Random discrete surfaces

Gilles Schaeffer

LIX, CNRS/École Polytechnique, France
http://www.lix.polytechnique.fr/~schaeffe

Based in part on joint works with P. Chassaing and M. Marcus.

An overview of the talk

A combinatorial model

Graphs, surfaces and maps

The number of planar maps

Tutte formulas and a bijection

Random maps on surfaces

A discrete model of random geometries

Higher genus maps

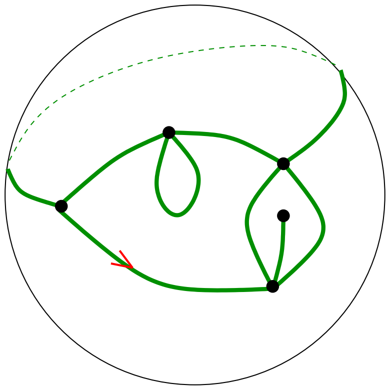
The scheme of a map

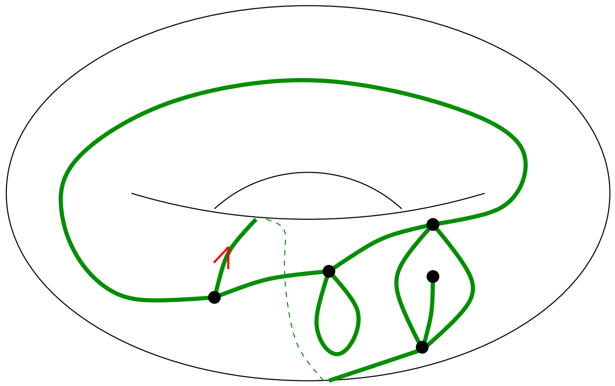
A combinatorial model

Graphs, surfaces and maps

Maps and surfaces. Definition

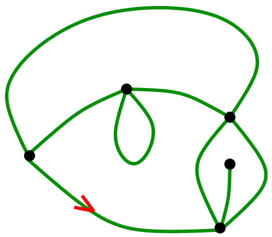
a map is an embedding of a graph in a surface with simply connected faces, considered up to homeomorphisms of the surface.





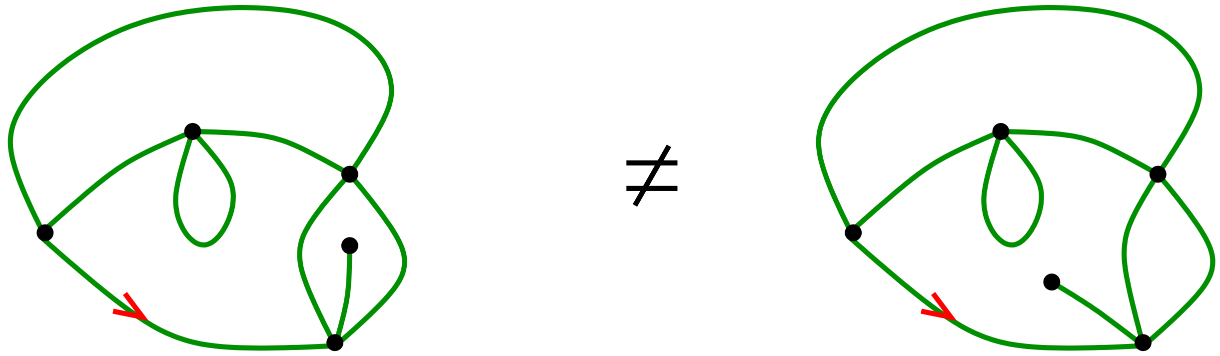
Rooted map = one edge is distinguished and oriented.

For the sphere, we make planar pictures, taking the infinite face on the right hand side of the root.



Maps and surfaces. Maps vs graphs

Distinct maps may share the same underlying graph.



map = graph $+$ cyclic order of edges around vertices.

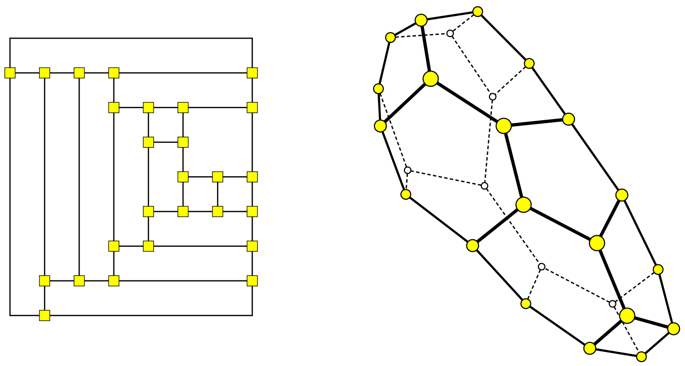
– Upon labelling $\frac{1}{2}$-edges, a map can be coded by these cyclic orders.

– The number of maps with n edges is finite

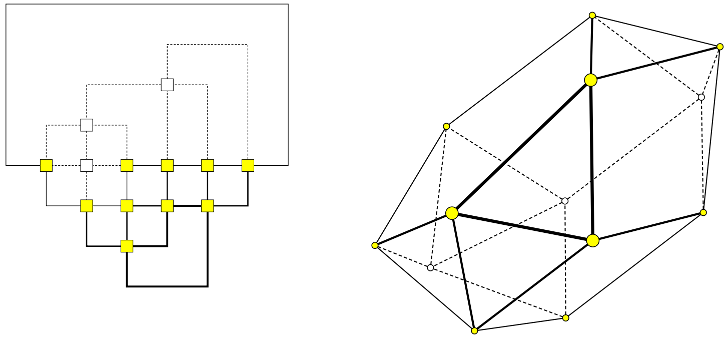
Unlike graphs, rooted maps are trivial to test for isomorphisms:
one can decide if $M_1 = M_2$ in linear time in the size.

Subfamilies. Maps as discrete surfaces

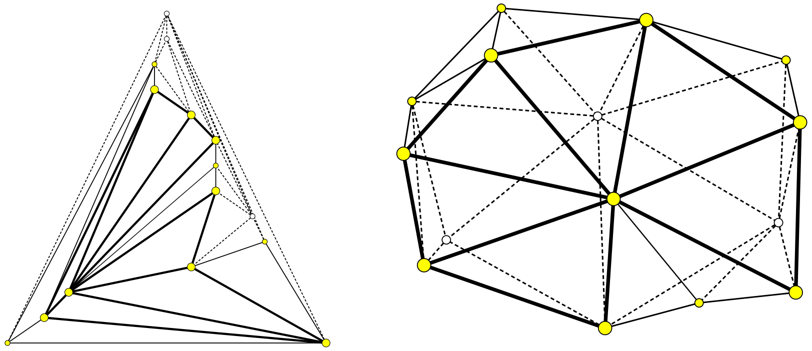
3-regular (or cubic) maps



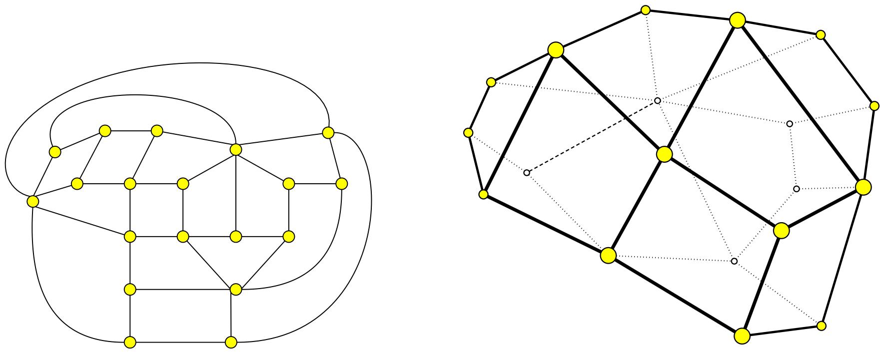
4-regular maps



Triangulations



Quadrangulations



The number of planar maps

Tutte formulas and a bijection

Enumeration.

– What is the number of rooted planar maps with n edges ?

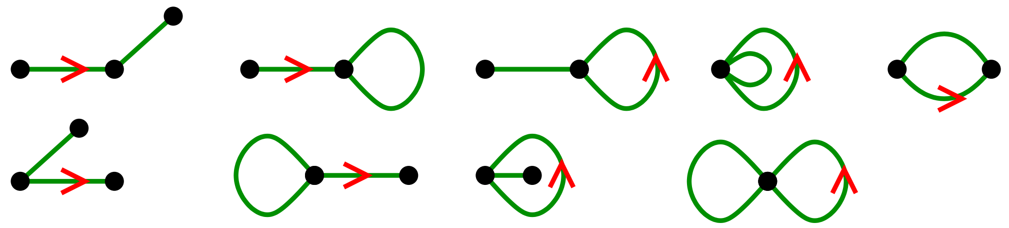
$$1,\ \ 2,\ \ 9,\ \ 54,\ \ 378, \ldots$$

– What is the number of rooted triangulations with n triangles ?

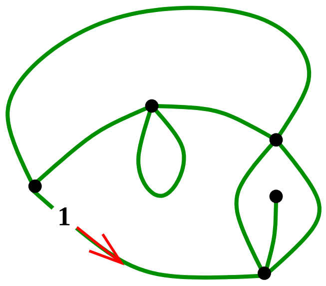
$$1,\ \ 3,\ \ 13,\ \ 68,\ \ 399,\ \ldots$$

Questions raised by Tutte (60's) in relation with the four color theorem.

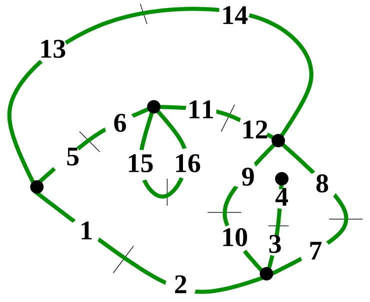
The smallest maps: $\mathcal{R}_0 = \{\ \bullet\ \}$, $\mathcal{R}_1 = \{$  $\}$

$\mathcal{R}_3 = \{$  $\}$

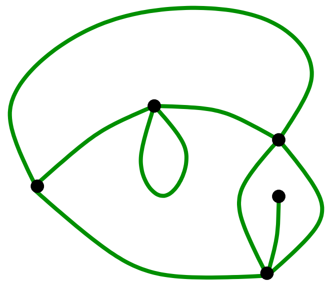
Maps, labels and roots.



$\mathcal{R}_n$: rooted;



$\mathcal{L}_n$: labelled;



$\mathcal{M}_n$: unrooted.

– A *rooted* map with n edges has $(2n-1)!$ distinct $\frac{1}{2}$-edge labellings

$$\Rightarrow \quad \text{rooted} \equiv \text{labelled}$$

– A map M with n edges has $\frac{2n}{\mathrm{Aut}(M)}$ possible roots

$$\Rightarrow \quad \text{rooted} \approx \text{unrooted}$$

In other terms:

$$|\mathcal{R}_n| \;=\; \frac{1}{(2n-1)!}|\mathcal{L}_n| \;=\; \sum_{M\in\mathcal{M}_n} \frac{2n}{\mathrm{Aut}(M)}$$

Enumeration. Surprinsing exact results in the planar case

Theorem (Tutte'62)

$$\#\{\text{planar maps, } n \text{ edges}\} \;=\; \frac{2\cdot 3^n(2n)!}{(n+2)!n!} \;\sim\; \frac{c_2}{n^{5/2}}\;12^n$$

$$\#\{\text{triangulations, } 2n \text{ faces}\} \;=\; \frac{2^{n+1}(3n)!}{(2n+2)!n!} \;\sim\; \frac{c_1}{n^{5/2}}\,(27/2)^n$$

and a few other nice formulas for other families.

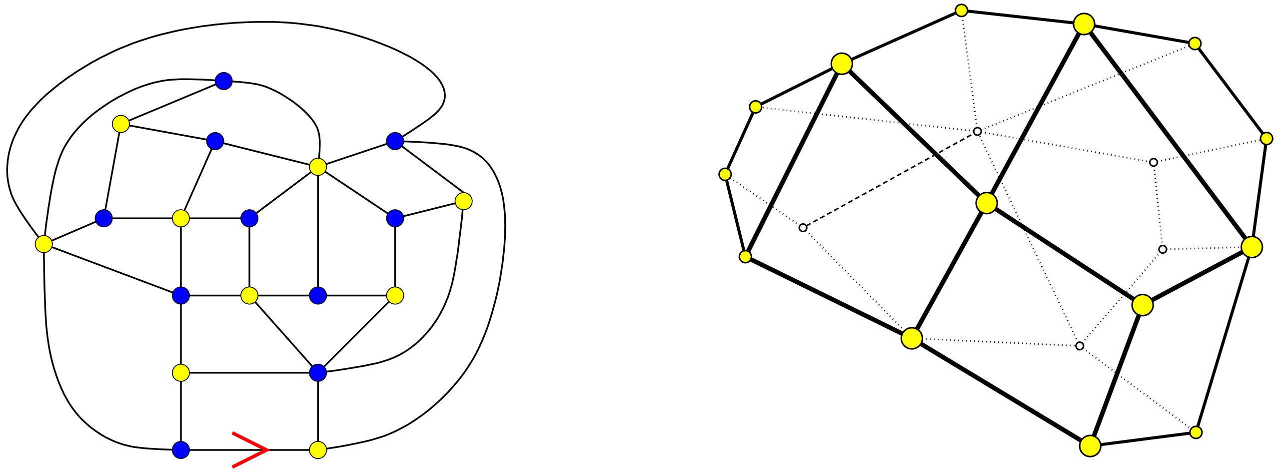
+ about twenty families of maps have algebraic generating functions.

→ *Planar constellations.* Bousquet-Mélou & S. '99.

→ *5-connected triangulations.* Gao & Wormald '01.

Remark. Planar graphs are much harder to count: their asymptotic number was only found this year (Noy & Gimenez '05).

Planar quadrangulations. Tutte's formula



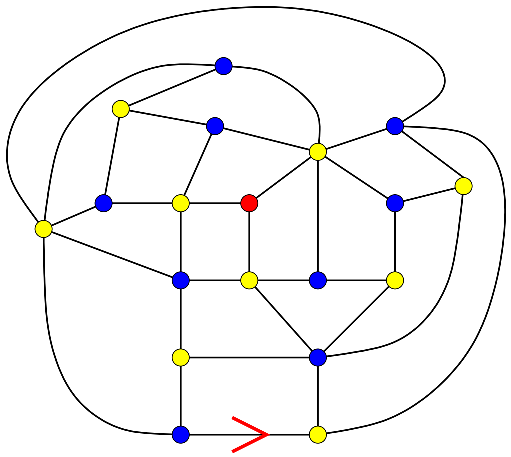
Theorem (Tutte 62). The number of rooted quadrangulations with n faces is

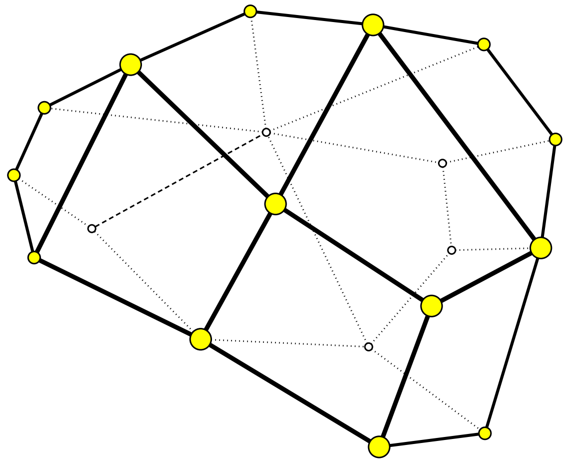
$$|\mathcal{Q}_n| = \frac{2}{n+2}\,\frac{3^n}{n+1}\binom{2n}{n}.$$

– Tutte's proof through recursions and algebra on GF.

– Bessis-Itzykson-Zuber'78: through perturbative expansion of matrix integrals.

Planar quadrangulations. Tutte's formula





Reformulation. The number of rooted **pointed** quadrangulations with n faces is

$$|\mathcal{Q}_n^\bullet| \;=\; (n+2)|\mathcal{Q}_n| \;=\; 2\,\frac{3^n}{n+1}\binom{2n}{n}.$$

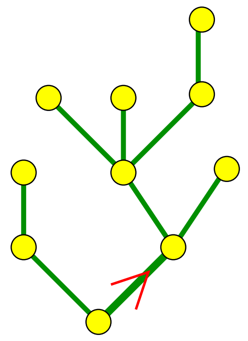
Observe the occurence of Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$..

Embedded trees.

A *rooted plane tree* is a rooted planar map with one face.

The number of rooted plane trees with n edges is

$$\frac{1}{n+1}\binom{2n}{n}$$



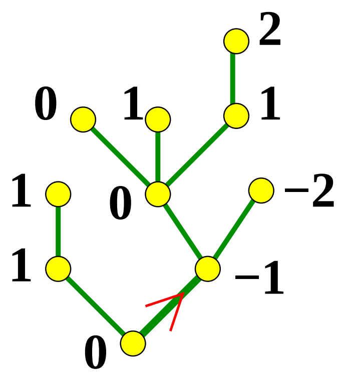
We miss a factor 3^n.

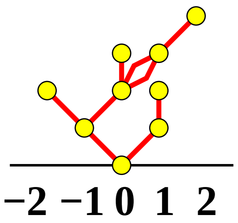
Embedded trees. Interpreting $\frac{3^n}{n+1}\binom{2n}{n}$.

An **embedded tree** is made of a rooted plane tree with vertices embedded in $\mathbb{Z}$.
More precisely, each edge is mapped independantly into $\{-1, 0, +1\}$.

The number of embedded trees with n edges is

$$\frac{3^n}{n+1}\binom{2n}{n}$$

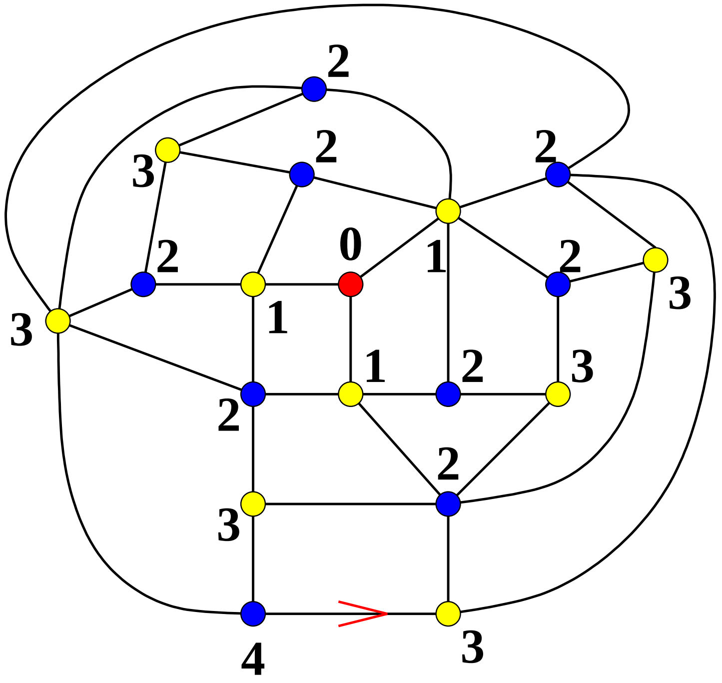


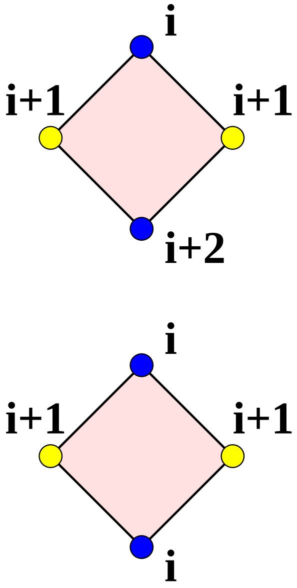


We shall exhibit a direct one-to-one correspondence between
– rooted **pointed** quadrangulations with n faces, and
– "twice" embedded trees with n edges.

A bijection. Distances..

Let us label vertices by distances to the red vertex.

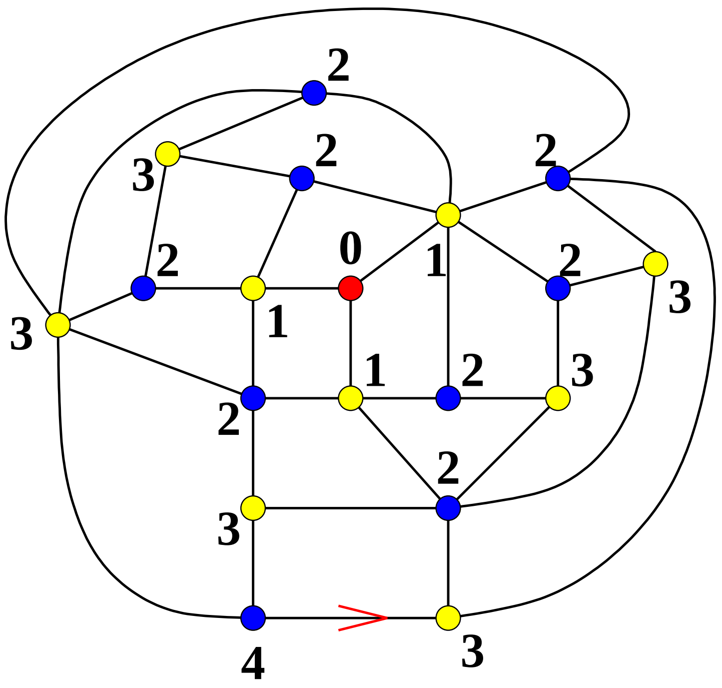


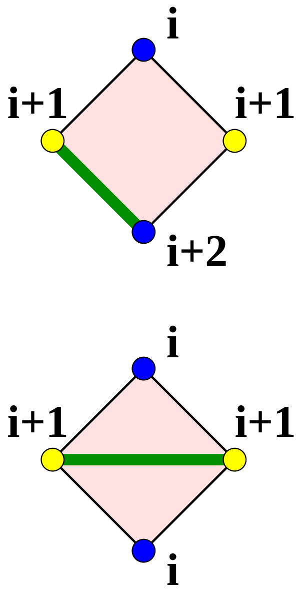


There are only two possible configurations around a face (bipartiteness).

A bijection. Local rules

Consider the following two local rules.

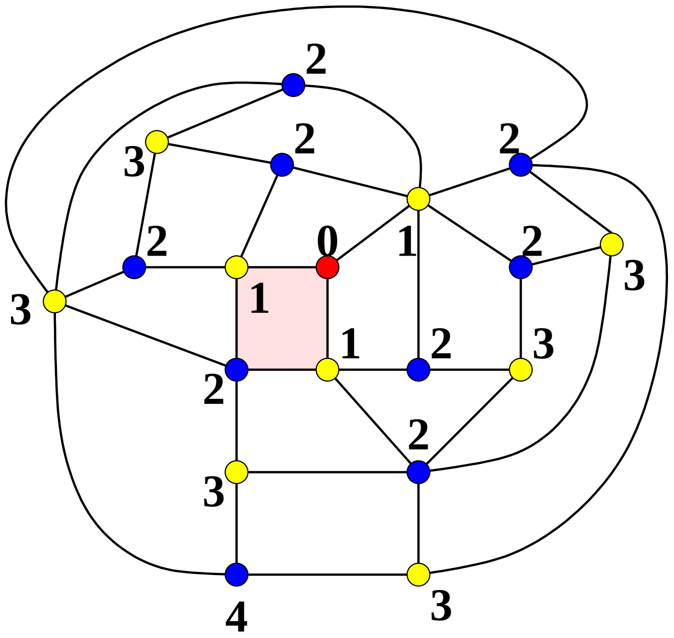


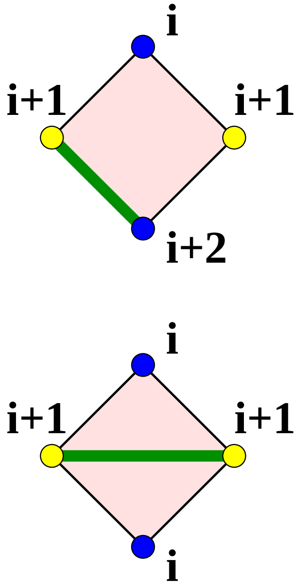


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

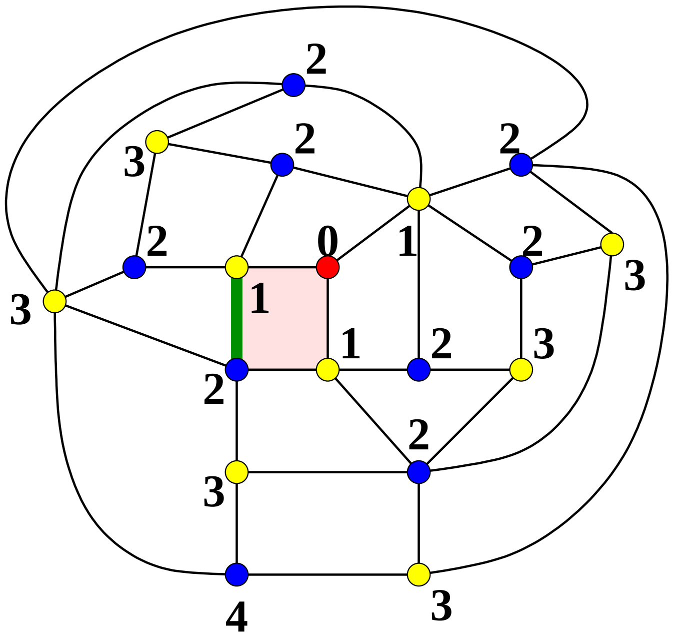


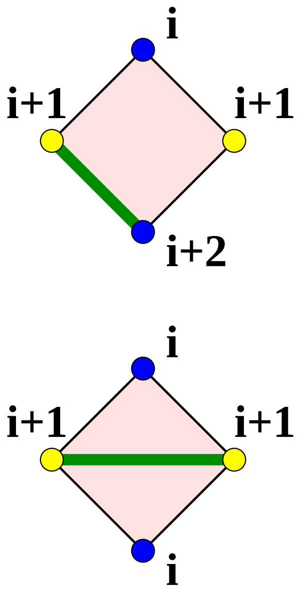


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

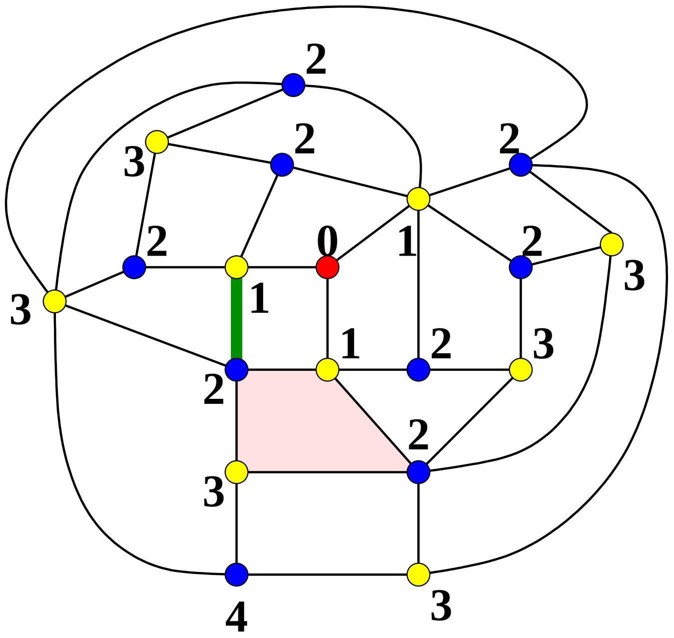


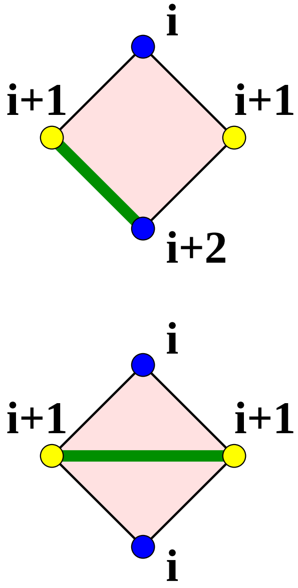


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

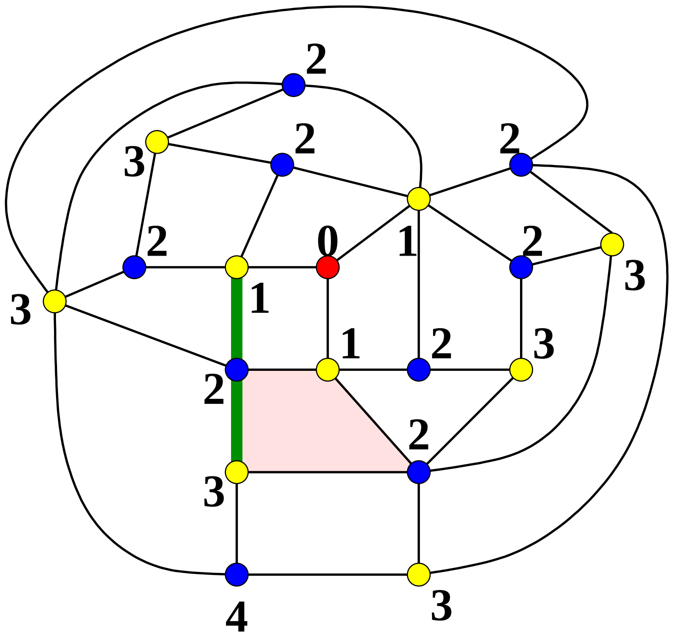


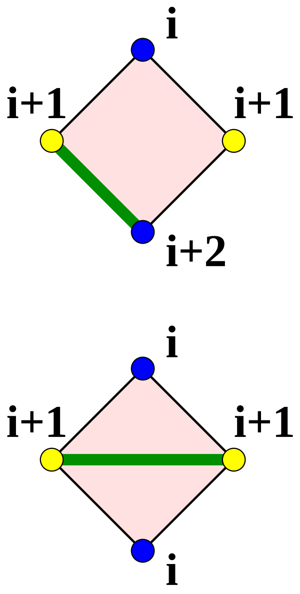


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

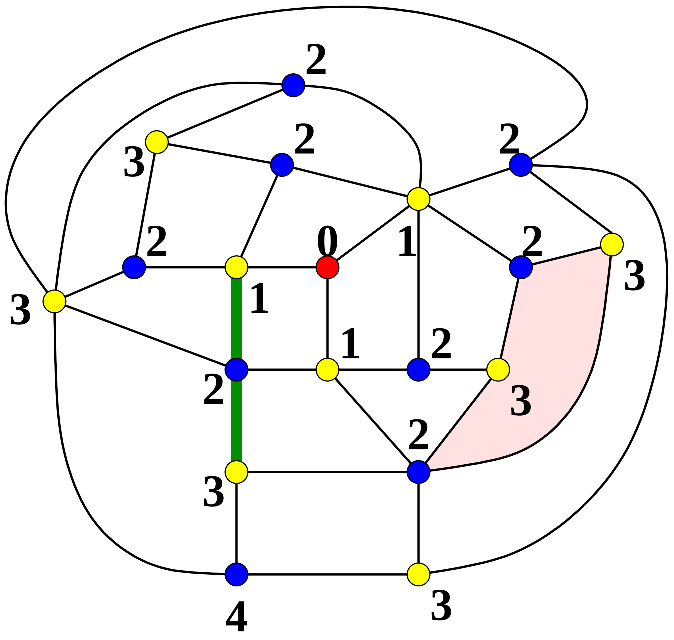


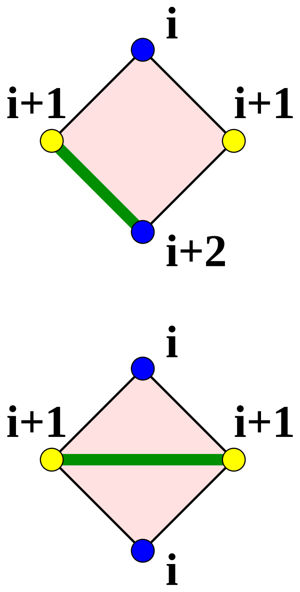


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

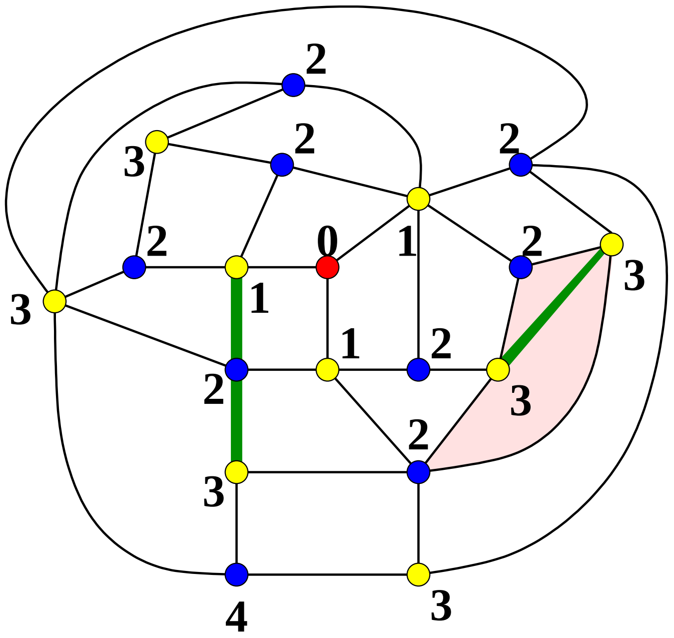


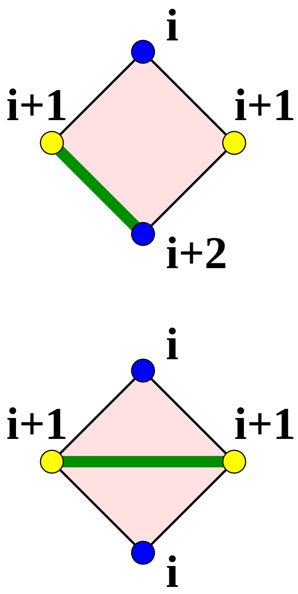


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

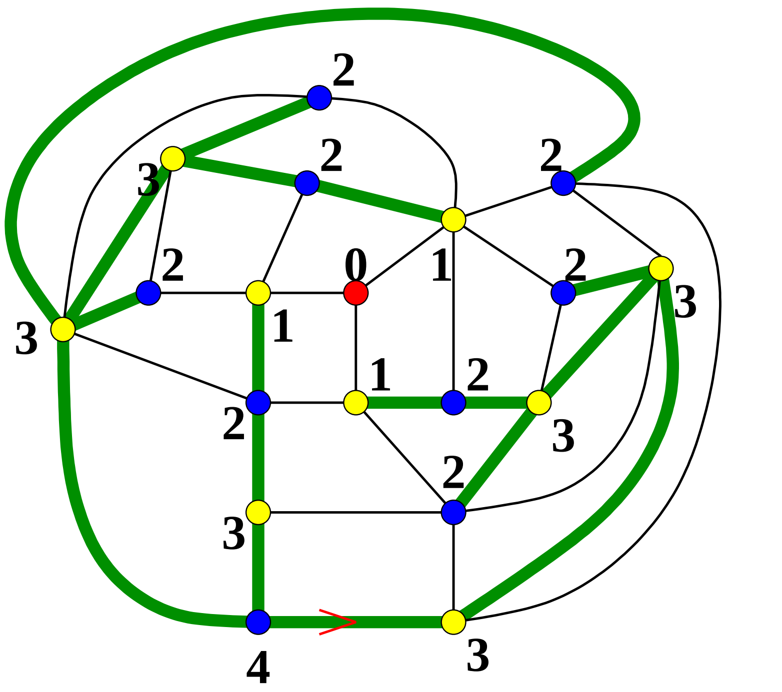


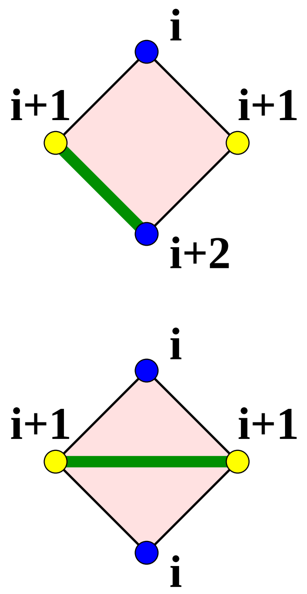


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

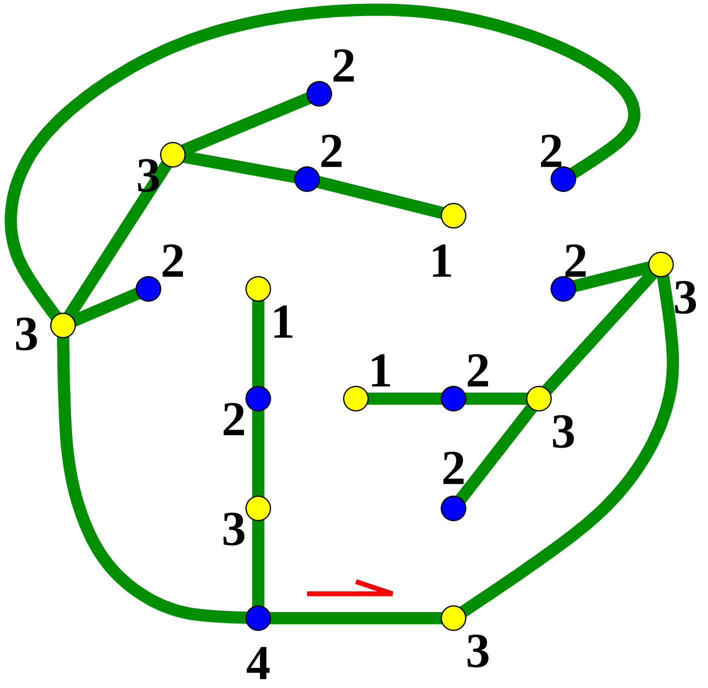


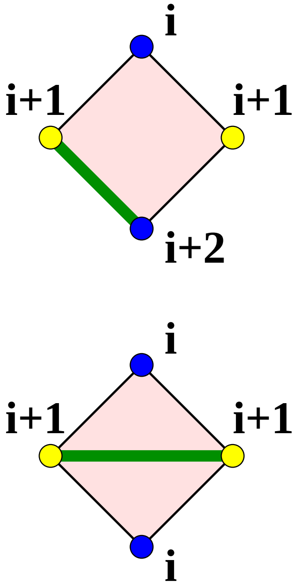


Apply these rules to all faces.

A bijection. Local rules

Consider the following two local rules.

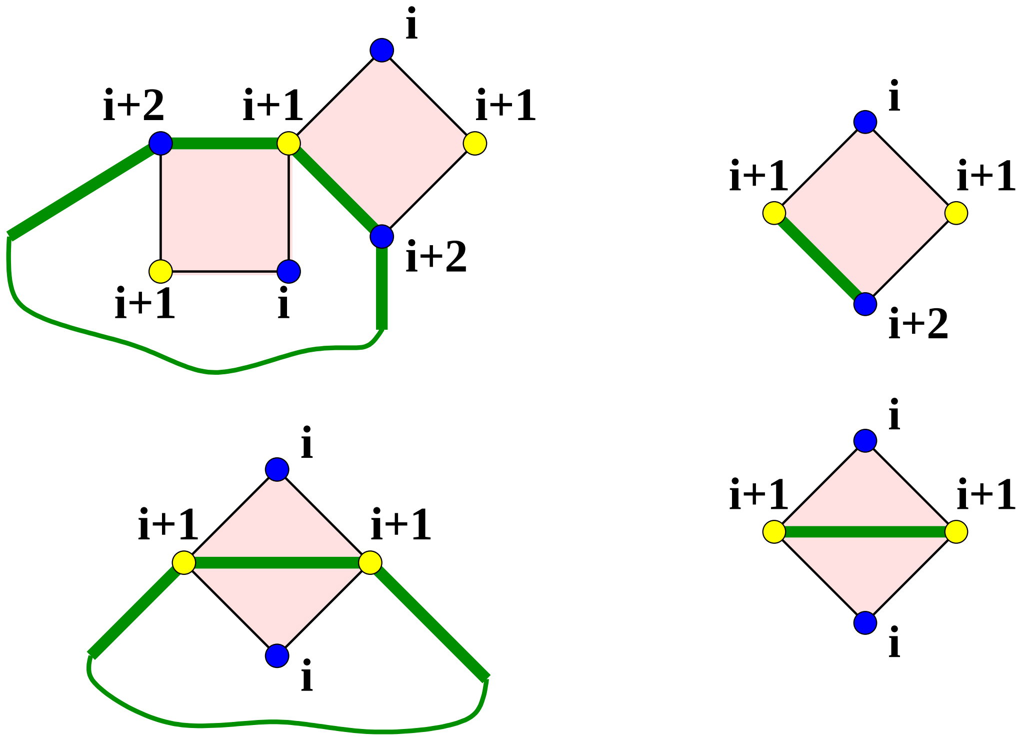




Proposition: the edges produced by local rules form a tree.

A bijection. Local rules

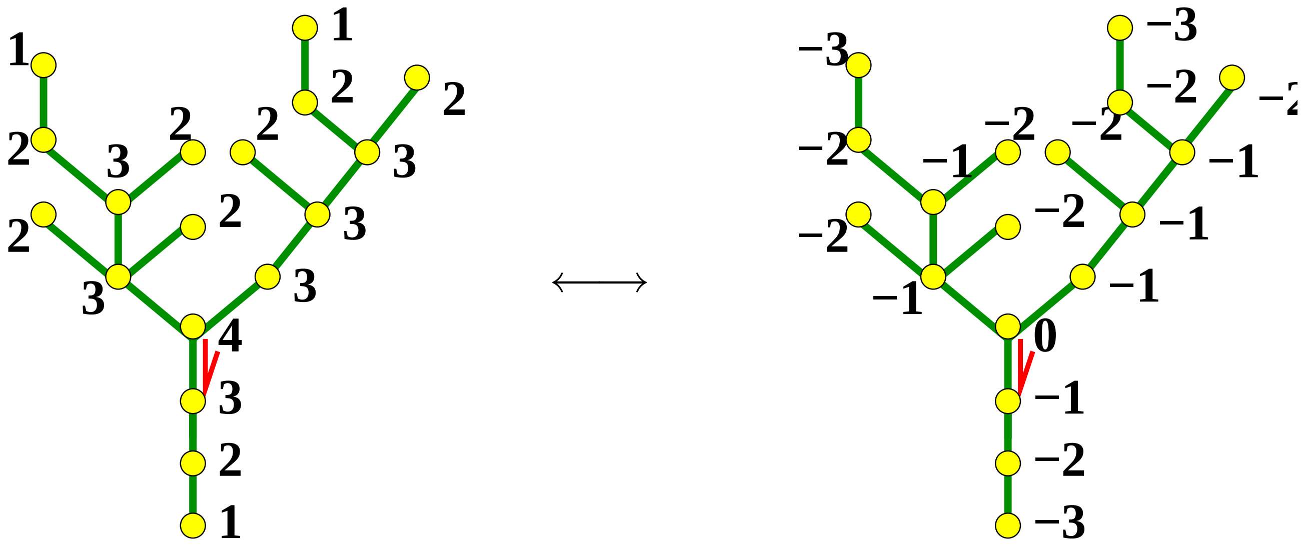
Proposition: the edges produced by local rules form a tree.



The root can be only in one of the two regions delimited by a cycle. Taking $i + 1$ minimal on the cycle, a contradiction is obtained between rules and labelling by distance.

A bijection. back from the tree.

By construction, labels in the tree differ at most by one along edges.

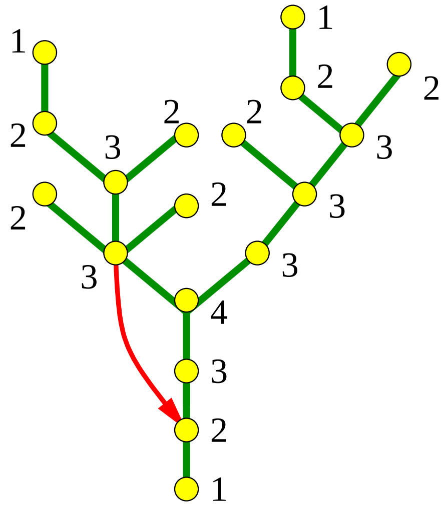


Upon translating all labels so that the root is 0, the resulting tree is an embedded tree.

Proposition. The quadrangulation can be recovered from the tree

A bijection. back from the tree.

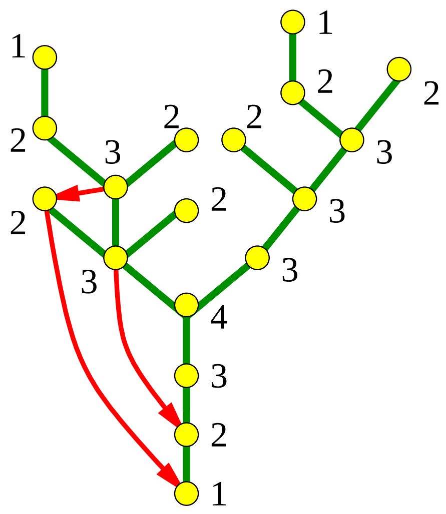
Starting from a translated embedded tree:



Missing edges are recovered by a greedy ($i \to i-1$) matching around the tree.

A bijection. back from the tree.

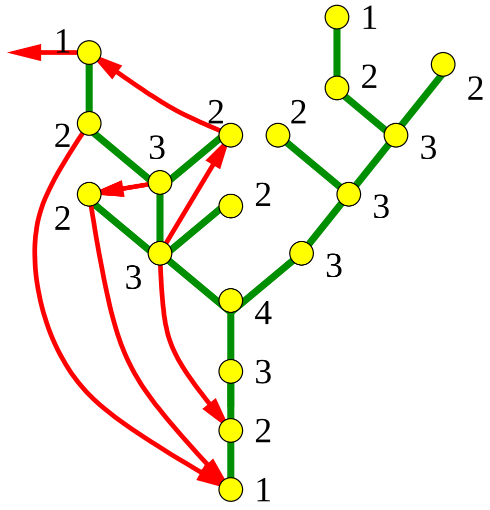
Starting from a shifted embedded tree:



Missing edges are recovered by a greedy $(i \to i-1)$ matching around the tree.

A bijection. back from the tree.

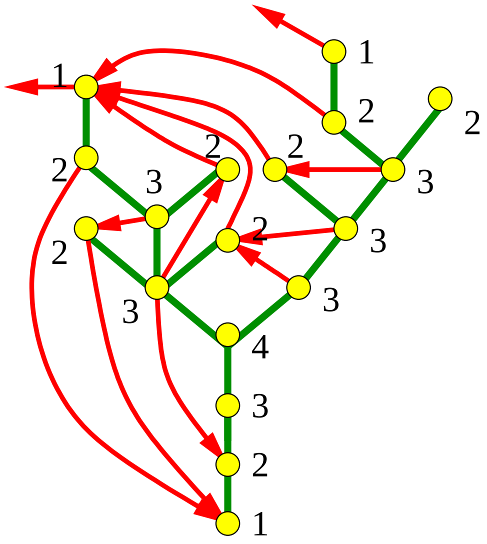
Starting from a shifted embedded tree:



Missing edges are recovered by a greedy $(i \to i-1)$ matching around the tree.

A bijection. back from the tree.

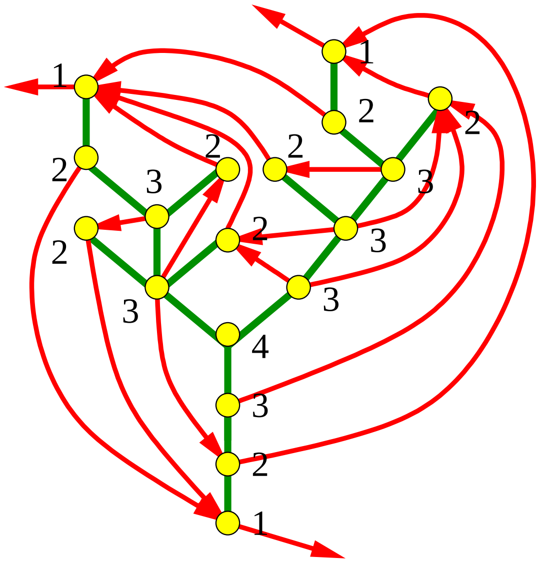
Starting from a shifted embedded tree:



Missing edges are recovered by a greedy $(i \to i-1)$ matching around the tree.

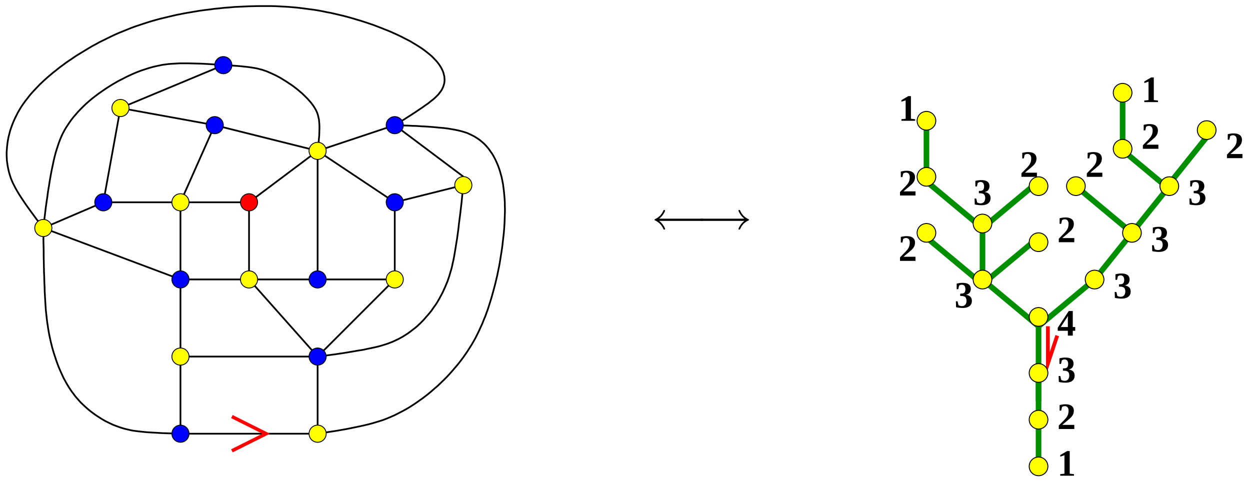
A bijection. back from the tree.

Starting from a shifted embedded tree:



Missing edges are recovered by a greedy $(i \rightarrow i-1)$ matching around the tree.

A bijection. Conclusion.

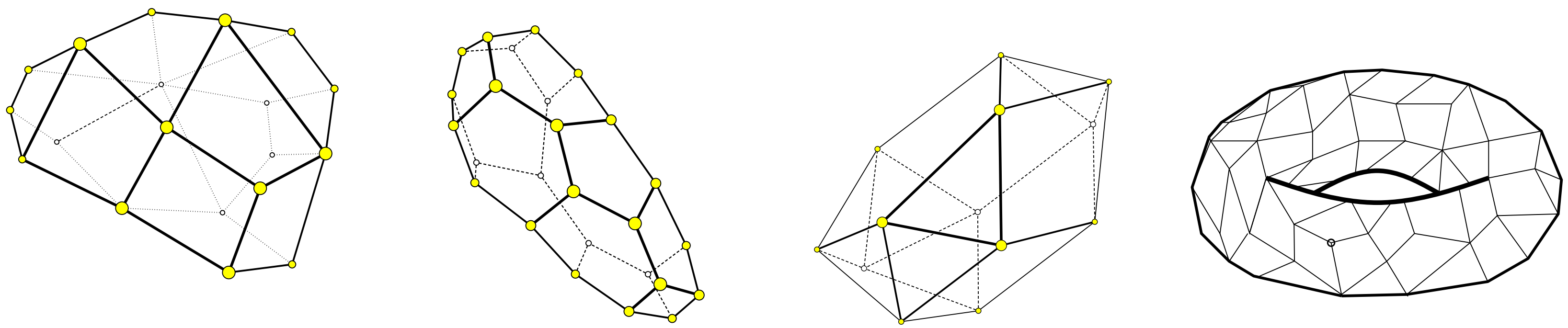


The bijection proves $|\mathcal{Q}_n^{\bullet}| = 2\frac{3^n}{n+1}\binom{2n}{n}$, and thus allows to recover Tutte's formula $|\mathcal{Q}_n| = \frac{2}{n+2}\frac{3^n}{n+2}\binom{2n}{n}$.

The local rule is a simplified variant of a recursive bijection (Cori-Vauquelin'83). It was recently extended to general bipartite planar maps (Bouttier et al'05.)

Random maps

as a discrete model of random geometries



Uniform random maps. Definition.

Let $\mathcal{R}_n$ be a family of rooted maps

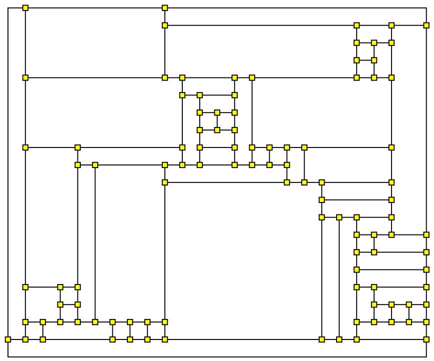
say, for instance $R_n = \{$planar quadrangulations with n faces$\}$.

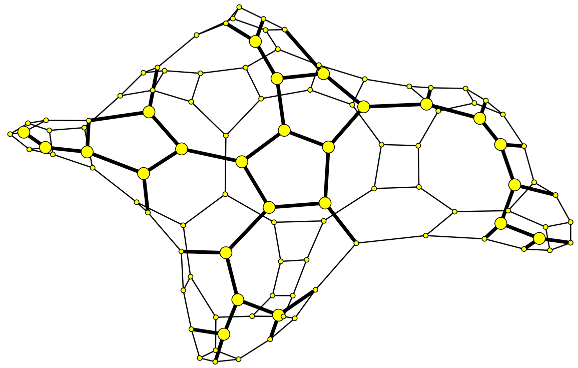
Consider a r.v. X_n with uniform distribution on $\mathcal{R}_n$:

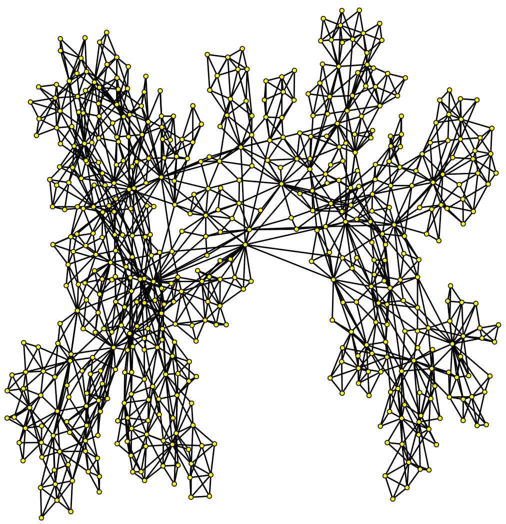
$$\Pr(X_n = R) \ = \ \frac{1}{|\mathcal{R}_n|}, \qquad \text{for all } R \in \mathcal{R}_n.$$

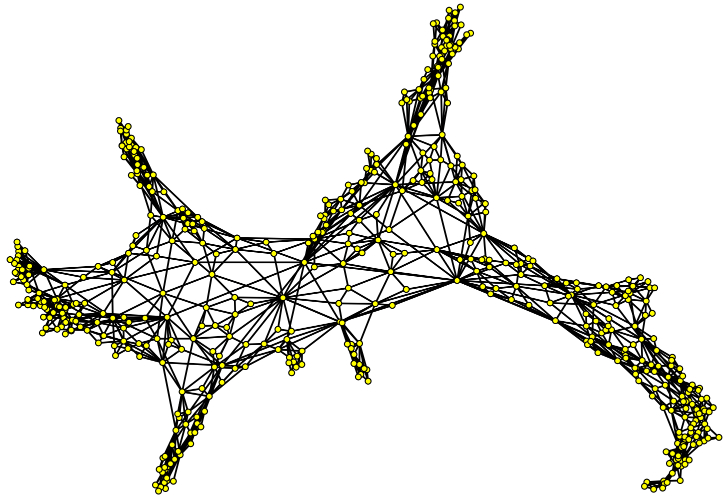
– This model is exactly equivalent to the *dynamical triangulations* that are used in statistical physics to modelize *2d discretized quantum geometry*.

A gallery of random maps



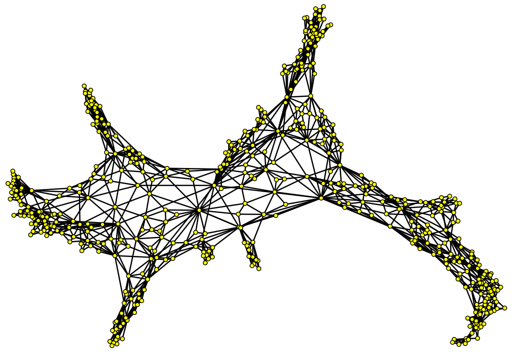






How is the intrinsic geometry of these random surfaces ?

Random maps appear to be quite different from regular lattices.

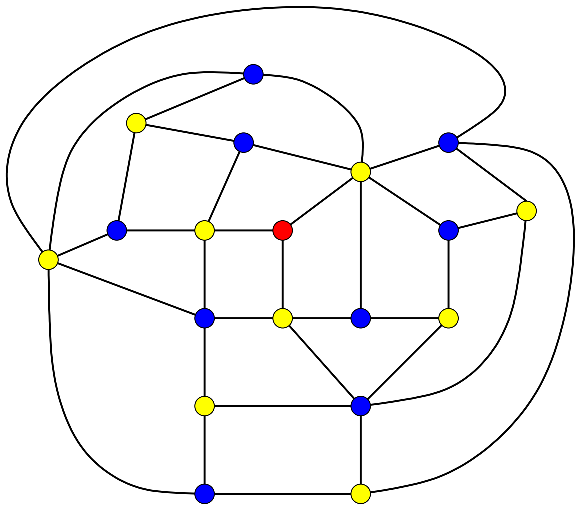


– a fat tree structure ?
– branchings into baby universes ?
– Hausdorf dimension ?
– short separators ?

These questions have raised a lot of interest in statistical physics.
Today we present some results on distances in random quadrangulations.

Profile and radius of a quadrangulation with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of a random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$

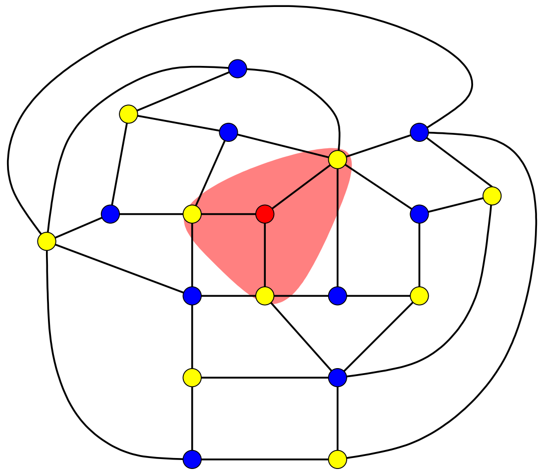


- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius of a quadrangulation with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of the random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



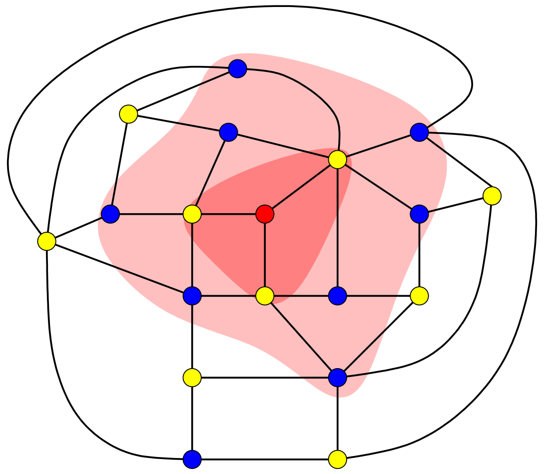
$X_n^{(1)} = 3$

- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius of a quadrangulation with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of the random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



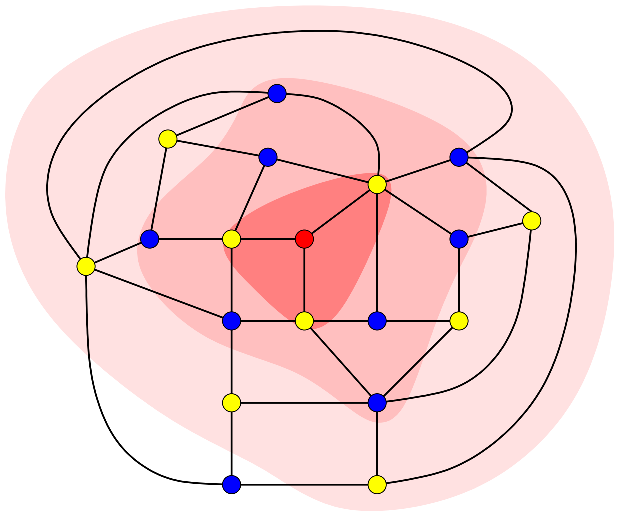
$$X_n^{(1)} = 3$$
$$X_n^{(2)} = 8$$

- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius of a quadrangulation with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of the random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



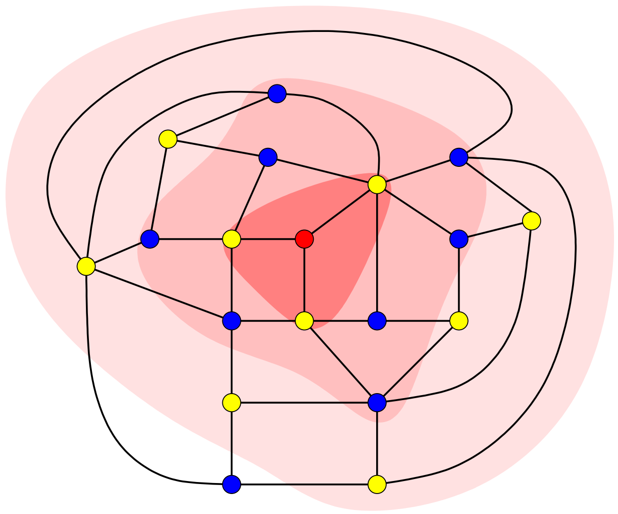
$$X_n^{(1)} = 3$$
$$X_n^{(2)} = 8$$
$$X_n^{(3)} = 6$$

- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius of a quadrangulation with n faces.

- $X_n^{(k)}$ is the number of vertices at distance k of the random vertex
- the *profile* is then $X_n = (X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}, \ldots)$



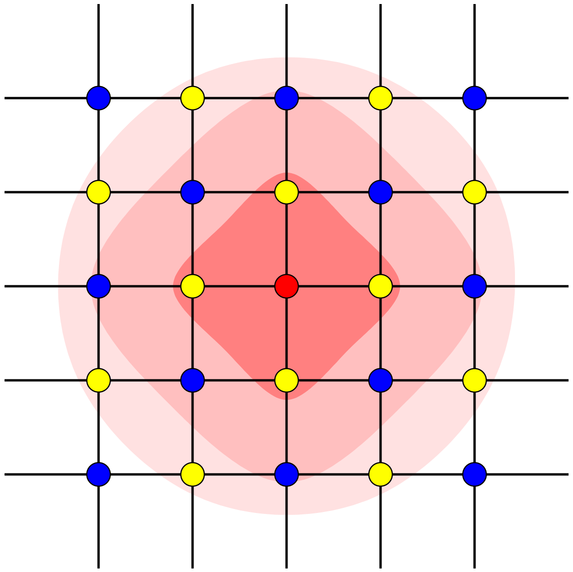
$$X_n^{(1)} = 3$$
$$X_n^{(2)} = 8$$
$$X_n^{(3)} = 6$$
$$X_n^{(4)} = 1$$
$$r_n = 4.$$

- r_n is the radius (maximal distance from the red vertex)

In particular $r_n \leq D_n \leq 2r_n$, where D_n is the diameter.

Profile and radius. On the grid ?

On a grid with n faces ($\sqrt{n} \times \sqrt{n}$), the behaviour is clear:



In particular,

$$X_n^{(k)} = \Theta(k) \text{ for } k < n^{1/2}, \quad \text{and } r_n \text{ grows like } n^{1/2}.$$

How do these parameters behave on random quadrangulations ?

Distances and embedded trees.

According to the previous bijection:

$$\text{Uniform distribution on quadrangulations with } n \text{ faces}$$
$$=$$
$$\text{Uniform distribution on embedded trees with } n \text{ edges}$$

Moreover the distribution of labels exactly encodes the profile!

In particular, the radius $r_n = \max(k \mid X_n^{(k)} > 0)$ is the difference between the min and max labels of a random embedded tree.

These are *identities in law*, not just asymptotic results.

Distances and random embedded trees. Typical labels.

Proposition. Branches of a uniform random plane tree with n edges have typically lenght $\Theta(\sqrt{n})$

Proposition. The labels along a branch form a random walk with uniform increments in $\{-1, 0, +1\}$.

$\Rightarrow$ labels on a length ℓ branch are $\Theta(\sqrt{\ell})$

Hence typical labels are of order $\Theta(n^{1/4})$, and a typical label is expected to be shared by $\Theta(n^{3/4})$ vertices.

In terms of random quadrangulations this means that the typical distance between two random vertices is $O(n^{1/4})$.

Profile and radius. Results

Much more precise results follow from the study of embedded trees.

For instance:

Theorem (Chassaing-S. 2002). The correct scaling is $k = tn^{1/4}$, and

– $n^{-3/4} X_n^{(tn^{1/4})} \xrightarrow{\text{law}} X(t)$, a process supported on $\mathbb{R}^+$,

– the radius satisfies $\mathbb{E}(r_n) \underset{n\to\infty}{\sim} cte \cdot n^{1/4}$.

The process underlying $X(t)$ is the Integrated Superbrownian Excusion, introduced by Aldous to describe the continuum limit of embedded trees.

Our theorem is based on a description of the ISE in terms of Brownian snakes due to Le Gall.

Profile and radius. Comparison.

The previous results are in agreement with previous predictions from physics.

For random triangulations:

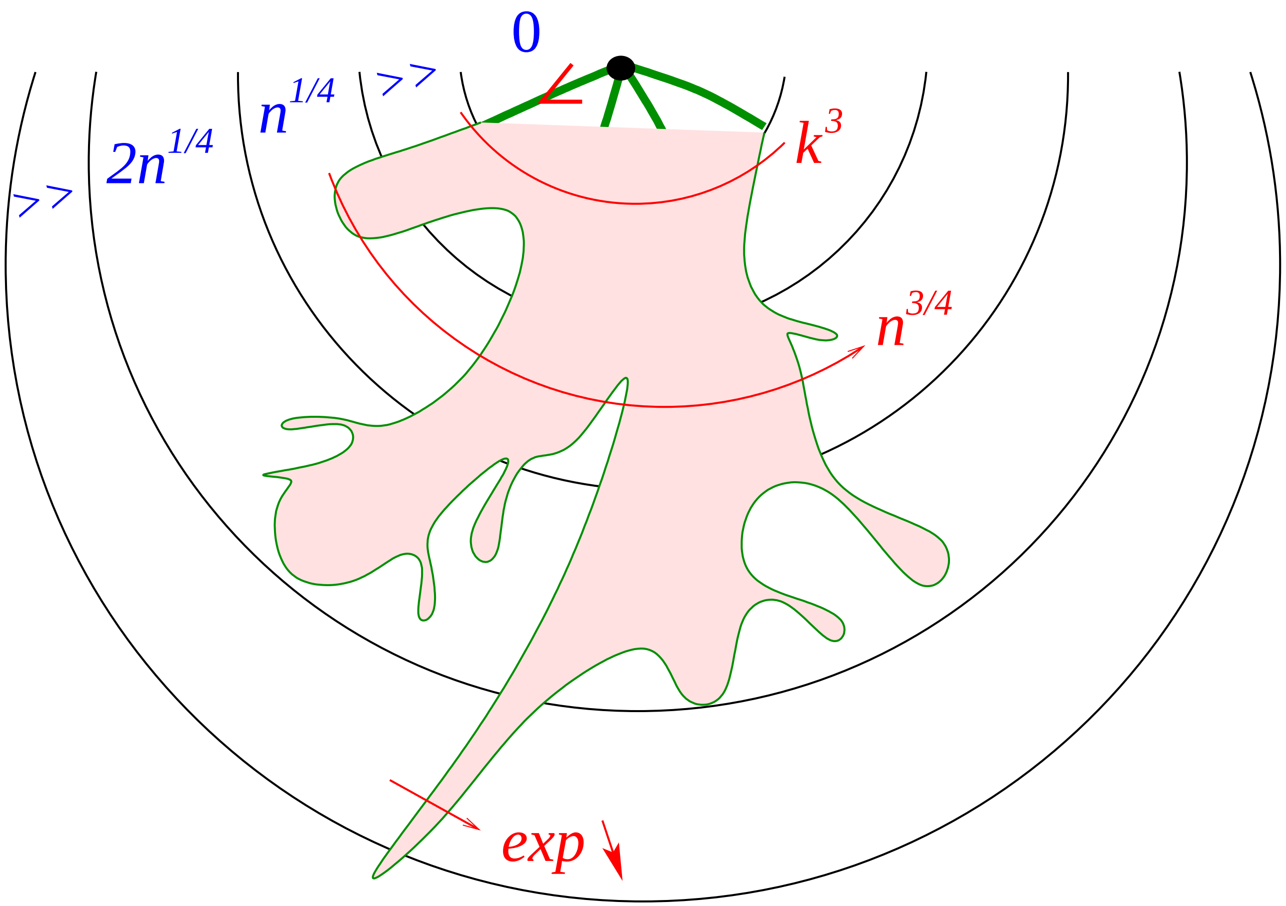
- Two beautyful heuristic calculations by physicists Watabiki, Ambjørn *et al.* (1994). *The Hausdorff dimension is* 4 *:*

$$\begin{array}{lll} \text{meaning} & \text{for } k \ll n^{1/4}, & \mathbb{E}(\int_0^k X_n^{(i)}) \sim k^4, \\ & \text{for } k \gg n^{1/4}, & \mathbb{E}(X_n^{(k)}) \text{ is exp. decreasing} \end{array}$$

  They had already proven the only possible scaling to be $k = tn^{1/4}$.

Our correspondence between quadrangulations and embedded trees has lead to many further results, see Marckert-Mokkadem'04, Bouttier et al.'04, Bousquet-Melou'05, Marckert-Miermont'05.

Random planar quadrangulations. A picture of distances.

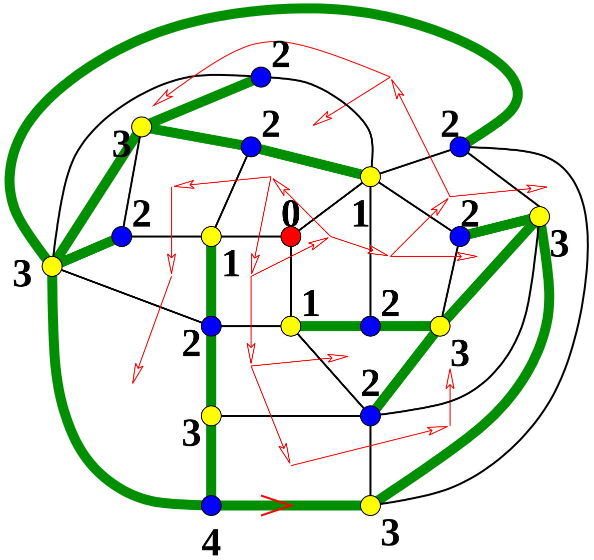


Maps on surfaces

The scheme of a map

Local rules for higher genus?

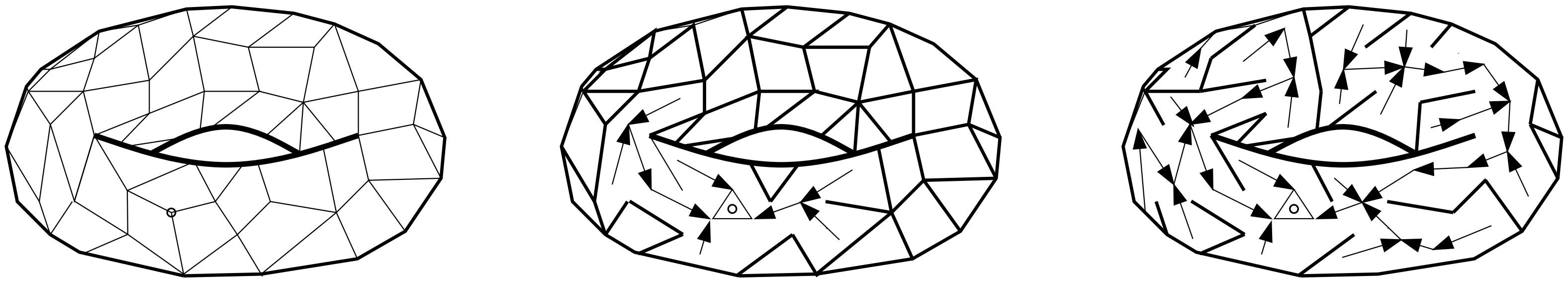
The fact that local rules produce a green tree in the plane is equivalent to saying that the dual red edges essentially form a tree:



In genus $g > 0$, local rules form cycles (the argument was based on planarity). However it remains true that red edges do not form cycles.

What happens on a surface?

During the growth of a dual tree on a surface of genus g, all faces are slowly merged into one big face.



The final result is not a tree but a map with one face.

Theorem (Marcus-S. 05) The local rule yields a bijection between:
– rooted pointed quadrangulations with n faces and genus g,
– and embedded one face maps with n edges and genus g.

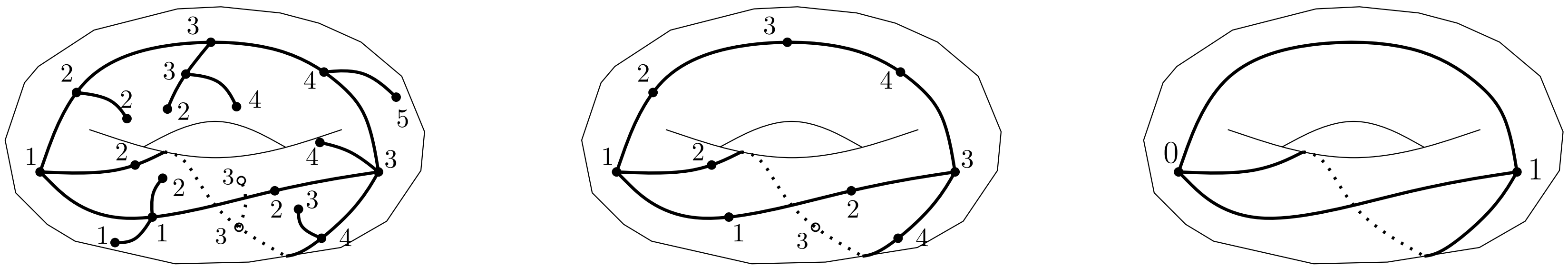
What happens on a surface?

Embedded one face maps can be decomposed as follows:

– remove recursively vertices of degree 1,

– replace chains by superedges and normalize labels.

The resulting **schemes** are one face maps with vertices of degree ≥ 3.



Proposition. The number of schemes of genus g is finite.

Proposition. The generating function of embedded one face maps having a given scheme is a simple rational function of the generating function of embedded trees.

What happens on a surface? Asymptotic enumeration

– The dominant schemes of genus g (those producing most maps) are made of $4g-2$ vertices of degree 3 and $6g-3$ edges.

– An embedded map of genus $g+1$ can be produced from an embedded map of genus g by gluying 3 points with the same label: this creates generically 4 new vertices in the dominant scheme.

– There are $\Theta(n^{1/4})$ labels and $\Theta(n^{3/4})$ points share a given label
$\Rightarrow$ there are $\Theta(n^{1/4}(n^{3/4})^3) = \Theta(n^{5/2})$ ways to increment the genus.

$$\#\{\text{quad. } n \text{ faces, genus } g\} = n^{\frac{5}{2}g} \cdot \#\{\text{planar quad. } n \text{ faces}\}.$$

What happens on a surface. Asymptotic results

We have hence rederived combinatorially the following result:

Theorem (Bender-Canfield'94 / also independantly in physics)
The family of quadrangulations on surfaces satisfies

$$\#\{\text{quadrangulations of genus } g \text{ with } n \text{ faces}\} \;=\; c_g n^{\frac{5}{2}(g-1)} \rho^n,$$

where $\rho = 12$.

The form $n^{\frac{5}{2}(g-1)} \rho^n$ of the asymptotic formula is typical of families of maps: the constant ρ depends on the family, but the polynomial correction is "always" driven by the same "universal critical exponent of pure 2d quantum gravity" $\frac{5}{2}(1-g)$.

Conclusion.

The previous approach yields the first rigorous proof that distances remain of order $\Theta(n^{1/4})$ in quadrangulations with n faces on higher genus surfaces.

Many questions remain open about the geometry of random (planar) maps. In particular:

- Is it possible to separate a map of size n in 2 roughly equal parts with a cycle of length $\ll n^{1/4}$?

This would help us to understand whether planar map have cut points in the continuum limit, or if there is a chance that they keep the topology of the sphere.