The combinatorics of Hurwitz numbers and increasing quadrangulations

Gilles Schaeffer   CNRS & École Polytechnique

Newton Institute, Cambridge, July 2018

Based in part on joined work with M. Bousquet-Mélou, E. Duchi and D. Poulalhon

Plan of the talk

Unlabeled VS Increasing quadrangulations...

Why increasing quadrangulations?

Hurwitz numbers and branched covers

A bijection, with Cayley type trees!

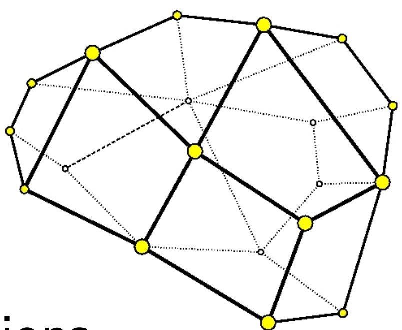
More evidences from higher genus maps...

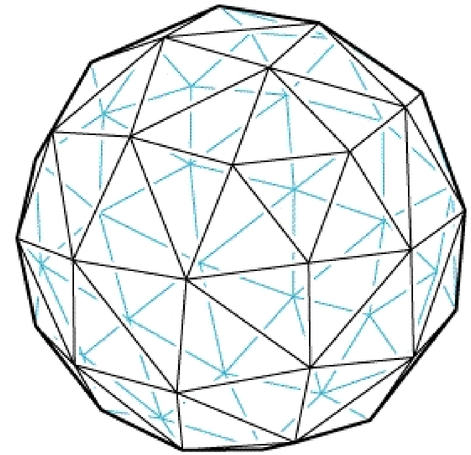
as a conclusion

Planar maps

Planar maps are graphs embedded on the sphere

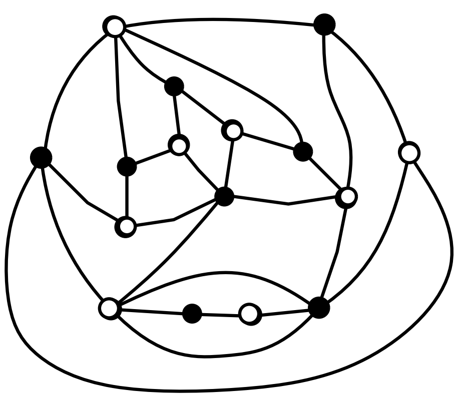
and considered up to homeomorphisms of the sphere





A triangulation

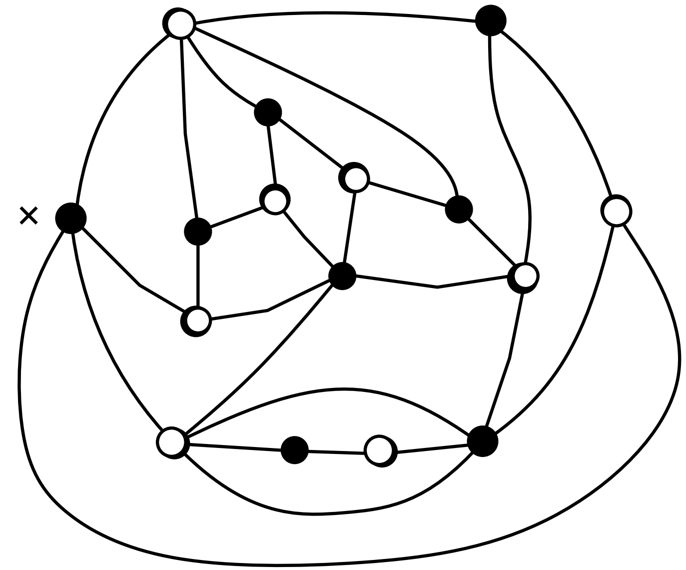
Two quadrangulations



Quadrangulations and their number

A rooted planar quadrangulation of size n is a rooted planar map with:

- n faces with degree 4
- $n+2$ vertices (that are bicolored, with black root)
- $2n$ edges (multiple edges allowed)



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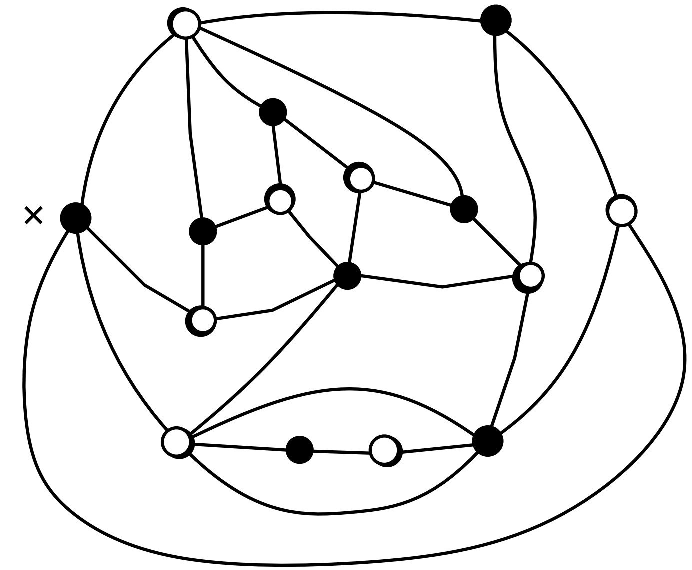

Theorem (Tutte, 1963): Let $\mathcal{Q} = \{$rooted planar quadrangulations $\}$

and let $Q(t) = 1 + \sum_{q\in\mathcal{Q}} t^{|q|}$ be the generating function where $|q| = \#$faces of q.

$$Q(t) = 1 + 2t + 9t^2 + \dots$$

Then $Q(t)$ is the unique formal power series solution of the system

$$\begin{cases} Q(t) = R(t) - tR(t)^3 \\ R(t) = 1 + 3tR(t)^2 \end{cases}$$

so that $Q(t) = \frac{(1-12t)^{3/2}-1+18t}{54t^2}$ and $|\mathcal{Q}_n| = \frac{2}{n+2}\frac{3^n}{n+1}\binom{2n}{n}$.

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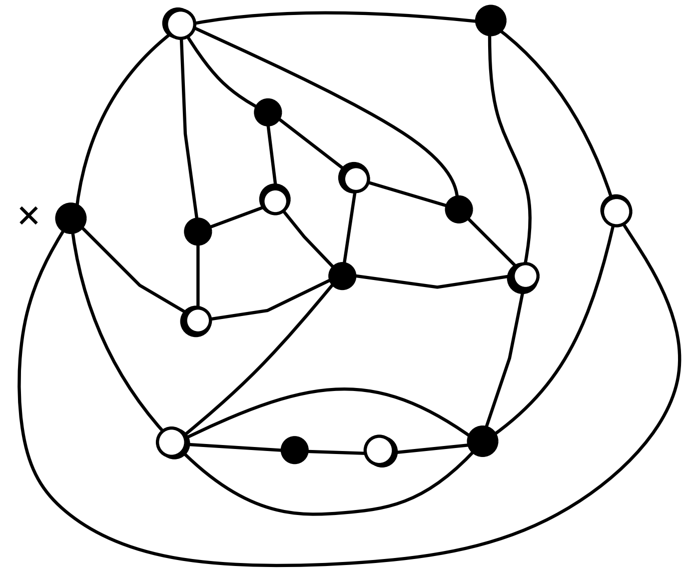

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algebraic equations

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explicit formula

Limits for uniform random quadrangulations

Uniform random rooted quadrangulations:

$$\Pr(Q_n = q) = \frac{1}{|\mathcal{Q}_n|} = \frac{1}{\frac{2\cdot 3^n (2n)!}{(n+2)!n!}} \quad \text{for all } q \in \mathcal{Q}_n$$

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I will stick with pure gravity, where there is still lots of work to do...

In particular I want to discuss an alternative discrete model of 2d pure quantum gravity.

Increasing quadrangulations

An increasing quadrangulation of size n is a bicolored planar map with:

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- $2n - 2$ faces of degree 4 having distinct labels such that
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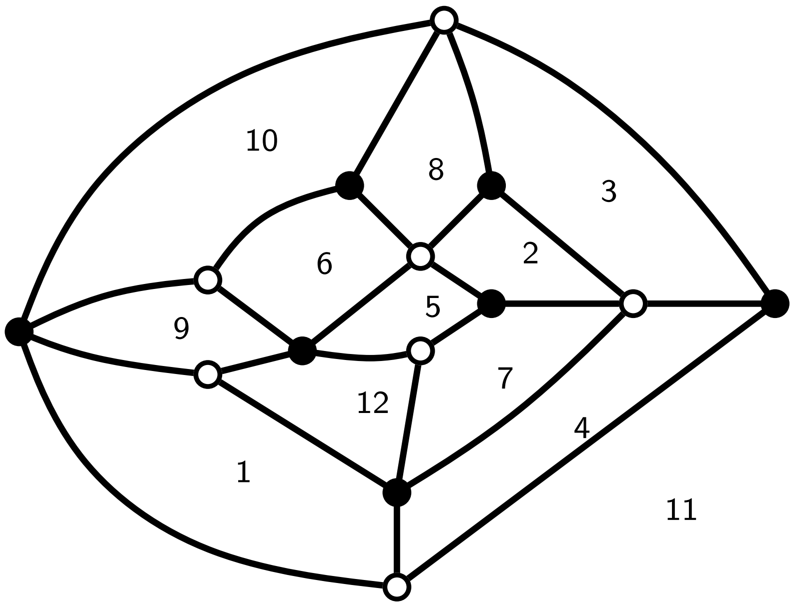

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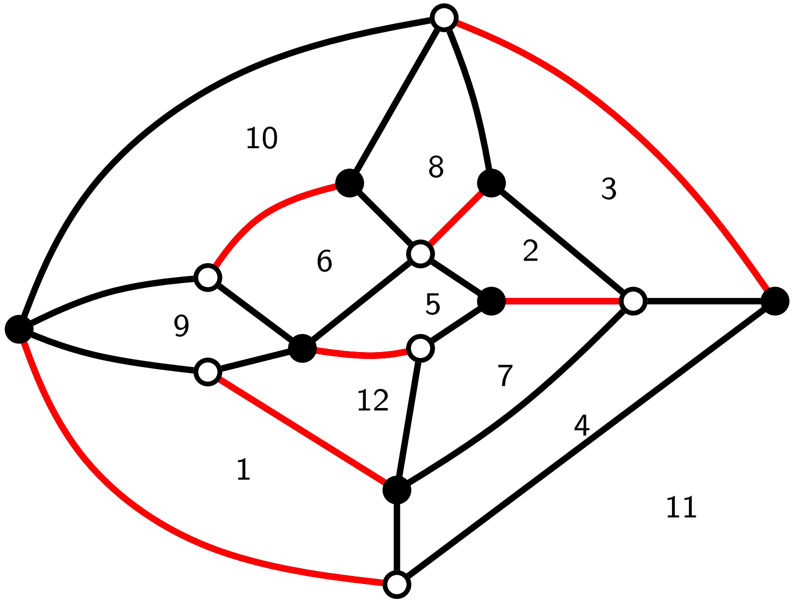

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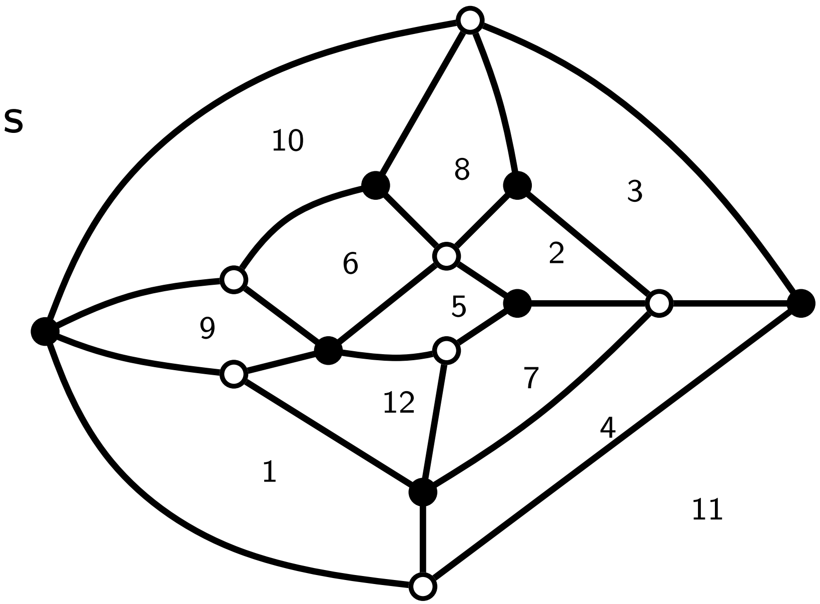

Theorem (Hurwitz 1891 / Strehl 1997, Goulden-Jackson 1997):

Let $\mathcal{Q}^+ = \{$increasing planar quadrangulations $\}$

and let $Q^+(t) = \sum_{q \in \mathcal{Q}^+} \frac{t^{|q|}}{|q|!}$ be the exponential gf where $|q| = \#$faces of q.

Then $Q^+(t)$ is solution of the system $\begin{cases} Q^+(t) = T(t) - \frac{1}{2}T(t)^2 \\ T(t) = t^2 \exp(T(t)) \end{cases}$

so that the nb of increasing planar quadrangulations of size n is

$$|\mathcal{Q}^+_n| = n^{n-3} \frac{(2n-2)!}{(n-1)!}.$$

Uniform random increasing quadrangulations?

Uniform random rooted increasing quadrangulations:

$$\Pr(Q_n^+ = q) = \frac{1}{|\mathcal{Q}_n^+|} = \frac{1}{n^{n-3}\frac{(2n-2)!}{(n-1)!}} \quad \text{for all } q \in \mathcal{Q}_n^+$$

Main conjecture.
The typical graph distances in a uniform random increasing quadrangulations with size n are of order $n^{1/4}$ and the scaling limit is the Brownian map.

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In this talk:

- Where do these increasing quadrangulations come from?
- Some available combinatorial tools...
- Some supporting evidences...

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a conjecture

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Hurwitz numbers and branched covers

A bijection with Cayley type trees!

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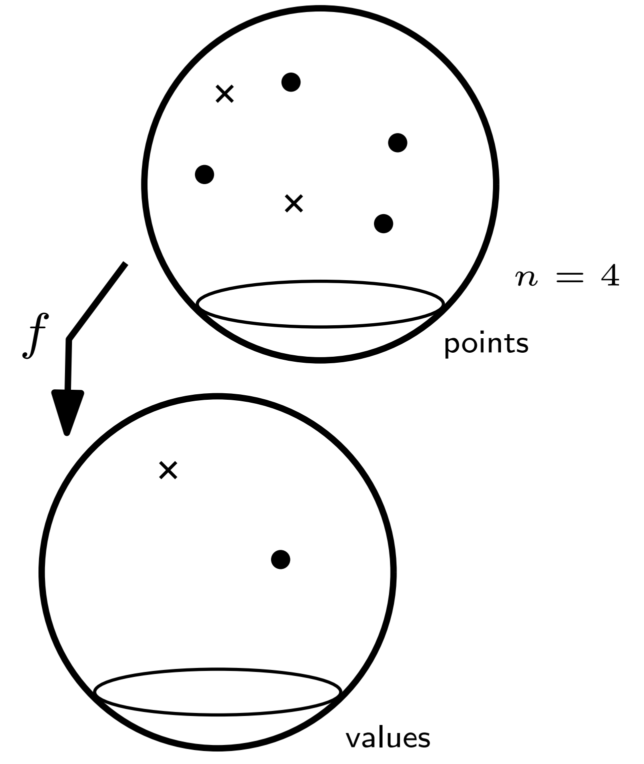
Branched covers and Hurwitz numbers

A mapping $f : \mathbb{S} \to \mathbb{S}$ is a branched cover of degree n of the sphere by itself if there is a finite set X of values on the image sphere such that



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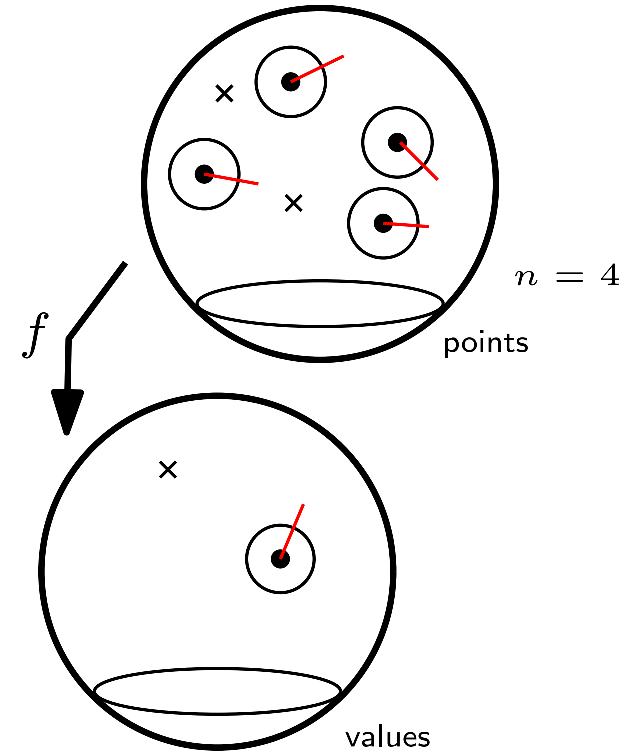

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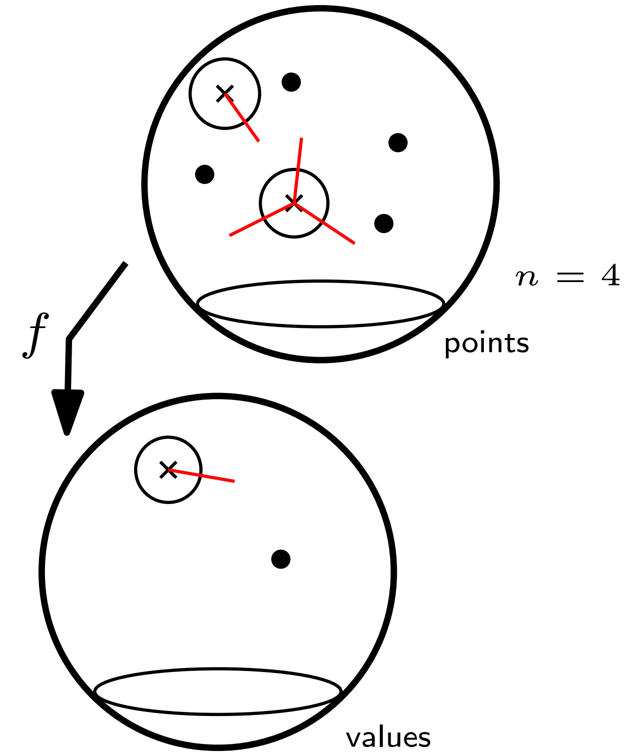
  z has n preimages $z_1, \ldots, z_n$, and
  f is homeomorphic to $y \mapsto y$ around z_i.

- each value z in X is critical:
  z has $p < n$ preimages $z_1, \ldots, z_p$, and
  f is homeomorphic to $y \mapsto y^{k_i}$ around z_i,
  where the orders k_i are positive integers
  such that $\sum k_i = n$.



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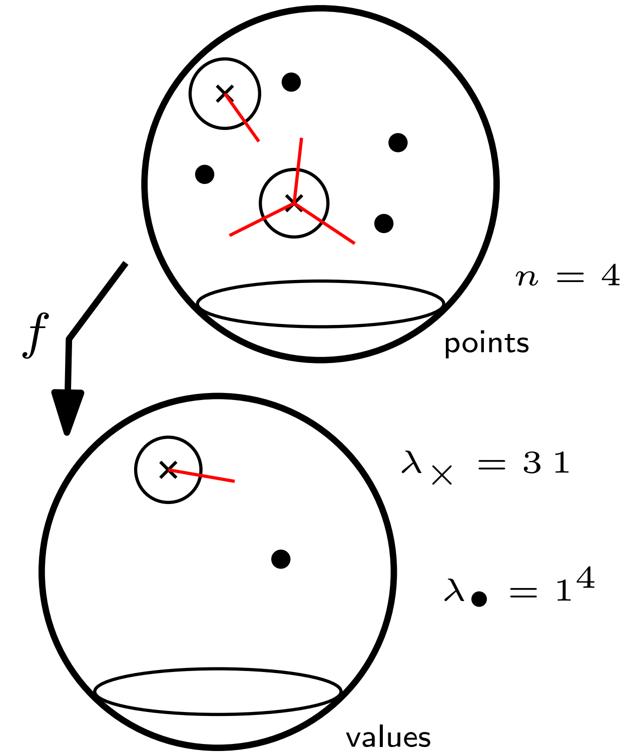
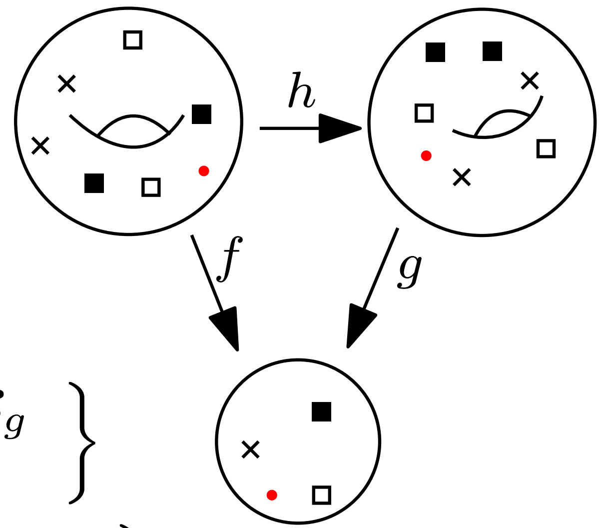


The type of a critical value is the partition whose parts are the order of its preimages

A critical value is simple if it has $n-1$ preimages, or equivalently if its type is $2\,1^{n-2}$

Branched covers and Hurwitz numbers

More generally Hurwitz considered branched covers of the sphere by a connected surface of genus g and raised in the 90's the question of counting these objects up to rooted homeomorphisms of the domain surface.



Typically the question is to compute

$$G^g_{m,n} = \#\left\{ \begin{array}{c} \text{(equiv. classes of) branched covers of } \mathbb{S} \text{ by } \mathcal{S}_g \\ \text{with degree } n \text{ and } m \text{ critical values} \end{array} \right\}$$

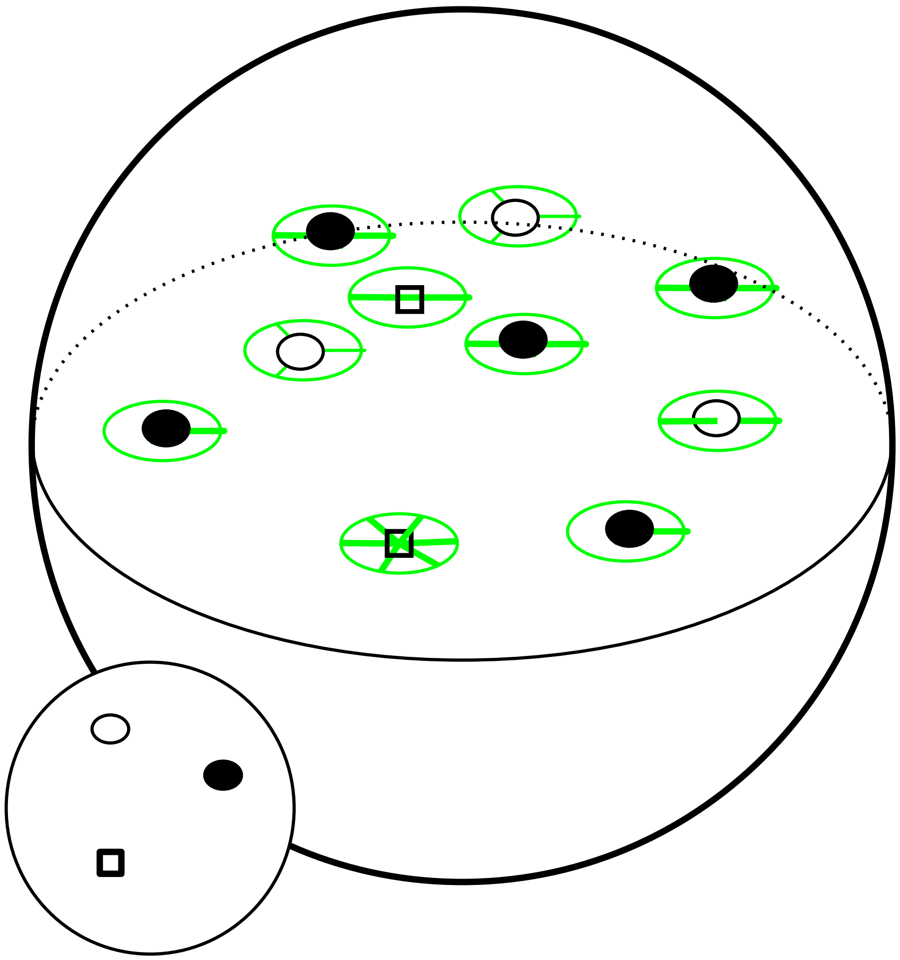
$$H^g_n = \#\left\{ \begin{array}{c} \text{(equiv. classes of) branched covers of } \mathbb{S} \text{ by } \mathcal{S}_g \\ \text{with degree } n \text{ and } 2n+2g-2 \text{ simple critical values} \end{array} \right\}$$

This enumerative study of branched covers was revitalized by the connections with moduli spaces of complex curves and Kontsevich theorem (Witten conjecture), that were raised by Okounkov in the 90's (1990's, while Hurwitz' 90's above were 1890's...)

In that context it was argued that branched covers with simple critical values should give an alternative model of 2d quantum gravity (Zvonkine 2004)

Amusingly an equivalent random sampling problem for rational functions was raised independantly by W. Thurston in the early 2000s.

Covers with 3 critical values and bipartite maps



3 critical values $\lambda^{\bullet} = 2^3 1^2$ $\lambda^{\circ} = 3^2 2$ $\lambda^{\square} = 62$

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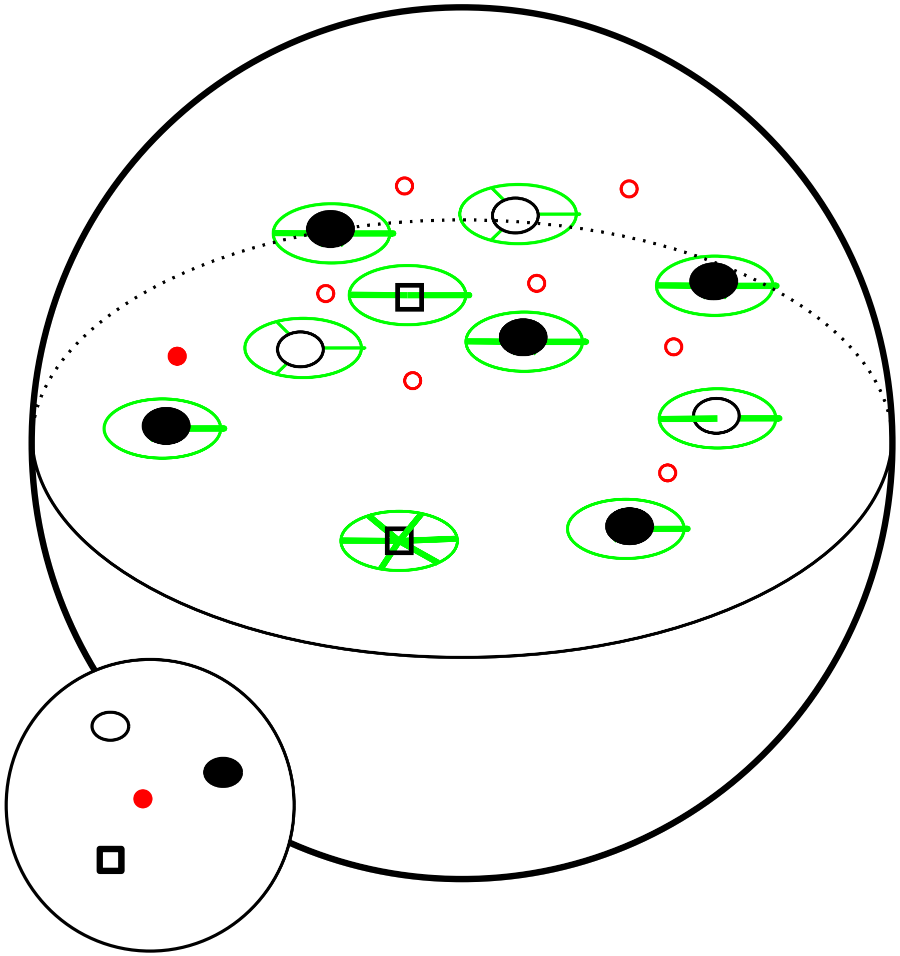

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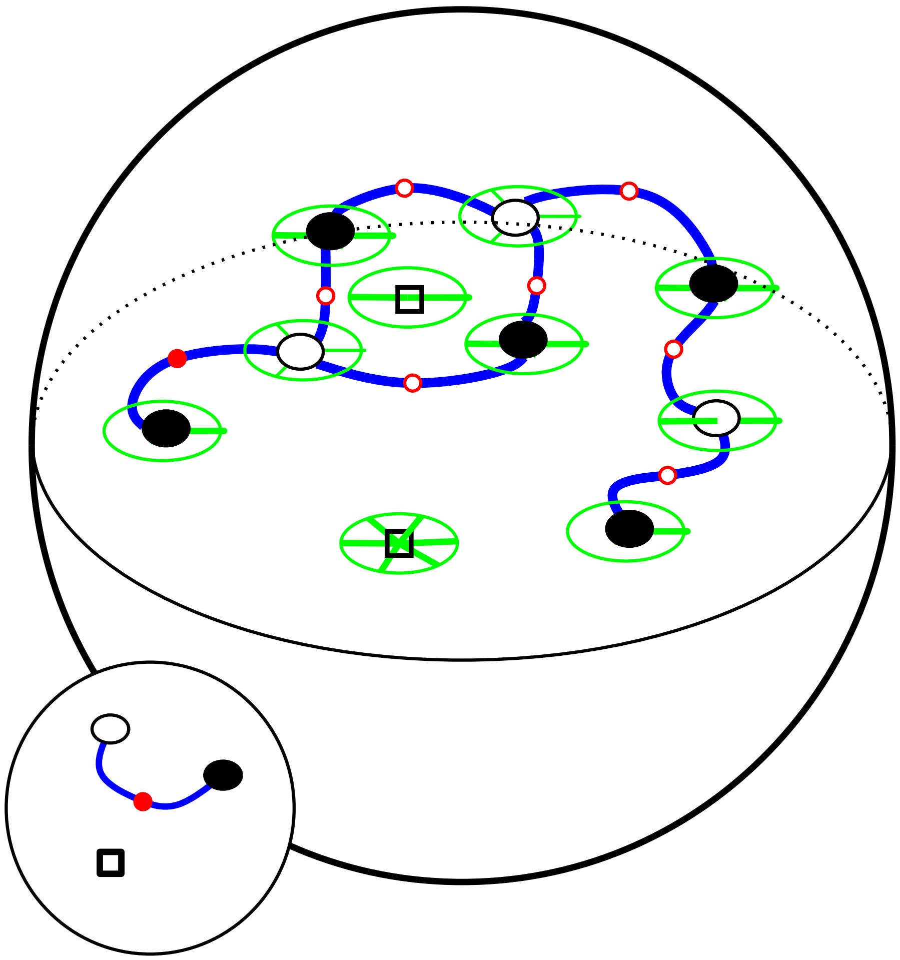

On the image sphere, draw an edge between • and ∘ via the basepoint

The pullback is a bipartite map:

we should check that faces are simply connected.

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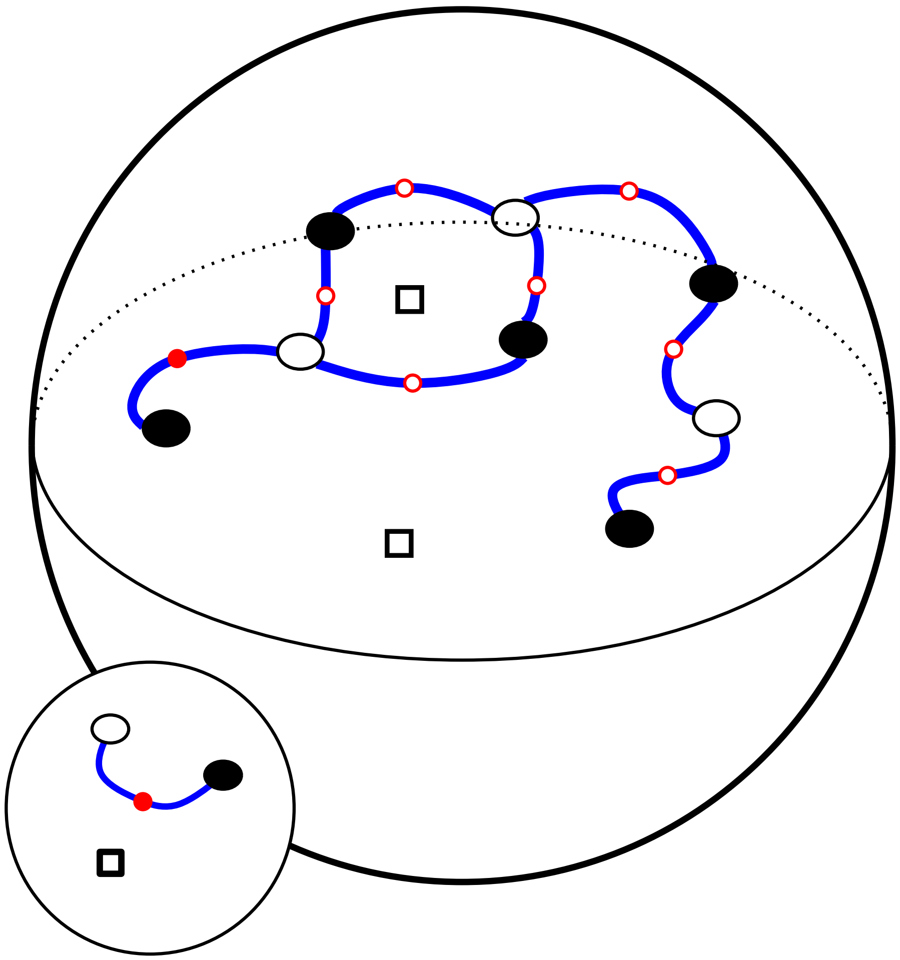

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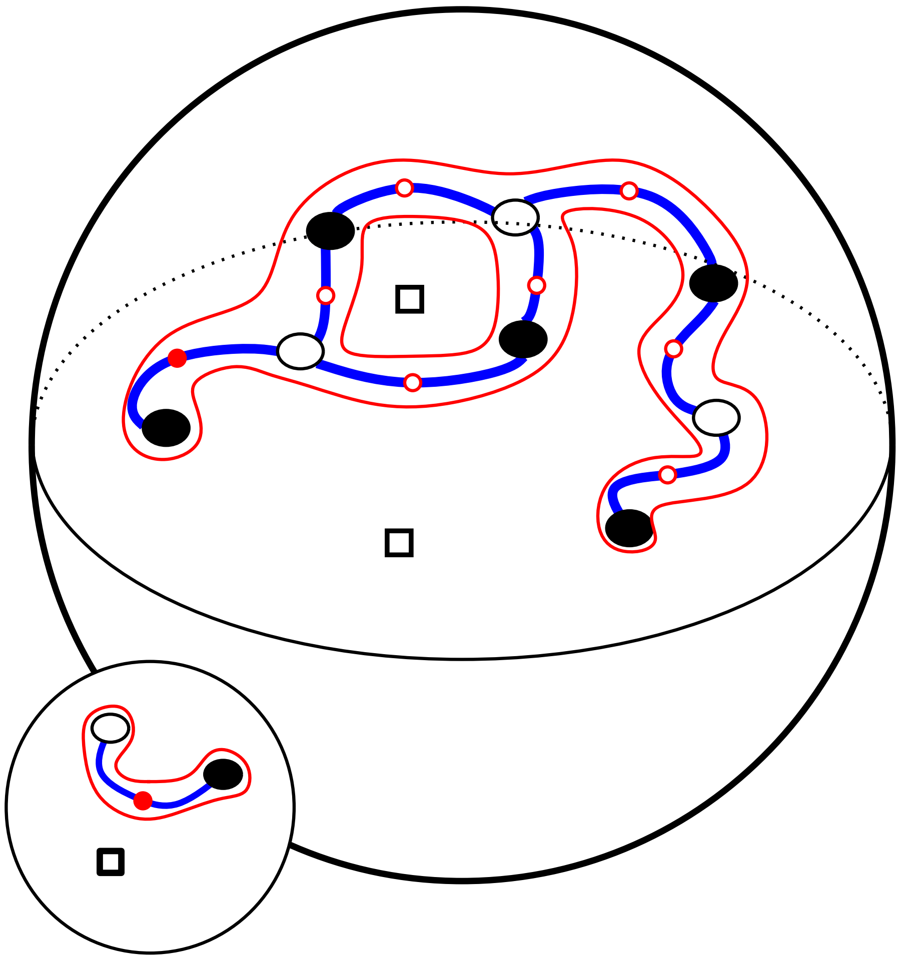

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Proof. A loop around the edge in the image sphere can be deformed to a loop around □

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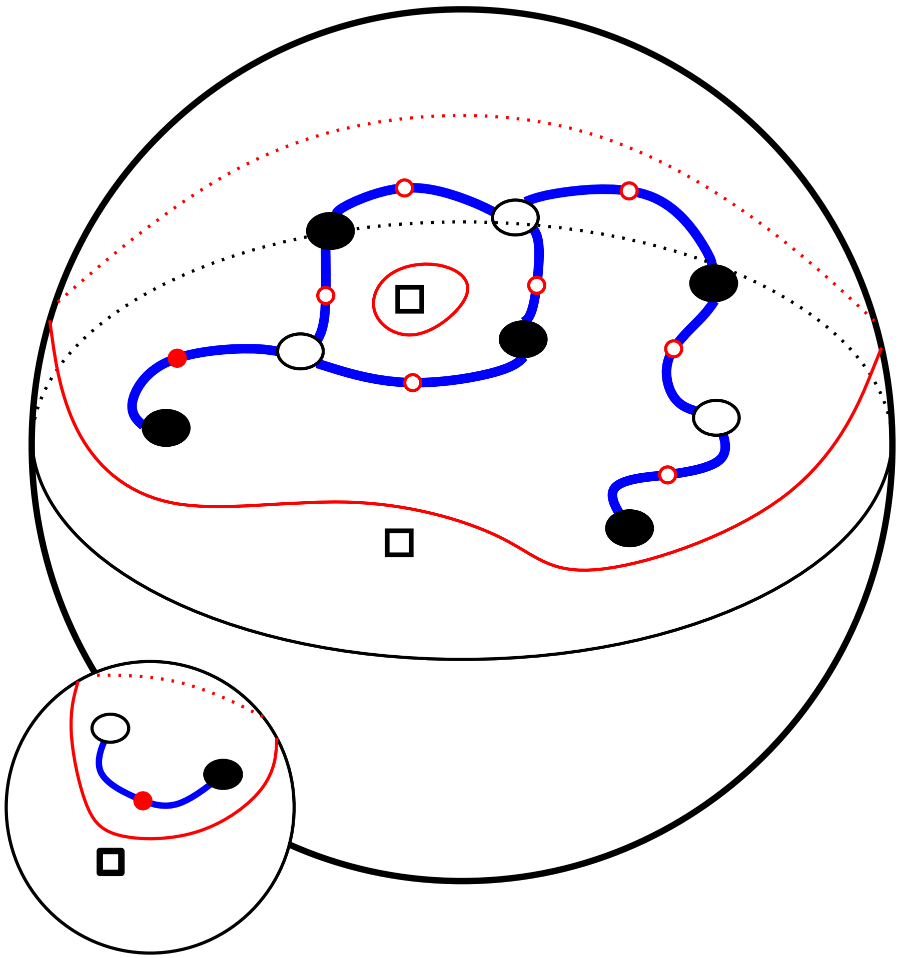

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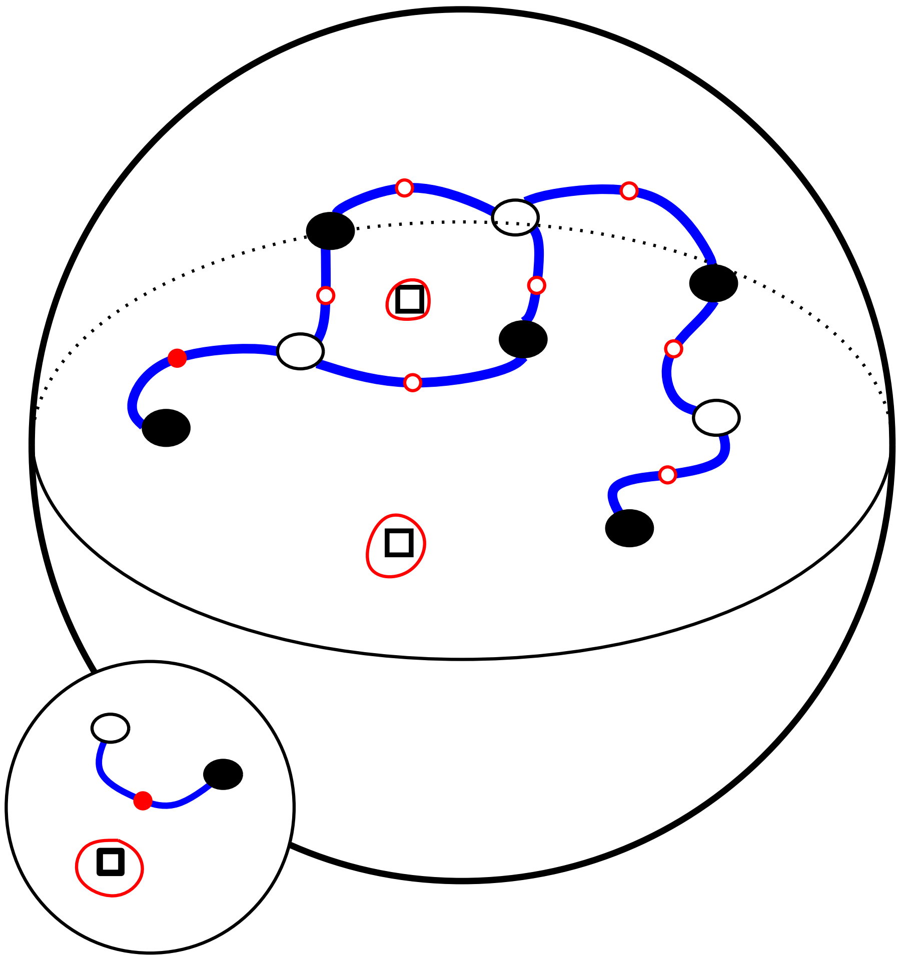

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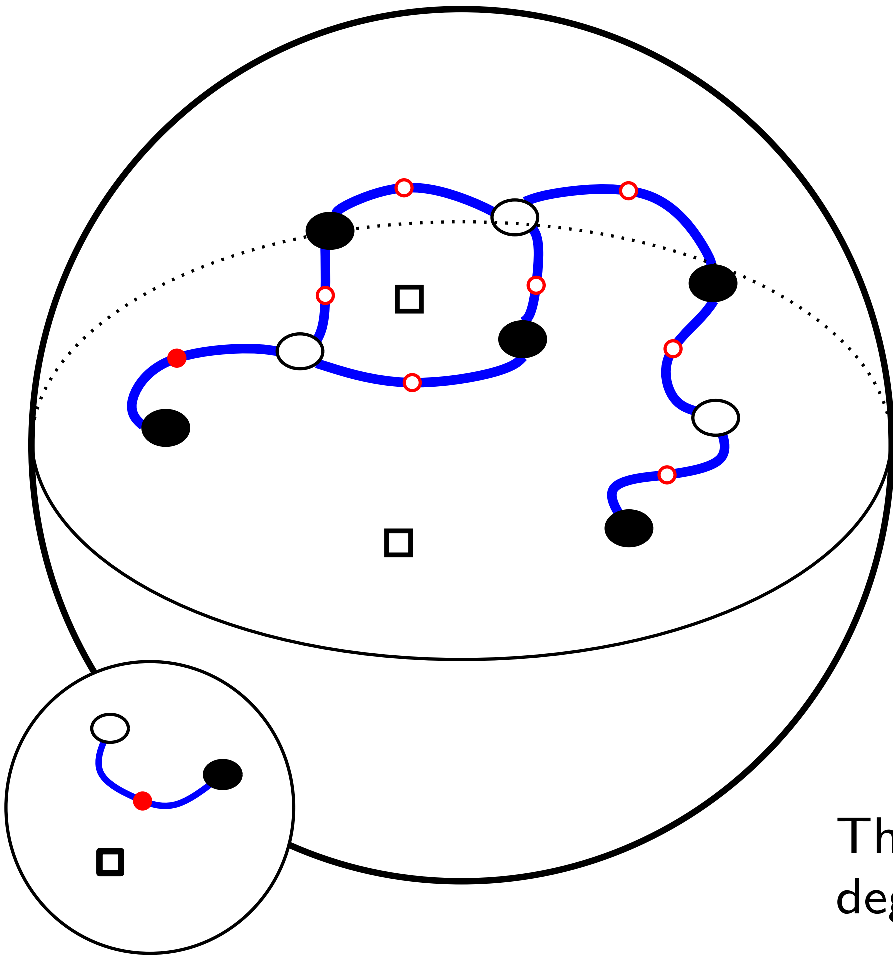

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Proposition (Folklore). This is a bijection between bipartite planar maps and branched covers of $\mathbb{S}$ by $\mathbb{S}$ with 3 critical values.

The partitions $\lambda_\bullet$, $\lambda_\circ$ and $2\lambda_\square$ gives respectively degrees of black and white vertices and faces

in particular for $\lambda_\square = 2^{n/2}$ all faces have degree 4 and we recover ordinary quadrangulations

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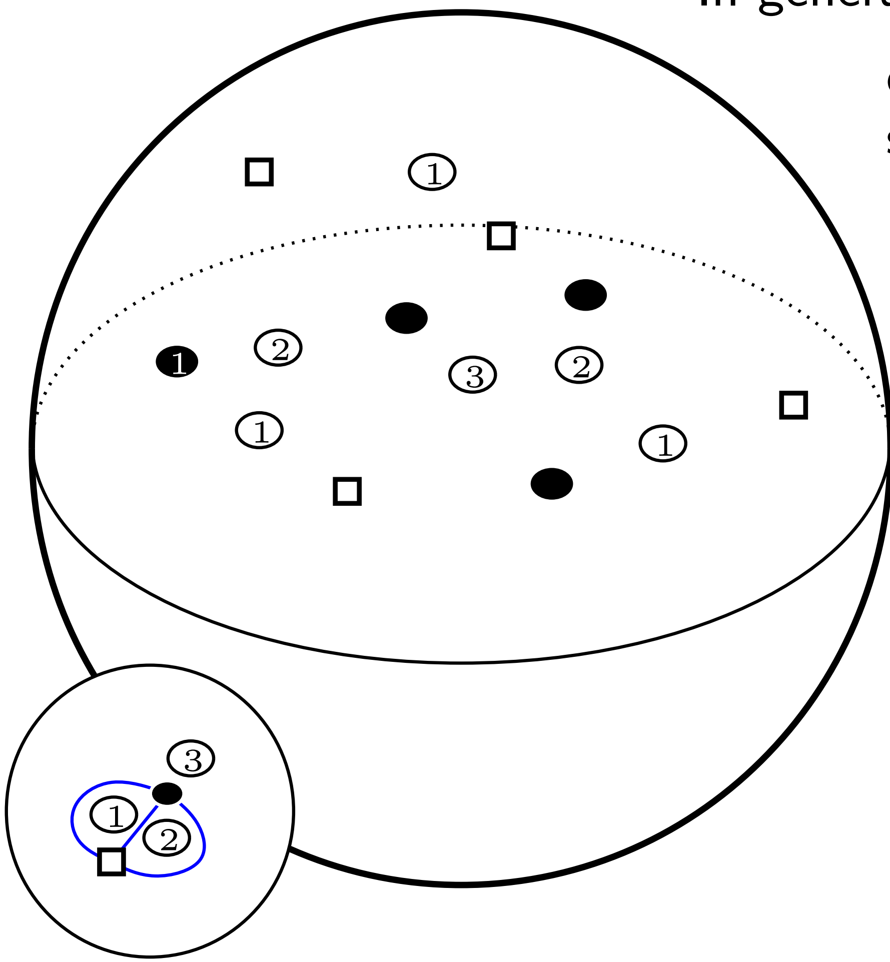
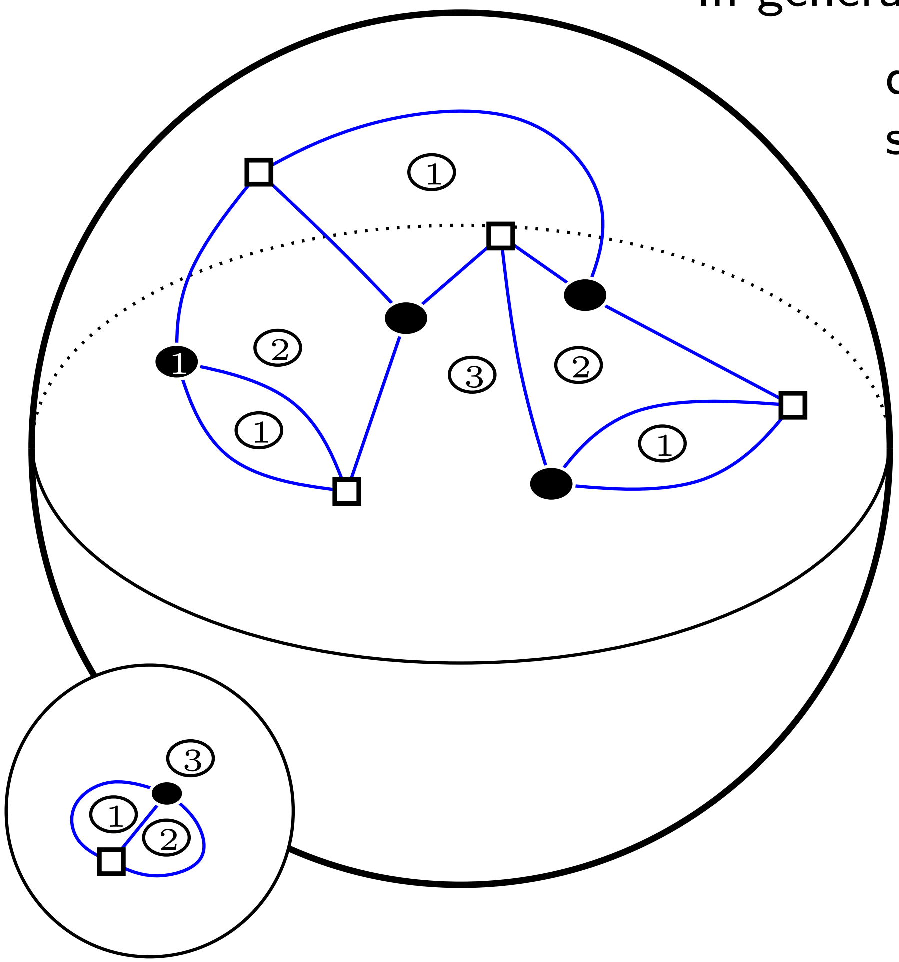
Simple branched covers, increasing quadrangulations



In general the general case of m critical points:

draw on the image $\mathbb{S}$ a fan of multiple edges separating the critical values, and take pullback,

Simple branched covers, increasing quadrangulations



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this gives a m-Eulerian map,
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- n black and n white vertices of degree m
- $n(m-2)+2$ labelled faces of arbitrary (even) degree such that
  - around black vertices, face labels form the cycle $(1, 2, \ldots, m)$ in ccw order
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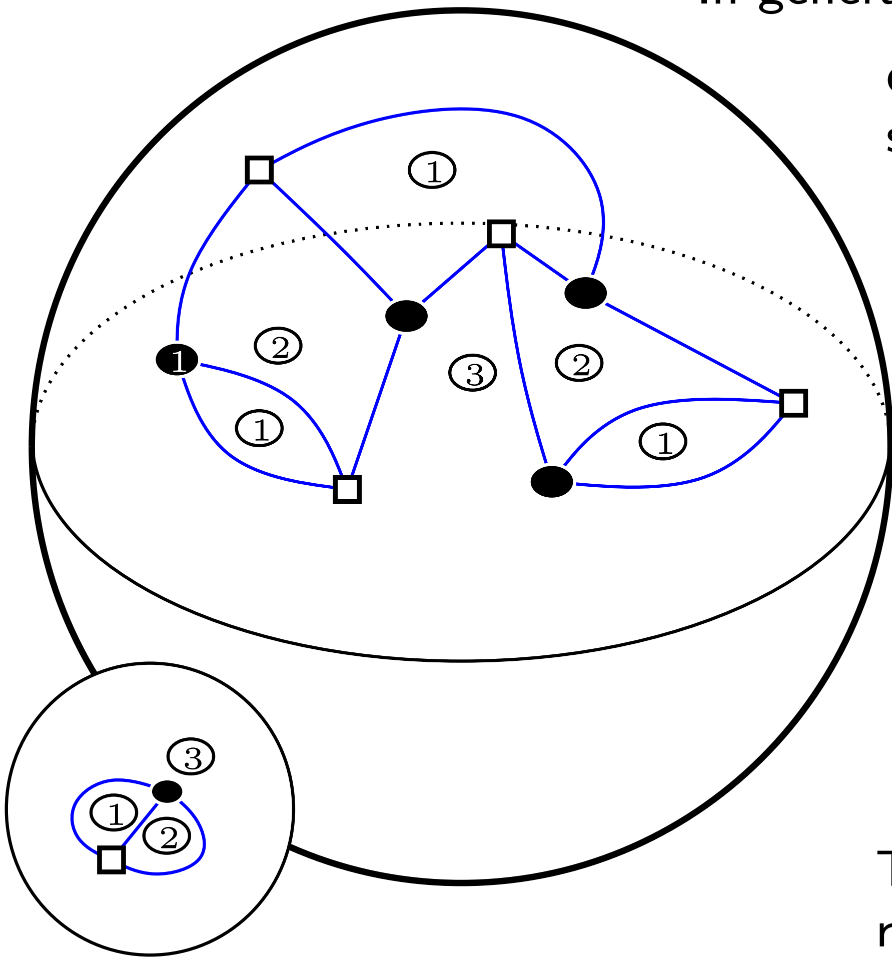
Simple branched covers, increasing quadrangulations



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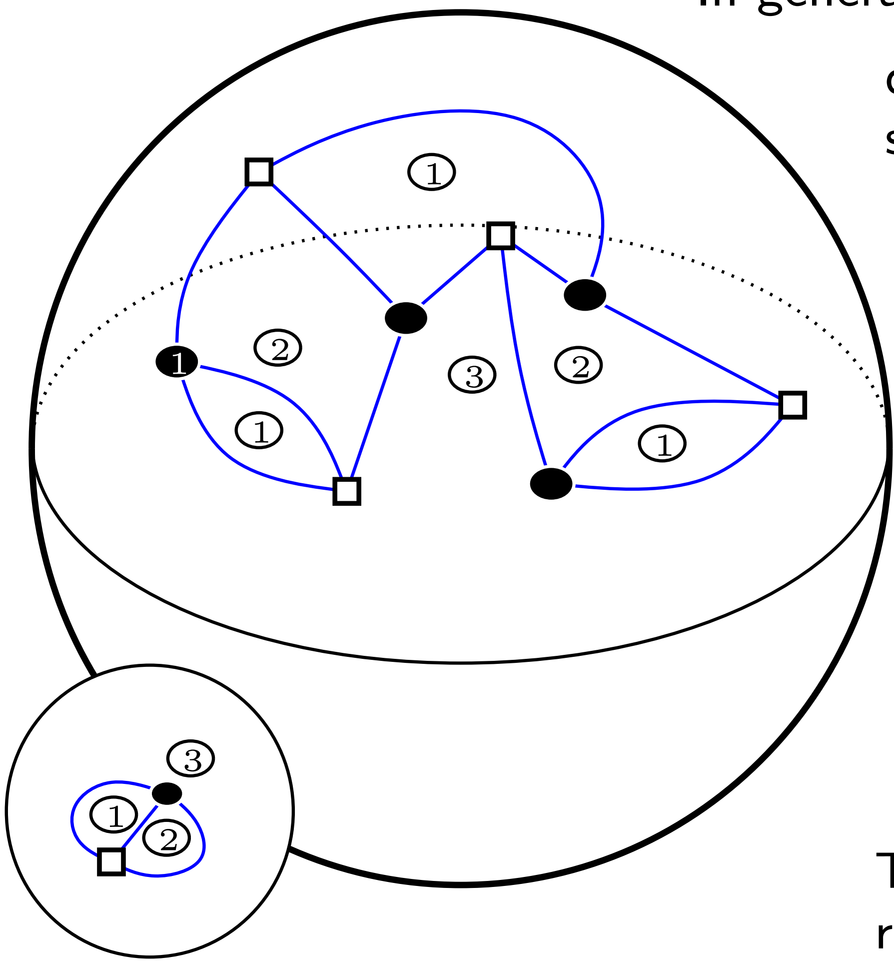

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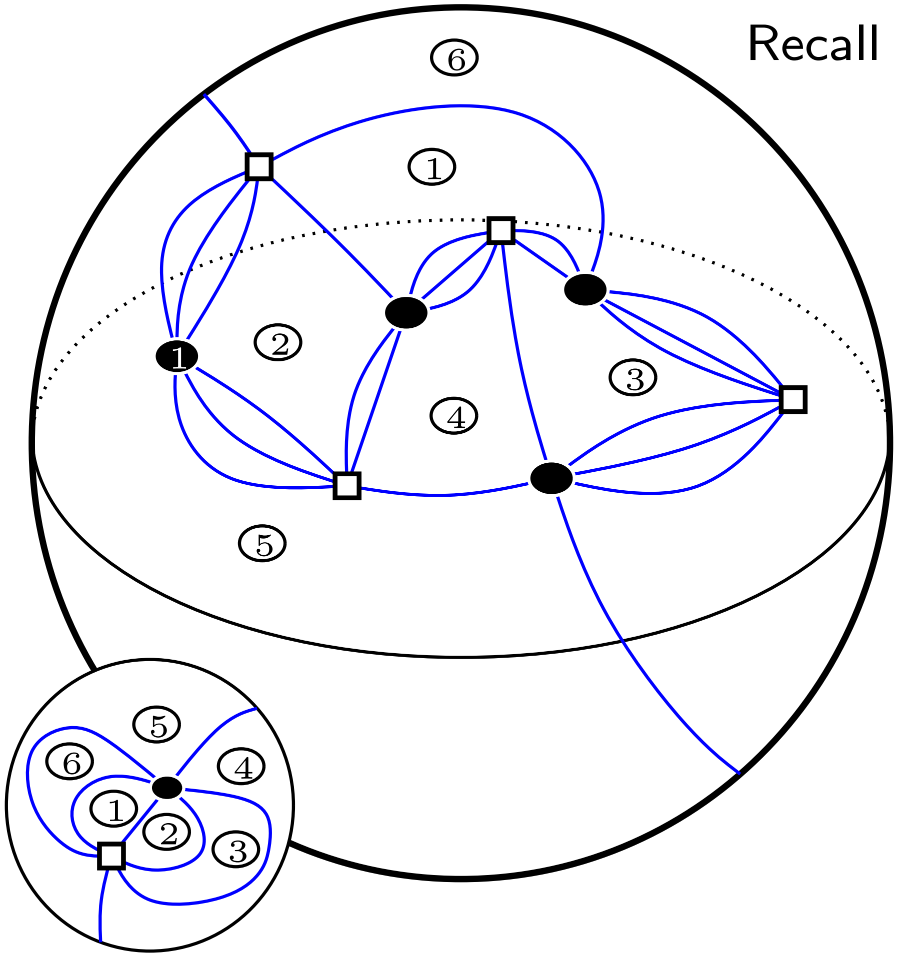
Corollary. There is a bijection between:

- branched covers of the sphere by itself with degree n and m simple critical values
- planar m-Eulerian maps with n black and m white vertices

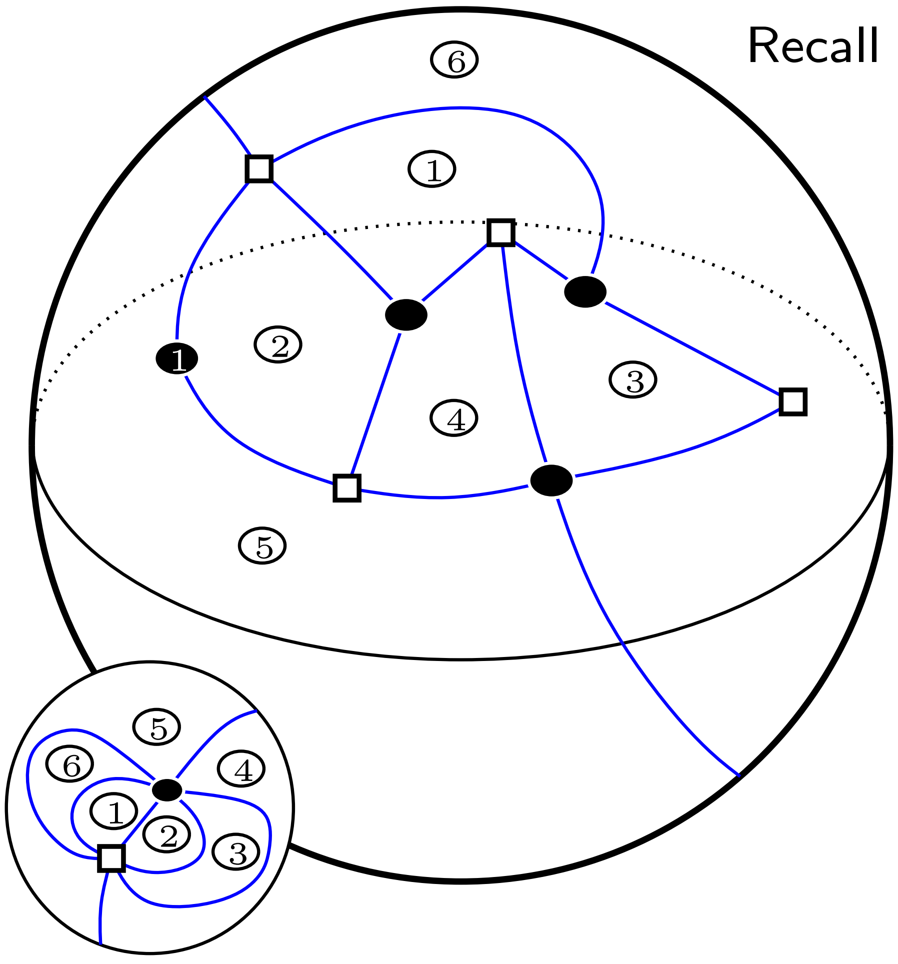
Simple branched covers, increasing quadrangulations



Recall that a critical value is simple if it has type 21^{d-2}.

Only simple critical values
⇒ only 1 quadrangle per label
⇒ after contraction of multiple edges,
an increasing quadrangulation

Simple branched covers, increasing quadrangulations



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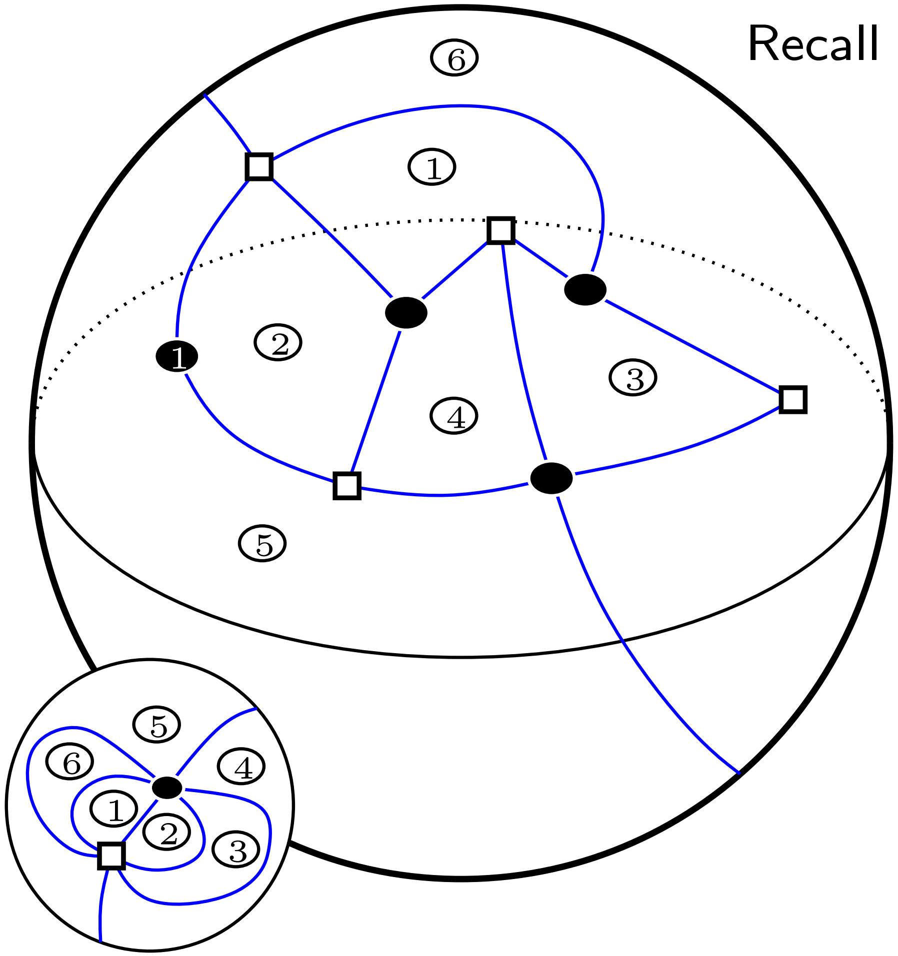
$\Rightarrow$ after contraction of multiple edges, an increasing quadrangulation

Simple branched covers, increasing quadrangulations



Recall that a critical value is simple if it has type 21^{d-2}.

Only simple critical values
$\Rightarrow$ only 1 quadrangle per label
$\Rightarrow$ after contraction of multiple edges, an increasing quadrangulation

Corollary. There is a bijection between:

- branched covers of the sphere by itself with $2n-2$ simple critical values
- increasing planar quadrangulations with $2n-2$ faces

Enumerative results

The original statement of Hurwitz is about simple branched covers:

Theorem (Hurwitz 1891 / Strehl 1997, Goulden and Jackson 97)

$$H_n^0 = \# \left\{ \begin{array}{c} \text{(equiv. classes of) branched covers of } \mathbb{S} \text{ by itself} \\ \text{with degree } n \text{ and } 2n-2 \text{ simple critical values} \end{array} \right\}$$

$$= \# \left\{ \text{increasing planar quadrangulations with } 2n-2 \text{ faces} \right\}$$

$$= n^{n-3} \frac{(2n-2)!}{(n-1)!}$$

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There is a companion statement for general branched covers

Theorem (Bousquet-Mélou and S. 00)

$$G_{m,n}^0 = \# \left\{ \begin{array}{c} \text{(equiv. classes of) branched covers of } \mathbb{S} \text{ by itself} \\ \text{with degree } n \text{ and } m \text{ critical values} \end{array} \right\}$$

$$= \# \{m\text{-eulerian planar maps with } n \text{ faces}\}$$

$$= m(m-1)^{n-1} \frac{((m-1)n)!}{(mn-2n+2)!n!}$$

Plan of the talk

Unlabeled VS Increasing quadrangulations...

a conjecture

Why increasing quadrangulations?

Hurwitz numbers and branched covers

A bijection with Cayley type trees!

(on ne se refait pas...)

More evidences from higher genus maps...

as a conclusion

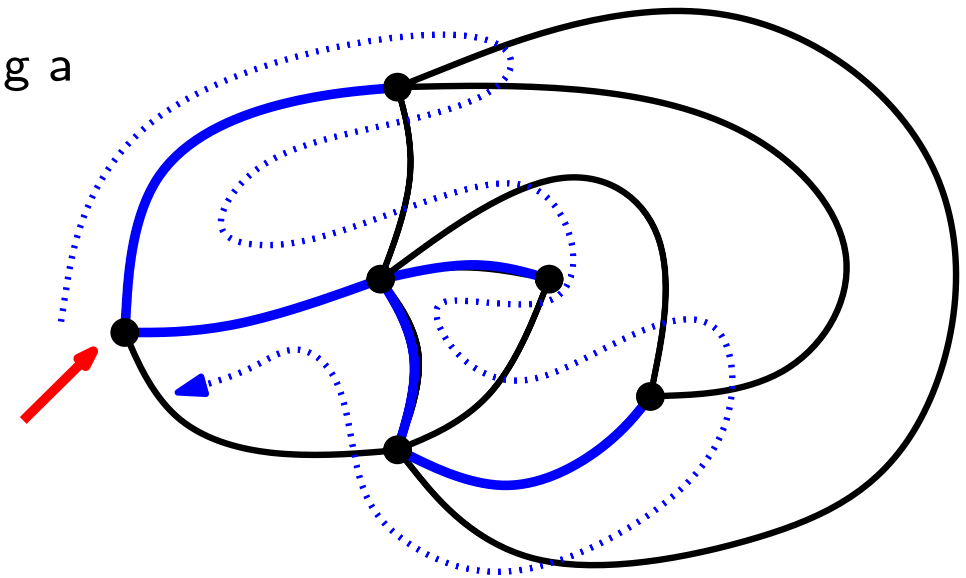
A general strategy for encoding maps by trees

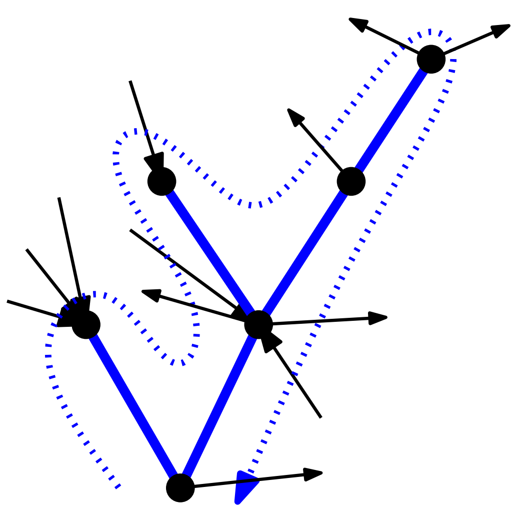
A tree-rooted map can be decomposed along a contour of its spanning tree:

– into two mating trees, *a la* Mullin

or

– into a blossoming tree: the spanning tree decorated with the start- and end-halves of the remaining edges



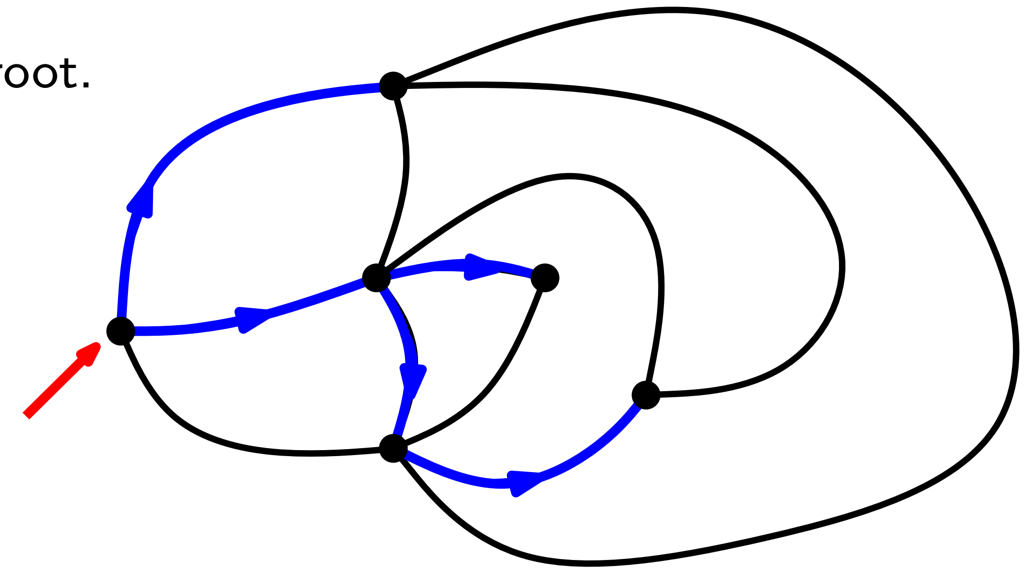


In order to use this approach we need to select a canonical spanning tree.

A general strategy for encoding maps by trees

Spanning trees can be replaced by orientations:

Orient the tree edges away from the root.



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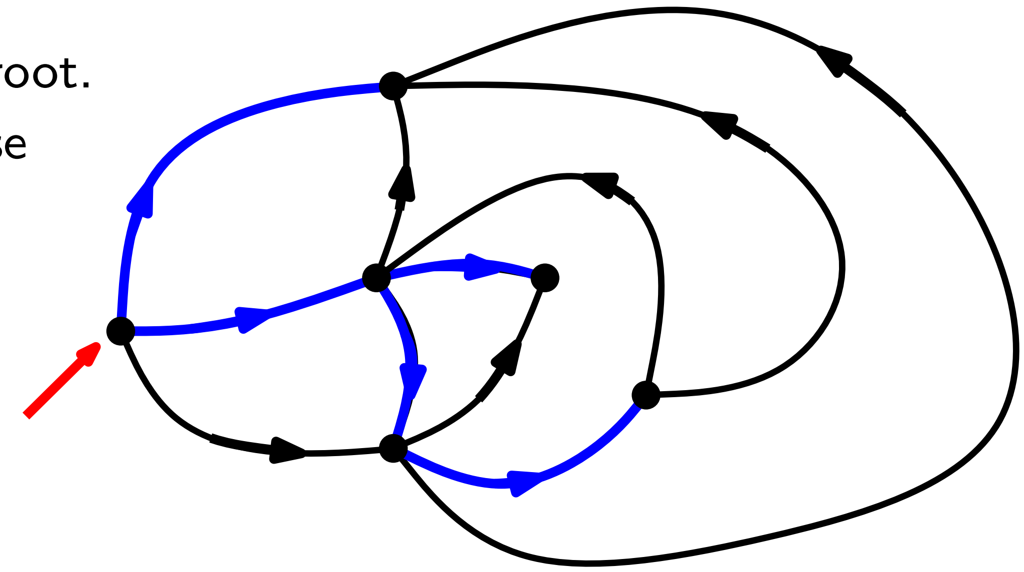

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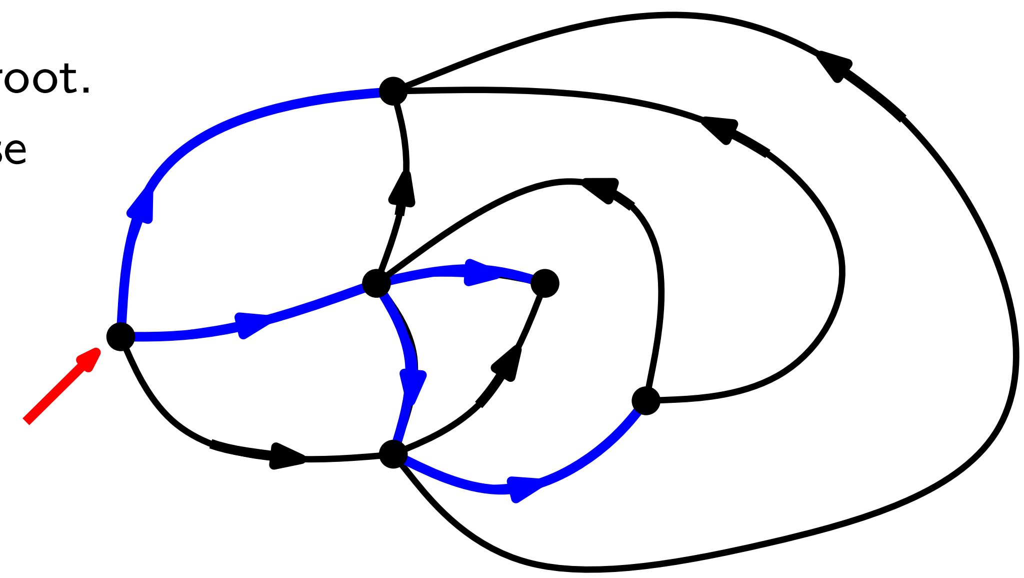
The resulting orientation is a minimal orientation: no clockwise circuit.



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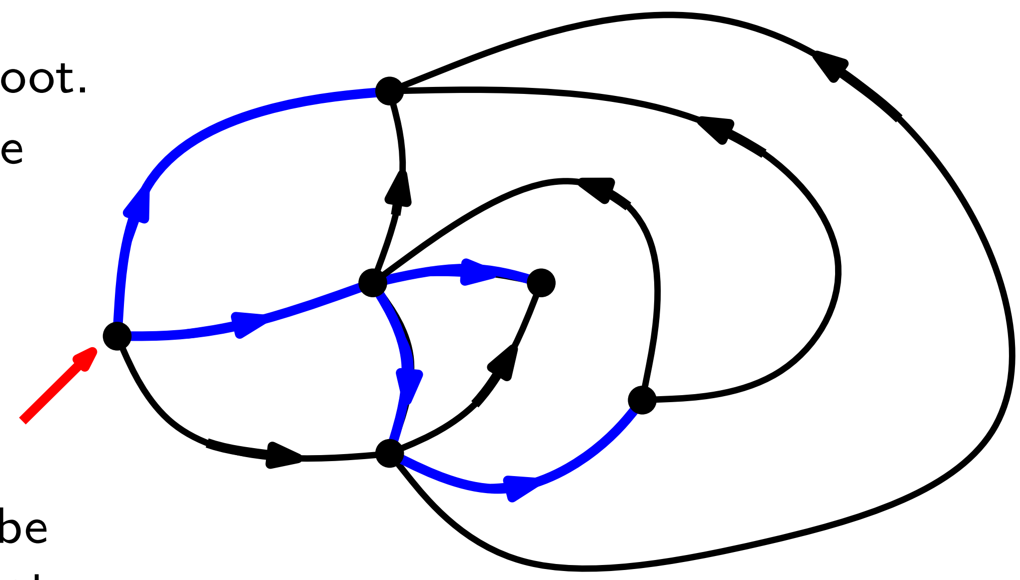

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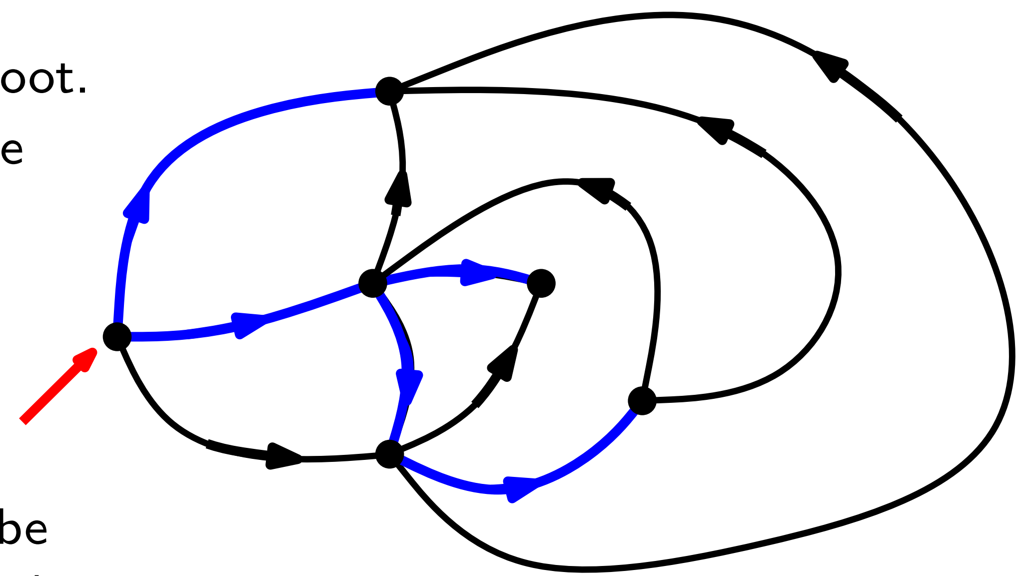

Theorem (Bernardi 2005) This is a bijection between tree-rooted maps with n edges and minimum accessible oriented maps with n edges

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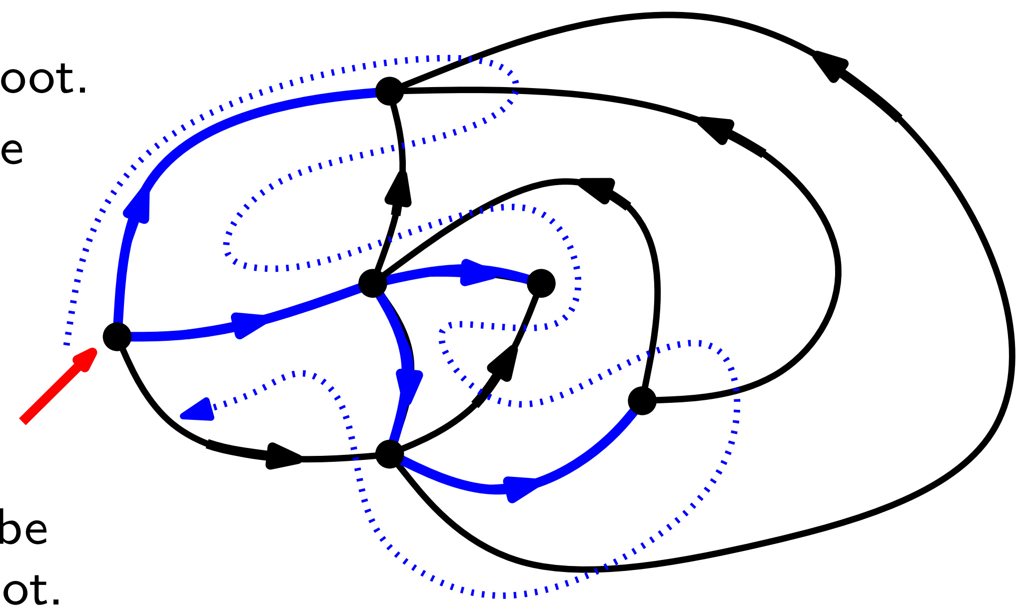

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The tree is recovered by reconstructing its contour .

The general blossoming strategy to design bijection with trees: find a natural accessible orientation, make it minimal, and use the associated spanning as blossoming tree.

Formalized by Albenque-Poulalhon (2015)

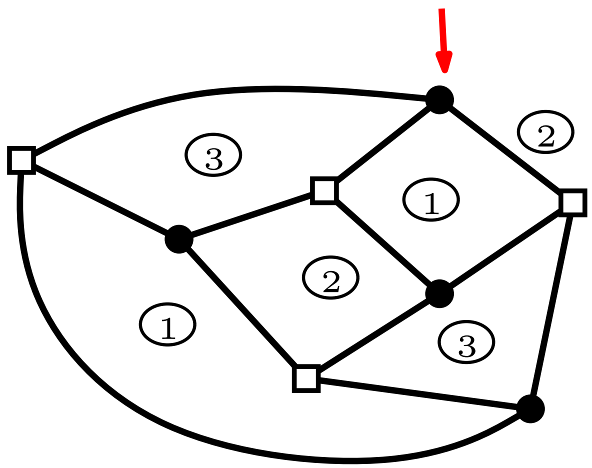
Application to m-Eulerian map: orientation?

Bipartite map with black and white vertices of degree m such that:

– faces with labels in $\{1, \ldots, m\}$

– around black vertices, face labels read $1, \ldots, m$ in cw order

– around white vertices, face labels read $1, \ldots, m$ in ccw order



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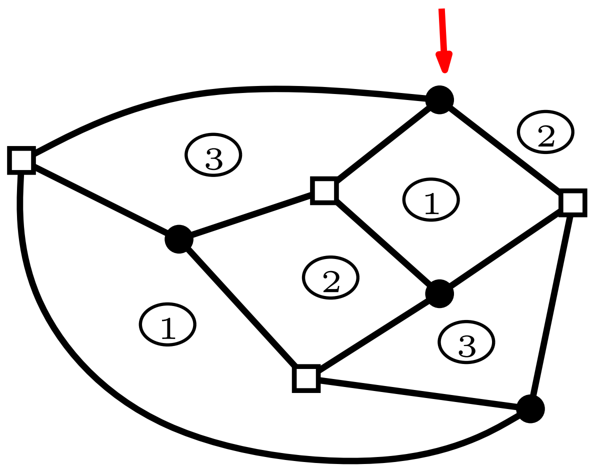

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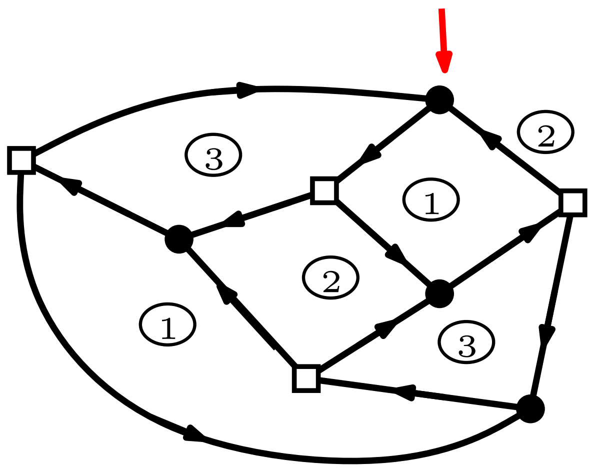

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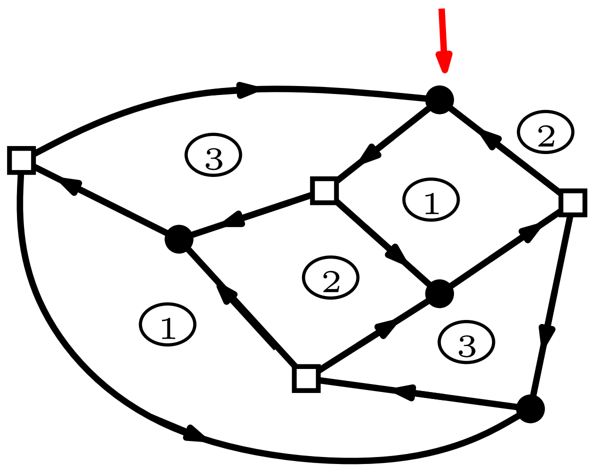

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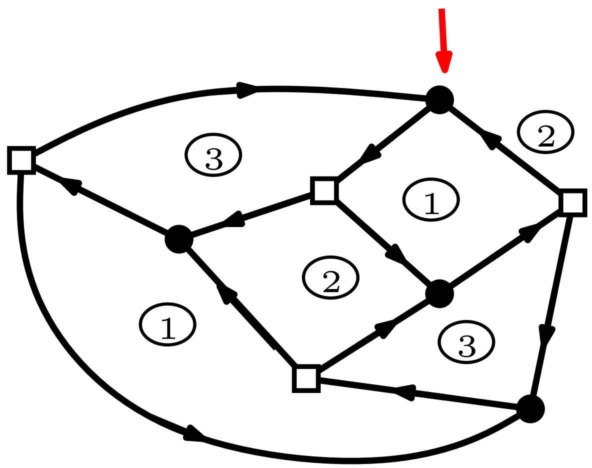

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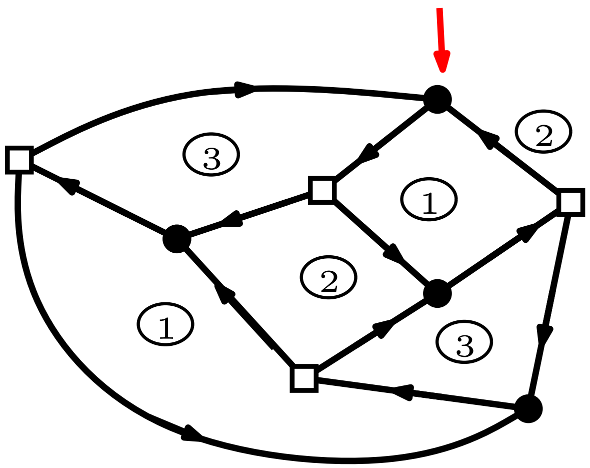
We can apply our strategy!

Opening a m-Eulerian map



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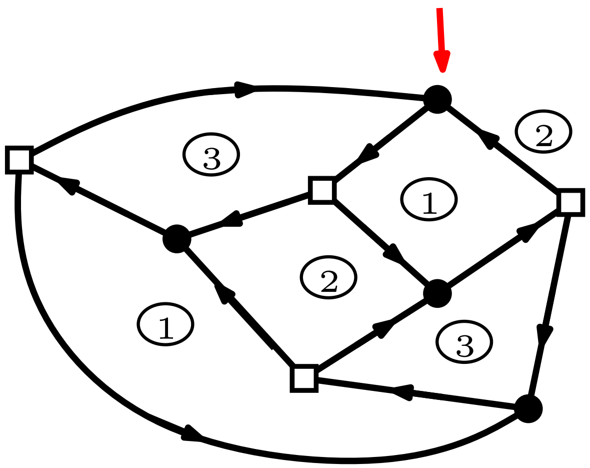
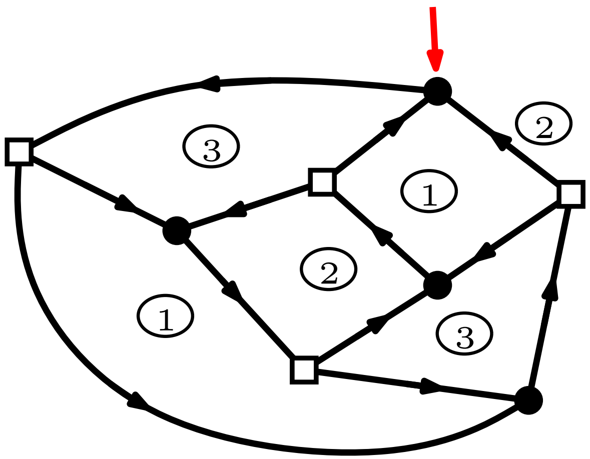


endow with minimal orientation

(return cycles)



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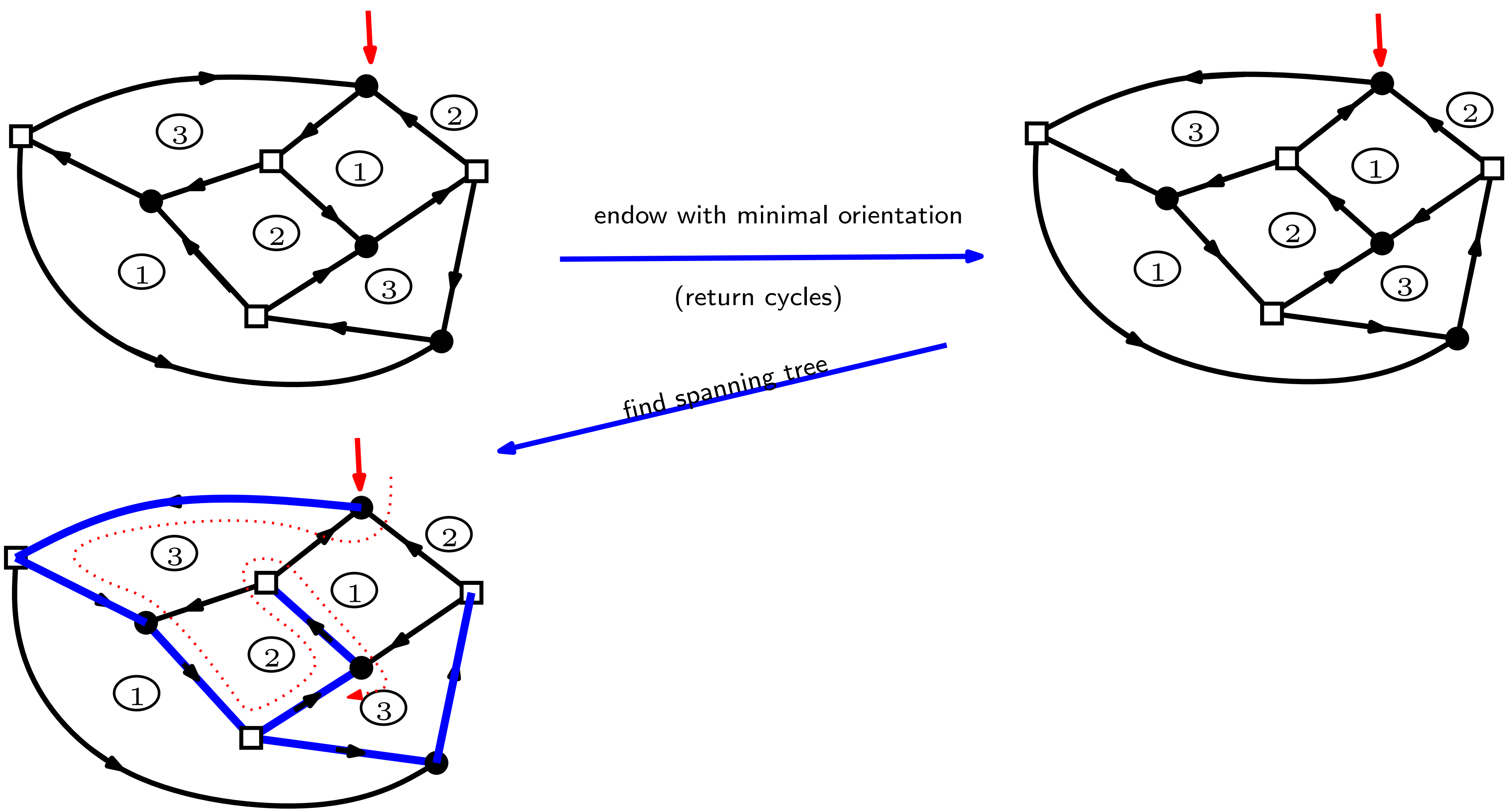

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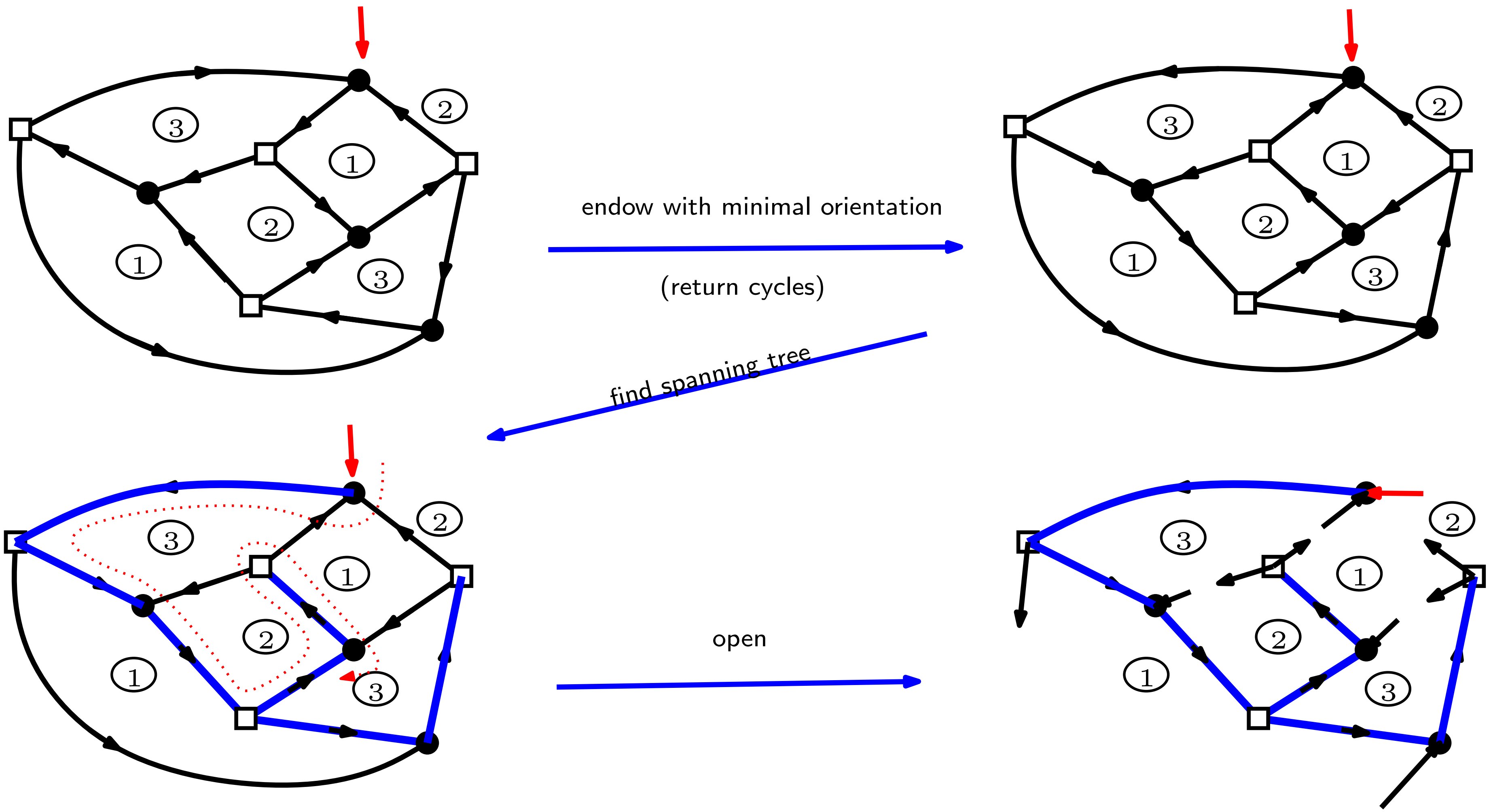

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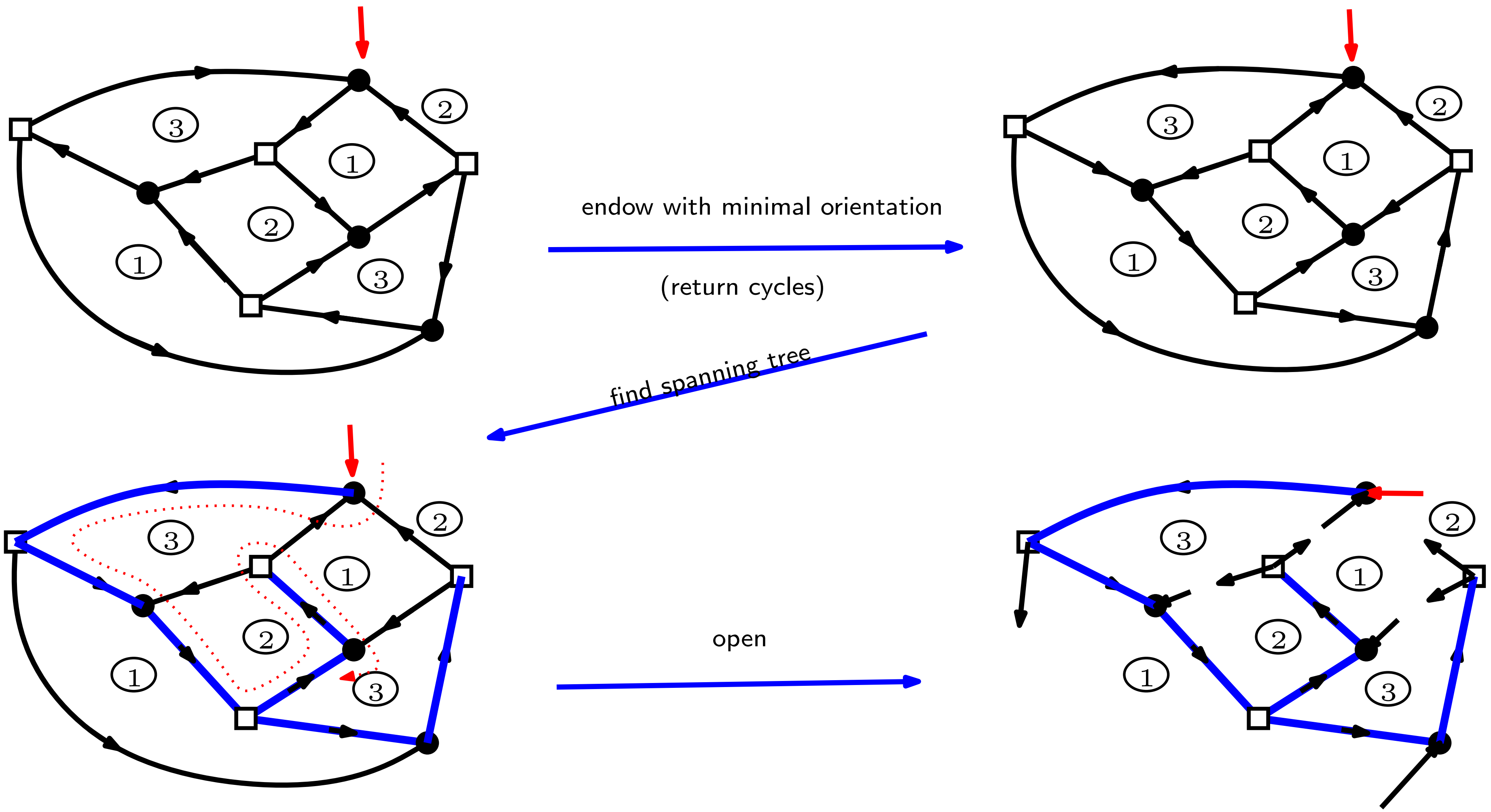

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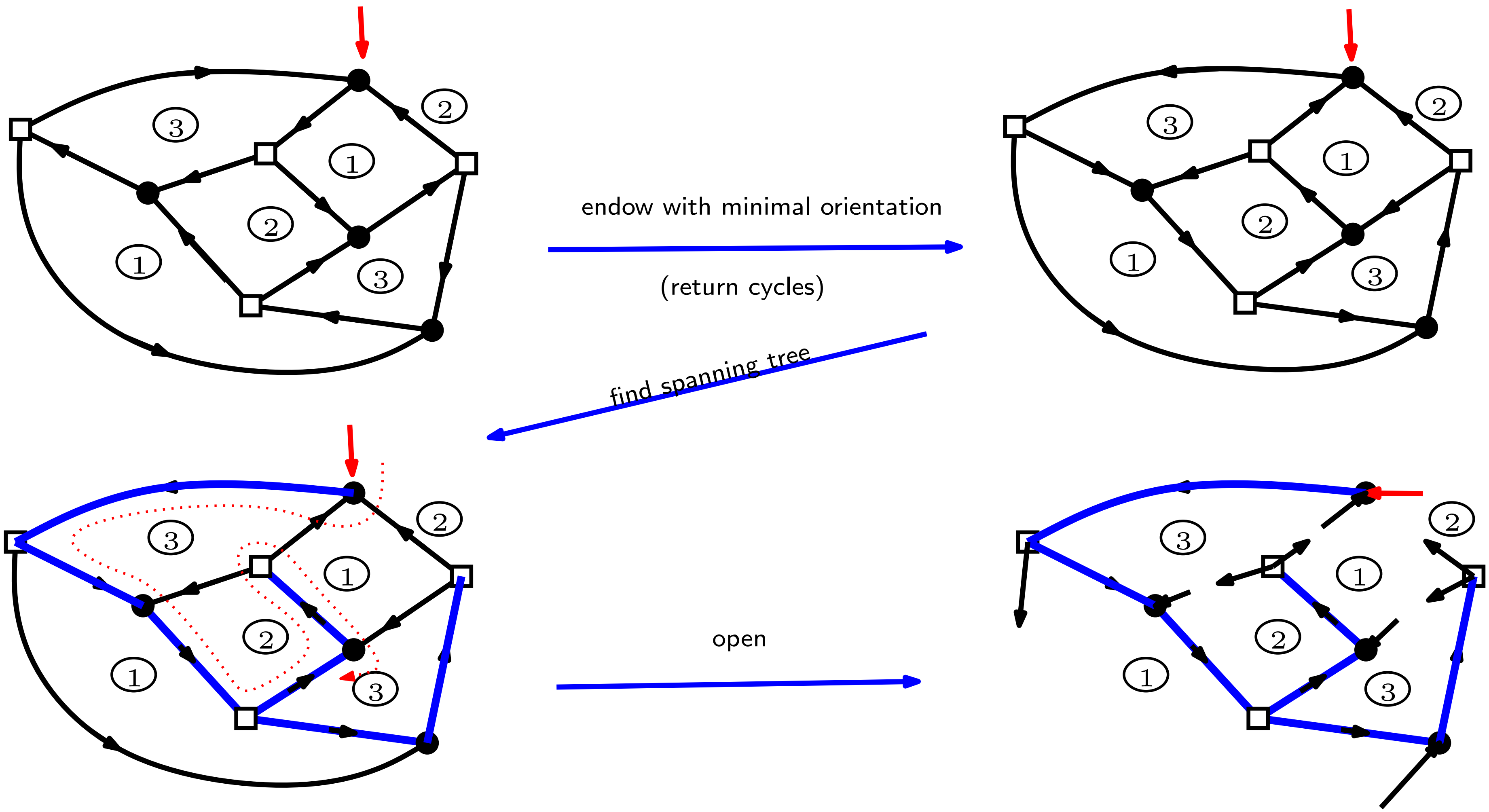
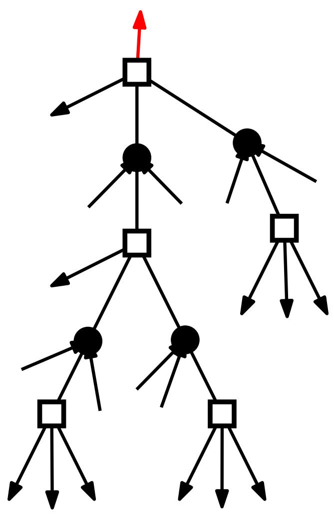


m-Eulerian maps and their blossoming trees

m-Eulerian trees: plane (ordered) trees such that:

– white vertices carry $m-1$ children
(black vertices or half-edges)

– black vertices carry $m-2$ half-edges
and a white child.



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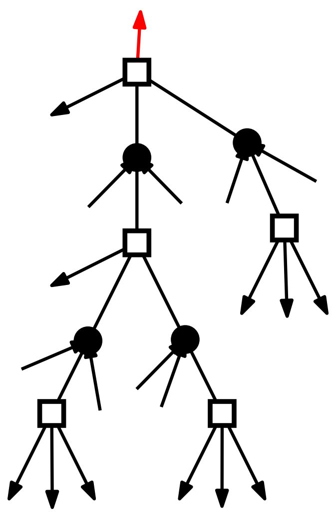

Counting rooted m-Eulerian trees: using a recursive decomposition

$$A_{\square}(t) = m(1 + A_{\bullet}(t))^{m-1}, \qquad A_{\bullet}(t) = (m-1) \cdot A_{\square}(t)$$

or observe directly that they are $(m-1)$-ary trees with $(m-1)$ types of edges

$$\Rightarrow \qquad \frac{1}{(m-2)n+1}\binom{(m-1)n}{n} \cdot (m-1)^{n-1}$$

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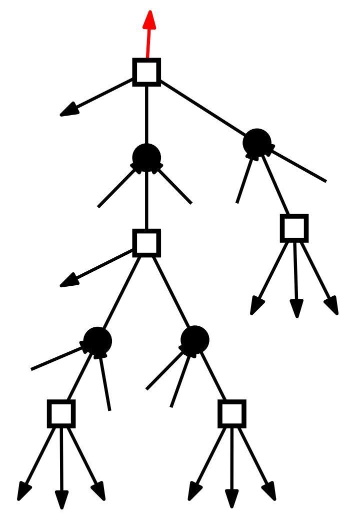

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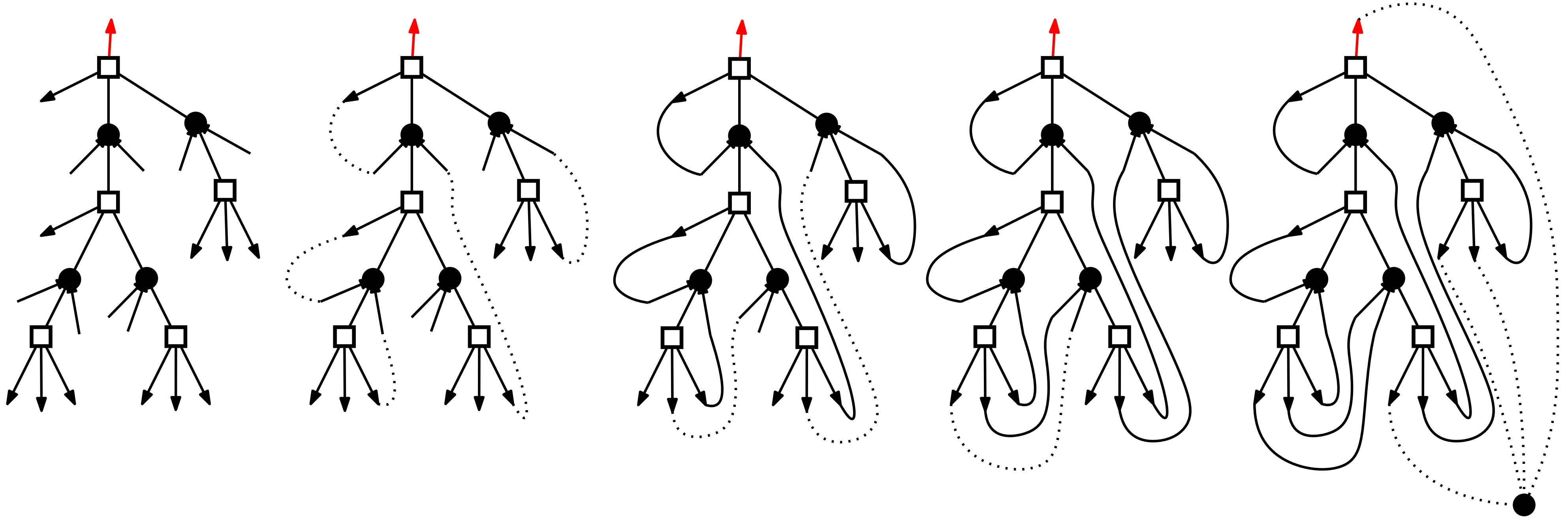
Proposition. The opening of an m-eulerian map is an m-eulerian tree with same vertex degree distributions.

The closure of a m-Eulerian tree



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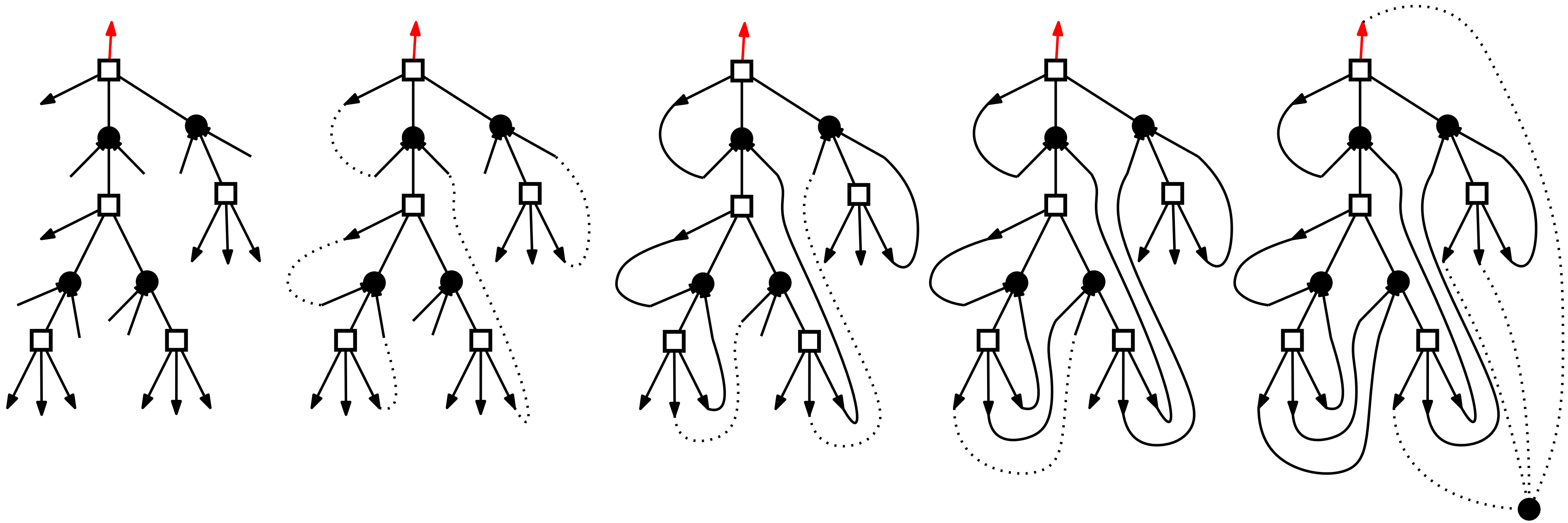

Theorem (BMS 00) Opening and closing are inverse bijections between well-rooted m-Eulerian trees and m-Eulerian maps with same number of nodes.

Here, well-rooted means that the root remains unmatched.

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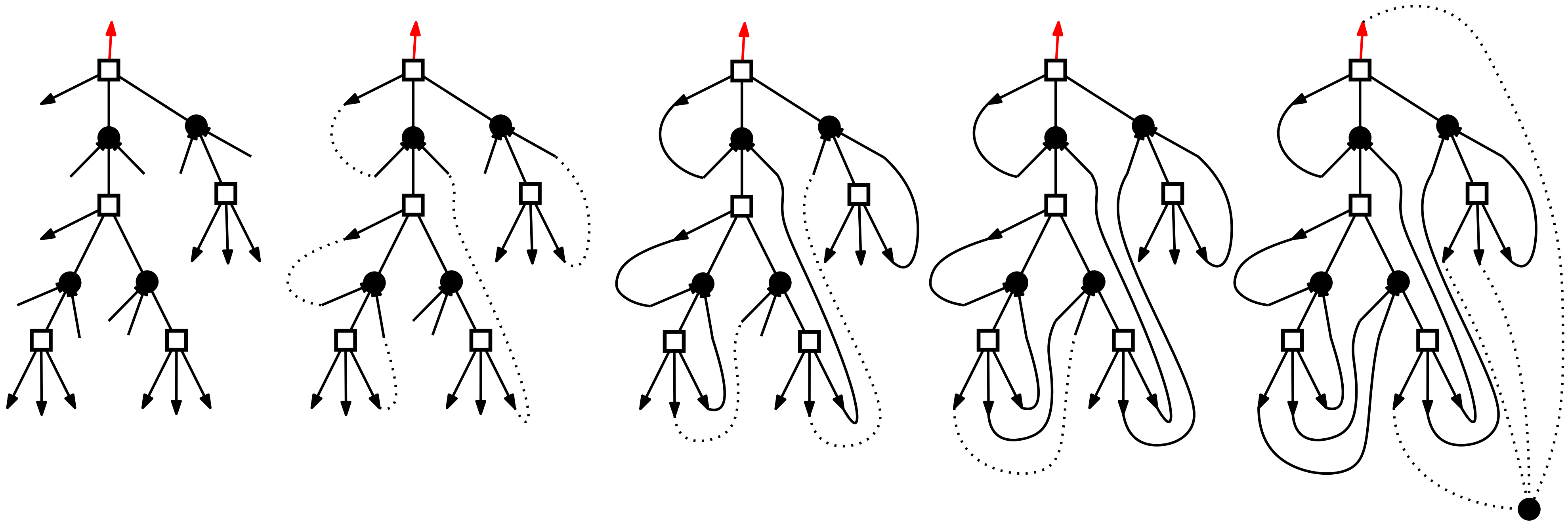

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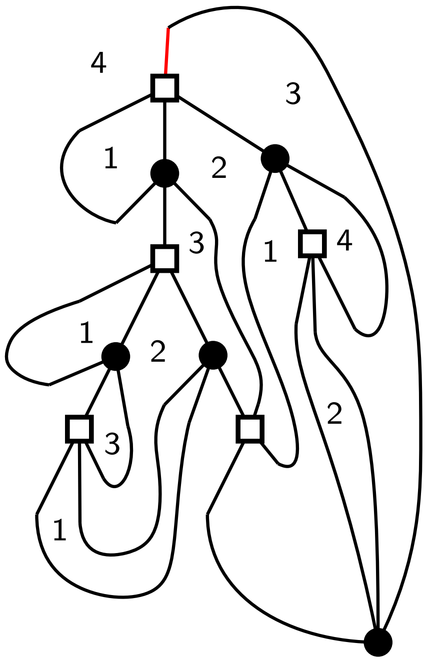
Up to rerooting, being well rooted occurs for m out of the $mn - 2(n-1)$ possible root of an unrooted tree...

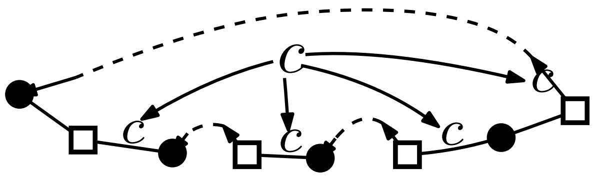
Corollary (BMS 2000) The number of m-eulerian maps with n black and n white vertices is

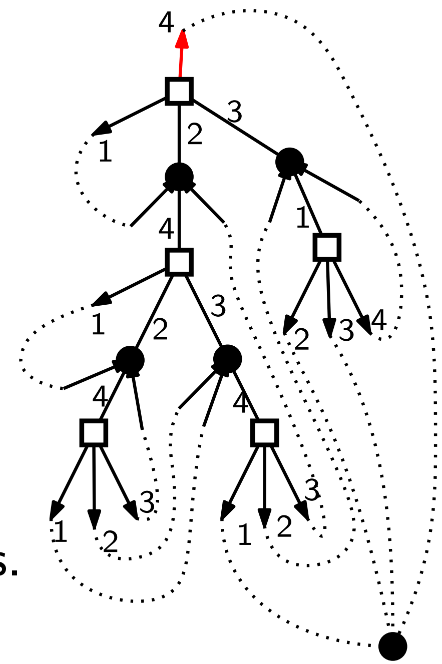
$$\frac{m}{(m-2)n+2}\frac{1}{(m-2)n+1}\binom{(m-1)n}{n}\cdot(m-1)^{n-1}$$

Face colors in the closure and Hurwitz trees

Recall that faces are colored in m colors:
transfert the colors to the tree







Each face of degree k yields

- 1 colored outgoing leaf,
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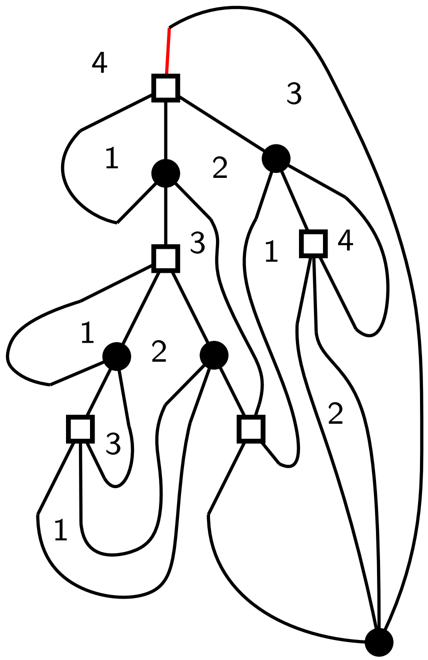

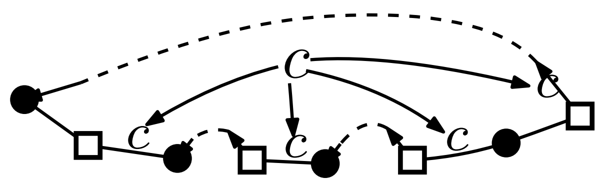

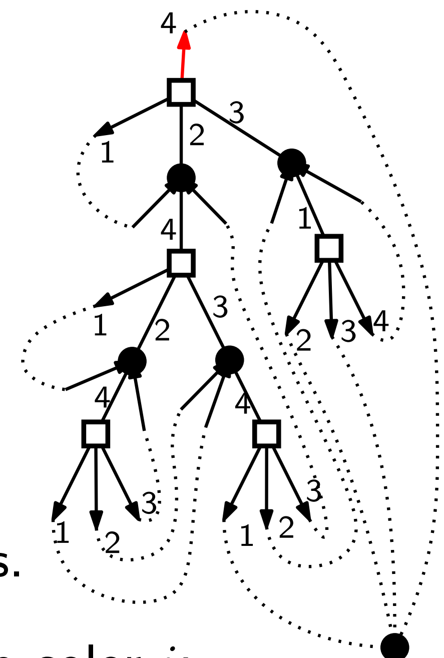

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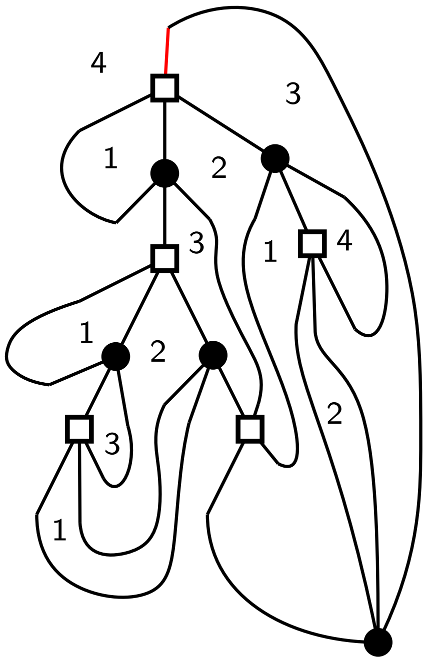

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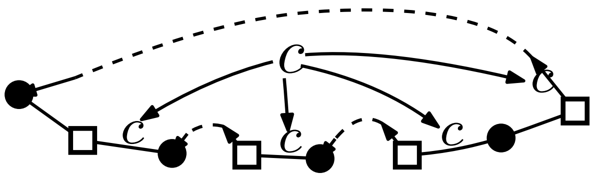

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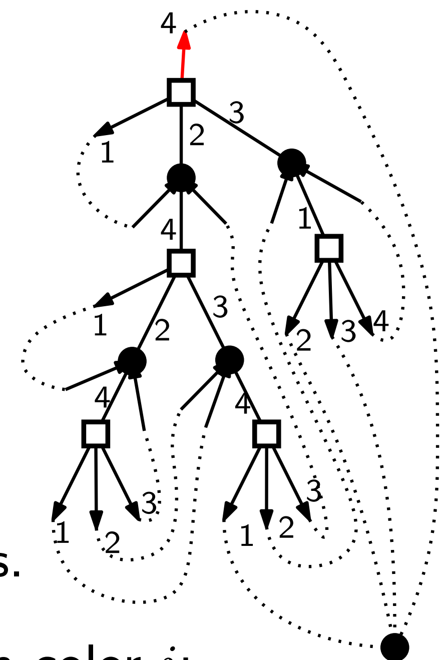

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Corollary. There is a bijection between:

- increasing quadrangulations with $2n-2$ faces
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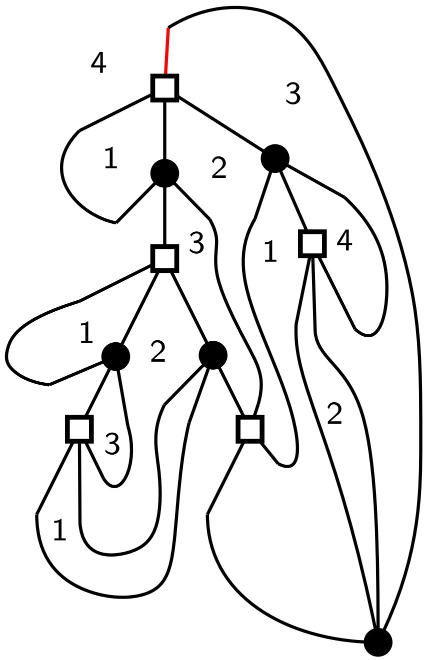
Face colors in the closure and Hurwitz trees



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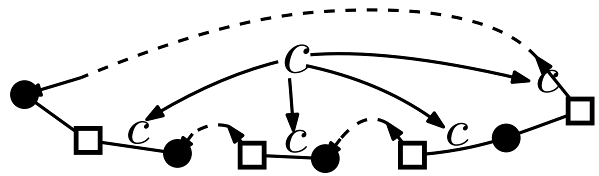

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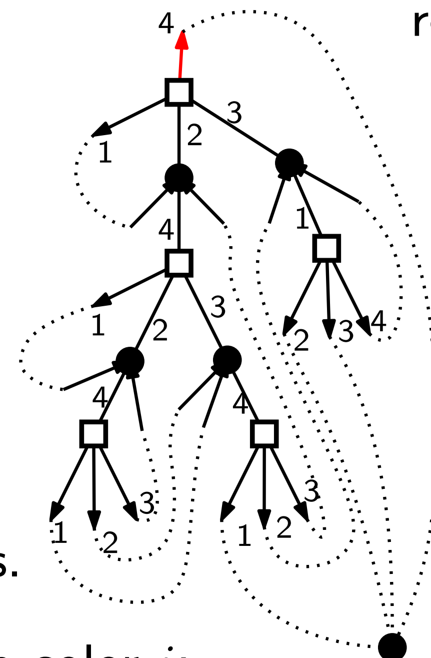
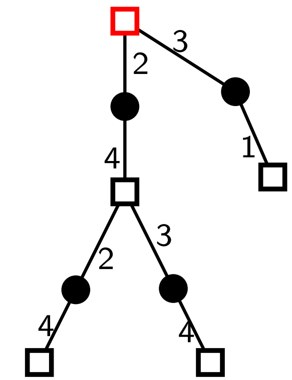


Half edge can be recovered from labels



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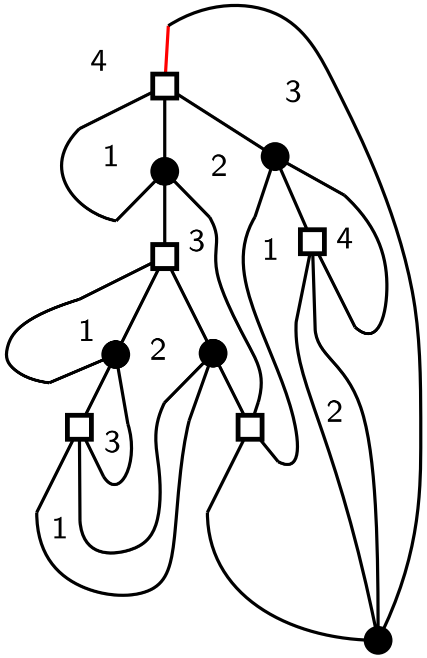
Hurwitz trees

Face colors in the closure and Hurwitz trees



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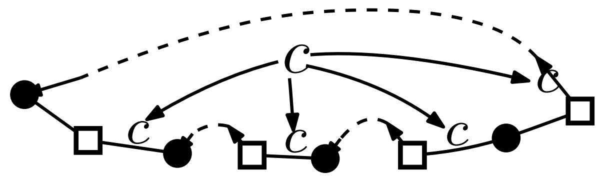

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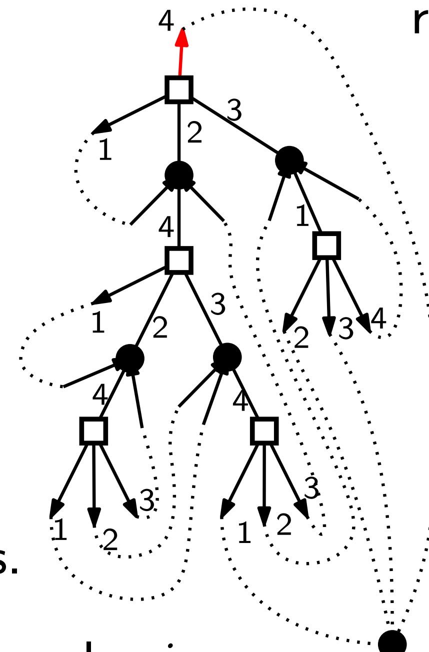

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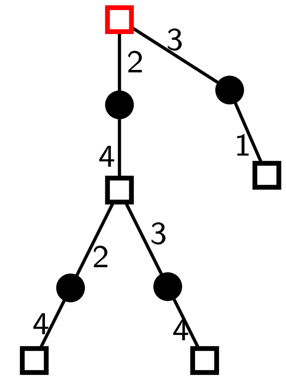

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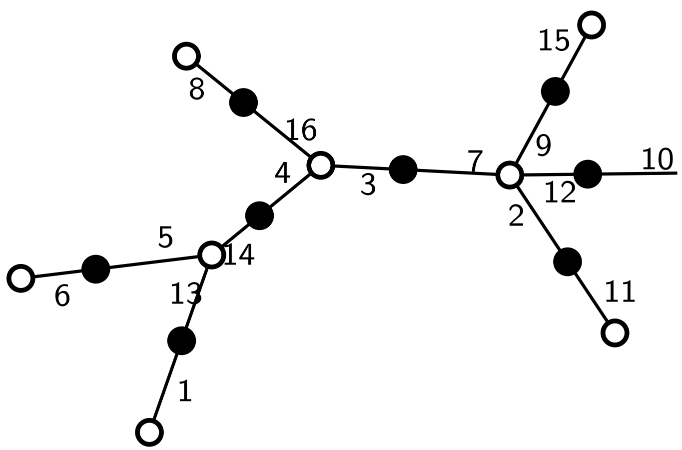
Hurwitz trees

3 ways to count: $m \to \infty$ / weight for faces / direct counting

Counting Hurwitz trees

Theorem (Duchi-Poulalhon-S. 2014) Increasing quadrangulations of size n are in bijection with simple Hurwitz trees having n white vertices, $n-1$ black vertices of degree 2, $2n-2$ labeled edges.



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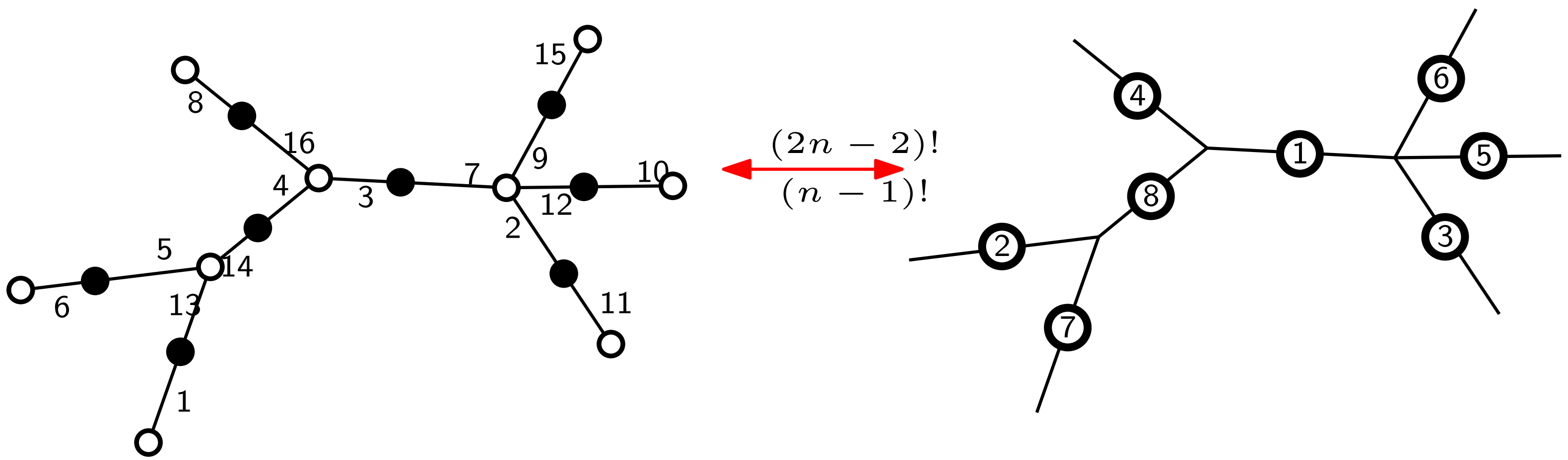

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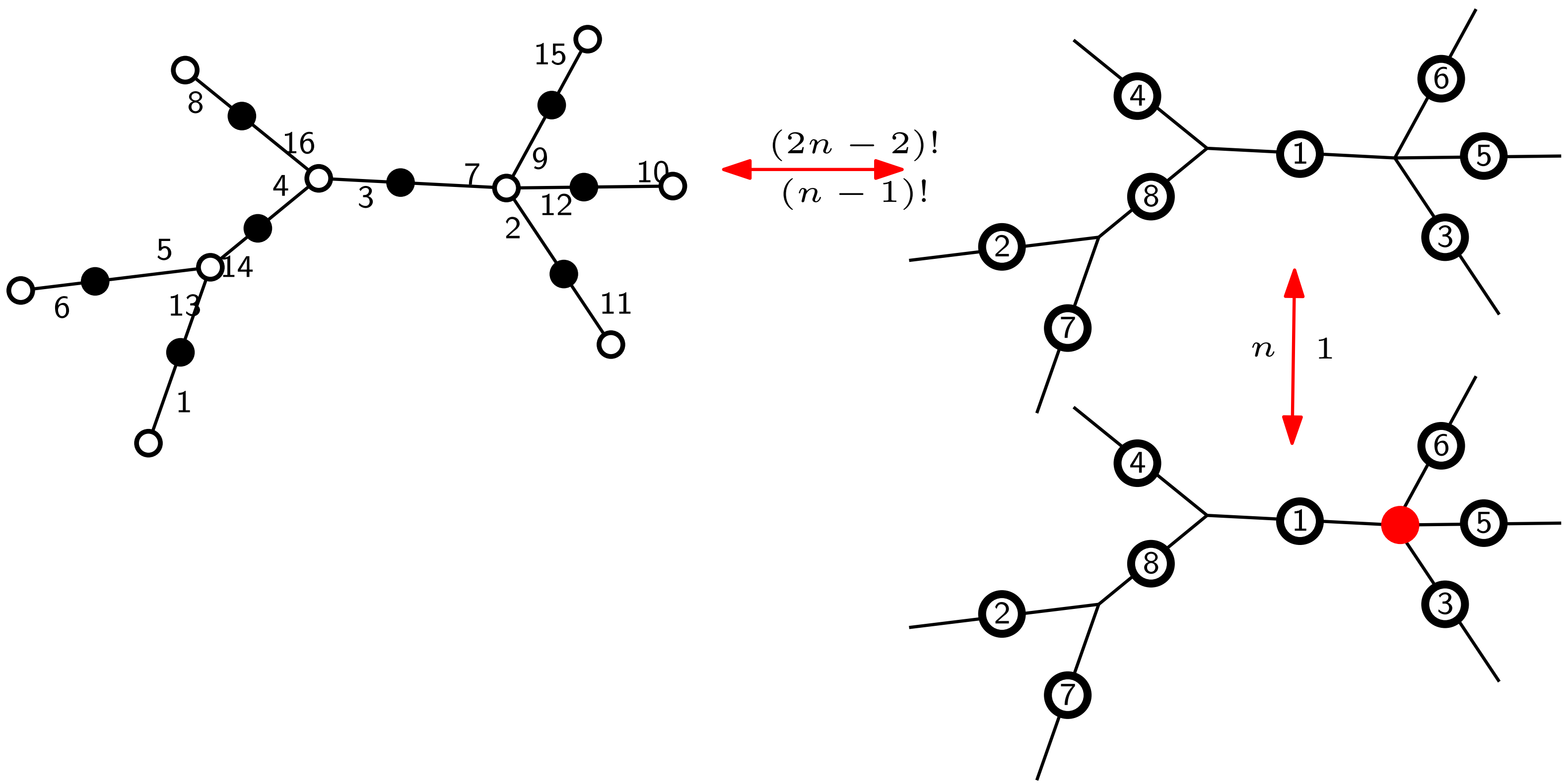

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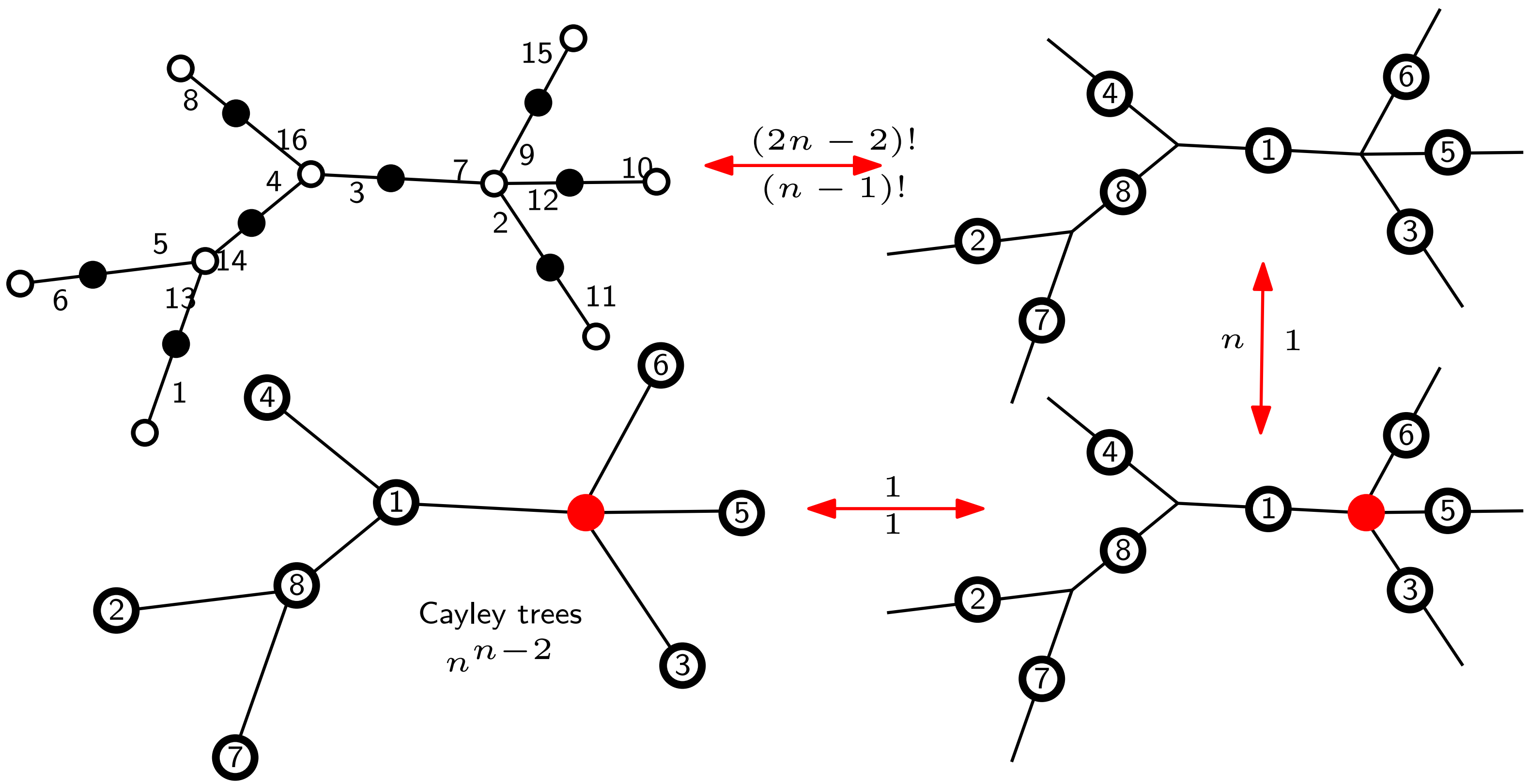

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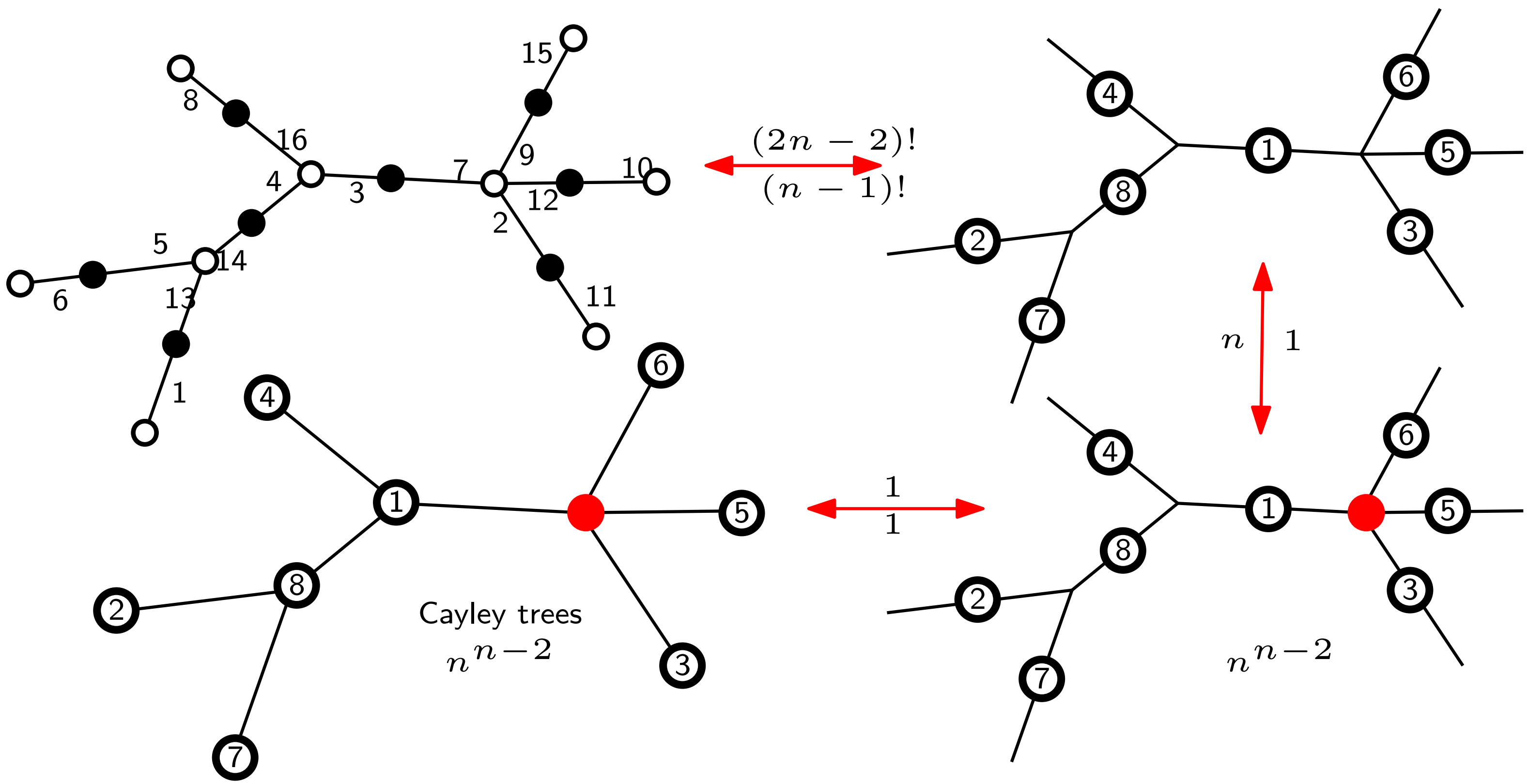

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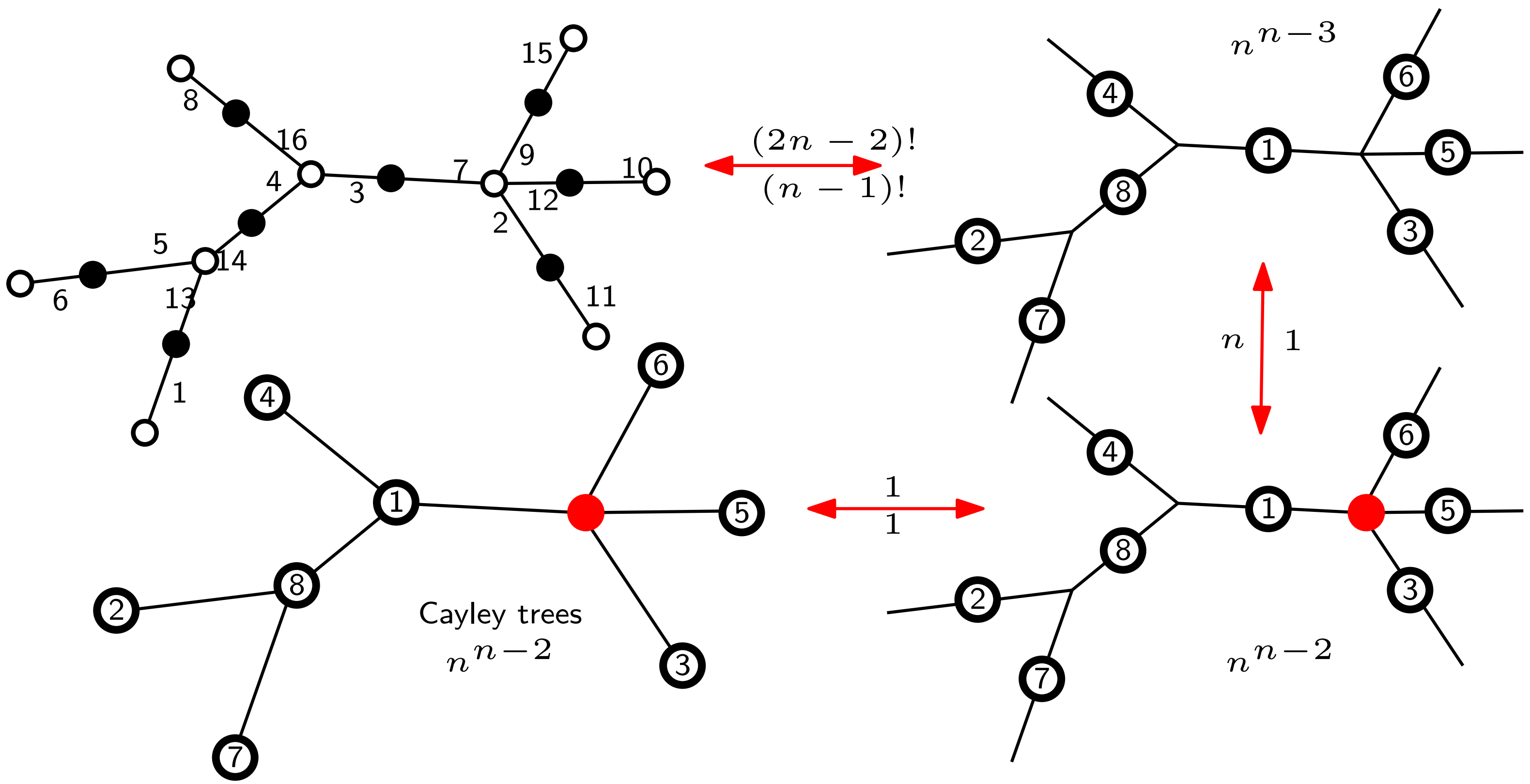

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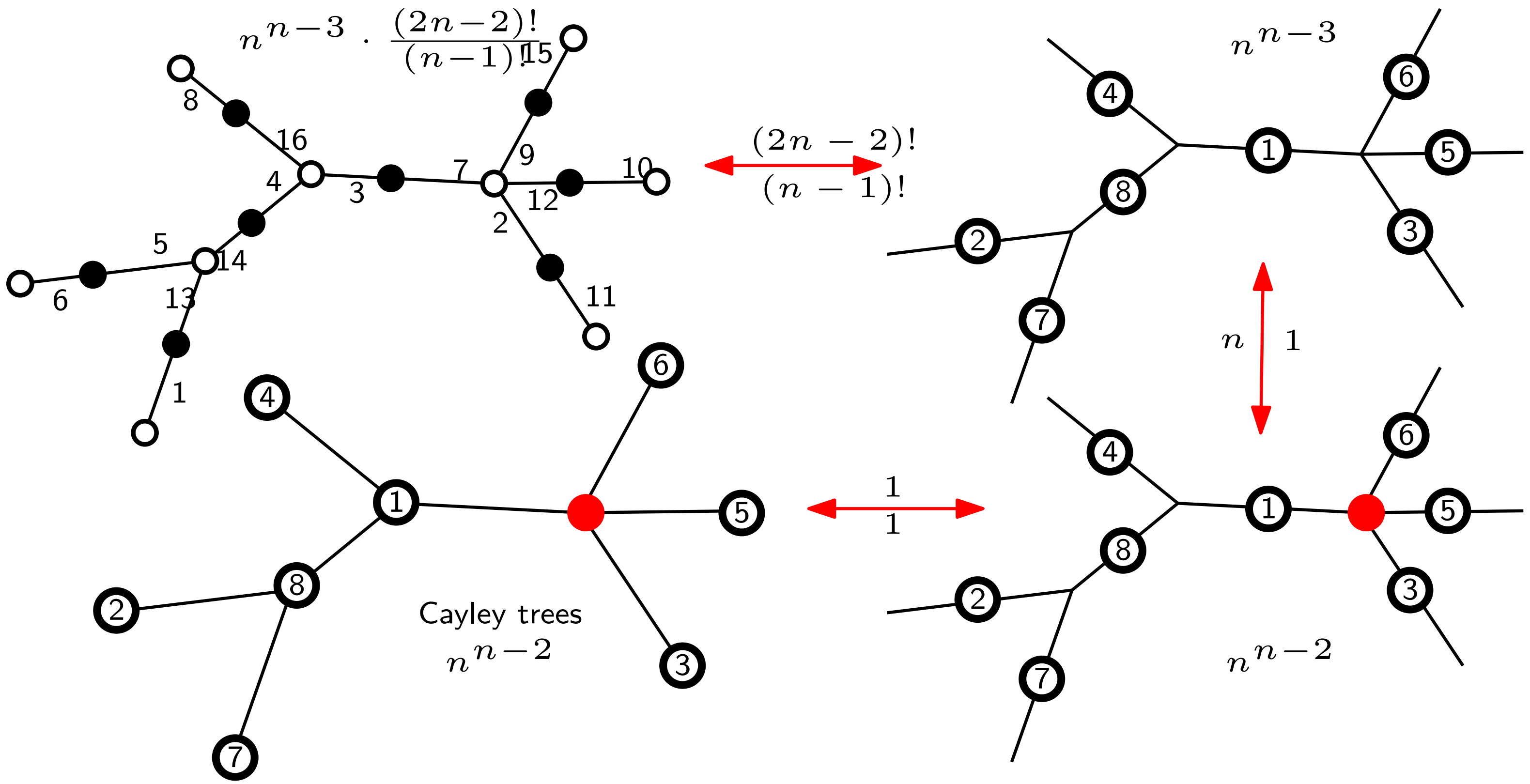
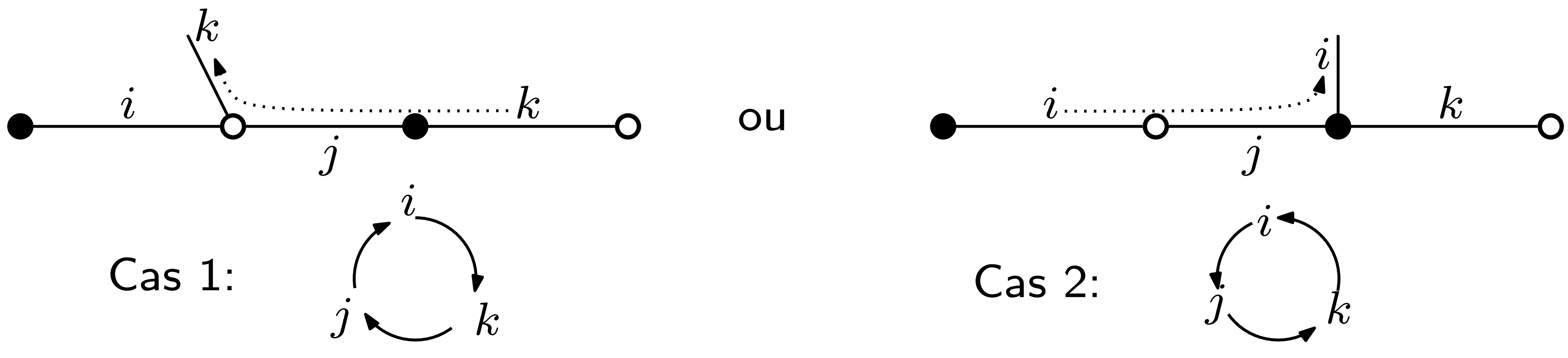


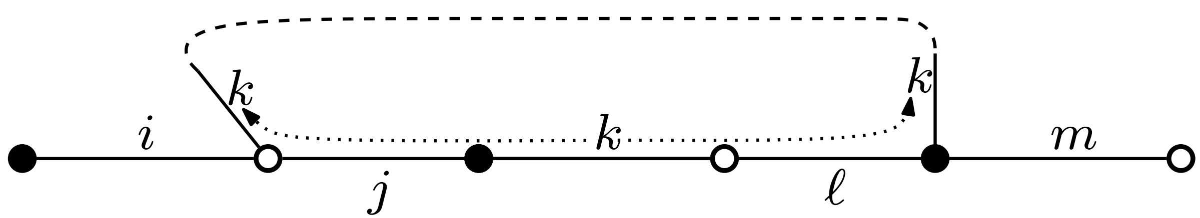
Hurwitz thm: $\#\{\text{increasing quadrangulations of size } n\} = n^{n-3}(2n-2)!/(n-1)!$.

From simple Hurwitz trees to increasing quadrangulations

A local rule to create increasing half edges

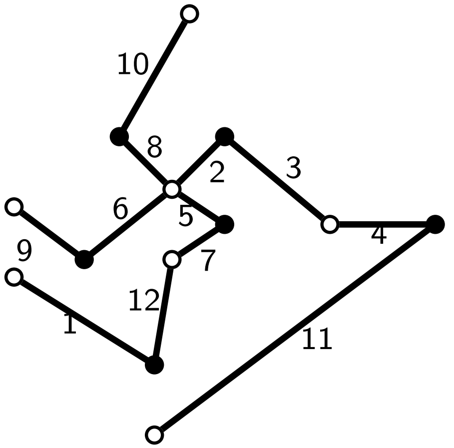


Two half-edges with same label ⇒ edge and face of degree 4



Iterate the local rules as long as possible...

From simple Hurwitz trees to increasing quadrangulations



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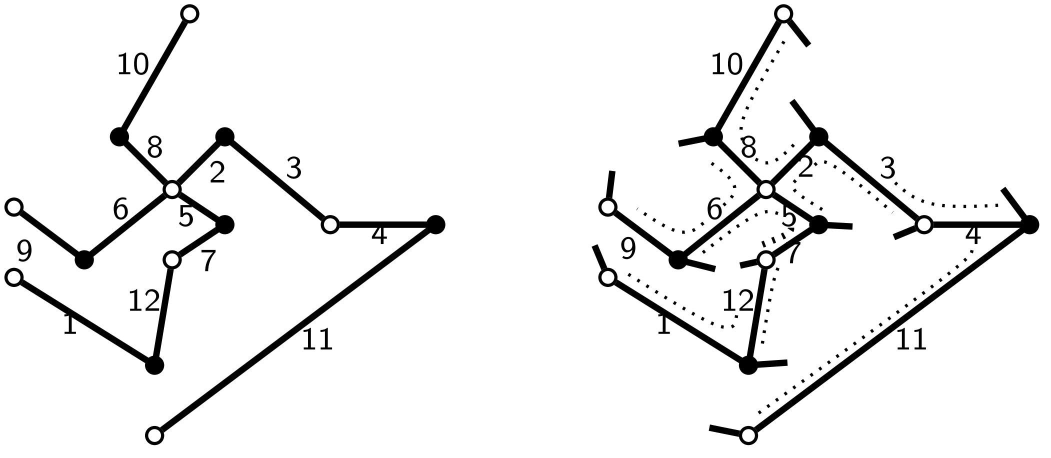
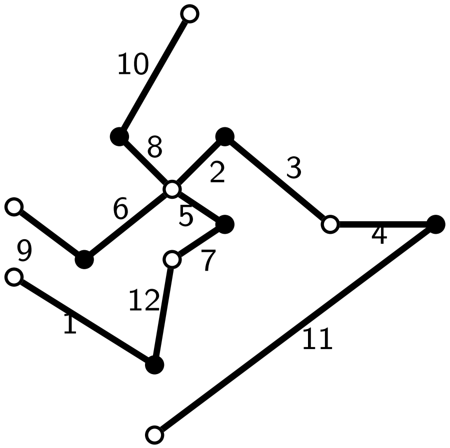
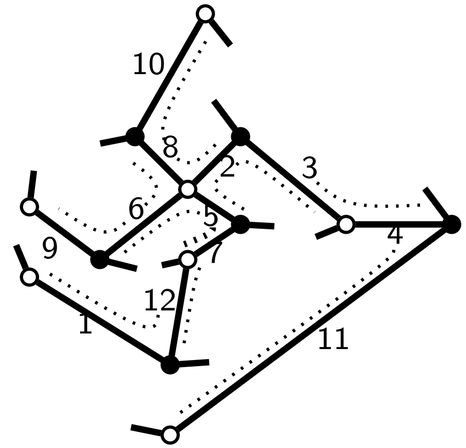


adding buds

From simple Hurwitz trees to increasing quadrangulations





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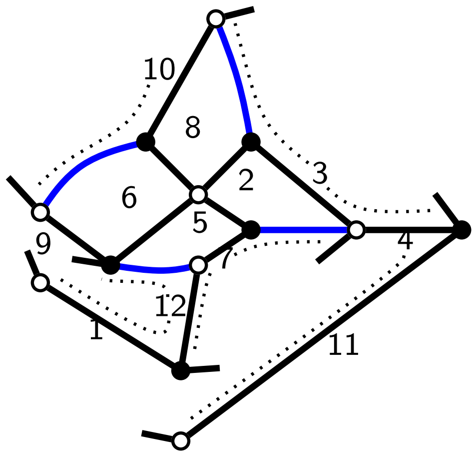
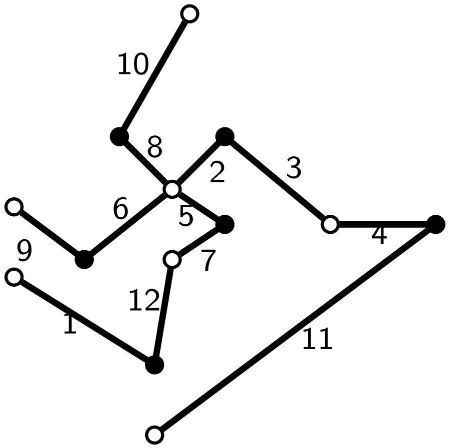
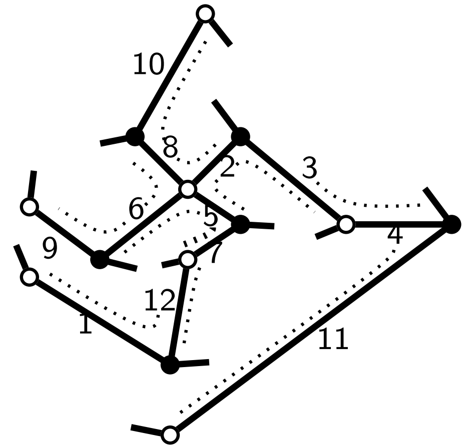


Parings and adding buds again

From simple Hurwitz trees to increasing quadrangulations





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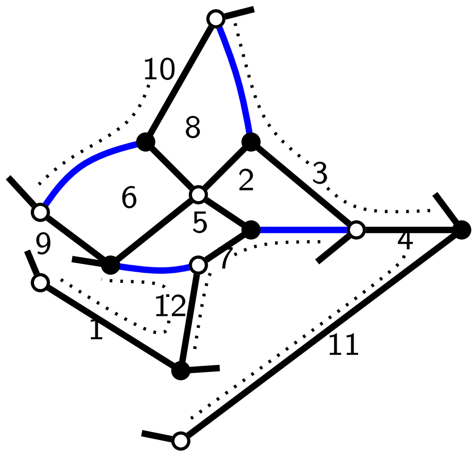

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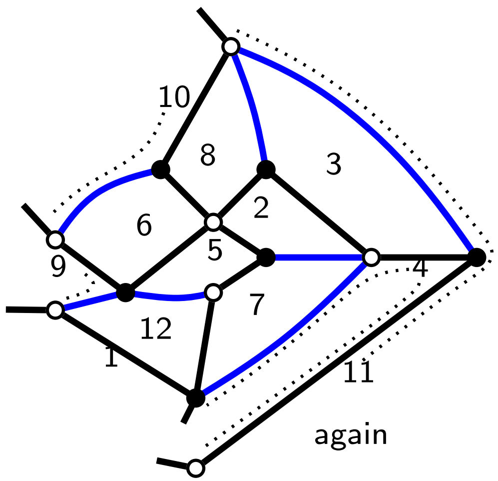

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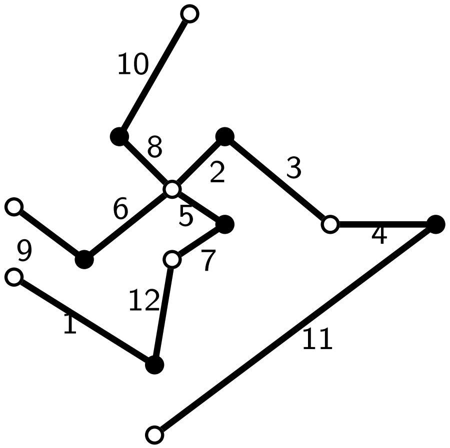

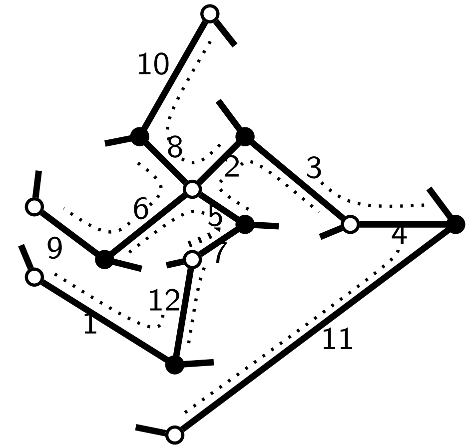

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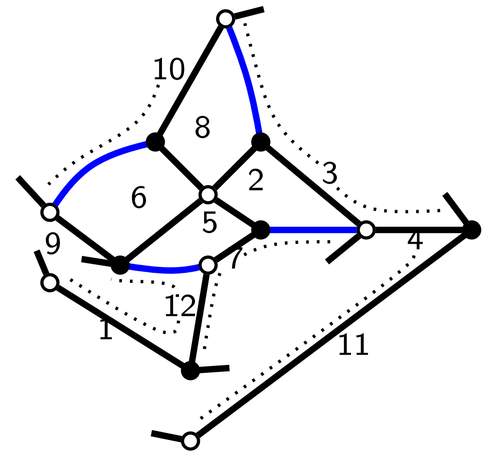

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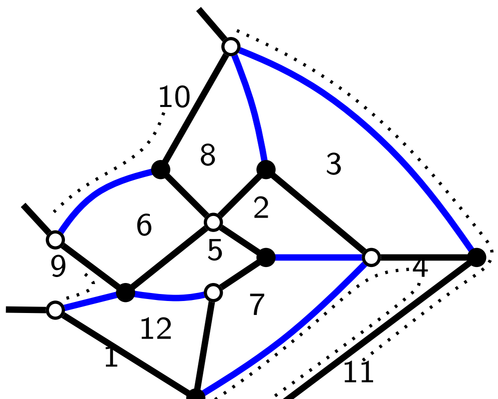

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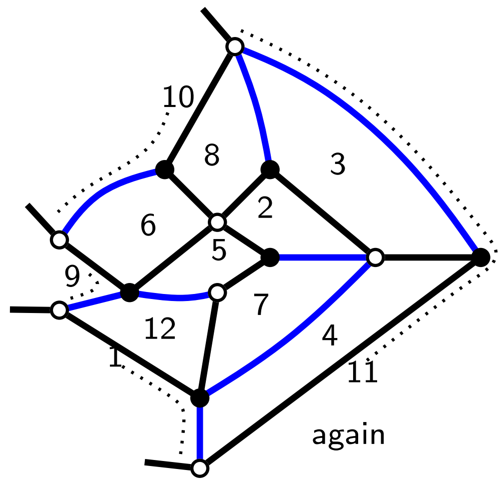

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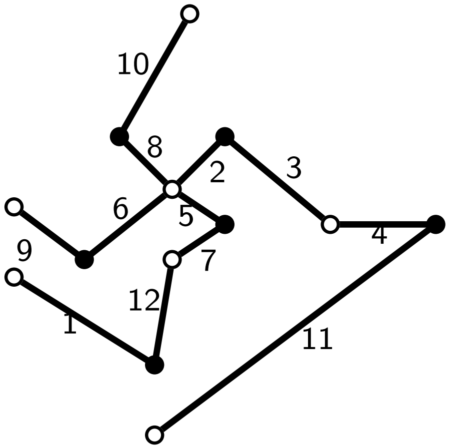

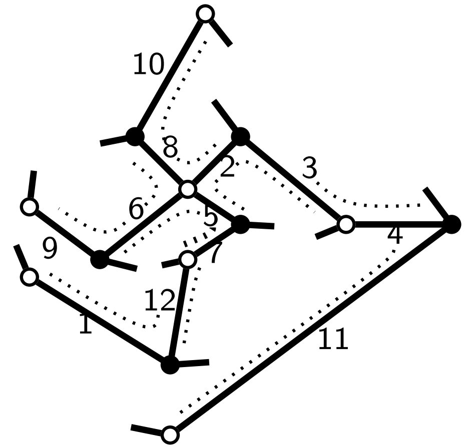

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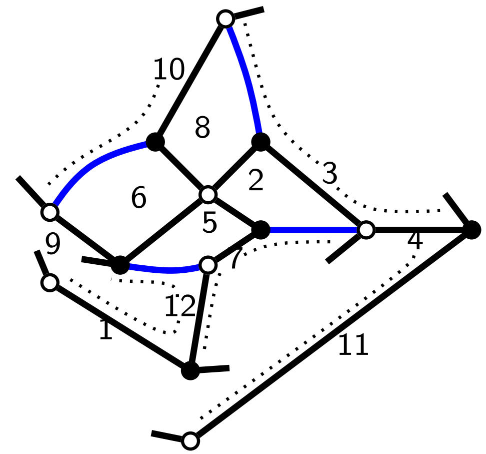

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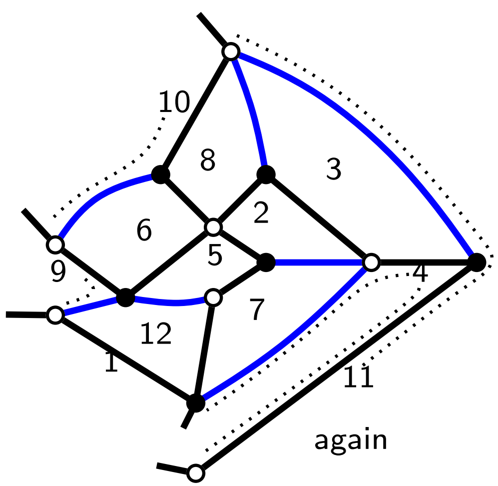

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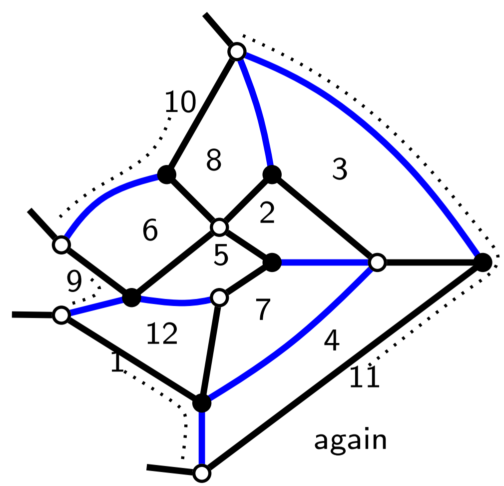
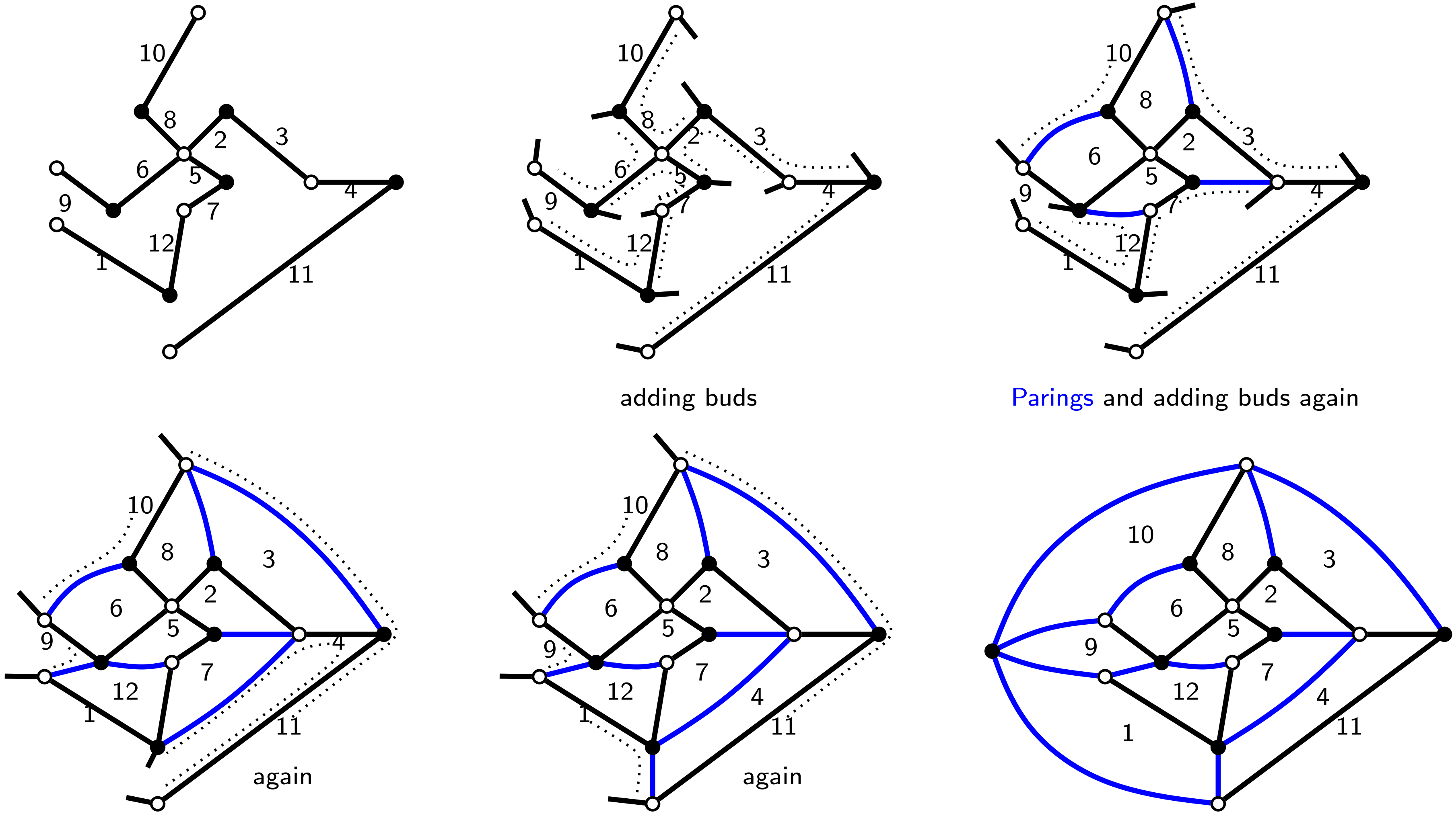


again

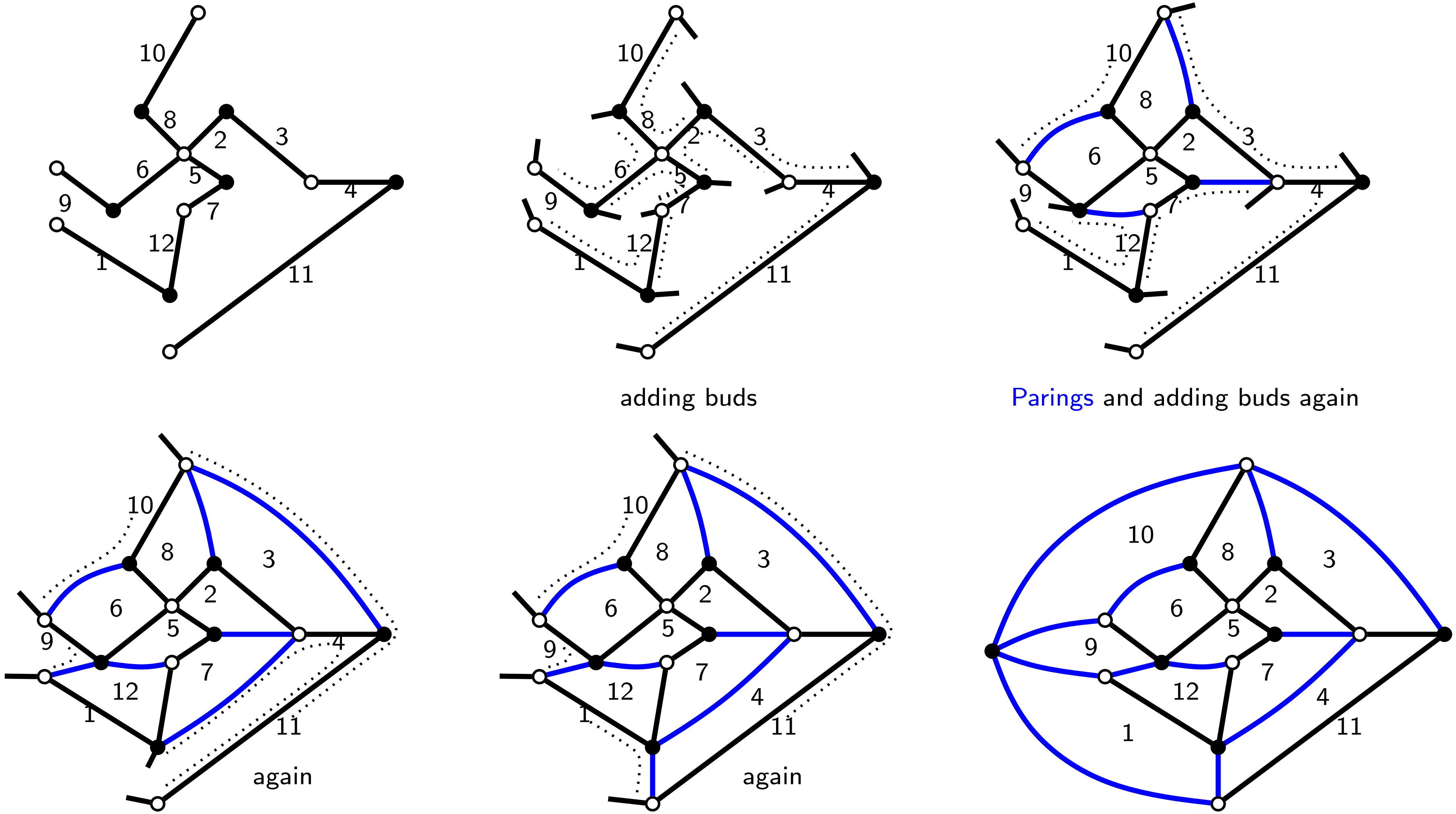
Lemma. When it stops, there are only white half-edges left.

From simple Hurwitz trees to increasing quadrangulations



Lemma. When it stops, there are only white half-edges left.
We connect them to a new black vertex.

From simple Hurwitz trees to factorizations



Theorem. Labelled closure is a bijection between

– simple Hurwitz trees of size n, and
– increasing quadrangulations of size n.

Plan of the talk

Unlabeled VS Increasing quadrangulations...

a conjecture

Why increasing quadrangulations?

Hurwitz numbers and branched covers

A bijection with Cayley type trees!

More evidences from higher genus maps...

as a conclusion

Pandharipande/Zagier's recurrence

Theorem (Carrell-Chapuy 14) The numbers Q_g^n of rooted bipartite quadrangulations of genus g with n faces satisfy the simple quadratic recurrence:

$$\frac{n+1}{6}Q_g^n = \frac{4n-2}{3}Q_g^{n-1} + \frac{(2n-3)(2n-2)(2n-1)}{12}Q_{g-1}^{n-2} + \frac{1}{2}\sum_{\substack{k+\ell=n \\ k,\ell\geq 1}}\sum_{\substack{i+j=g \\ i,j\geq 0}}(2k-1)(2\ell-1)Q_i^{k-1}Q_j^{\ell-1}.$$

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Both results arise similarly from the fact that the generating series satisfies a set of differential equations, called KP hierarchy

for quadrangulations: pre2000 physics + Goulden-Jackson 2006
for Hurwitz: Pandharipande 2000, Okounkov 2000

see surveys of Kazarian and Lando 2015 and Chapuy 2018 (habilitation)

A similar quadratic recurrence exists for m-Eulerian maps (B. Louf 2018++)

Corollary of PZ recurrence for gf

The Carrell-Chapuy recurrence allows to recover Tutte's expression for the gf $Q(z)$ of planar quadrangulations, and to rederive directly the following corollary:

Corollary (Bressis-Itzykson-Zuber 80, Bender-Canfield 91) The fixed genus gf $Q_g(z)$ of quadrangulations is a rational function of the planar gf $Q(z)$.

See Lepoutre 2018 for a bijective proof.

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Similarly the Pandharipande-Zagier recurrence allows to recover Hurwitz's expression for the gf $H(z)$ of increasing planar quadrangulations, and to rederive directly the following corollary:

Corollary (Goulden-Jackson-Vakil 2001) The fixed genus gf $H_g(z)$ of increasing quadrangulations is a rational function of the planar gf $H(z)$.

No bijective proof is known.

$\Rightarrow$ adapt Lepoutre, or use BDG bijection?

Corollary of PZ recurrence for asymptotic

The Carrell-Chapuy recurrence also allows to recover the asymptotic behavior of Q_n^g:

$$Q_n^g \underset{n\to\infty}{\sim} t_g \cdot n^{(5g-1)/2} \cdot 12^n$$

where $\tau_g = t_g \cdot 2^{5g-2} \cdot \Gamma(\frac{5g-1}{2})$ satisfies simple a quadratic recurrence related to a Painlevé I equation.

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These two similar behaviors are characteristic of 2d pure quantum gravity models and form the main supporting evidence for our conjecture.

Conclusion

I have been concentrating in the talk on simple Hurwitz numbers

This is just the tip of the Iceberg...

- explicit formulas and bijective proofs extends to single Hurwitz numbers $H^0(\lambda)$ and partially to double Hurwitz numbers $H^0(\lambda, \mu)$

- the BDFG bijection can be used instead of blossoming trees

  $\Rightarrow$ leads to Hurwitz mobiles instead of Hurwitz trees
  $\Rightarrow$ extends to higher genus but not clear how to get explicit counting results

- in both cases one can track an "oriented pseudo distance" but it has $\Theta(n)$ increments!

- the results can be rephrased in terms of transitive factorizations of permutations in products of transpositions

  $\Rightarrow$ leads to a cut and join equation that plays the role of Tutte's equations but it is not clear how to use this for peeling

- analog questions arise for inequivalent or monotone factorizations, and for the weighted Hurwitz numbers that generalize them.

Thank you for your attention!