Positive trees and equations with one catalytic variable and one small unknown

GILLES SCHAEFFER
LIX, CNRS, Institut Polytechnique de Paris

based on joined work with ENRICA DUCHI,
IRIF, UNIVERSITÉ DE PARIS

Pour les 40 ans de l'article *Planar Maps are Well Labeled Trees*
Robert Cori et Bernard Vauquelin, Canad. J. Math. 1981
11 octobre 2021, Bordeaux

Summary of the talk

Cori-Vauquelin's bijection, reloaded

A simple case study: Bicolored binary trees

Equations with one catalytic variable and one small unknown
& systematic algebraic decompositions

Examples and applications

Extrait 1: talk at Séminaire Hypathie 2001

distances in quadrangulations and local rules, applications to random maps

Extrait 2: talk in honor of Robert Cori, 2009

from Cori-Vauquelin's "éclatement" to the local rule

Extrait 3: talk at AofA 2014

local rules, Miermont's roundup rule and 'patrons'

These various reformulations aim at explaining why we get well labeled trees from maps

Currently the best explanation is given by the slice decompositions, as explained by Grégory.

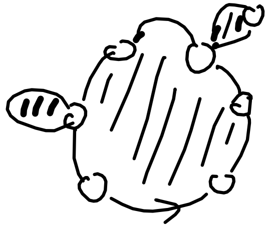
Moreover the reformulation explain how one could deduce the local rules from the 'éclatement'

not really how we could have found the éclatement without Bernard and Robert...

So, how could we have found the éclatement without Bernard and Robert... ?

Before that, Robert had obtained various encodings of rooted planar maps with words in differences of algebraic languages... even more mysterious to me...

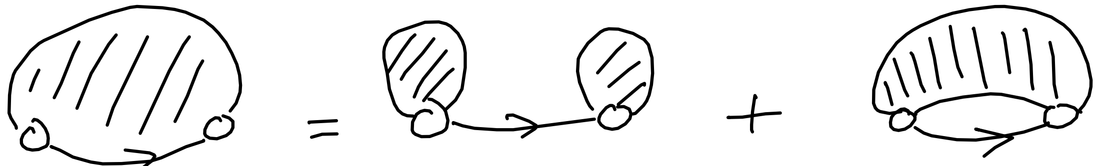
Even before W.T. Tutte had given recursive decompositions using catalytic parameters:



$|T| = \#$ arêtes

$d(T) =$ external degree

$$F(u) \equiv F(t,u) = \sum_T t^{|T|} u^{d(T)}$$



$$F(u) = tu^2(1+F(u))^2 + t\,F(u^k \mapsto u+\dots+u^{k+1})$$

It is now well understood why Tutte's decomposition "easily" imply the final algebraic equations, thanks to Bousquet-Melou–Jehanne theorem (more later)

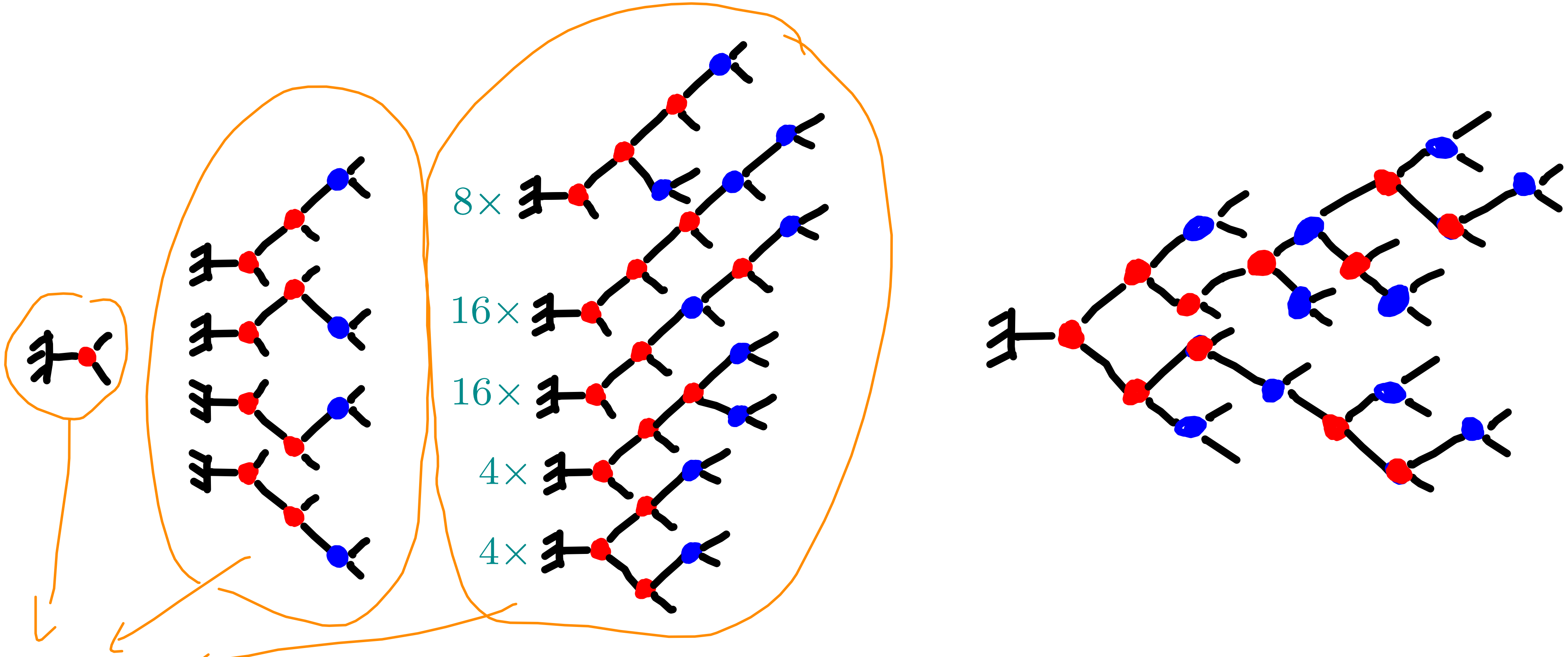
Could we have deduced the bijection with trees, or at least some direct algebraic decompositions from Tutte's equation?

A simple case study: Bicolored binary trees

Dyck-Łukasiewicz trees

$\mathcal{B} = \{$Bicolored trees$\}$: rooted binary trees with blue and red inner vertices.

$\mathcal{D} = \{$Dyck-Łukasiewicz trees$\}$: one more red vertex than blue and no more red vertices than blue in each strict subtree



1, 4, 48, 832, 17408, 408576, 10362880, 277954560, 7777026048, 224908017664

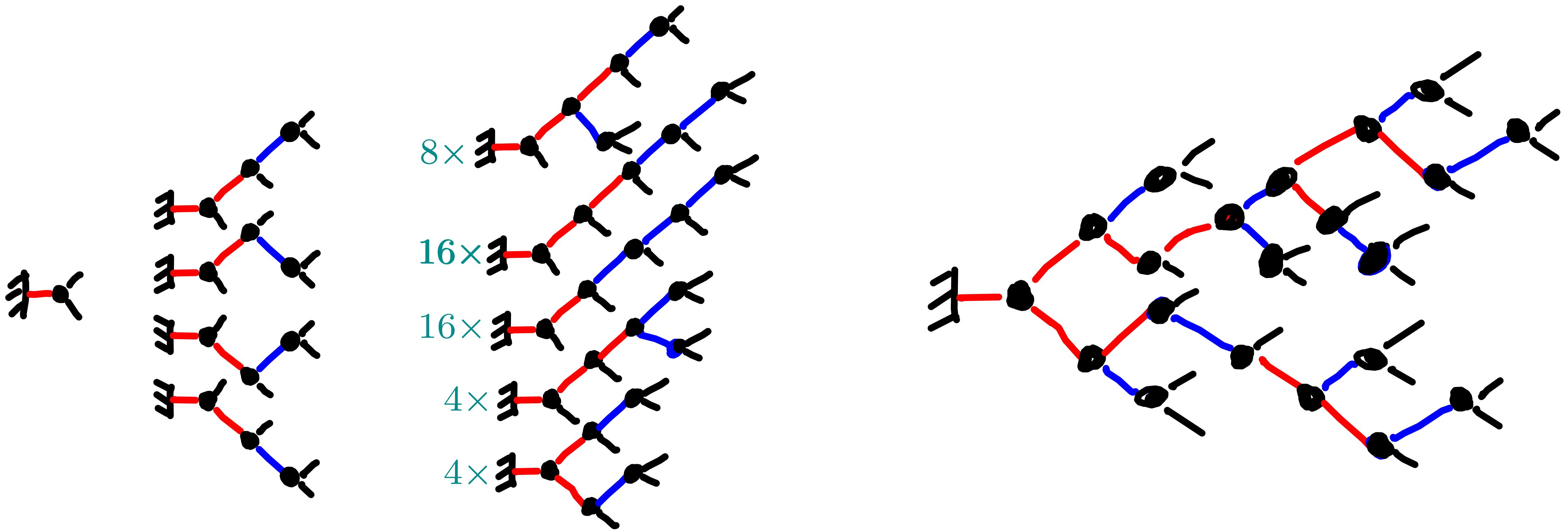
(fun game if you are tired of listening: guess formula... you have 10 min before I give it)

Reformulation as edge-bicolored trees

$\mathcal{B} = \{$blue/red binary trees$\}$: planted binary tree with blue and red edges

$\mathcal{P} = \{$Positive bicolored trees$\}$: no more red than blue in each planted subtree

$\mathcal{D} = \{$Dyck-Łukasiewicz trees$\}$: positive + one more red edge than blue



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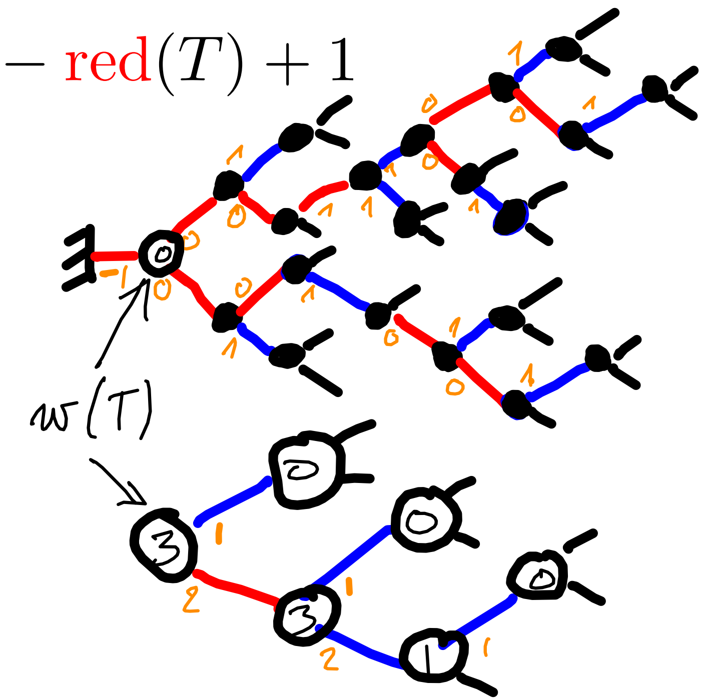
A catalytic decomposition for positive bicolored trees

Let $F(u) \equiv F(u,t) = \sum_{T \in \mathcal{P}} u^{w(T)} t^{|T|}$, with $w(T) = \textcolor{blue}{\text{blue}}(T) - \textcolor{red}{\text{red}}(T) + 1$

so that $f \equiv f(t) = [u^0]F(u) = \sum_{T \in \mathcal{D}} t^{|T|}$ is the gf of Dyck trees

and more generally $F_m = [u^m]F(u)$ is the gf of positive tree with root vertex weight m.



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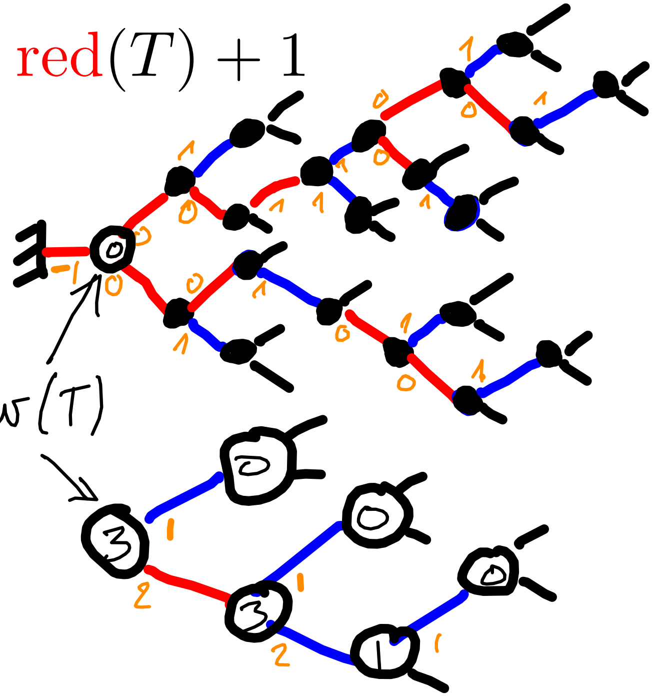
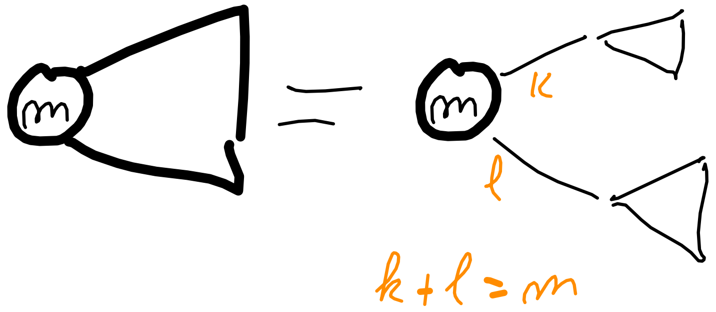
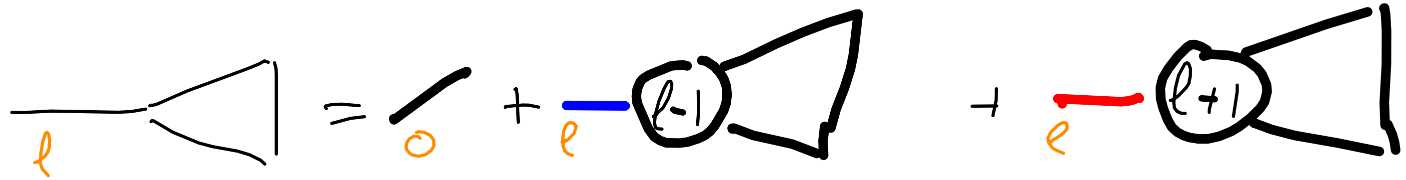


Then:

$$F(u) = tX(u)^2 \qquad \text{with} \qquad X(u) = 1 + u \cdot F(u) + \frac{F(u) - f}{u}$$





One variable / one function catalytic equations are easy

Bousquet-Mélou–Jehanne's trick gives an algebraic system

$\frac{\partial}{\partial u}$ applied to $$F(u) = t\left(1 + u\,F(u) + \frac{F(u) - f}{u}\right)^2$$

yields $$\frac{\partial}{\partial u}F(u) = \frac{\partial}{\partial u}F(u) \cdot 2t\ \left(u + \tfrac{1}{u}\right)\left(1 + u\,F(u) + \tfrac{F(u)-f}{u}\right)$$
$$+2\left(F(u) - \tfrac{1}{u}\ \tfrac{F(u)-f}{u}\right)\left(1 + u\,F(u) + \tfrac{F(u)-f}{u}\right)$$

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cancels

Let $U \equiv U(t)$ be the unique fps s.t. $U = 2t\,\left(U^2 + 1\right)\left(1 + U\,F(U) + \frac{F(U)-f}{U}\right)$

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$$\begin{cases} U &= 2t\,(U^2+1)\,(1+U\,V+W) \\ 0 &= U\,V - W \\ V &= t\,(1+U\,V+W)^2 \end{cases}$$

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Equations for Dyck trees are particularly simple!

The system can be further simplified

$$\begin{cases} U = 2t\left(U^2+1\right)(1+2\,U\,V) \\ V = t\,(1+2\,U\,V)^2 \end{cases}$$

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$$\Rightarrow \left\{ \begin{array}{lcl} U & = & \frac{2t\,(1+2U\,V)}{1-2tU\,(1+2U\,V)} \\ V & = & \frac{t\,(1+2U\,V)}{1-2tU\,(1+2U\,V)} \end{array} \right. \Rightarrow \quad U = 2\,V$$

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Theorem: The gf $f = f(t)$ of Dyck trees with n vertices satisfies:

$$f = V - 4V^3 \qquad \text{where} \qquad V = t(1+4V^2)^2$$

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$$[t^n]V = \frac{1}{n}[x^{n-1}](1 + 4x^2)^{2n} = \begin{cases} \frac{4^m}{2m+1}\binom{4m+2}{m} & \text{if } n = 2m+1, \\ 0 & \text{otherwise} \end{cases}$$

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$$[t^n]f = \frac{1}{n}[x^{n-1}](x-4x^3)'(1+4x^2)^{2n} = \begin{cases} \frac{4^m}{(m+1)(2m+1)}\binom{4m+2}{m} & \text{if } n = 2m+1, \\ 0 & \text{otherwise} \end{cases}$$

Marking and identification of V

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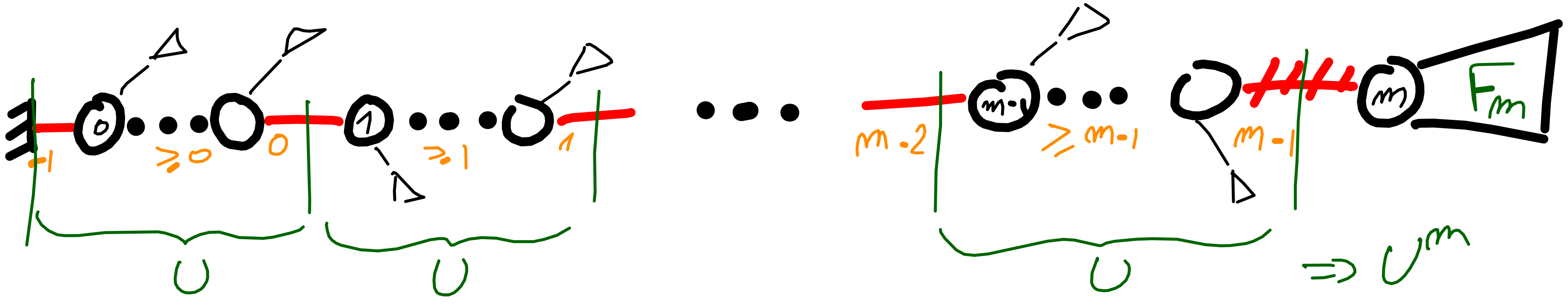
Observe that $$[t^{2m+1}]V = (m+1)[t^{2m+1}]f = [t^{2m+1}]f^{-\bullet}$$

$\Rightarrow$ V is the gf of (rooted) Dyck trees with a marked red edge

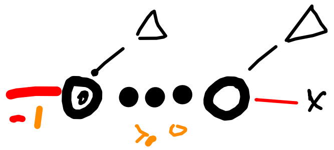
Last passage decomposition and identification of U

The series V is the gf of (rooted) Dyck trees with a marked red edge

Consider a Łukasiewicz (or last passage) factorization of the weight sequence along the branch toward the root.



Now recall we defined $V = F(U) = \sum_{m \geq 0} U^m [u^m] F(u)$

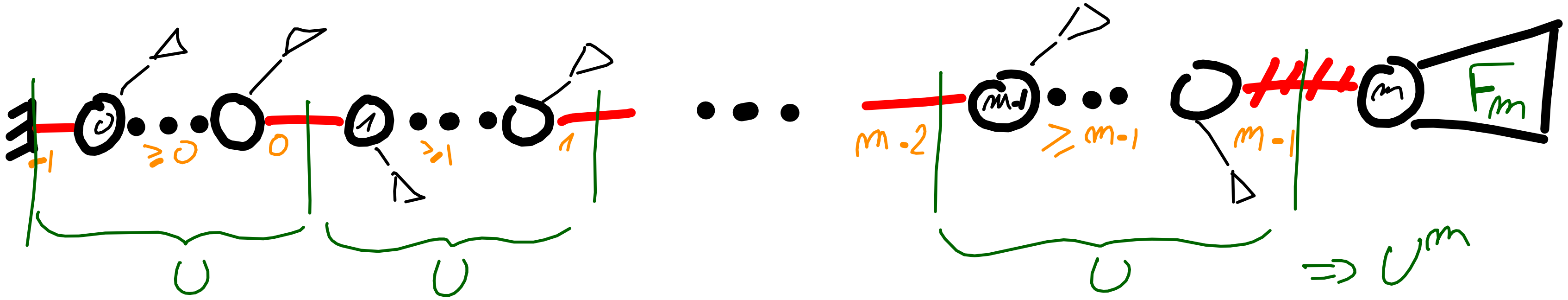
so that $U =$ 

$\Rightarrow$ our series U is the gf of Dyck trees with a marked leaf !

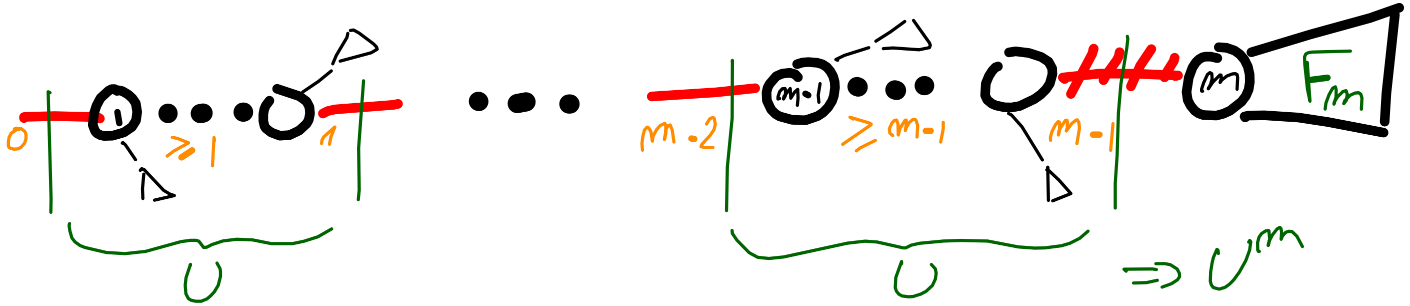
The core of a balanced tree and identification of W

The series V is the gf of (rooted) Dyck trees with a marked red edge
The series U is the gf of Dyck trees with a marked leaf

Recall $V = \sum_{m\geq 0} U^m F_m =$



hence $W = \sum_{m\geq 1} U^{m-1} F_m =$



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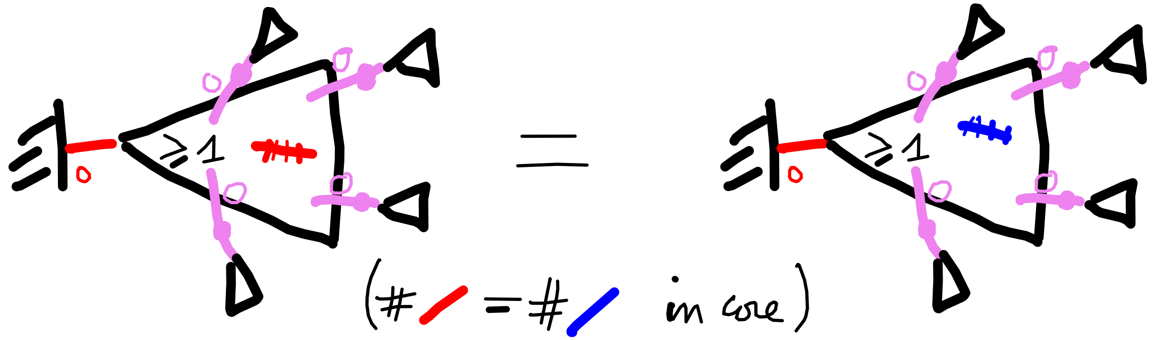
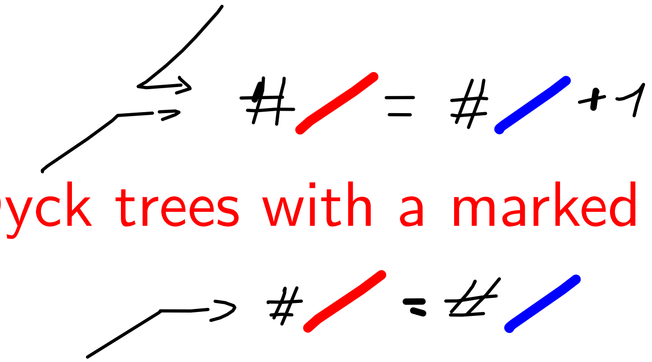


$\Rightarrow$ W is the gf of balanced positive trees with a marked blue edge in their internally positive core.

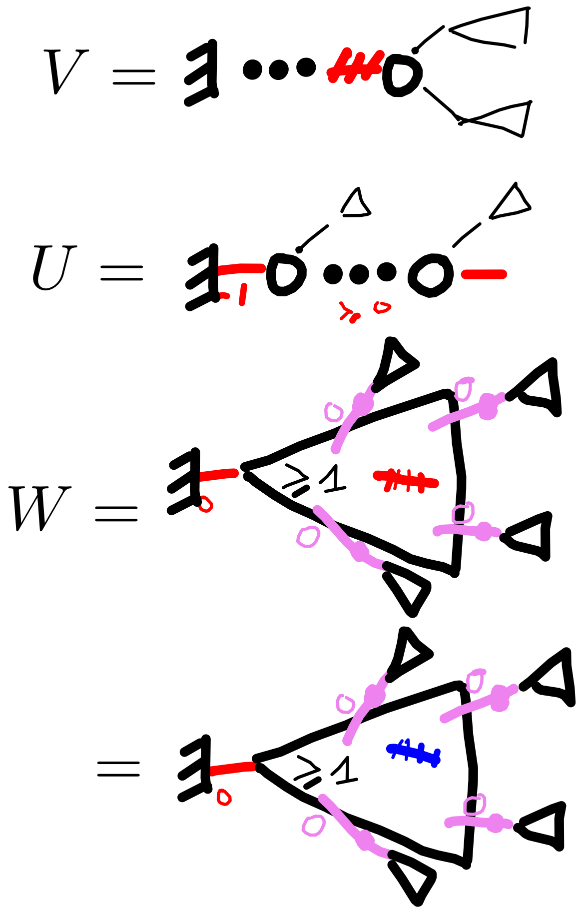
Decomposing marked Dyck-Łukasiewicz trees

Let's now restart from the combinatorial interpretations: let

- V denote the gf of (rooted) Dyck trees with a marked red edge

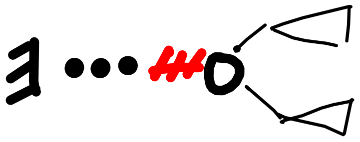


- U denote the gf of Dyck trees with a marked leaf
- W denote the gf of balanced positive trees with a marked red edge in their internally positive core.
  W is also the gf of balanced positive trees with a marked blue edge in their internally positive core.

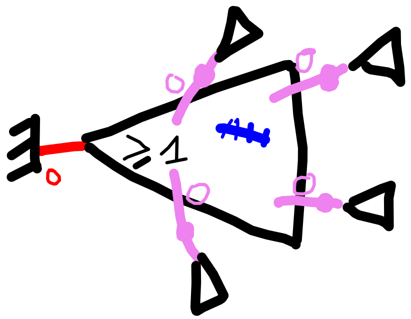


We would like a direct quaternary decomposition of these marked rooted trees to reprove directly that $V = t(1 + 4V^2)^2$.

Combinatorial derivation of $U = 2V$ and $W = U\,V$

$V =$ 

$U =$ 

$W =$ 

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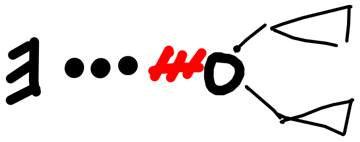
$V =$ 

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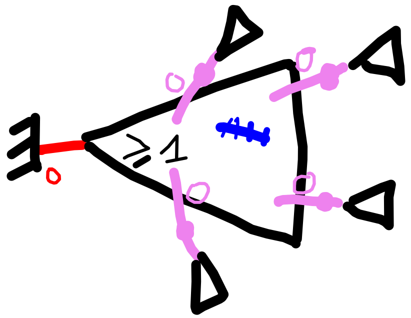
$W =$ 

Claim: There is a 2-to-1 correspondance between Dyck trees with a marked leaf and Dyck trees with a marked red edge with the same size

Immediate since a Dyck tree with $2n+1$ vertices has $n+1$ red edges and $2n+2$ leaves.

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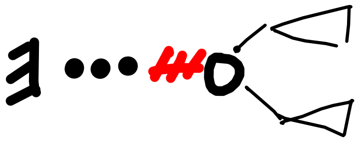
$V =$ 

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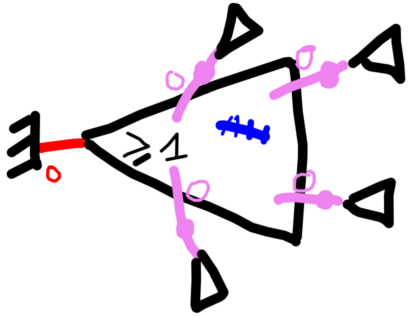
$W =$ 

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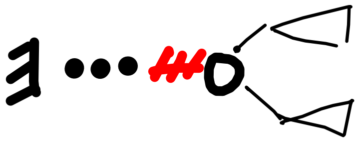
$V =$ 

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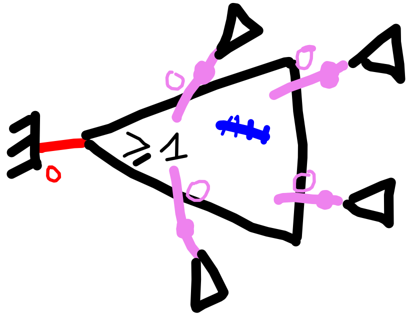
$W =$ 

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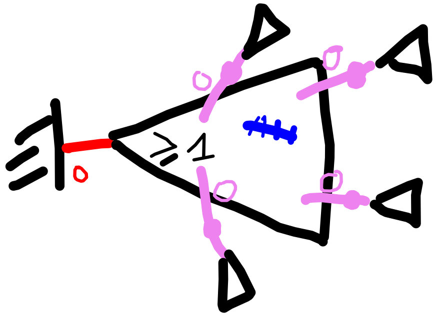
$W =$ 

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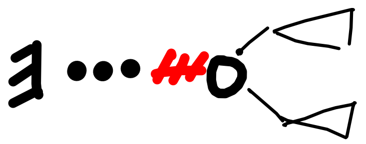
$V =$ 

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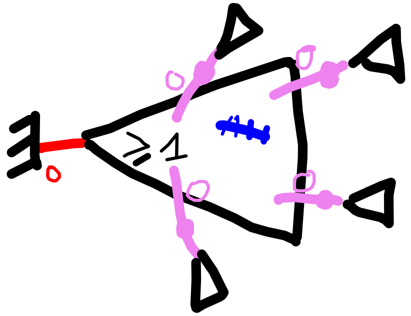
$W =$ 

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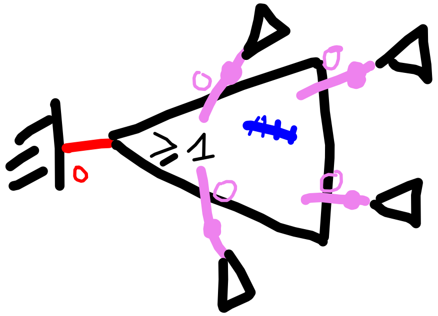
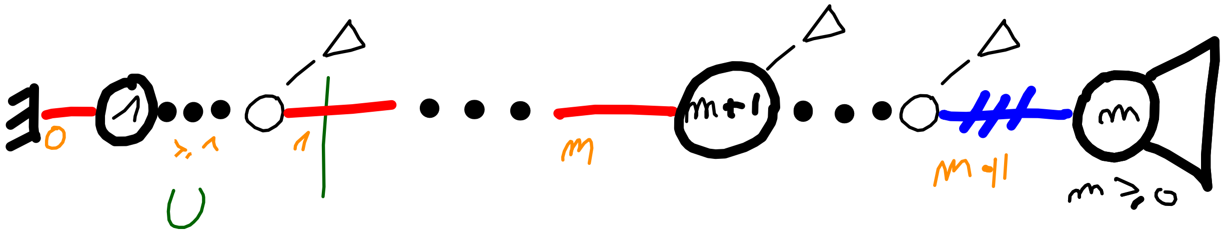
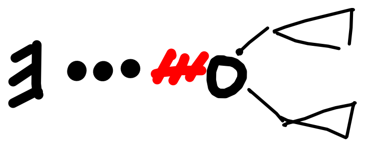
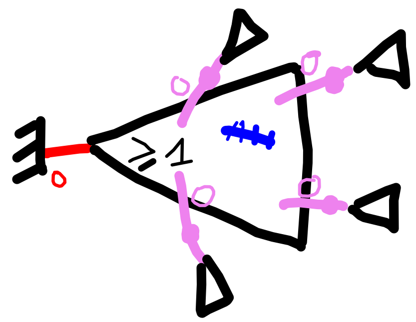
$W =$  $=$ 

Combinatorial derivation of $U = 2V$ and $W = U\,V$

$V =$ 

$U =$ 

$W =$ 

Claim: There is a 2-to-1 correspondance between Dyck trees with a marked leaf and Dyck trees with a marked red edge with the same size

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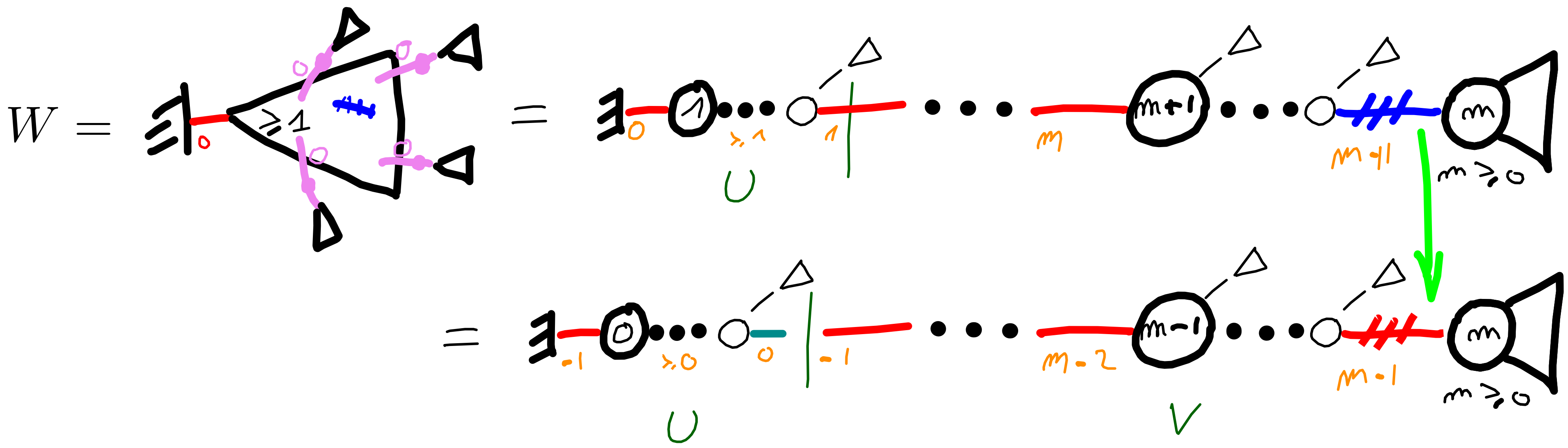
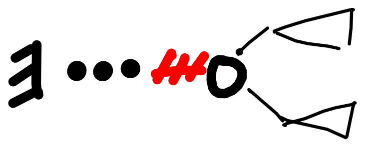
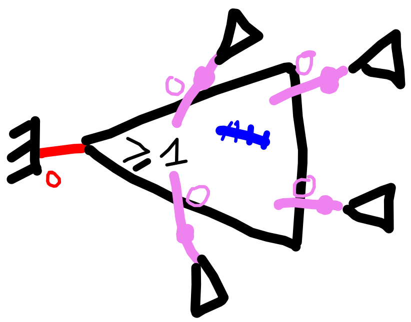


Combinatorial derivation of $U = 2V$ and $W = U\,V$

$V =$ 

$U =$ 

$W =$ 

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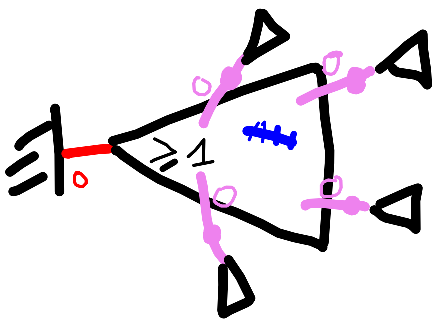
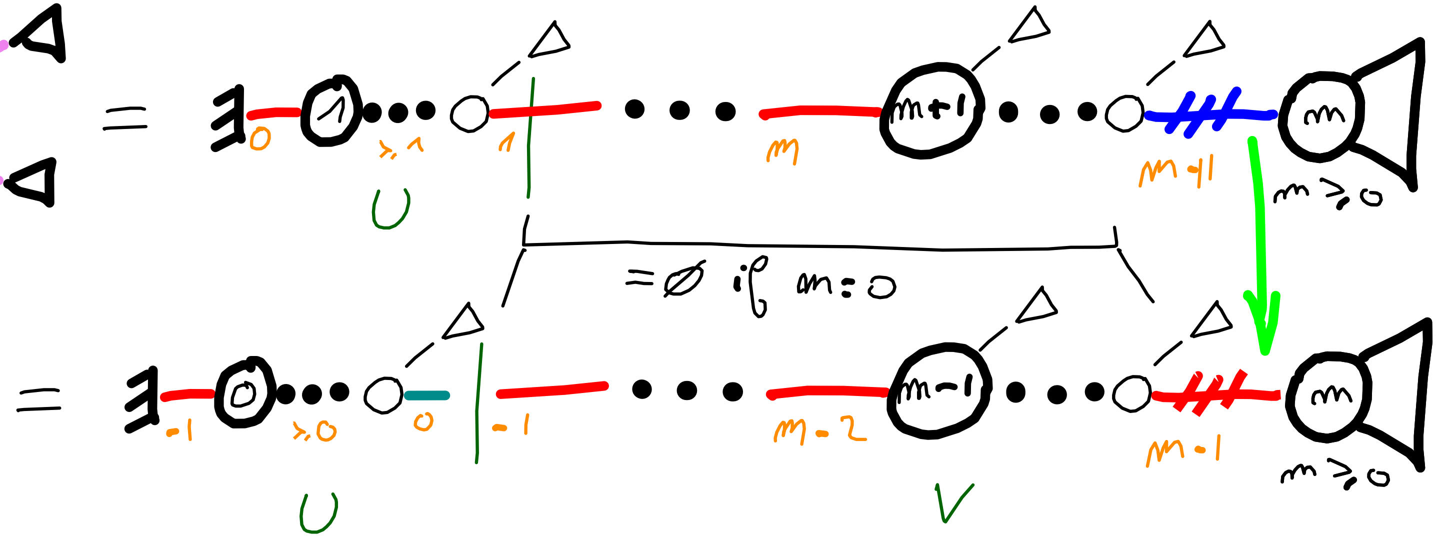
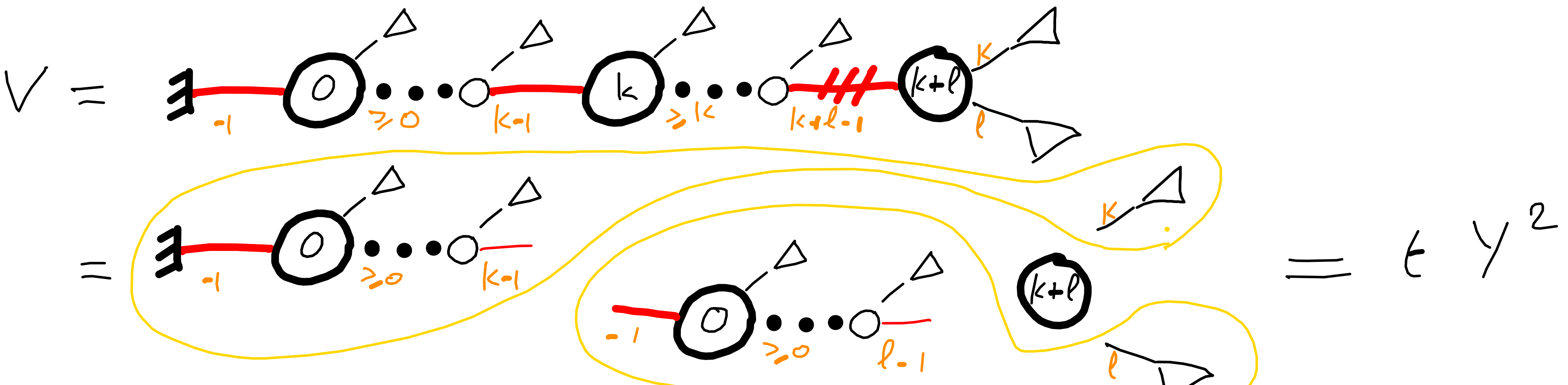
$W =$ 



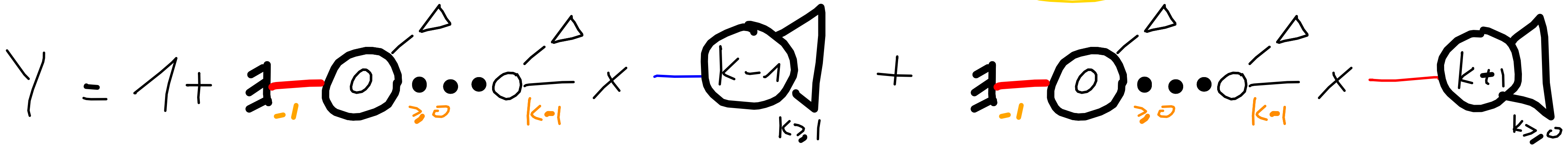
$\Rightarrow W = U\,V$

Finally, a quaternary decomposition of marked Dyck trees

Theorem: The class of marked Dyck trees admit the following decomposition:



where



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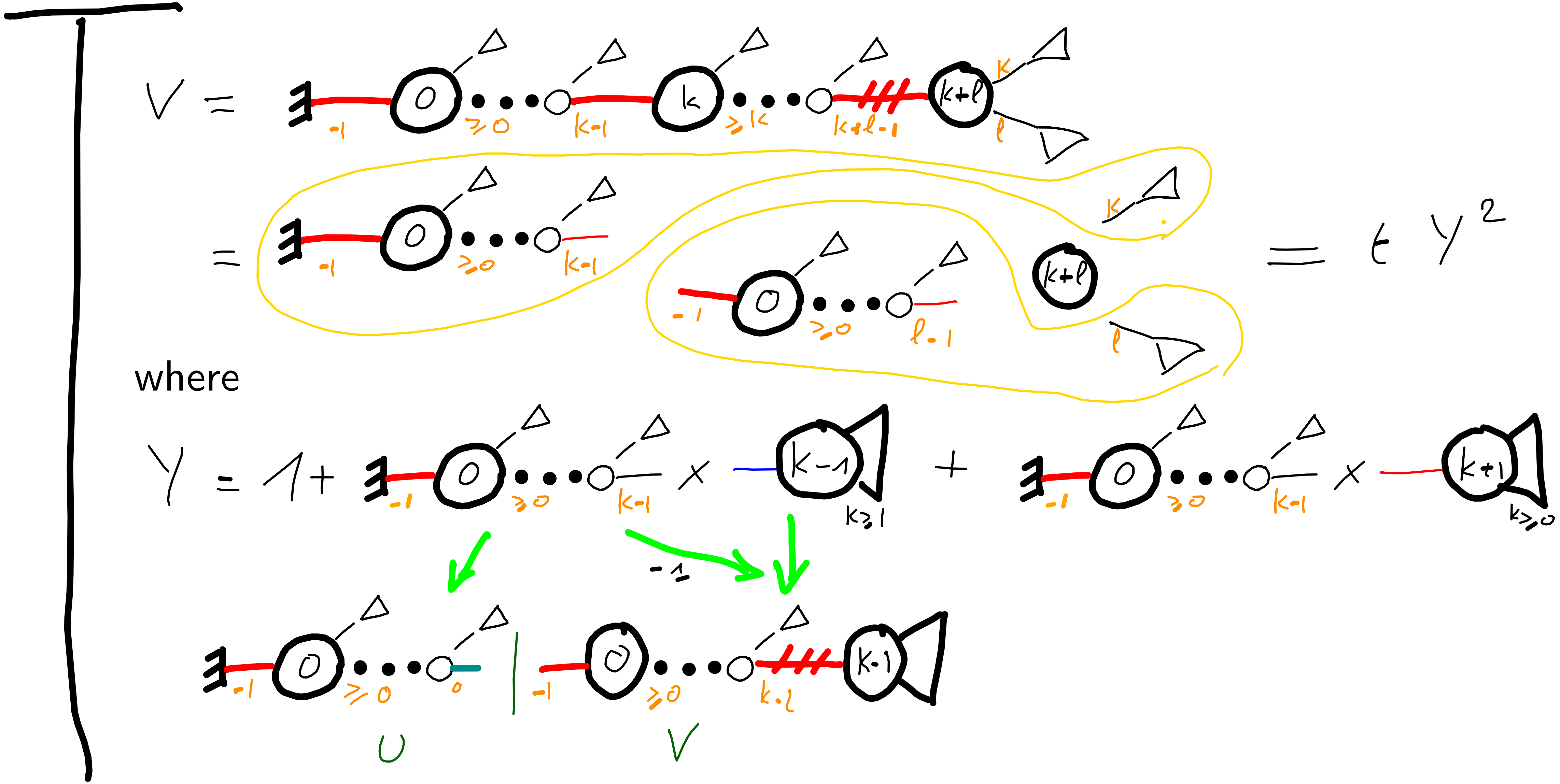

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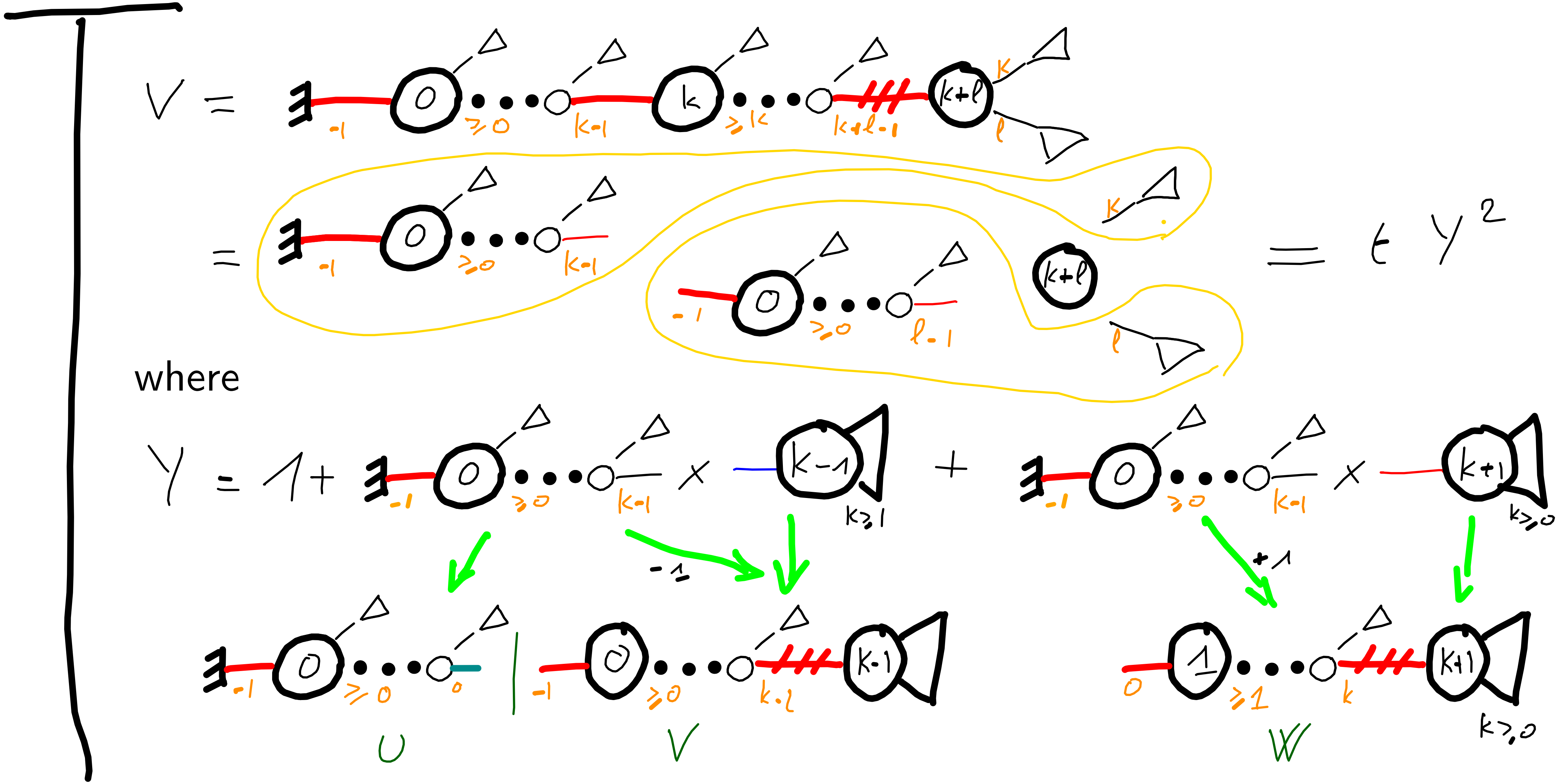

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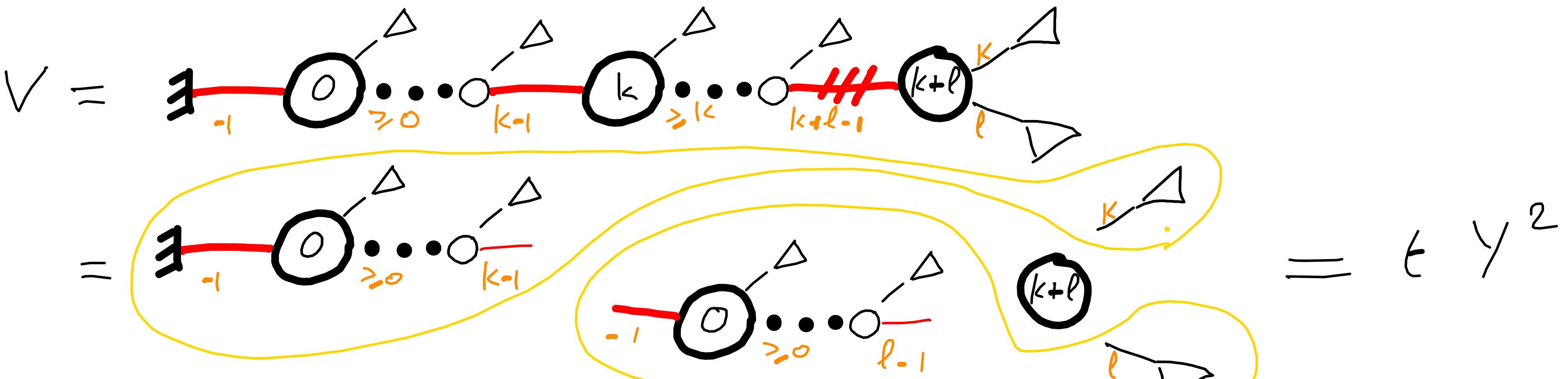

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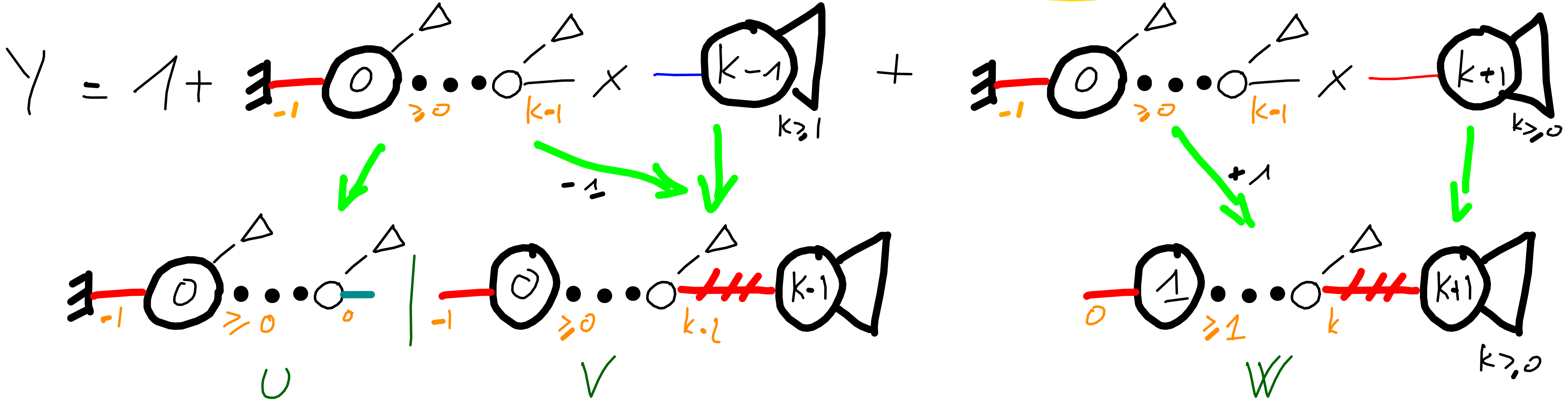

$$\Rightarrow V = t\,(1 + U\,V + W)^2$$

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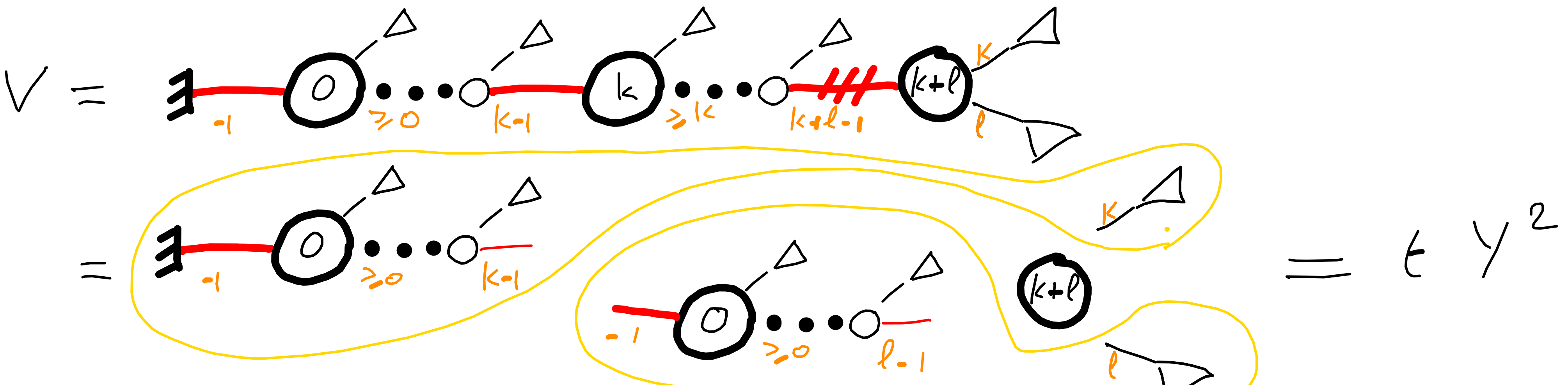

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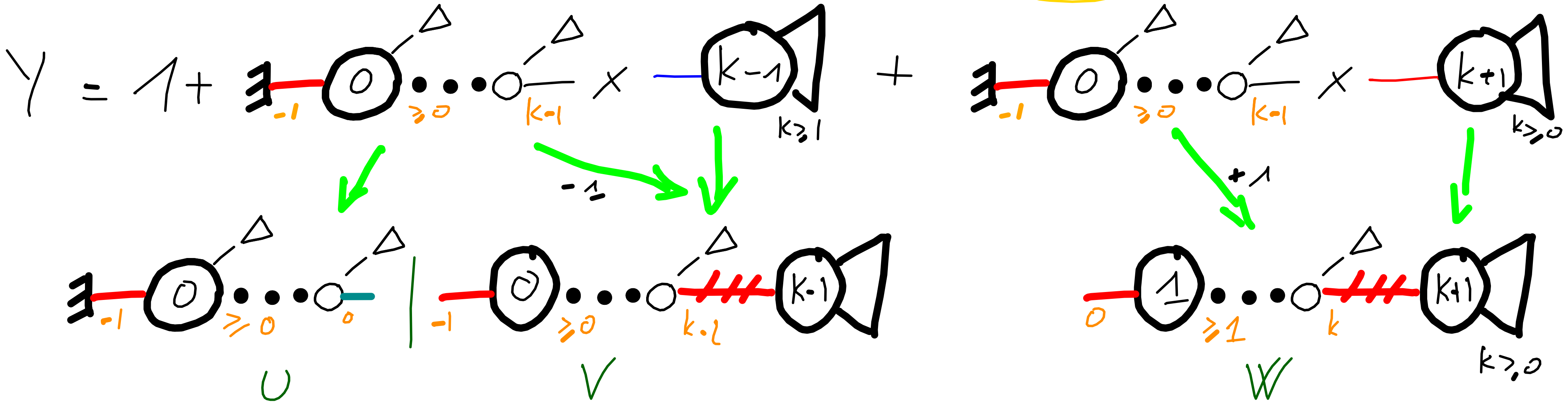


$\Rightarrow V = t\,(1 + U\,V + W)^2$

$\Rightarrow V = t\,(1 + 2 \cdot 2\,V \cdot V)^2$

Generic equations with 1 catalytic variable and 1 small function.

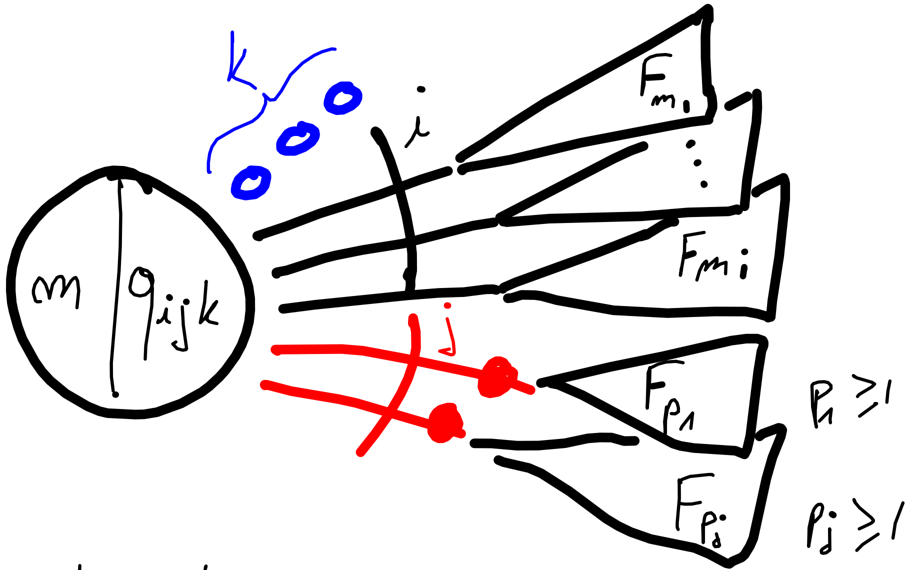
The general case

$$Q(v, w, u) = \sum_{i,j,k \geq 0} q_{ijk} v^i w^j u^k$$ a formal power series

$F(u) \equiv F(u, a, b, t)$ the unique fps* solution of

$$F(u) = t\,Q\left(F(u), \frac{b}{u}(F(u) - f), a\,u\right)$$

$$F_m =$$



$$F(u) = \sum_{m \geq 0} F_m$$

$$m = k + (m_1 + \dots + m_i) + ((p_1 - 1) + \dots + (p_j - 1))$$

The general case

$$Q(v,w,u) = \sum_{i,j,k\geq 0} q_{ijk} v^i w^j u^k \text{ a formal power series}$$

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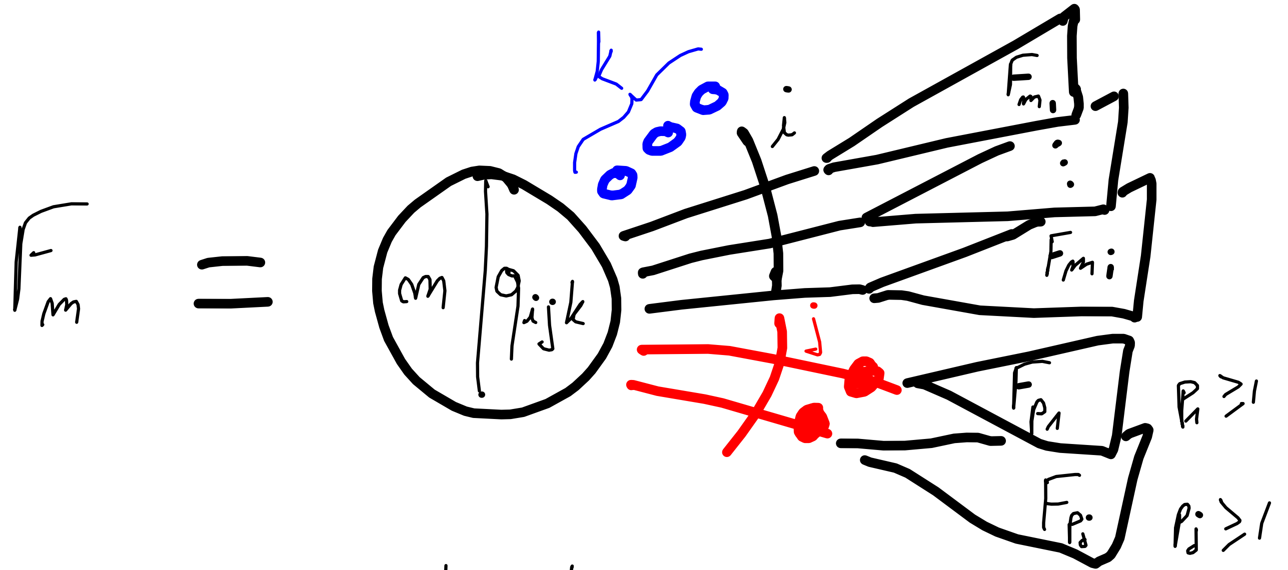
(assume $Q(v,v,1)$ non linear and $Q(1,1,u)$ non constant)

Similar results hold for

$$F(u) = Q\left(F(u), \frac{b}{u}(F(u)-f), a\,u, t\right)$$



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The general case: Bousquet-Mélou–Jehanne's trick

$\frac{\partial}{\partial u}$ applied to $\quad F(u) = t\,Q\left(F(u), \frac{b}{u}(F(u)-f), a\,u\right)$

$$F'_u(u) = F'_u(u)\cdot t\,\left(Q'_v(\ldots) + \tfrac{b}{u}Q'_w(\ldots)\right) - t\,\tfrac{b}{u}\,\tfrac{F(u)-f}{u}Q'_w(\ldots) + t\,aQ'_u(\ldots)$$

Let $U \equiv U(t)$ be the unique fps s.t. $U = t\,U\,Q'_v\left(F(U), b\tfrac{F(U)-f}{U}, aU\right) + t\,b\,Q'_w\left(F(U), b\tfrac{F(U)-f}{U}, aU\right)$

Then U, $V = F(U)$, $W = \frac{F(U)-f}{U}$ and f satisfy the system

$$\begin{cases} U &= t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ 0 &= -t\,\frac{b}{U}\,W\,Q'_w(V, bW, aU) + t\,aQ'_u(V, bW, aU) \\ V &= t\,Q(V, bW, aU) \\ f &= V - U\,W \end{cases}$$

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Used by Chapuy in his M1 to derive singular behavior of f when Q has **positive coefficients** and **is linear** in w:

Drmota-Lalley-Woods give square root singular behavior for U, V, W

and delicate computations show that there is a cancellation in $f = V - U\,W$

so that f systematically has $(1 - t/\rho)^{3/2}$ as singular behavior for proper polynomial Q

The general case: Drmota, Noy, Yu's trick*

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Use Line 1 to replace Q'_w by Q'_v in Line 2:

Then U, $V = F(U)$, $W = \frac{F(U)-f}{U}$ and f are the unique fps satisfying the system

$$\begin{cases} U &= t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W &= t\,W\,Q'_v(V, bW, aU) + t\,aQ'_u(V, bW, aU) \\ V &= t\,Q(V, bW, aU) \\ f &= V - U\,W \end{cases}$$

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Used by DNY to generalize Chapuy's result on the singular behavior of f to arbitrary polynomial positive Q:

$f = V - U\,W$ still requires delicate 2nd order computations to check square root cancellation.

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$f = V - U\,W$ still requires delicate 2nd order computations to check square root cancellation.

The system for U, V, W is $\mathbb{N}$-algebraic if Q has $\mathbb{N}$ coeffs but *a priori* no clear combinatorial relation to F and f.

The general case: singular behavior via marking

$$F(u) = t\,Q\left(F(u), \frac{b}{u}(F(u) - f), a\,u\right)$$

$\frac{\partial}{\partial u}$: $\quad F'_u(u) = F'_u(u) \cdot t\, \left(Q'_v(\ldots) + \frac{b}{u} Q'_w(\ldots)\right) - t\, \frac{b}{u}\, \frac{F(u)-f}{u} Q'_w(\ldots) + t\, a Q'_u(\ldots)$

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canceled by $u = U$

$$\frac{\partial}{\partial t}: \quad F'_t(u) = F'_t(u)\cdot t\left(Q'_v(\ldots) + \frac{b}{u}Q'_w(\ldots)\right) - t\,\frac{b}{u}f'_t\,Q'_w(\ldots) + Q(\ldots)$$

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$$\frac{\partial}{\partial t}: \quad \underbrace{F'_t(u) = F'_t(u)\cdot t\left(Q'_v(\ldots)+\tfrac{b}{u}Q'_w(\ldots)\right)}_{u=U \qquad 0} - t\,\tfrac{b}{u}f'_t\,Q'_w(\ldots) + Q(\ldots)$$

$$\Rightarrow \boxed{t f'_t = \frac{U}{b}\,\frac{Q(\ldots)}{Q'_w(\ldots)}}$$

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canceled by u = U

$\frac{\partial}{\partial t}$: $\underbrace{F'_t(u) = F'_t(u)\cdot t\left(Q'_v(\dots) + \frac{b}{u}Q'_w(\dots)\right)} - t\,\frac{b}{u}f'_t\,Q'_w(\dots) + Q(\dots)$

u = U   0

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$$\Rightarrow \boxed{t f'_t = \frac{t\,Q(\ldots)}{1-t\,Q'_v(\ldots)} = \frac{V}{1-t\,Q'_v(V,bW,aU)}}$$

Immediately implies without computations that $t f'_f$ has generic square root singularity, and thus that f_t has $(1-t/\rho)^{3/2}$ singularity.

The general case: further useful observations!

$$F(u) = t\,Q\left(F(u), \frac{b}{u}(F(u)-f), a\,u\right)$$

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canceled by $u = U$

$$\frac{\partial}{\partial t}: \quad F'_t(u) = F'_t(u)\cdot t\,\left(Q'_v(\ldots) + \tfrac{b}{u}Q'_w(\ldots)\right) - t\,\tfrac{b}{u}\,f'_t\,Q'_w(\ldots) + Q(\ldots)$$

$$\frac{\partial}{\partial b}: \quad F'_b(u) = F'_b(u)\cdot t\,\left(Q'_v(\ldots) + \tfrac{b}{u}Q'_w(\ldots)\right) + t\,\left(\tfrac{F(u)-f}{u} - \tfrac{b}{u}\,f'_b\right)\,Q'_w(\ldots)$$

The general case: further useful observations!

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$$\Rightarrow \boxed{WU = bf'_b = af'_a \quad \text{and} \quad V = UW + f = b(bf)'_b}$$

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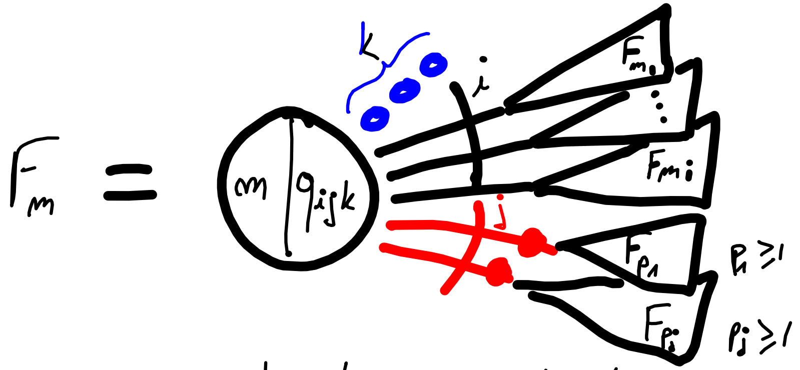
Systematic combinatorial interpretation of V as marked trees!

The general case: combinatorial interpretation of V, U and W

$$\begin{cases} V = t\,Q(V, bW, aU) = b f'_b \\ U = t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W = t\,W\,Q'_v(V, bW, aU) + t\,a Q'_u(V, bW, aU) \\ (tf'_t) = t\,(tf'_t)\,Q'_v(V, bW, aU) + V \end{cases}$$



$$m = k + (m_1 + \dots + m_i) + ((p_1 - 1) + \dots + (p_j - 1))$$

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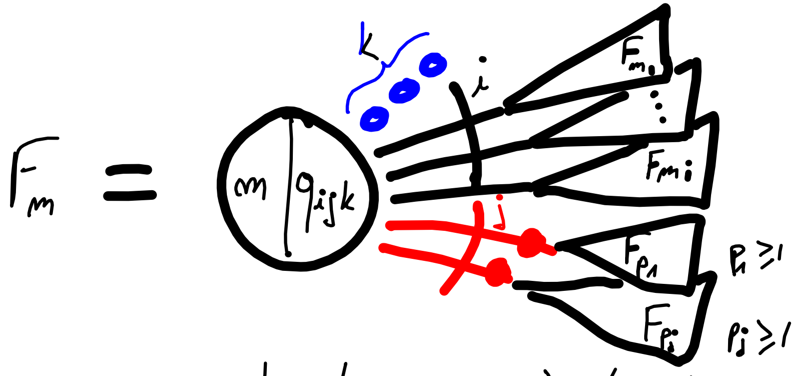
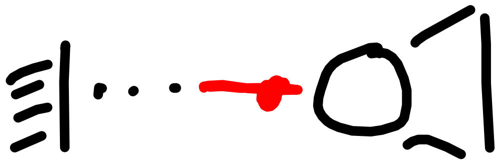
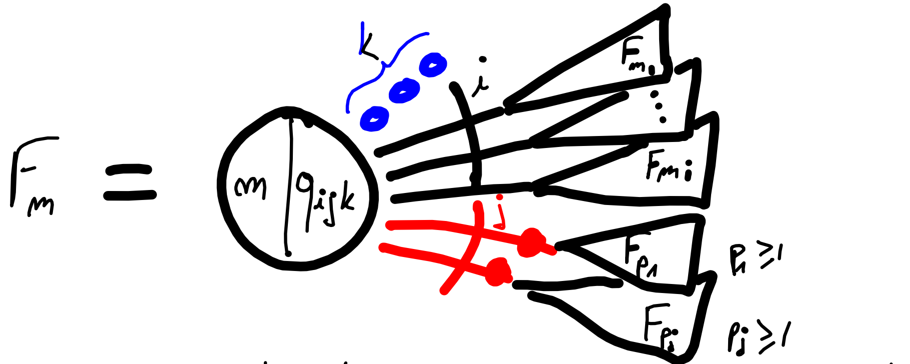


$$m = k + (m_1 + \dots + m_i) + ((p_1 - 1) + \dots + (p_j - 1))$$

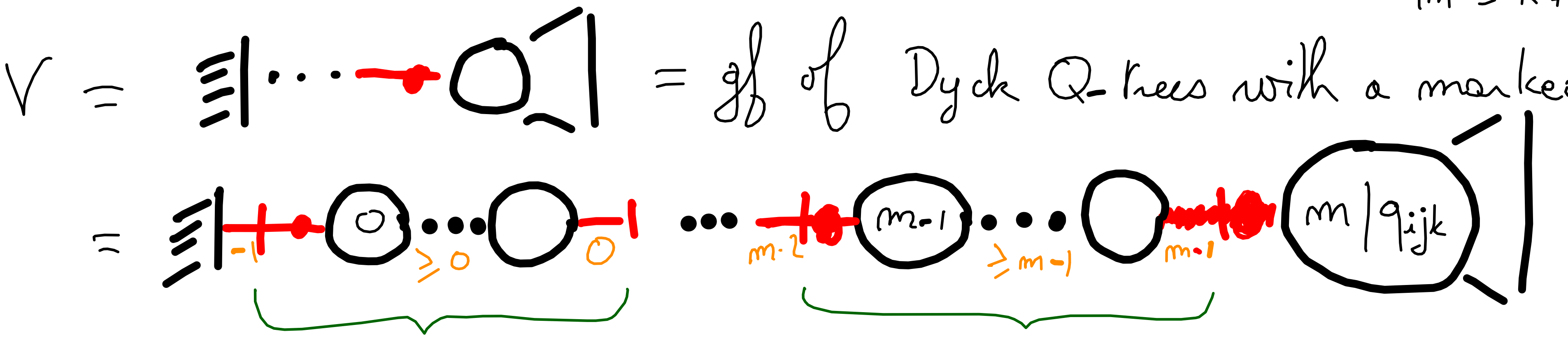
$V =$  $=$ gf of Dyck Q-trees with a marked red edge.

The general case: combinatorial interpretation of V, U and W

$$\begin{cases} V &= t\,Q(V, bW, aU) \quad = \quad b f'_b \\ U &= t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W &= t\,W\,Q'_v(V, bW, aU) + t\,a Q'_u(V, bW, aU) \\ (t f'_t) &= t\,(t f'_t)\,Q'_v(V, bW, aU) + V \end{cases}$$

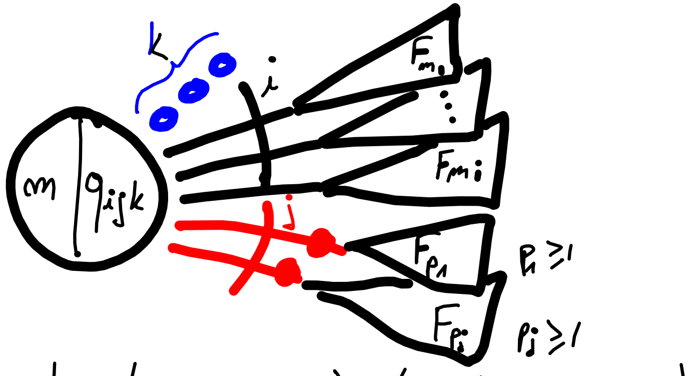


$$m = k + (m_1 + \dots + m_i) + ((p_1 - 1) + \dots + (p_j - 1))$$

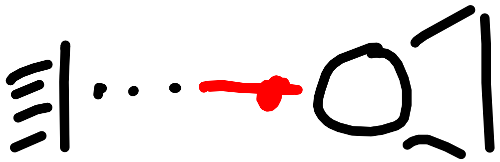


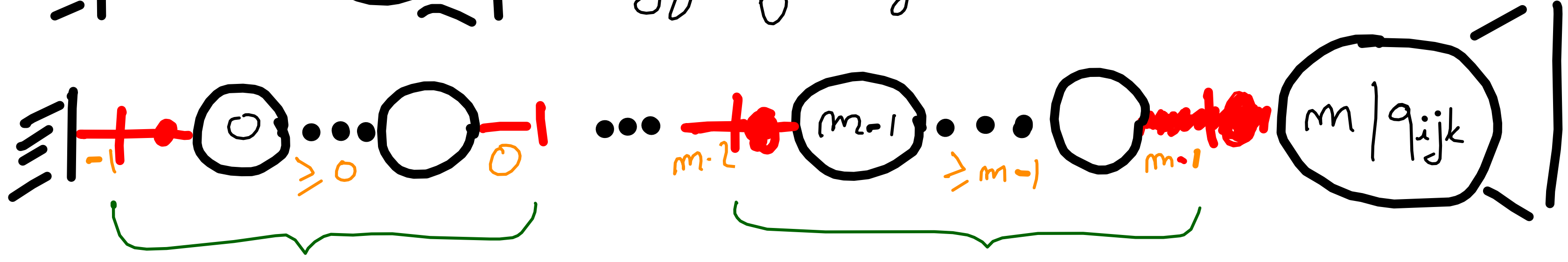
The general case: combinatorial interpretation of V, U and W

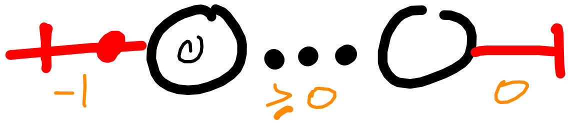
$$
\begin{cases}
V &= t\,Q(V, bW, aU) \;=\; b f'_b \\
U &= t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\
W &= t\,W\,Q'_v(V, bW, aU) + t\,a Q'_u(V, bW, aU) \\
(t f'_t) &= t\,(t f'_t)\,Q'_v(V, bW, aU) + V
\end{cases}
$$

$F_m =$ 

$m = k + (m_1 + \dots + m_i) + ((p_1 - 1) + \dots + (p_j - 1))$

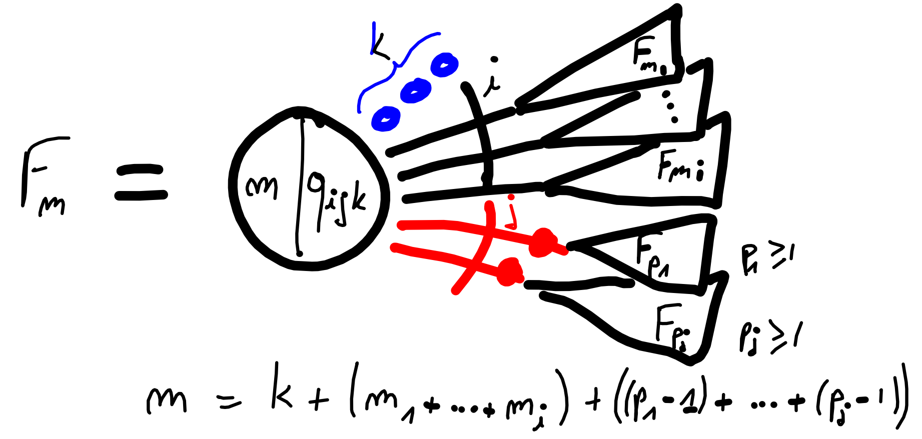
$V =$  = gf of Dyck Q-trees with a marked red edge.

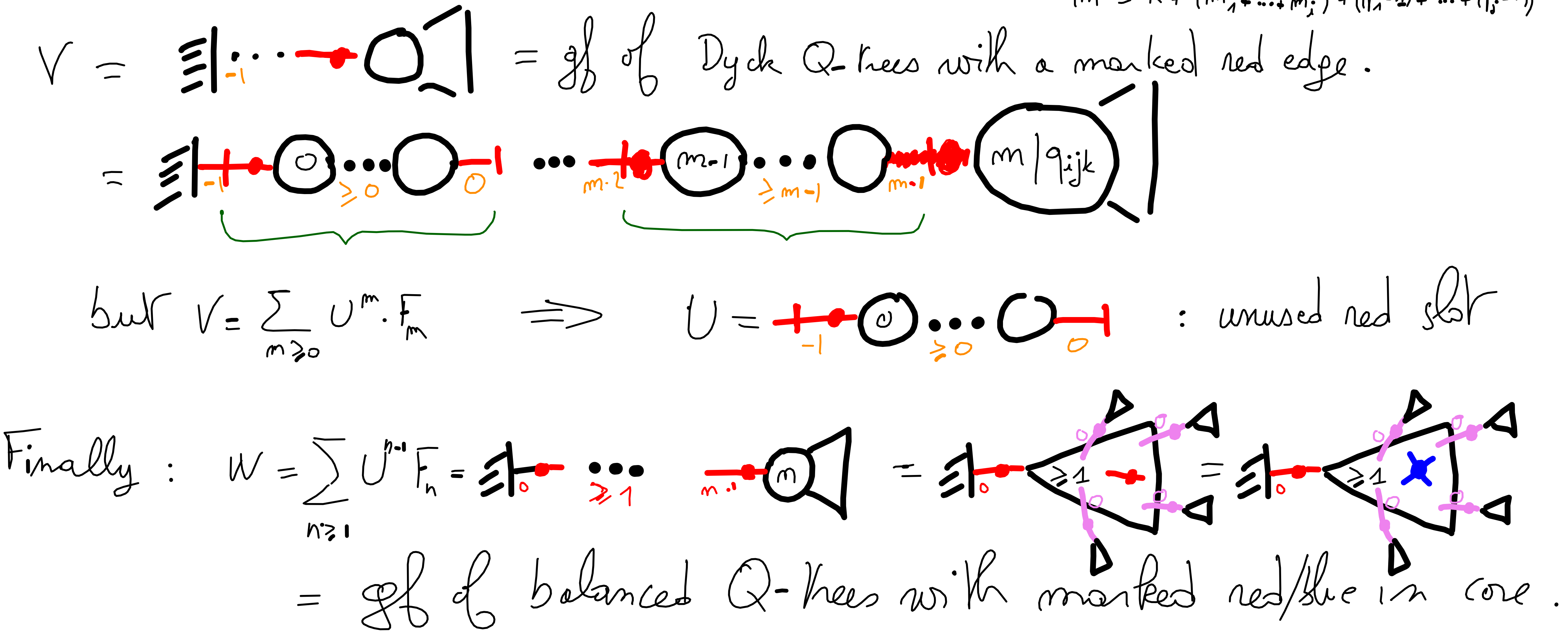
$=$ 

but $V = \sum_{m \geq 0} U^m \cdot F_m \implies U =$  : unused red slot

The general case: combinatorial interpretation of V, U and W

$$\left\{\begin{array}{rcl} V &=& t\,Q(V, bW, aU) \;=\; b f_b' \\ U &=& t\,U\,Q_v'(V, bW, aU) + t\,b\,Q_w'(V, bW, aU) \\ W &=& t\,W\,Q_v'(V, bW, aU) + t\,a Q_u'(V, bW, aU) \\ (t f_t') &=& t\,(t f_t')\,Q_v'(V, bW, aU) + V \end{array}\right.$$

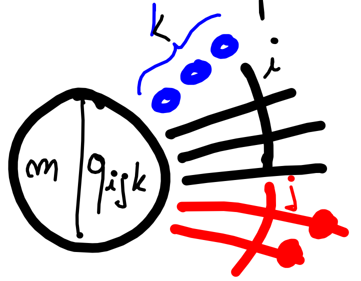


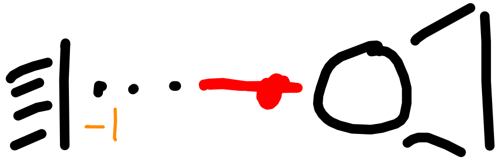


The general case: combinatorial interpretation of V, U and W

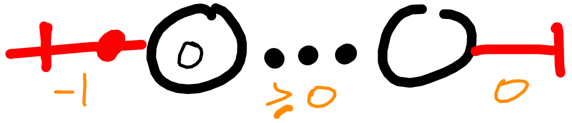
Now we restart from the combinatorial description:

$Q(v, w, u)$ = node g.f :

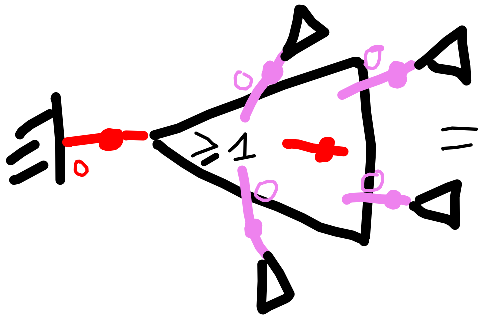


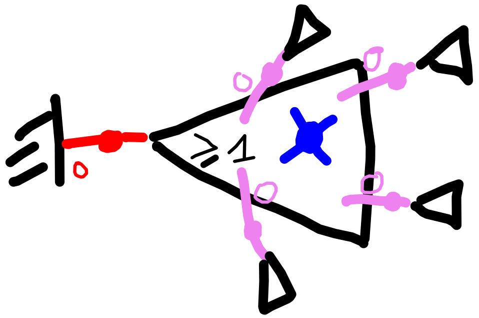
V = 

Dyck Q-trees with a marked red edge.

U = 

Dyck Q-trees with an unused red slot

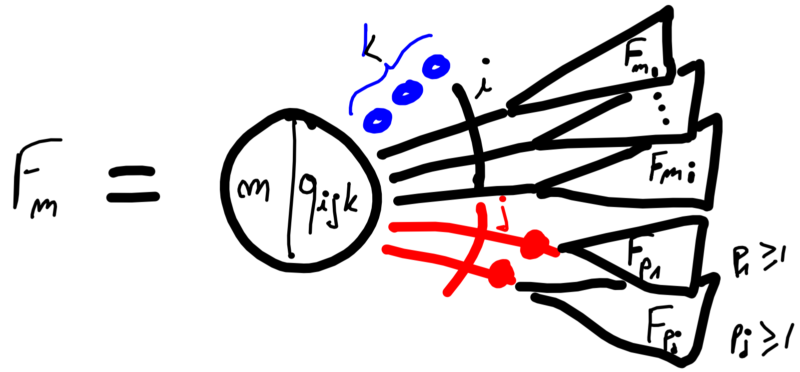
W = 

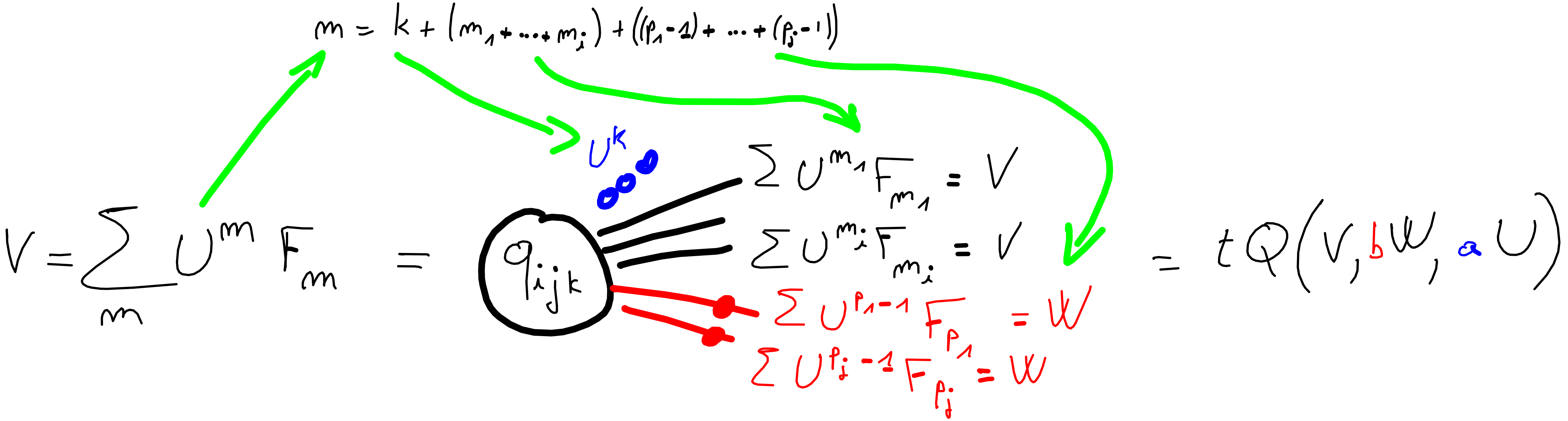
= 

Balanced Q-trees with a marked red edge / blue dot in the positive core

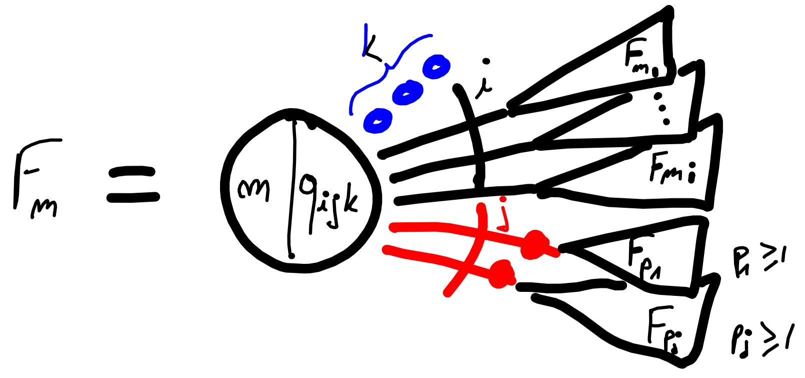
The general case: combinatorial derivation of the algebraic equations

$$\begin{cases} V = t\,Q(V, bW, aU) \\ U = t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W = t\,W\,Q'_v(V, bW, aU) + t\,a Q'_u(V, bW, aU) \\ (tf'_t) = t\,(tf'_t)\,Q'_v(V, bW, aU) + V \end{cases}$$

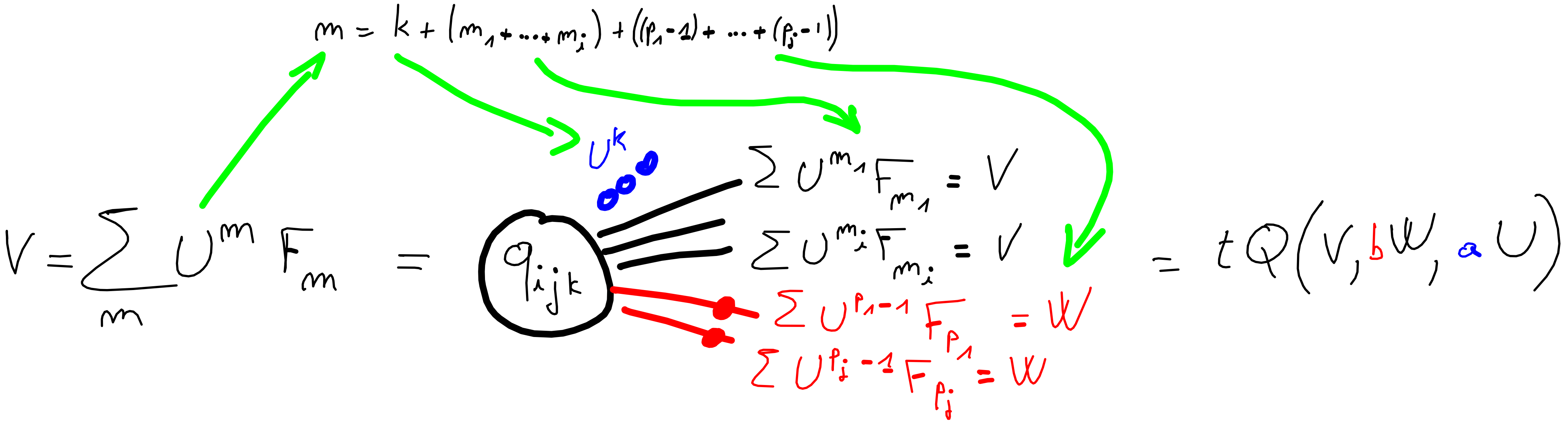




The general case: combinatorial derivation of the algebraic equations

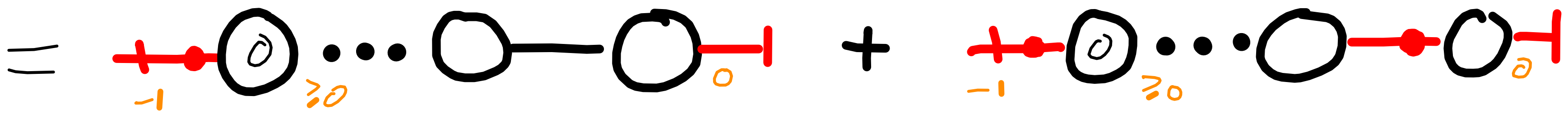


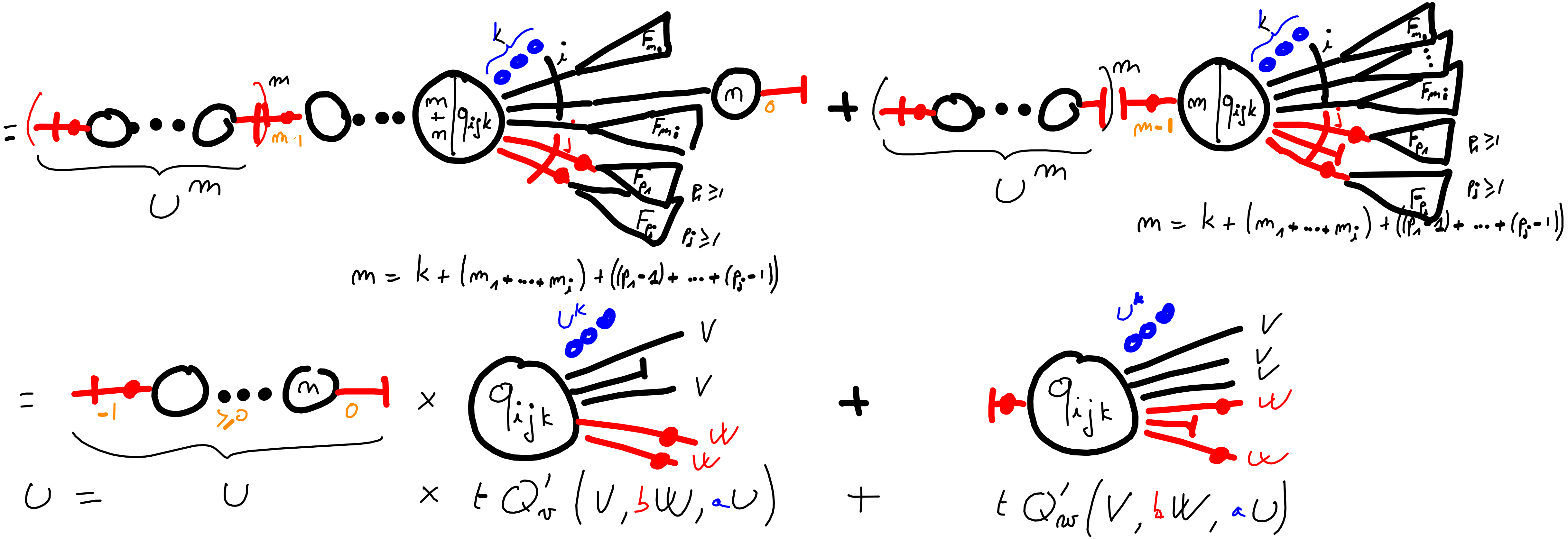
$$\left\{\begin{array}{rcl} V & = & t\,Q(V, bW, aU) \\ U & = & t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W & = & t\,W\,Q'_v(V, bW, aU) + t\,a Q'_u(V, bW, aU) \\ (t f'_t) & = & t\,(t f'_t)\,Q'_v(V, bW, aU) + V \end{array}\right.$$



The general case: combinatorial derivation of the algebraic equations

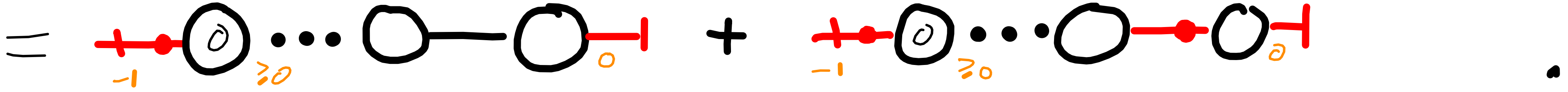
$$
\begin{cases}
V = t\,Q(V, bW, aU) \\
U = t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\
W = t\,W\,Q'_v(V, bW, aU) + t\,a\,Q'_u(V, bW, aU) \\
(tf'_t) = t\,(tf'_t)\,Q'_v(V, bW, aU) + V
\end{cases}
$$

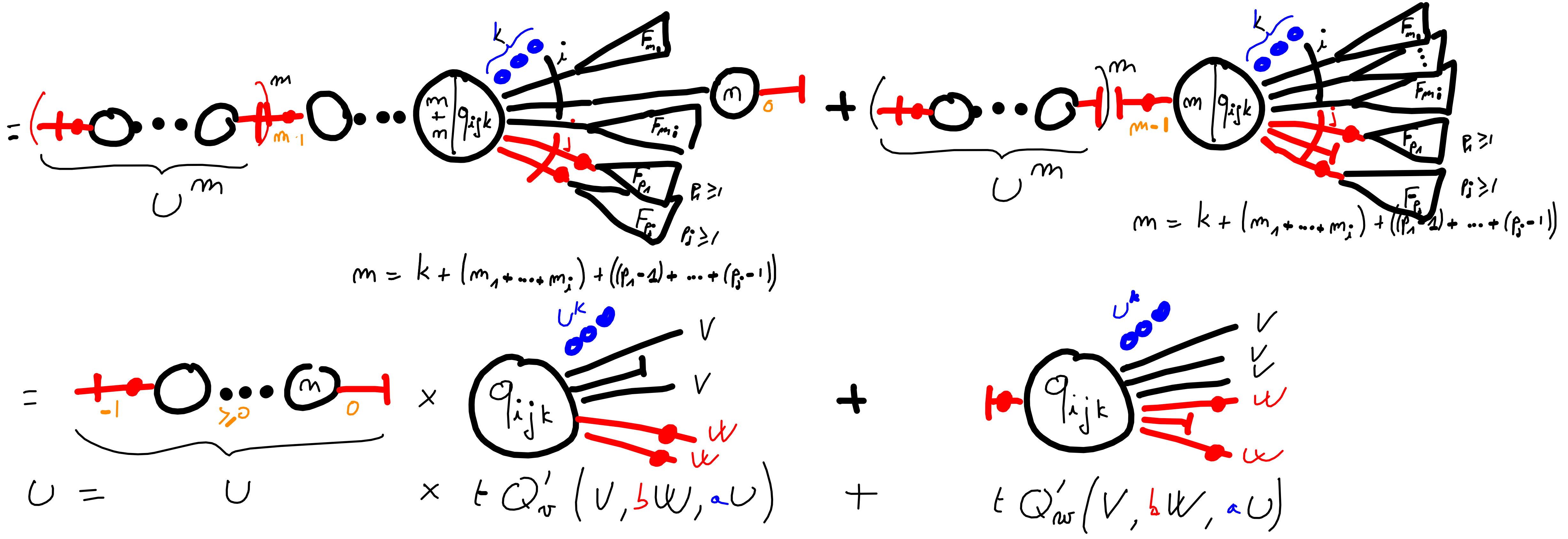




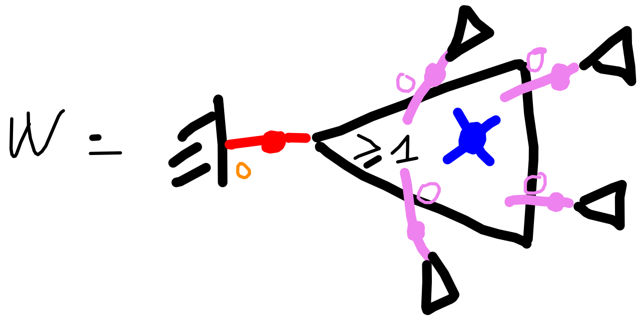
The general case: combinatorial derivation of the algebraic equations

$$\begin{cases} V &= t\,Q(V, bW, aU) \\ U &= t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W &= t\,W\,Q'_v(V, bW, aU) + t\,aQ'_u(V, bW, aU) \\ (tf'_t) &= t\,(tf'_t)\,Q'_v(V, bW, aU) + V \end{cases}$$

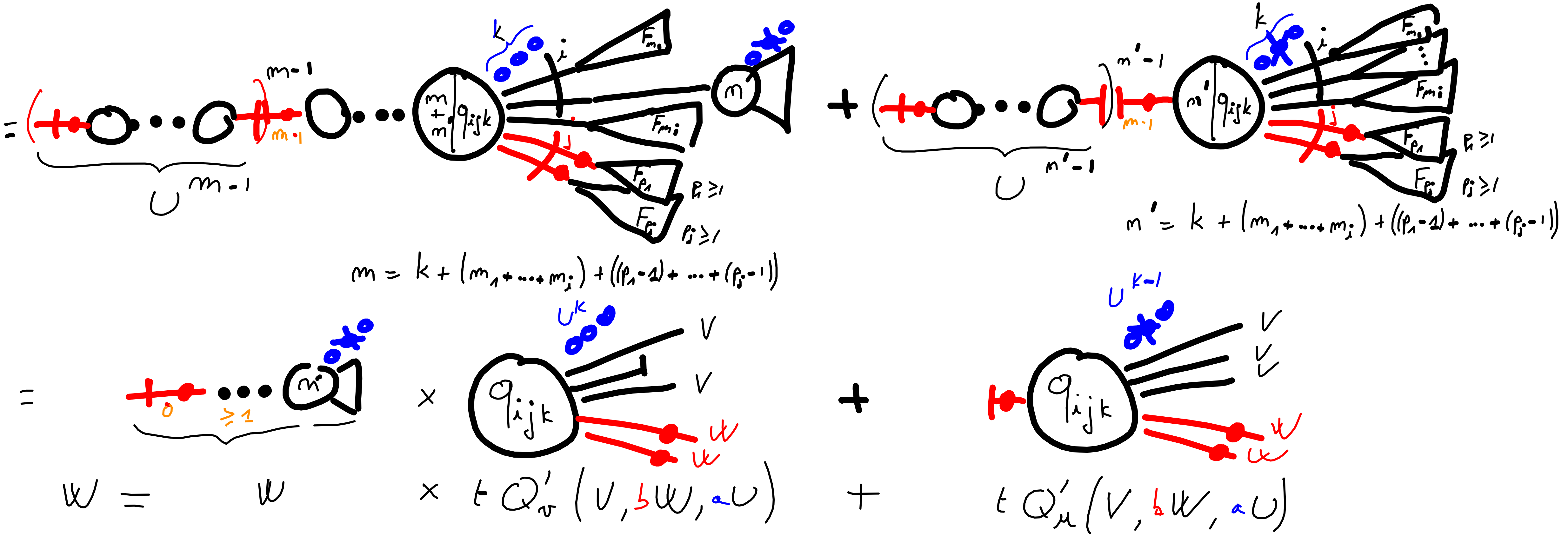




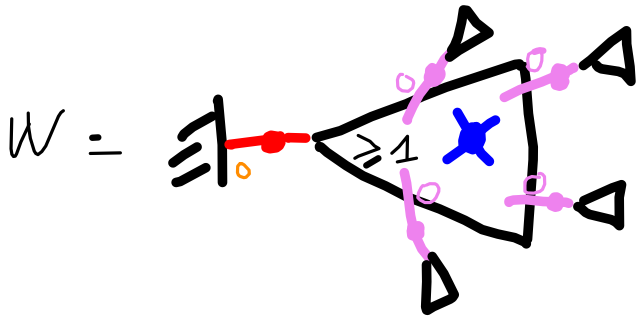
The general case: combinatorial derivation of the algebraic equation



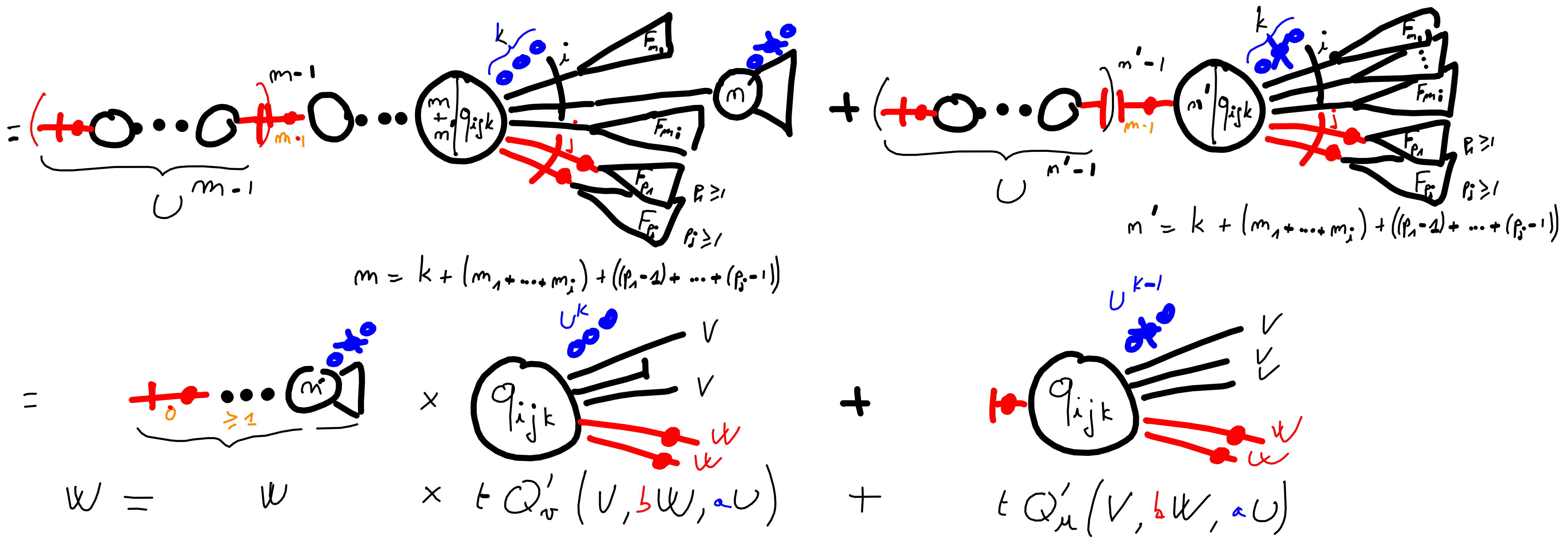
$$\begin{cases} V &= t\,Q(V, bW, aU) \\ U &= t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W &= t\,W\,Q'_v(V, bW, aU) + t\,a Q'_u(V, bW, aU) \\ (tf'_t) &= t\,(tf'_t)\,Q'_v(V, bW, aU) + V \end{cases}$$



The general case: combinatorial derivation of the algebraic equation

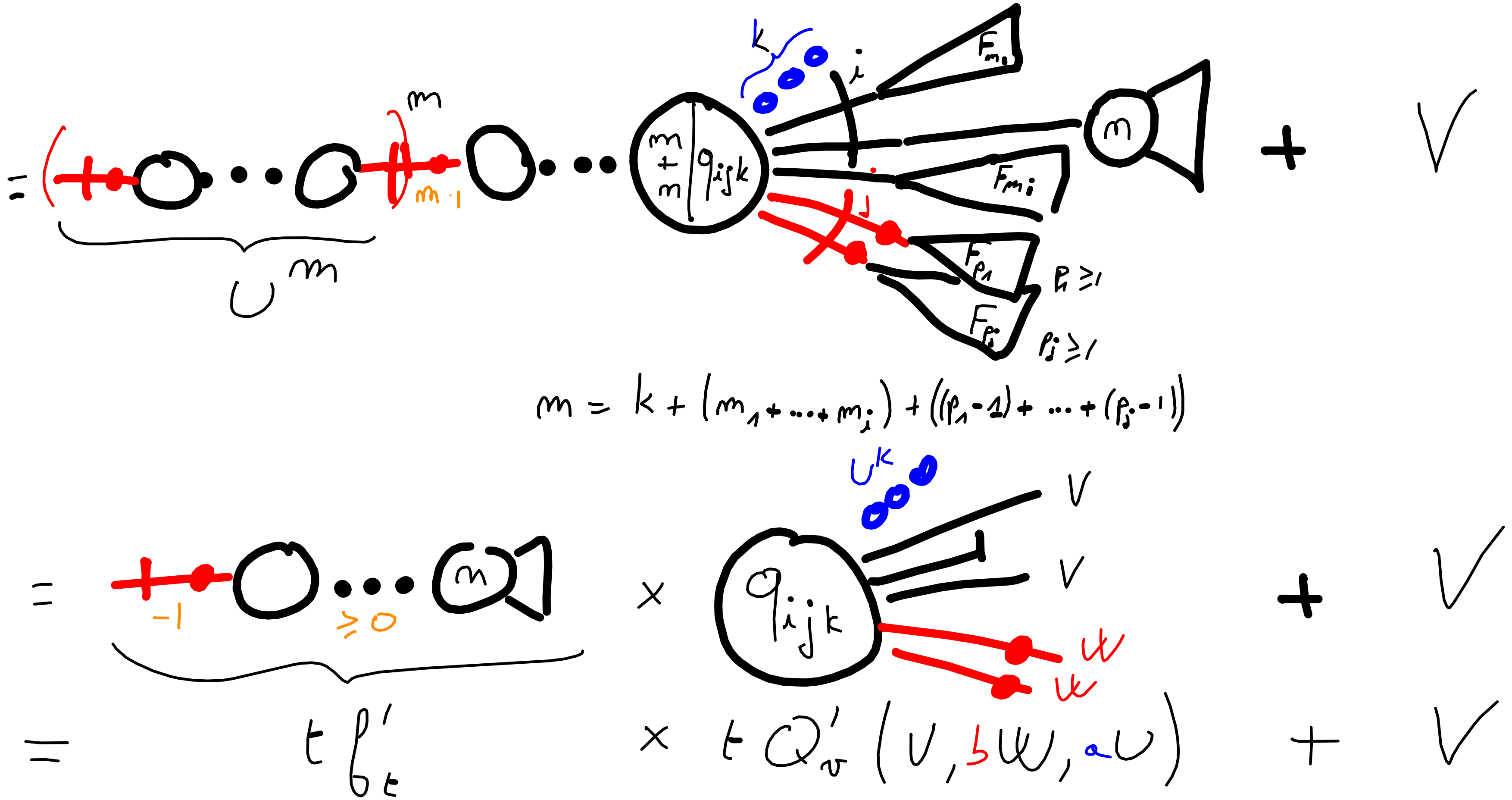


$$\begin{cases} V = t\,Q(V, bW, aU) \\ U = t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W = t\,W\,Q'_v(V, bW, aU) + t\,a\,Q'_u(V, bW, aU) \\ (tf'_t) = t\,(tf'_t)\,Q'_v(V, bW, aU) + V \end{cases}$$



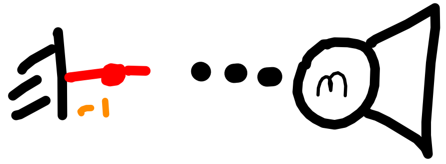
The general case: combinatorial derivation of the algebraic equation

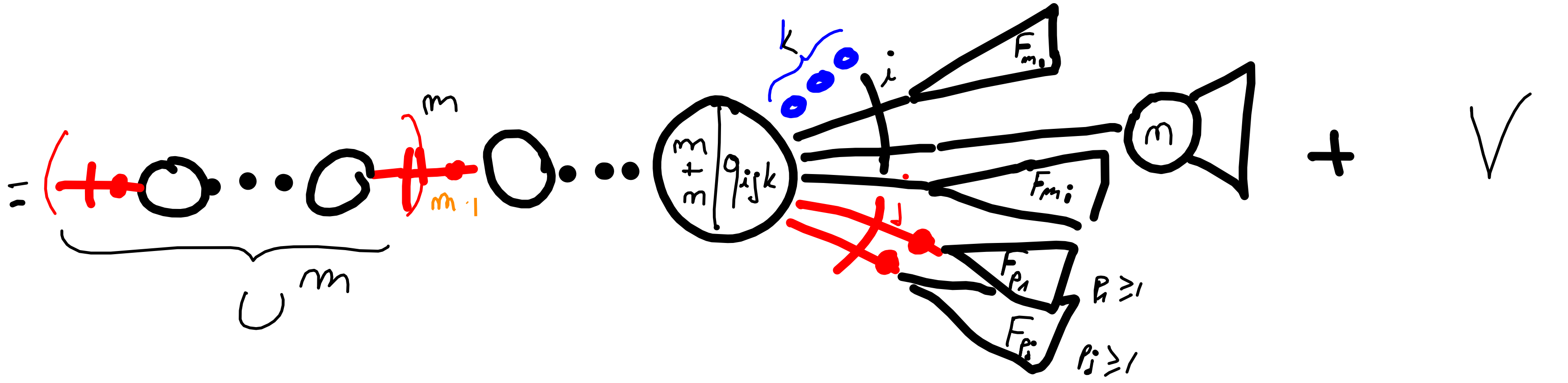
$$
\begin{cases}
V = t\,Q(V, bW, aU) \\
U = t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\
W = t\,W\,Q'_v(V, bW, aU) + t\,aQ'_u(V, bW, aU) \\
(tf'_t) = t\,(tf'_t)\,Q'_v(V, bW, aU) + V
\end{cases}
$$



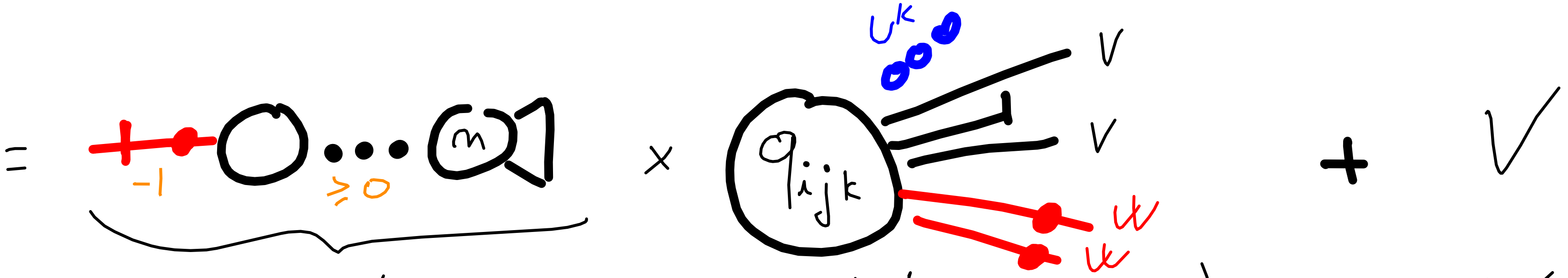
The general case: combinatorial derivation of the algebraic equation

$$\begin{cases} V &= t\,Q(V, bW, aU) \\ U &= t\,U\,Q'_v(V, bW, aU) + t\,b\,Q'_w(V, bW, aU) \\ W &= t\,W\,Q'_v(V, bW, aU) + t\,a\,Q'_u(V, bW, aU) \\ (tf'_t) &= t\,(tf'_t)\,Q'_v(V, bW, aU) + V \end{cases}$$

$t f'_t =$ 

$=$ 

$$m = k + (m_1 + \dots + m_i) + ((p_1 - 1) + \dots + (p_j - 1))$$

$=$ 

$$t f'_t = t f'_t \times t\,Q'_v(V, bW, aU) + V$$

Application of the general result

Random sampling:

$\Rightarrow$ the system is an irreducible algebraic decomposition in the terminology of [Drmota-Lalley-Woods] hence amenable to Sportiello's Bolzman sampling algorithm (linearity depends on the specific decomposition operations)

Special cases: this yields algebraic decompositions for

- Linxiao Chen's fully parked trees (2021)
- Duchi et al.'s fighting fish and variants (2016)
- Various families of permutations (West's two-stack sortable) (1990)
- Tutte's map decomposition (60's)

Works as well with exponential series: Dyck Cayley trees.

However in most of the cases combinatorial intuition is still needed to simplify the resulting decompositions, and express it in terms of the original structures.

Thanks you !

and long life to bijective combinatorics!