Distances in random surfaces with fixed genus

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Based in part on joint works with P. Chassaing and M. Marcus.

An overview of the talk

A combinatorial model

Graphs, surfaces and maps

Counting maps

Algebraic generating functions

Random maps on surfaces

A discrete model of random geometries

Higher genus maps

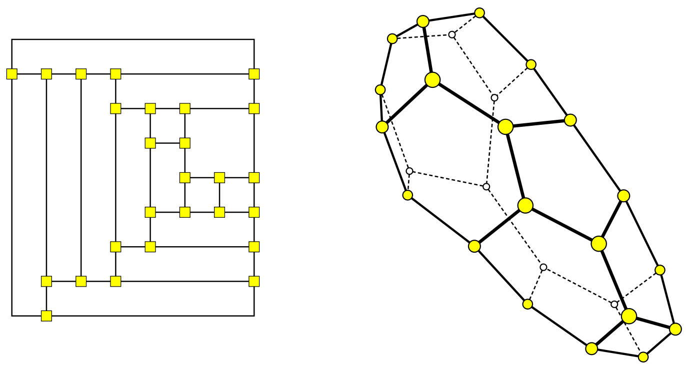
The scheme of a map

What are we talking about

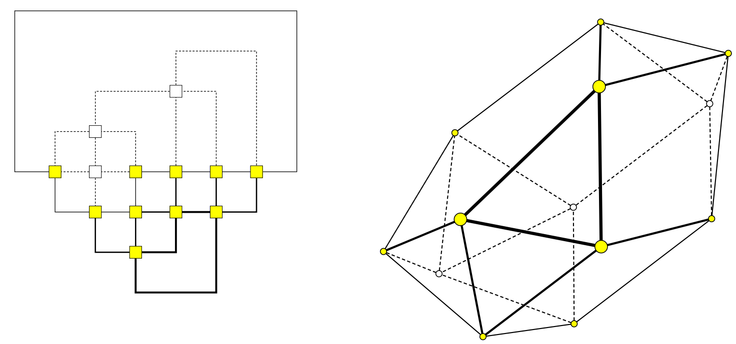
Graphs, surfaces and maps

Discretized surfaces and graphs

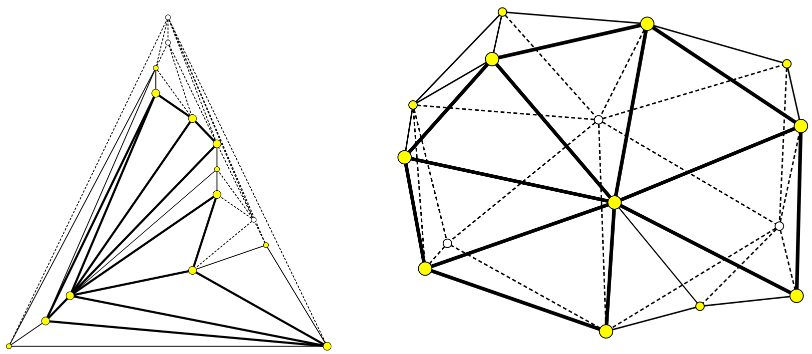
3-regular (cubic, ϕ^3) maps



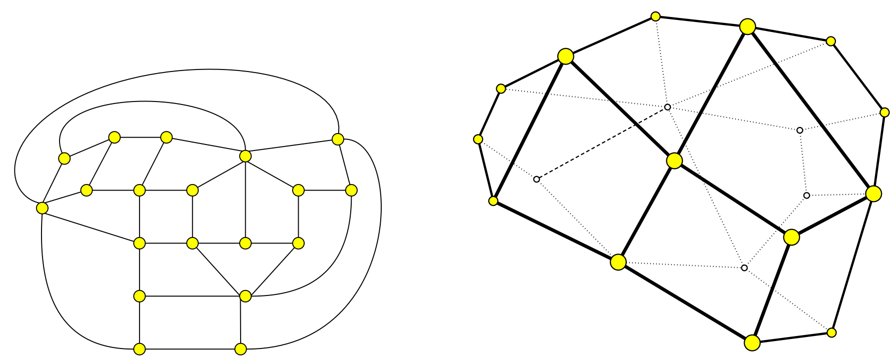
4-regular (ϕ^4) maps



Triangulations

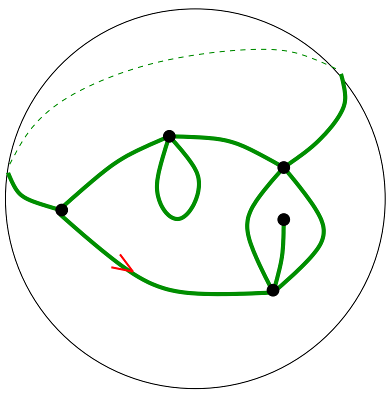


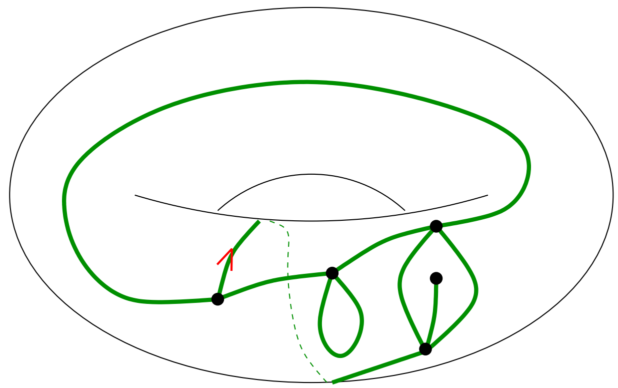
Quadrangulations



Maps and surfaces. Definition

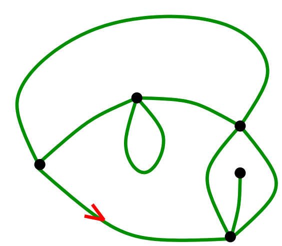
a map is an embedding of a graph in a surface with simply connected faces, considered up to homeomorphisms of the surface.





Rooted map = one edge is distinguished and oriented.

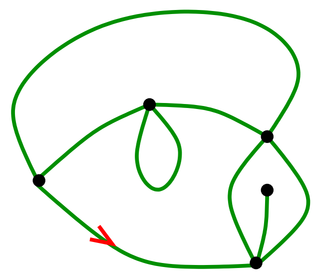
For the sphere, we make planar pictures, taking the infinite face on the right hand side of the root.



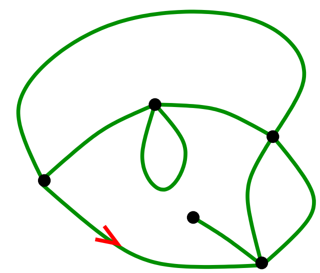
Maps and surfaces. Maps vs graphs

Graph = adjacency matrix.
Distinct maps may share the same underlying graph,

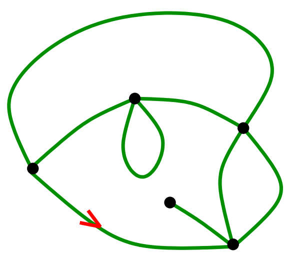
map $\neq$ graph



$\neq$



map = fat graph = graph + cyclic order of edges around vertices.



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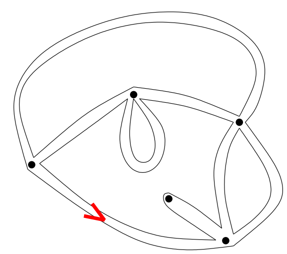
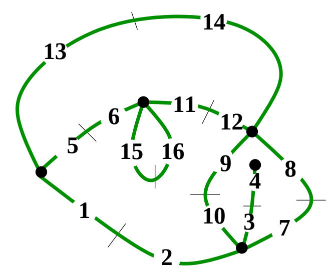




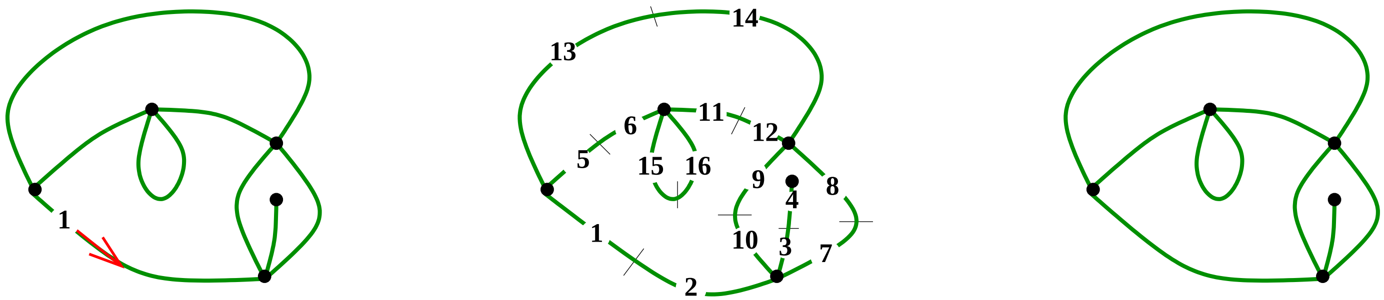
– Upon labelling $\frac{1}{2}$-edges, a map can be encoded by permutations:

$$\sigma_\bullet = (1,5,13)(2,7,3,10)(4)(8,14,12,9)(6,15,16)$$

$$\alpha = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12)(13,14)(15,16)$$

The genus is encoded in here: $c(\sigma_\bullet) + c(\sigma_\bullet\alpha) = c(\alpha) + 2 - 2g$.

Labels and roots. Counting and symmetry factors



$\mathcal{R}_n$: rooted; $\mathcal{L}_n$: labelled; $\mathcal{M}_n$: unrooted.

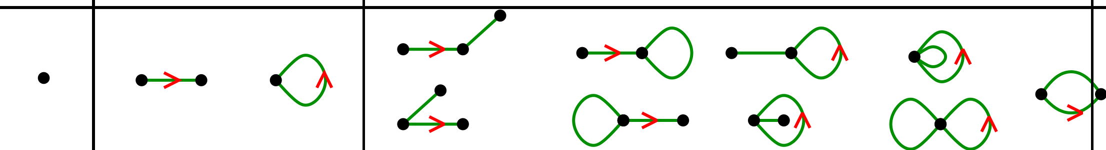
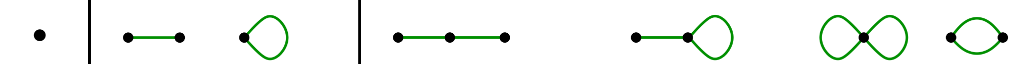
– A *rooted* map with n edges has $(2n-1)!$ distinct $\frac{1}{2}$-edge labellings

– A map M with n edges has $\frac{2n}{\mathrm{Aut}(M)}$ possible roots

In particular: $|\mathcal{R}_n| = \frac{1}{(2n-1)!}|\mathcal{L}_n| = \sum_{M\in\mathcal{M}_n} \frac{2n}{\mathrm{Aut}(M)}$.

$\Rightarrow$ Rooted, labelled or unrooted with symmetry factor are the same.

Counting planar maps.

n	0	1	2	3	4 …
				…	…
$\lvert\mathcal{R}_n\rvert$	1	2	9	54	378 …
				…	…
$\frac{2n}{\text{Aut}}$	$\frac{1}{1}$	$\frac{2}{2}$ $\frac{2}{2}$	$\frac{4}{2}$ $\frac{4}{1}$ $\frac{4}{2}$ $\frac{4}{4}$	…	……

The **generating function** $R(t)$ of rooted maps with edges marked by t is the formal power series:

$$R(t) \;=\; \sum_{R\in\mathcal{R}} t^{|R|} \;=\; 1 + 2t + 9t^2 + 54t^3 + 378t^4 + \dots$$

In physics, $R(t) = 2t\frac{\partial}{\partial t}F^{(0)}(t)$ where $F^{(0)}(t)$ is the free energy of some one matrix model $Z = \int_{H_N} dM e^{NTr(V(M)-tM^2)}$:

$$F^{(0)}(t) \;=\; \sum_{M\in\mathcal{M}} \frac{t^{|M|}}{\text{Aut}(M)} \;=\; \lim_{N\to\infty} \frac{1}{N^2}\log(Z)$$

Counting maps

the enumerative point of view

Combinatorial motivations. Nice exact formulas.

Theorem (Tutte'62)

$$\#\{\text{rooted planar maps, } n \text{ edges}\} = \frac{2 \cdot 3^n (2n)!}{(n+2)!n!} \sim \frac{c_2}{n^{5/2}}\ 12^n$$

$$\#\{\text{rooted triangulations, } 2n \text{ faces}\} = \frac{2^{n+1}(3n)!}{(2n+2)!n!} \sim \frac{c_1}{n^{5/2}}\,(27/2)^n$$

and a few other nice formulas for other families of maps.

– Tutte's proofs are based on recursive decompositions and computations with generating functions.

– These results can be recovered from matrix integrals (BIPZ'78).

But: a simple formula should be explained by a simple proof...

Combinatorial motivations. Algebraic generating functions.

In terms of generating functions:

$$R(t) = \sum_{n\geq 0} |\mathcal{R}_n| t^n = \sum_{M \in \mathcal{R}} t^{|M|} = \frac{-1 + 8t + (1-12t)^{3/2}}{54t^2}.$$

Equivalently R is a root of the polynomial

$$P(r,t) \;=\; 1 - 16t - r + 18tr - 27t^2r^2.$$

Many subclasses of maps have such algebraic generating functions.

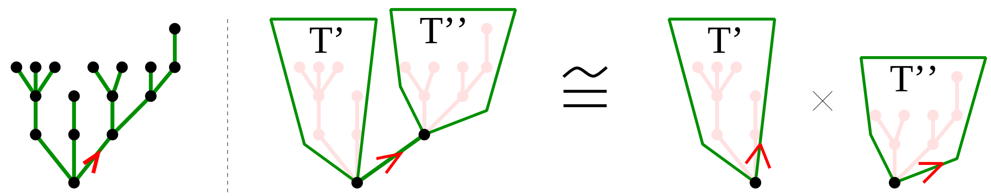
→ *5-connected triangulations.* Gao & Wormald '01.

→ *bipartite cubic with hard particles.* Bouttier et al '03.

Combinatorial motivations. Algebraic generating functions.

Algebraic g.f. are naturally associated to tree-like structures.

Let $\mathcal{T}_n$ denote the set of plane trees with n edges, and $T(t) = \sum_{T \in \mathcal{T}} t^{|T|}$.



Then $T(t) = \displaystyle\sum_{T=(T',T'')} t^{|T'|+|T''|+1} = 1 + t(\sum_{T' \in \mathcal{T}} t^{|T'|})^2$

so that $T(t) = 1 + t(1 + T(t))^2$, T is algebraic.

Any algebraic generating function should reflect such an algebraic or *context-free* structure.

Combinatorial motivations. Algebraic generating functions.

Let $\mathcal{R}_n^{(g)}$ be the set of rooted maps on a surface of genus g.

Theorem (Bender et al. 92). There exists a rational function $r^{(g)}(a,t)$ such that

$$R^{(g)}(t) = r^{(g)}(A(t), t)$$

where is the unique formal power series s.t. $A = 3t(1+A)^2$.
Moreover

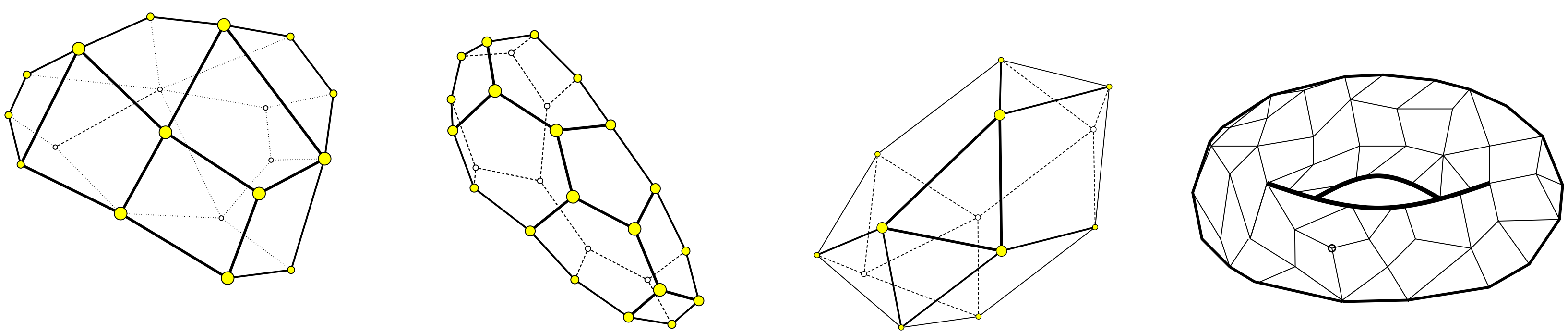
$$|R_n^{(g)}| \sim C_g \cdot n^{\frac{5}{2}(g-1)} 12^n,$$

where C_g is a constant depending only on g.

This result was independantly obtained in physics via perturbative expansion of matrix integrals: $\log Z = N^2 R^{(0)} + R^{(1)} + \frac{1}{N^2} R^{(2)} + \dots$

Random maps

as a discrete model of random geometries



Uniform random maps. Definition.

Let $\mathcal{R}_n$ be a family of rooted maps

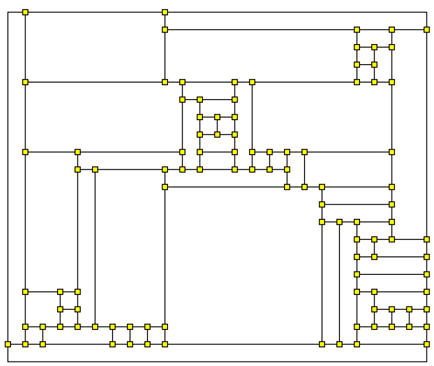
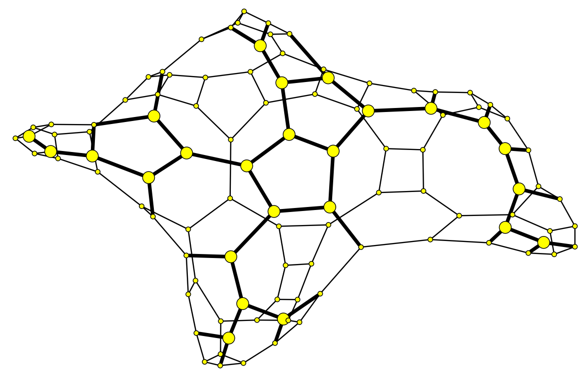
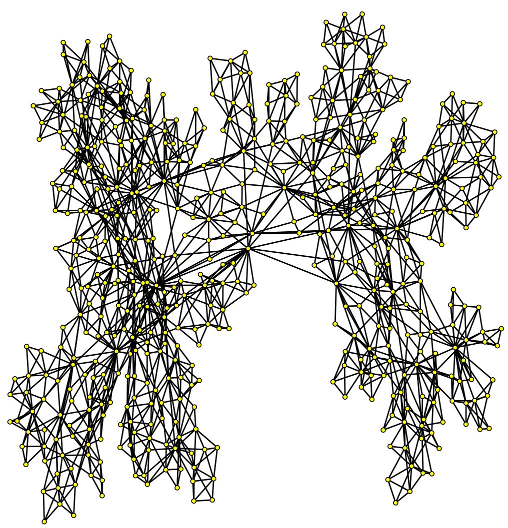
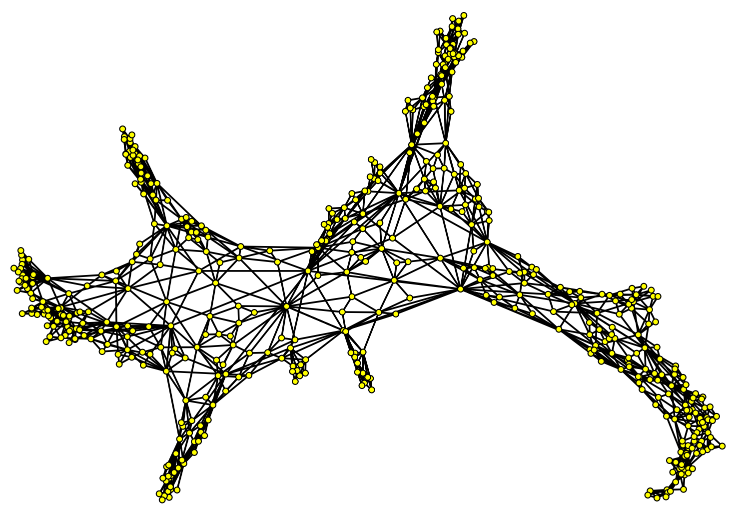
say, for instance $R_n = \{$all planar maps with n faces$\}$.

or $R_n = \{$planar quadrangulations with n faces$\}$.

Consider a r.v. X_n with uniform distribution on $\mathcal{R}_n$:

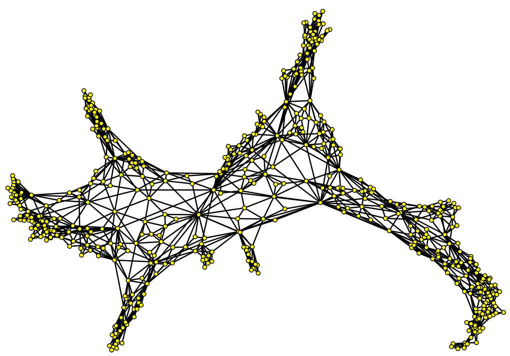
$$\Pr(X_n = R) \;=\; \frac{1}{|\mathcal{R}_n|}, \qquad \text{for all } R \in \mathcal{R}_n.$$

A gallery of random maps

How is the intrinsic geometry of these random surfaces ?

Random maps appear to be quite different from regular lattices.



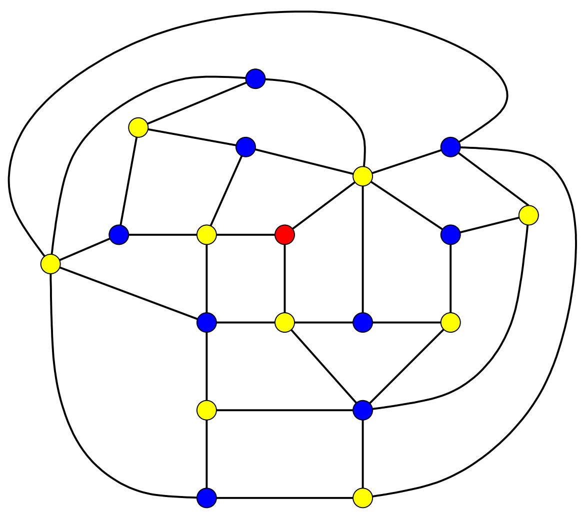
– a fat tree structure ?
– branchings into baby universes ?
– Hausdorf dimension ?
– short separators ?

Several of these questions have been answered in the physics and combinatorial literature (not all though).

Let us concentrate on distances in random map.

Profile and radius of a map with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of a random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

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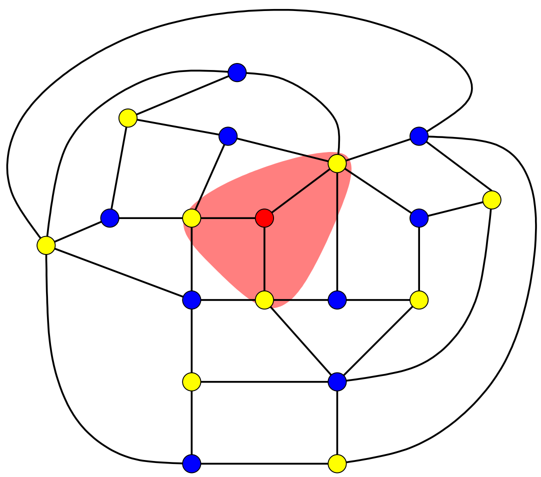

$X_n^{(1)} = 3$

- r_n is the radius (maximal distance from the red vertex)

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Profile and radius of a map with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of a random vertex
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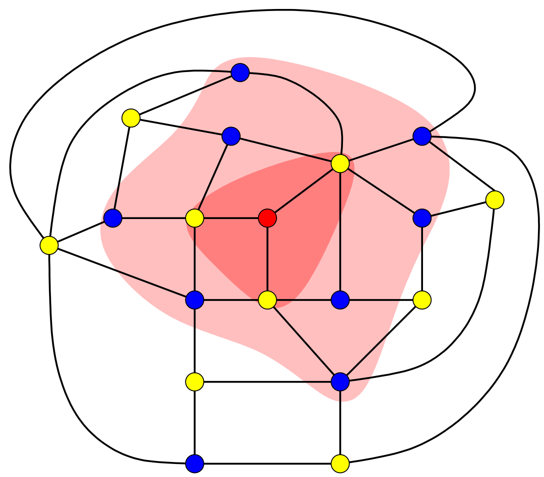


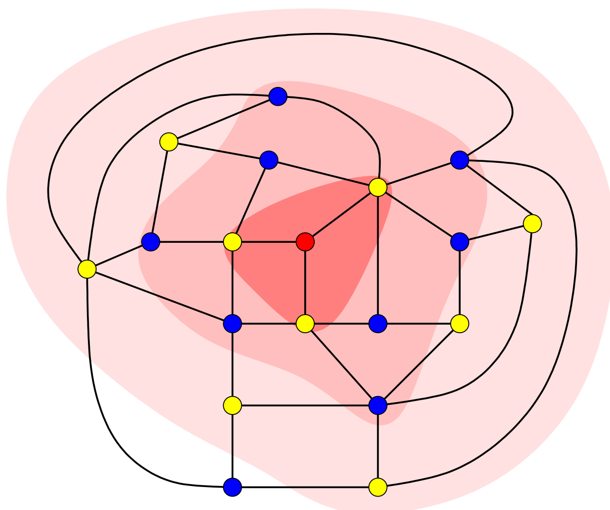
$X_n^{(1)} = 3$
$X_n^{(2)} = 8$

- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius of a map with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of a random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



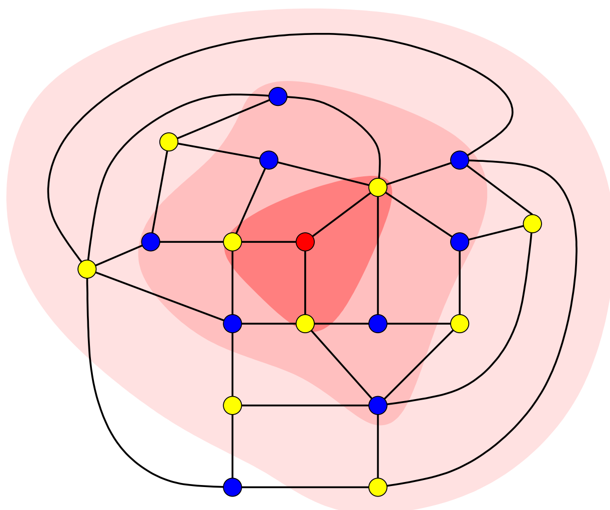
$X_n^{(1)} = 3$
$X_n^{(2)} = 8$
$X_n^{(3)} = 6$

- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius of a map with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of a random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



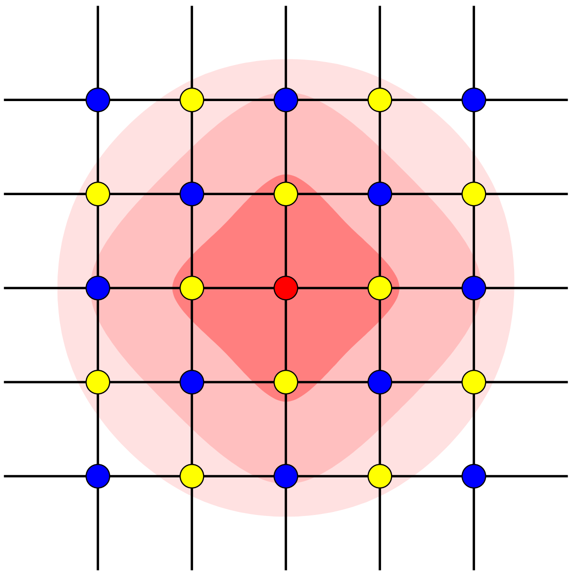
$X_n^{(1)} = 3$
$X_n^{(2)} = 8$
$X_n^{(3)} = 6$
$X_n^{(4)} = 1$
$r_n = 4.$

- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius. On the grid ?

On a grid with n faces ($\sqrt{n} \times \sqrt{n}$), the behaviour is clear:



In particular,

$$X_n^{(k)} = \Theta(k) \text{ for } k < n^{1/2}, \quad \text{and } r_n \text{ grows like } n^{1/2}.$$

How do these parameters behave on random maps ?

Profile and radius. Known results in the planar case

For random planar triangulations:

- Watabiki, Ambjørn *et al.* 1994. *The Hausdorff dimension is* 4

$$\text{meaning} \quad \text{for } k \ll n^{1/4}, \qquad \mathbb{E}\left(\int_0^k X_n^{(i)}\right) \sim k^4,$$
$$\text{for } k \gg n^{1/4}, \qquad \mathbb{E}(X_n^{(k)}) \text{ is exp. decreasing}$$

  They have **proved** that the only possible scaling is $k = tn^{1/4}$ and computed some functional transform of the limit profile.

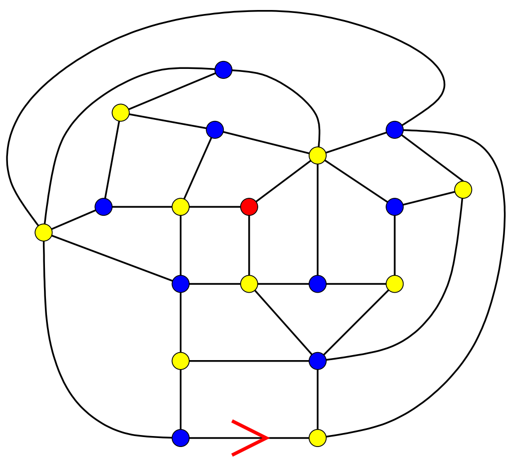
From a purely mathematical point of view the computation is not completely satisfying. Besides it does not allow for exact computation in the discrete domain.

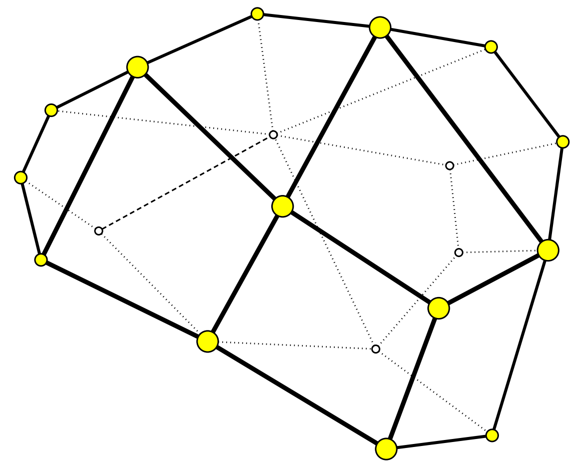
We describe another approach using an encoding of maps by trees.

Distances and trees

Quadrangulations.

We restrict our attention to quadrangulations (dual 4-regular maps).



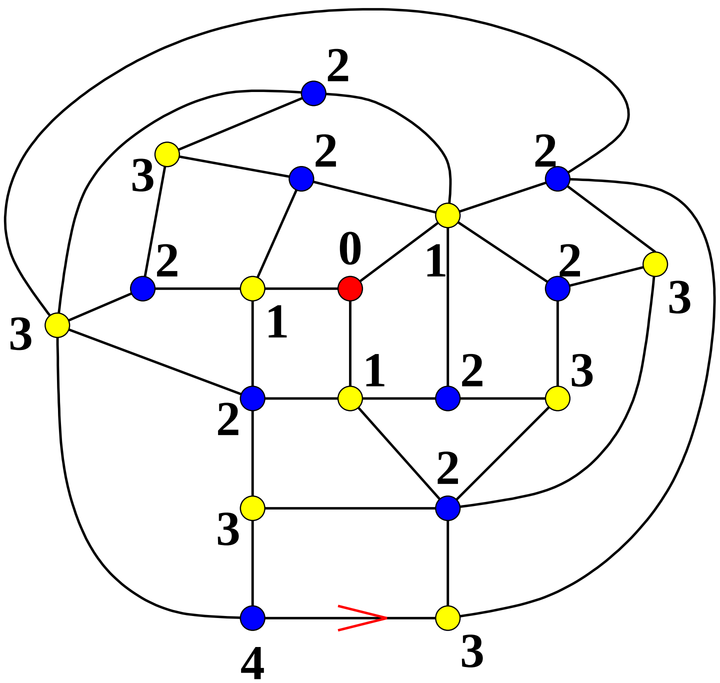


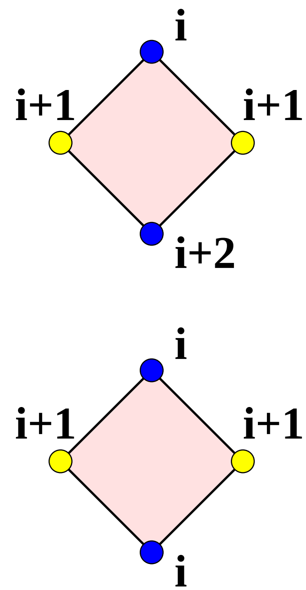
In the planar case, these maps are bipartite (blue/red).

Given a red vertex we label other vertices by their distance to this base point.

A bijection. Distances.

Let us label vertices by distances to the red vertex.

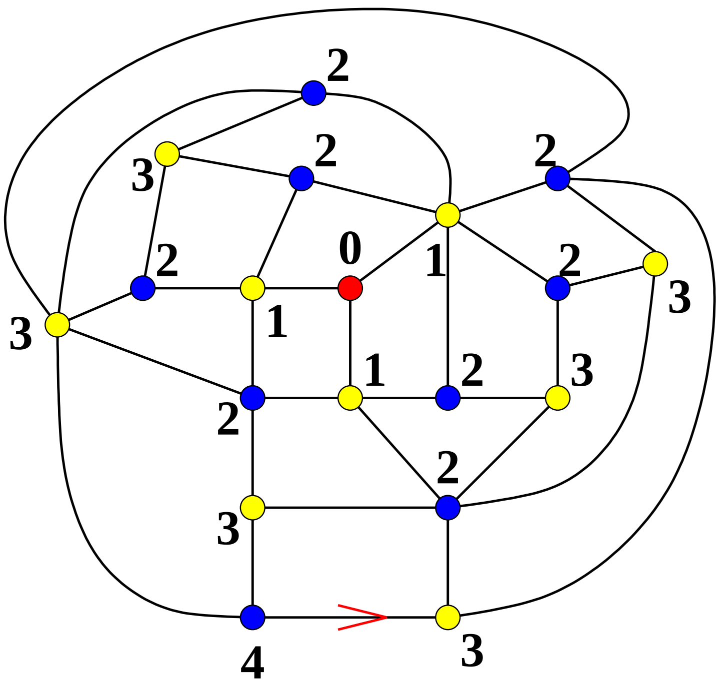


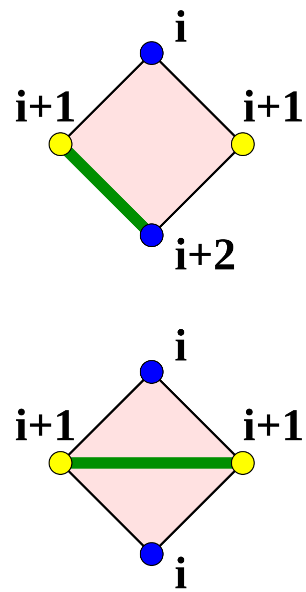


There are only two possible configurations around a face (bipartiteness).

A bijection. Local rules

Consider the following two local rules:





Apply these rules to all faces.

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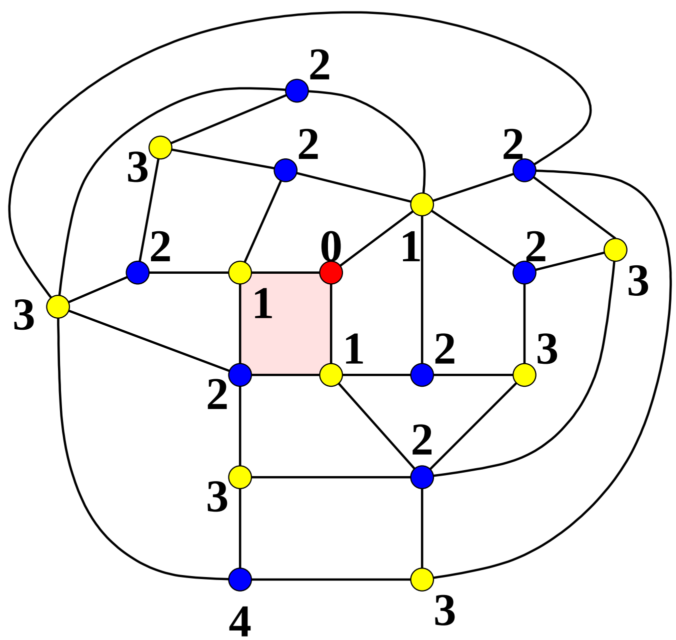

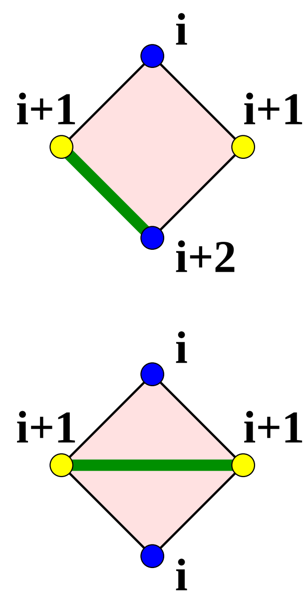

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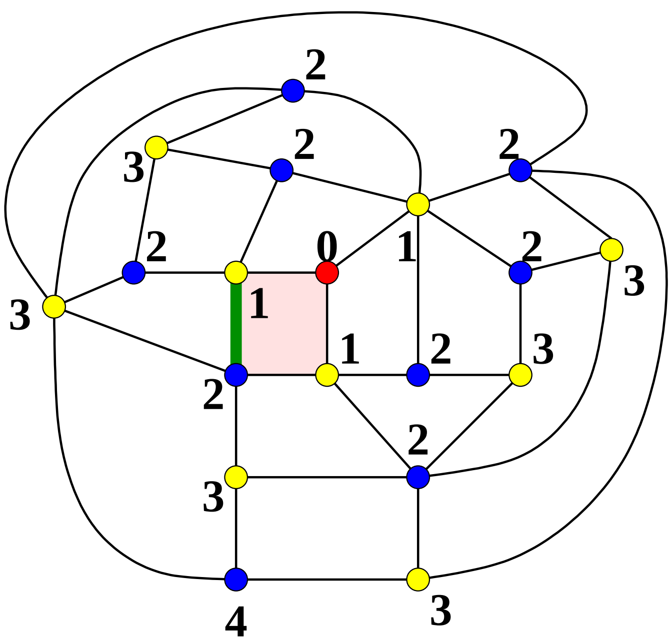

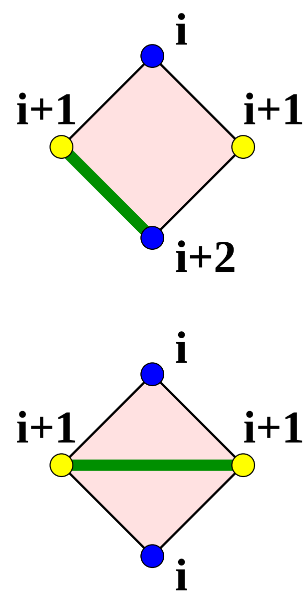

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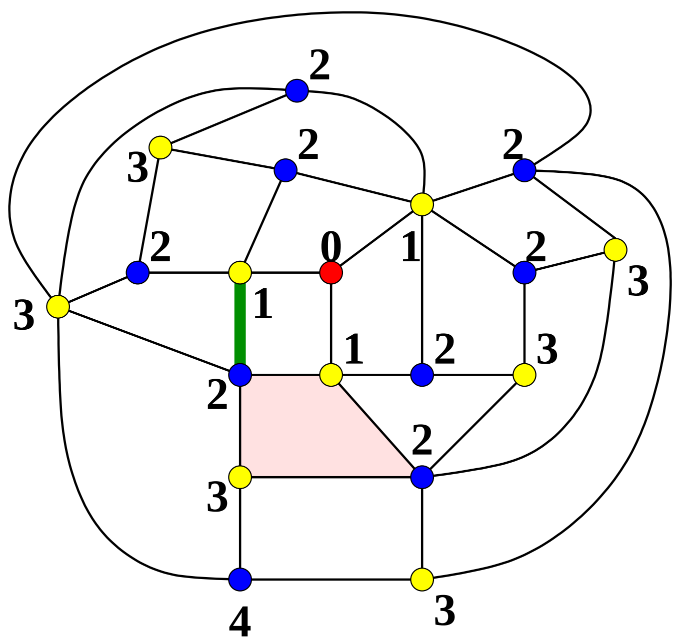

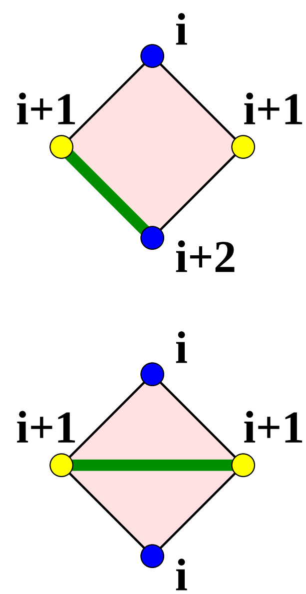

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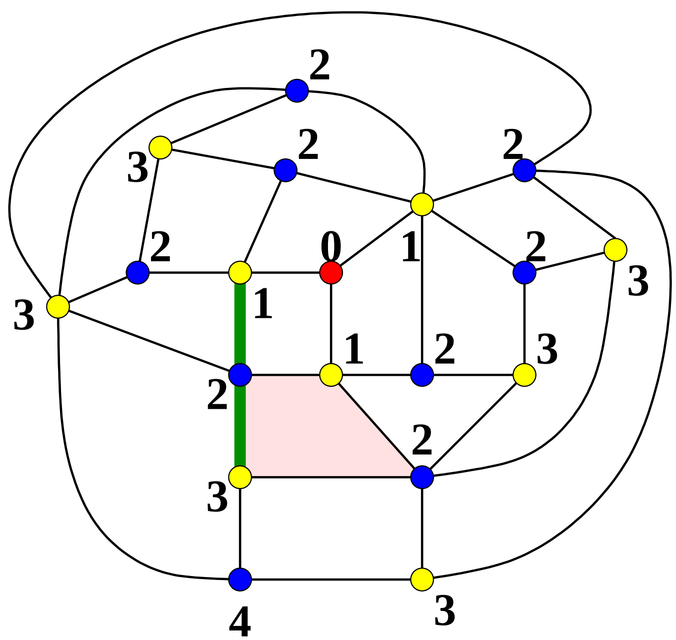

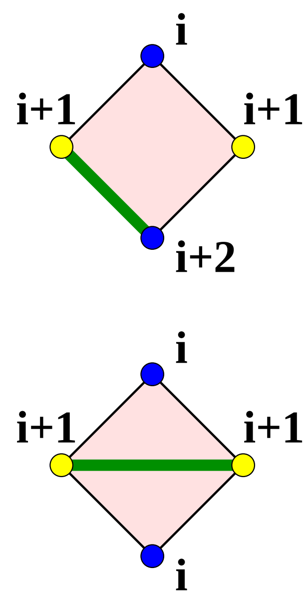

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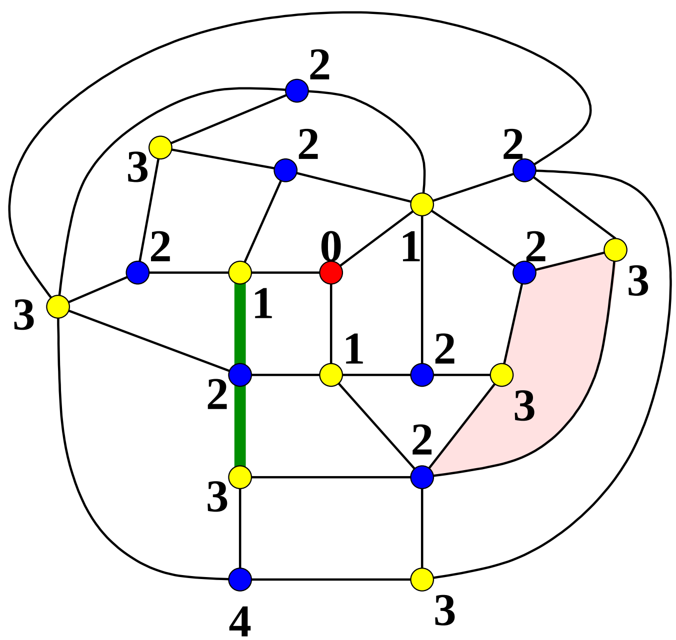

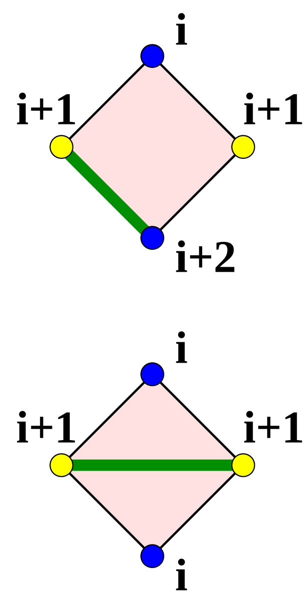

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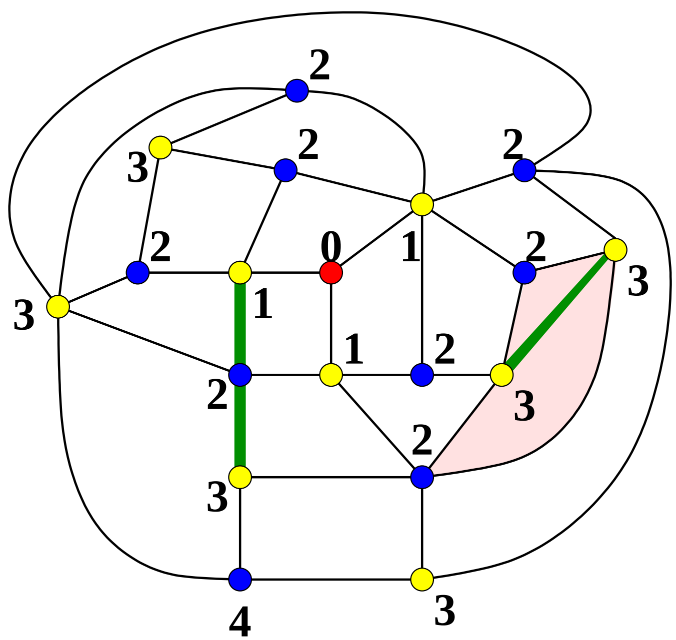

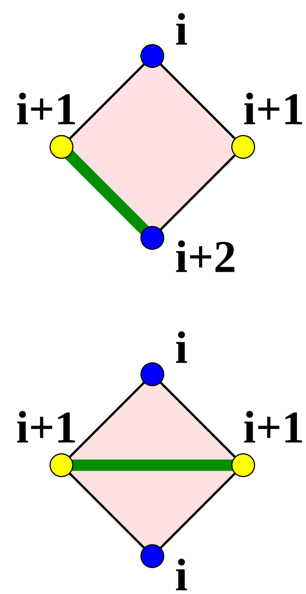

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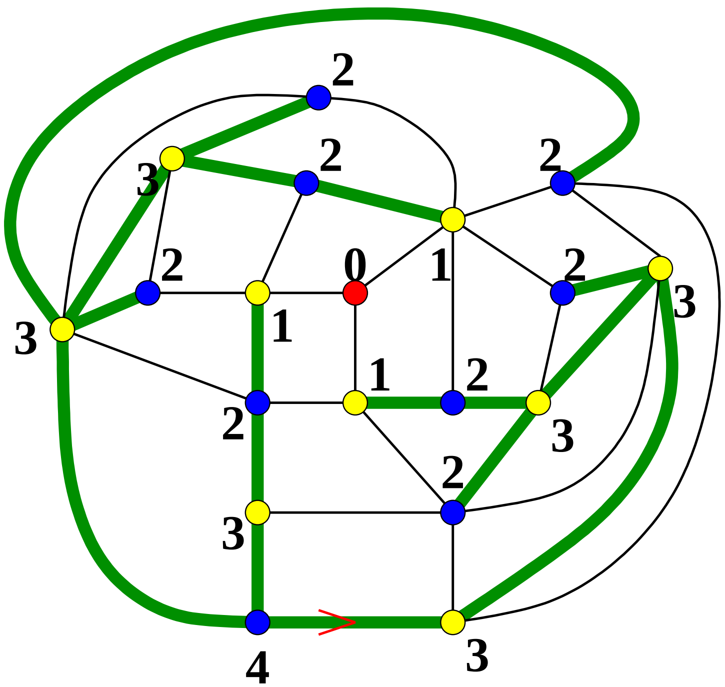

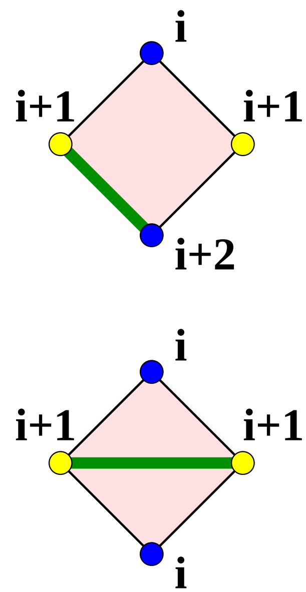

Apply these rules to all faces.

A bijection. Local rules

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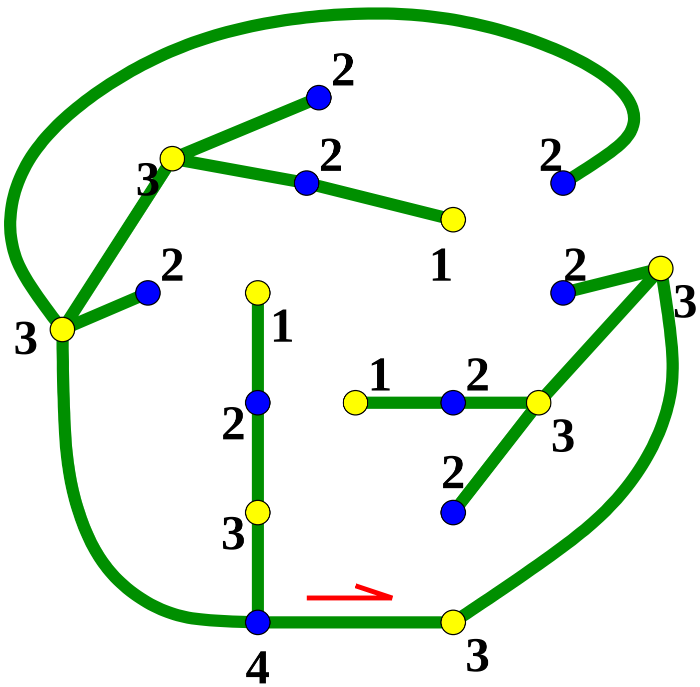

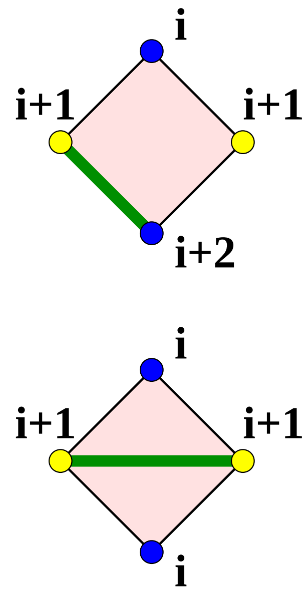

Proposition: the edges produced by local rules form a tree.

A bijection. Local rules

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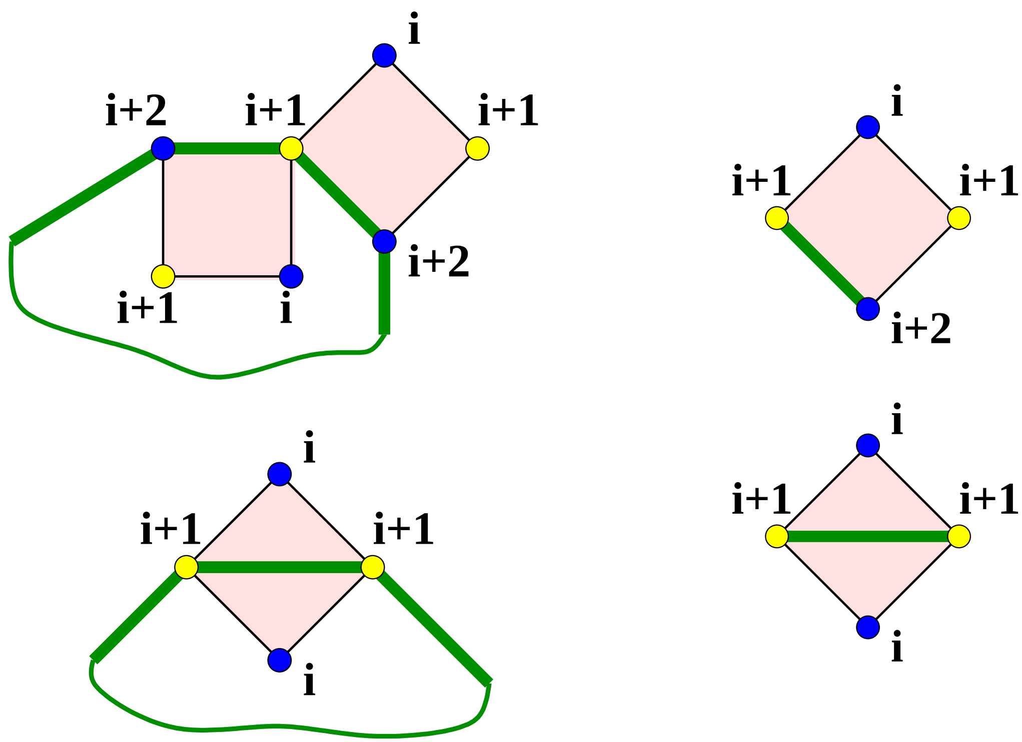
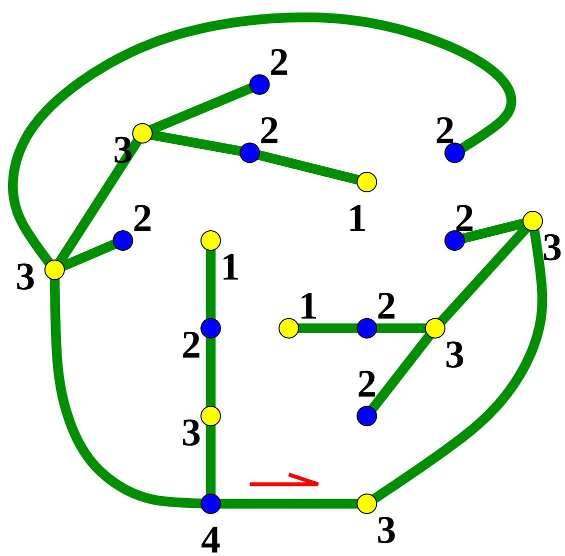
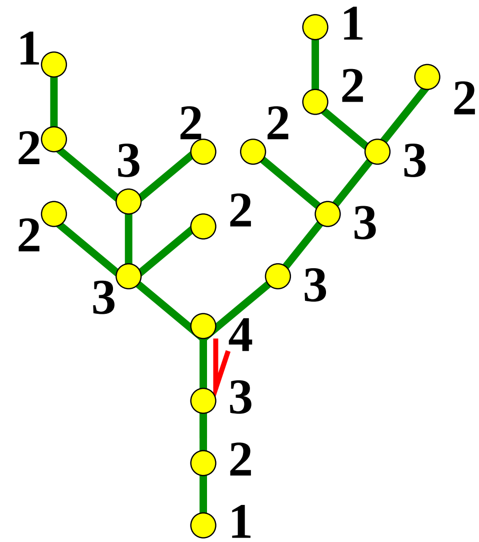


The root can be only in one of the two regions delimited by a cycle. Taking $i+1$ minimal on the cycle, a contradiction is obtained between rules and labelling by distance.

A bijection. back from the tree.

By construction, labels in the tree differ at most by one along edges, and the minimum label is 1.

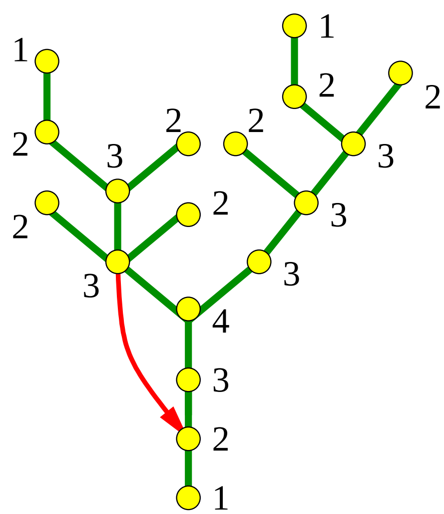




Proposition. The quadrangulation can be recovered from the labelled tree

A bijection. back from the tree.

Starting from a labelled tree:



Since all faces must be of the form $(i, i-1, i, i+1)$ or $(i, i-1, i)$, missing edges are recovered by a greedy $(i \to i-1)$ matching around the tree.

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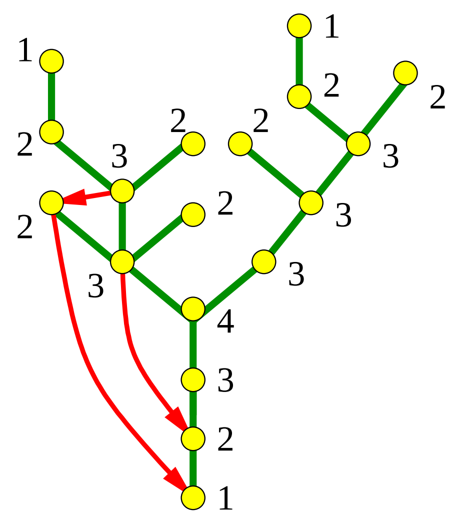

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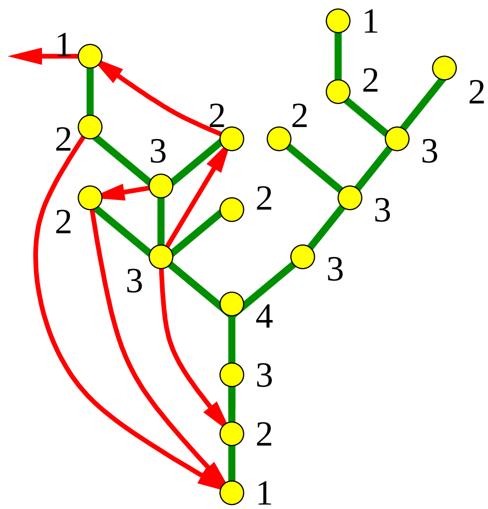

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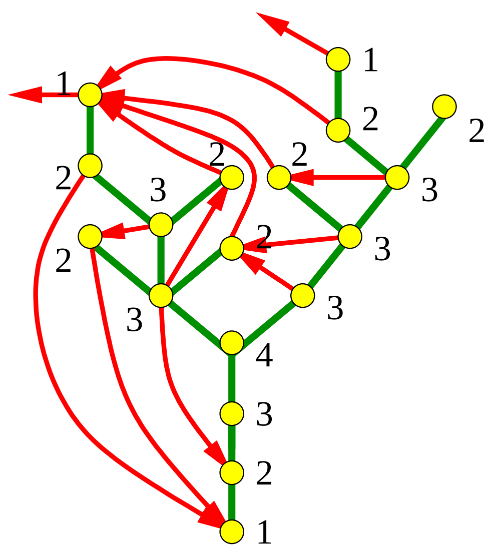

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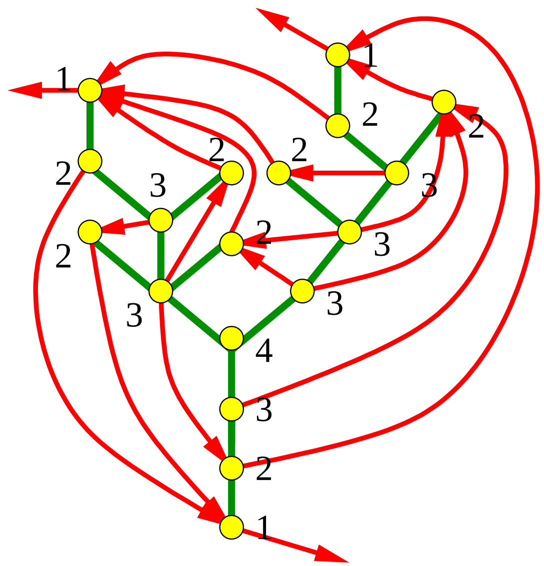

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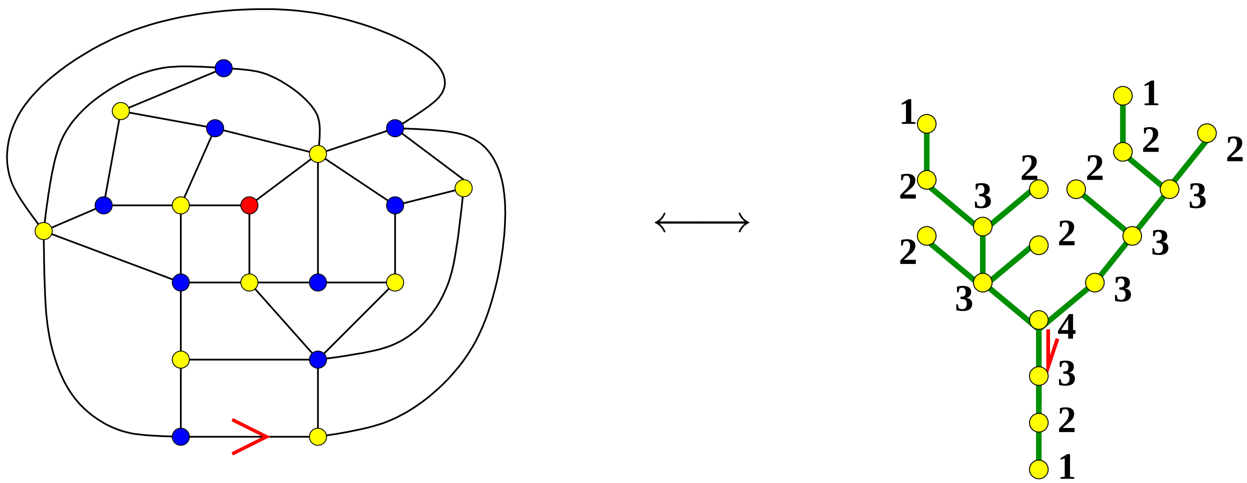
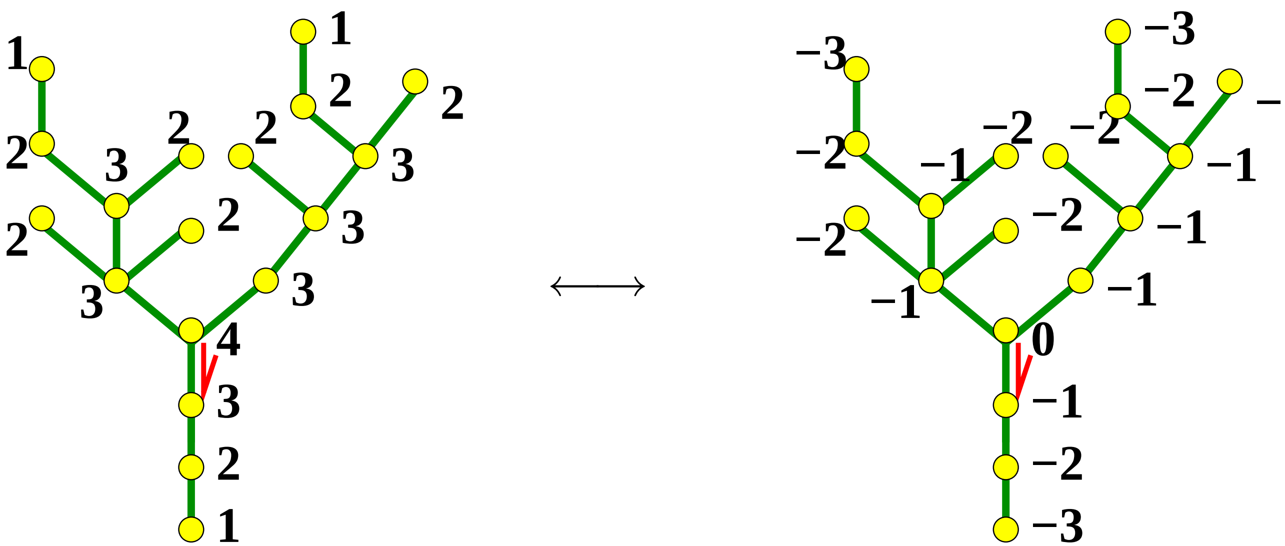
A bijection. Conclusion.



Theorem. The local rule is a one-to-one correspondence between
– rooted pointed quadrangulations with n faces, and
– “twice” labelled trees with n edges and minimum label 1.

A bijection. Labelled and embedded trees.

Since the minimal label must be 1, it is sufficient to record the variation $\{+1, 0, -1\}$.

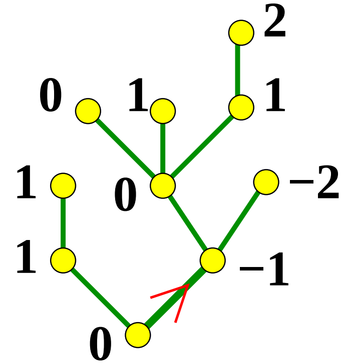


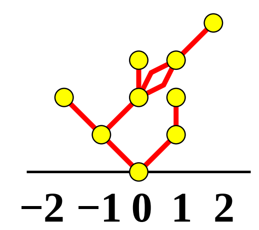
An **embedded tree** is a rooted tree with vertices embedded in $\mathbb{Z}$: the root is set at 0 and each edge is mapped into $\{-1, 0, +1\}$.

A bijection. Labelled and embedded trees.

Embedded trees have a context-free decomposition: $A = 3t(1+A)^2$.
Equivalently, the number of embedded trees with n edges is

$$3^n \cdot \frac{1}{n+1}\binom{2n}{n}$$





Corollary. The local rule is one-to-one between

– rooted pointed quadrangulations with n faces, and

– "twice" embedded trees with n edges.

The number of these objects is thus $2 \cdot 3^n \cdot \frac{1}{n+1}\binom{2n}{n}$.

Distances and embedded trees.

According to the previous bijection:

Uniform distribution on quadrangulations with n faces

$$=$$

Uniform distribution on embedded trees with n edges

Moreover the distribution of labels exactly encodes the profile!

In particular, the radius $r_n = \max(k \mid X_n^{(k)} > 0)$ is the difference between the min and max labels of a random embedded tree.

These are *identities in law*, not just asymptotic results.

Distances and random embedded trees. Typical labels.

Proposition. Branches of a uniform random plane tree with n edges have typically lenght $\Theta(\sqrt{n})$

Proposition. The labels along a branch form a random walk with uniform increments in $\{-1, 0, +1\}$.

$\Rightarrow$ labels on a length ℓ branch are $\Theta(\sqrt{\ell})$

Hence typical labels are of order $\Theta(n^{1/4})$, and a typical label is expected to be shared by $\Theta(n^{3/4})$ vertices.

In terms of random quadrangulations this means that the typical distance between two random vertices is $O(n^{1/4})$.

Profile and radius. Results

Much more precise results follow from the study of embedded trees.

For instance:

Theorem (Chassaing-S. 2002). The correct scaling is $k = tn^{1/4}$, and

– $n^{-3/4} X_n^{(tn^{1/4})} \xrightarrow{\text{law}} X(t)$, a process supported on $\mathbb{R}^+$,

– the radius satisfies $\mathbb{E}(r_n) \underset{n\to\infty}{\sim} cte \cdot n^{1/4}$.

The process underlying $X(t)$ is the Integrated Superbrownian Excusion, introduced by Aldous to describe the continuum limit of embedded trees.

Our theorem is based on a description of the ISE in terms of Brownian snakes due to Le Gall.

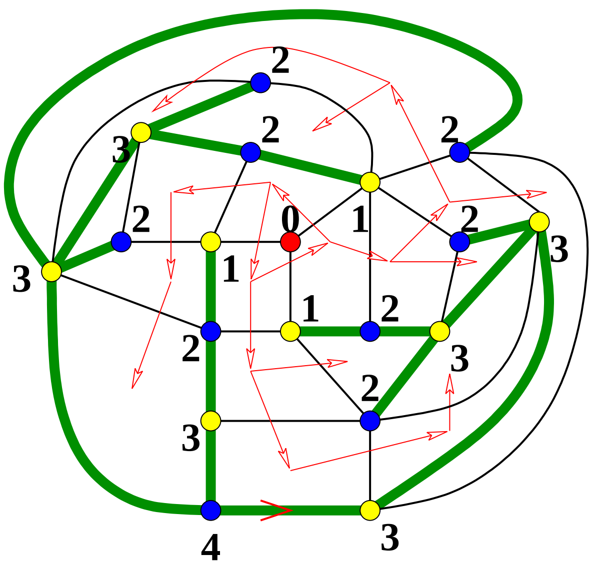
Our correspondence between quadrangulations and embedded trees has lead to many further results, see Marckert-Mokkadem'04, Bouttier et al.'04 '05 '06, Bousquet-Melou'05, Marckert-Miermont'05.

Maps on surfaces

The scheme of a map

Local rules for higher genus?

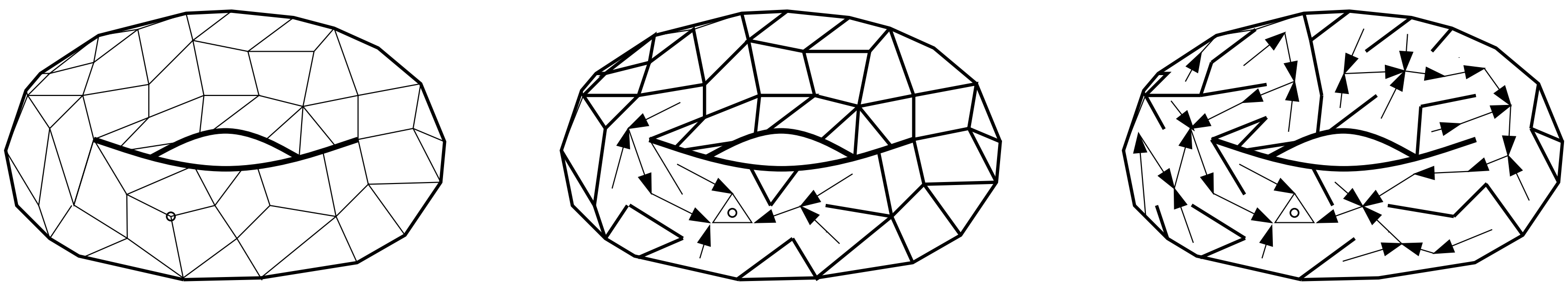
The fact that local rules produce a green tree in the plane is equivalent to saying that the dual red edges essentially form a tree:



In genus $g > 0$, local rules form cycles (the argument was based on planarity). However it remains true that red edges do not form cycles.

What happens on a surface?

During the growth of a dual tree on a surface of genus g, all faces are slowly merged into one big face.

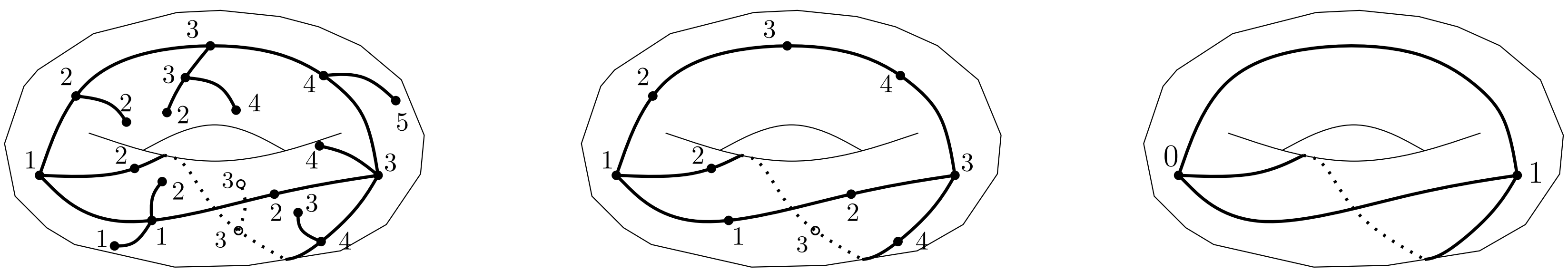


The final result is not a tree but a map with one face.

Theorem (Marcus-S. 05) The local rule yields a bijection between:
– rooted pointed quadrangulations with n faces and genus g,
– and embedded one face maps with n edges and genus g.

What happens on a surface?

Embedded one face maps can be decomposed as follows:
– remove recursively vertices of degree 1,
– replace chains by superedges and normalize labels.
The resulting **schemes** are one face maps with vertices of degree ≥ 3.



Proposition. The number of schemes of genus g is finite.

Proposition. The generating function of embedded one face maps having a given scheme is a simple rational function of the generating function of embedded trees.

What happens on a surface? Asymptotic enumeration

– The dominant schemes of genus g (those producing most maps) are made of $4g-2$ vertices of degree 3 and $6g-3$ edges.

– An embedded map of genus $g+1$ can be produced from an embedded map of genus g by gluying 3 points with the same label: this creates generically 4 new vertices in the dominant scheme.

– There are $\Theta(n^{1/4})$ labels and $\Theta(n^{3/4})$ points share a given label $\Rightarrow$ there are $\Theta(n^{1/4}(n^{3/4})^3) = \Theta(n^{5/2})$ ways to increment the genus.

$$\#\{\text{quad. } n \text{ faces, genus } g\} = n^{\frac{5}{2}g} \cdot \#\{\text{planar quad. } n \text{ faces}\}.$$

What happens on a surface. Asymptotic results

We have hence rederived combinatorially the following result:

Theorem (Bender-Canfield'94 / also independantly in physics)
The family of quadrangulations on surfaces satisfies

$$\#\{\text{quadrangulations of genus } g \text{ with } n \text{ faces}\} \;=\; c_g n^{\frac{5}{2}(g-1)} \rho^n,$$

where $\rho = 12$.

The form $n^{\frac{5}{2}(g-1)}\rho^n$ of the asymptotic formula is typical of families of maps: the constant ρ depends on the family, but the polynomial correction is "always" driven by the same "universal critical exponent of pure 2d quantum gravity" $\frac{5}{2}(1-g)$.

Distances in quadrangulations of genus g.

This approach also yields the proof that distances remain of order $\Theta(n^{1/4})$ in quadrangulations with n faces on higher genus surfaces.

Indeed consider embedded maps constructed out of embedded trees by the previous generic construction:

– take a random embedded tree with n edges;

– take $3g$ random vertices and glue them 3 by 3 in a random way;

– check that the resulting map has coherent labels, that it has genus g, and that the gluing order is canonical.

This process gives equal chances to each embedded maps of genus g and has probability $\Theta(n^{-\frac{5}{2}g})$ to succeed.

Therefore an exponentially rare event on embedded trees (like $r_n \gg n^{\frac{1}{4}+\varepsilon}$) has to be exponentially rare on embedded maps of genus g.

Conclusion.

We have described distances in maps using embedded trees.

Is it possible to “see the distance” in the matrix integral framework?

Many questions other remain open about the geometry of random (planar or not) maps. In particular:

- Is it possible to separate a map of size n in 2 roughly equal parts with a cycle of length $\ll n^{1/4}$?

This would help us to understand whether maps have cut points in the continuum limit, or if there is a chance that map keep their topology in this limit.