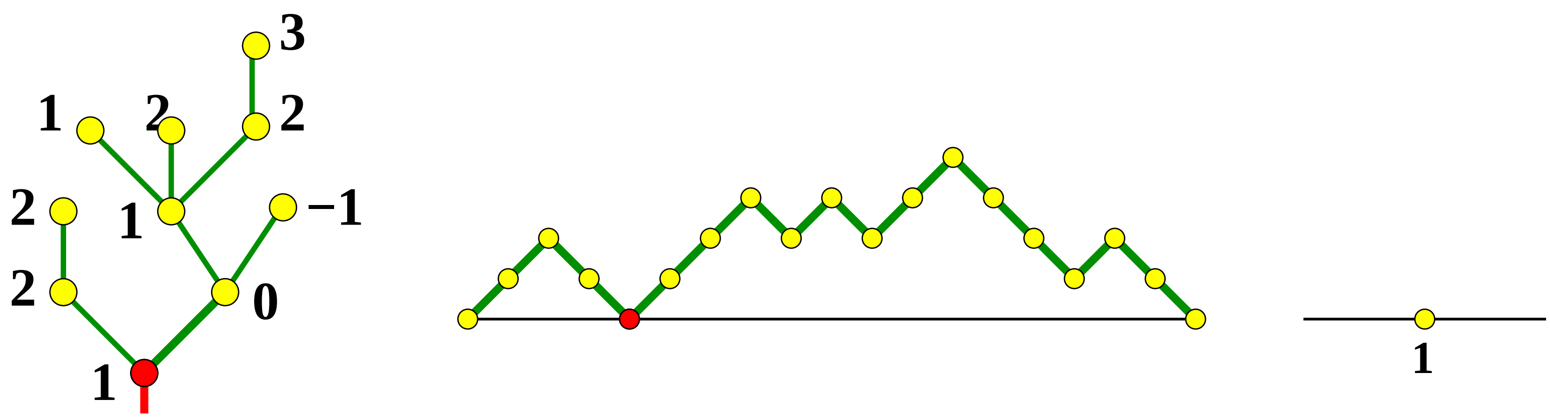

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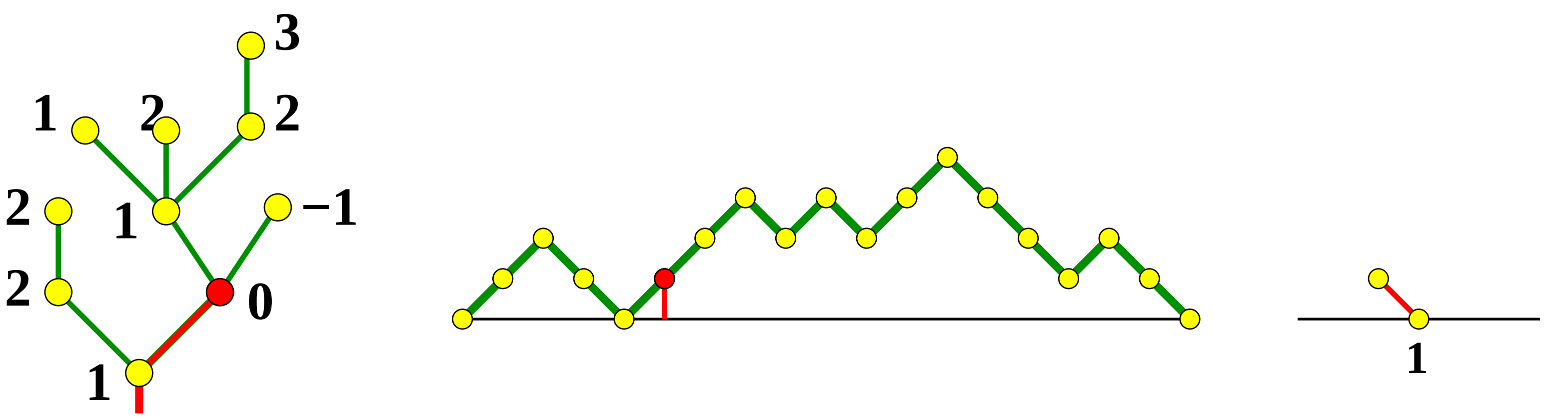

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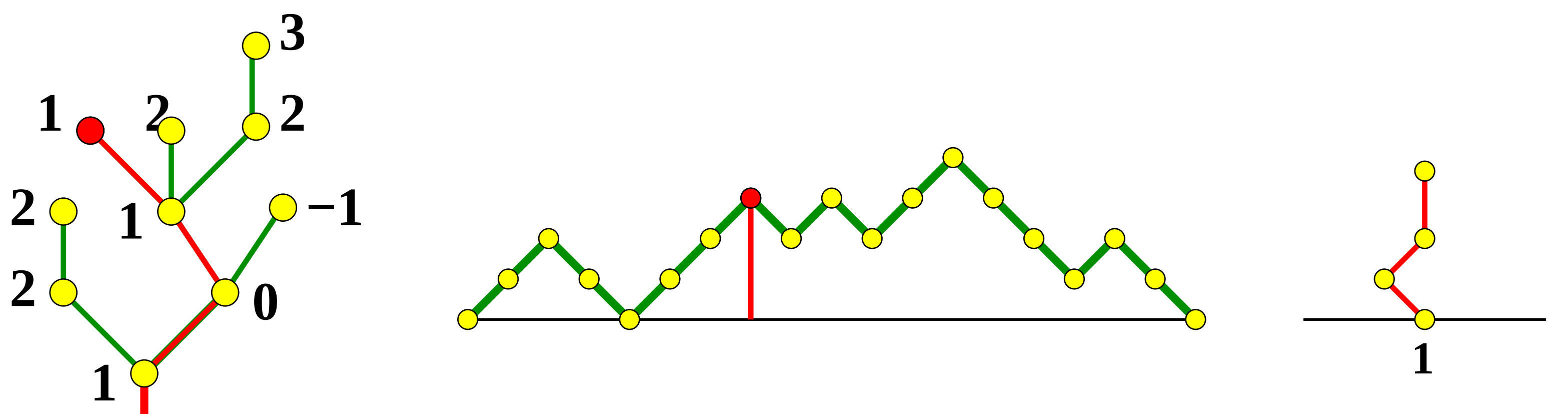

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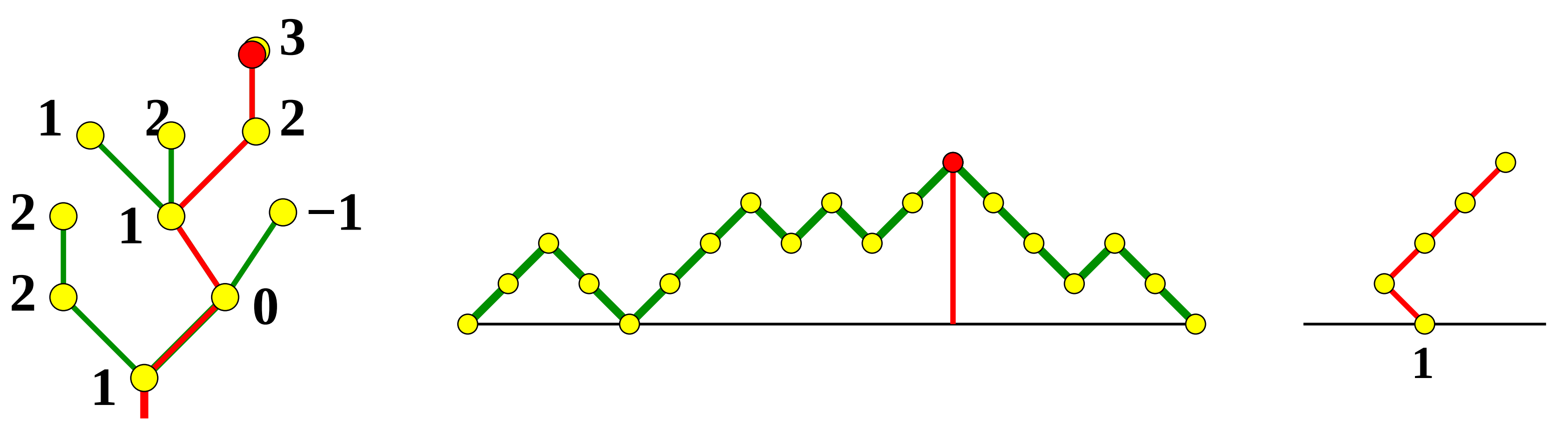

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Embedded trees. Contour walks and convergences

Theorem (Chassaing & S. '02, see also Markert & Mokkadem '02)

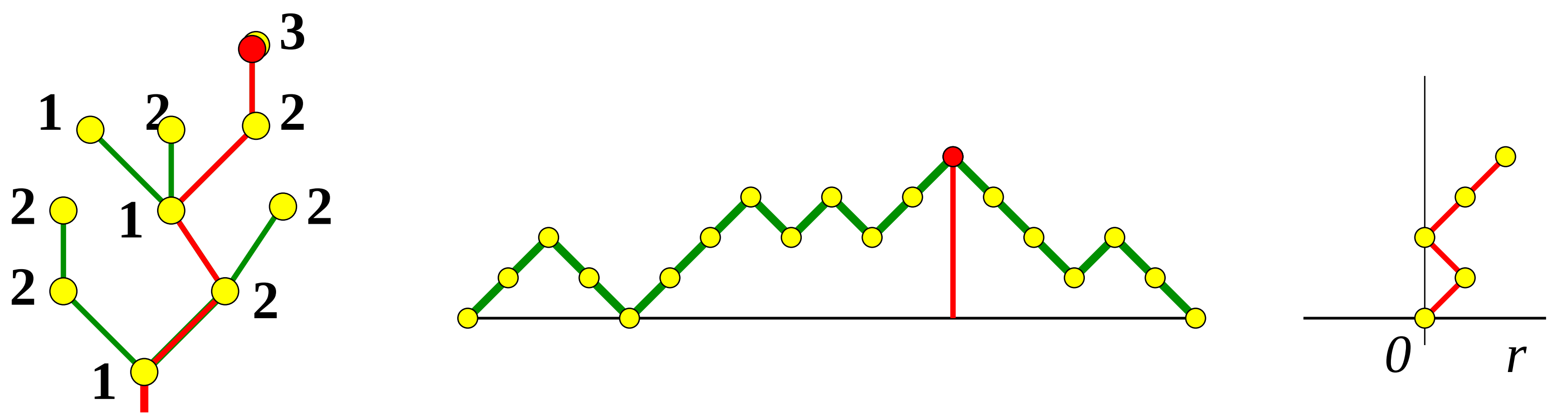
The normalised contour encoding $(\frac{E_n(tn)}{n^{1/2}}, \frac{V_n(tn)}{n^{1/4}})$ of (U_n, ψ_n) weakly converges to $(e(t), \hat{W}_t)$.

Well labelled vs embedded trees

A proof of the radius conjecture for quadrangulations

Well labelled vs embedded trees. Positivity

Well labelled trees = embedded trees in $\mathbb{Z}$, conditioned to positivity.



Well labelled vs embedded trees. Statement

Well labelled trees = embedded trees in $\mathbb{Z}$, conditioned to positivity.

Theorem. Positivity conditioning can be relieved.

> There is a coupling $(T_n, \phi_n) \times (U_n, \psi_n)$ such that the largest label r_n of (T_n, ϕ_n) and the support $[L_n, R_n] \subset \mathbb{Z}$ of (U_n, ψ_n) satify
>
> $$|r_n - (R_n - L_n)| \leq 2.$$

Intuition: a "Vervaat's like" construction for embedded trees, using the *conjugation of trees* discussed in Part I.

The radius theorem

Theorem (Chassaing & S. '02)

The r.v. $n^{-1/4} r_n$ weakly converges to $(8/9)^{1/4} r$,

where $\quad r = R - L \quad$ is the width of ISE.

Furthermore, convergence of all moments holds true.

In particular $\quad \mathbb{E}(r_n) \sim c\, n^{1/4}, \quad$ where $c = (8/9)^{1/4} \mathbb{E}(r)$.

Observe that the numerical value of c is not know...

The radius theorem

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Furthermore, convergence of all moments holds true.

Remark on the proof: we really needed convegence of contour encondings to the Brownian snake, because the width r of ISE is not a continuous functional of the random mesure $\mathcal{J}$.
Instead $r = \max(\hat{W}_t) - \min(\hat{W}_t)$ is a continuous functional of $\hat{W}_t$.

Conclusion

I hope that the connection with Brownian snakes and ISE will lead to precise definition, statement and proof of the following guess:

The Continuum Random Map is a Brownian snake conditioned to positivity.

$$\text{CRM} \quad =?= \quad \text{ISE}^+$$

Many thanks for your attention !