ON THE ASYMPTOTIC NUMBER OF PLANE CURVES AND ALTERNATING KNOTS

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Abstract. We present a conjecture for the power-law exponent in the asymptotic number of types of plane curves as the number of self-intersections goes to infinity. In view of the description of prime alternating links as flype equivalence classes of plane curves, a similar conjecture is made for the asymptotic number of prime alternating knots.

The rationale leading to these conjectures is given by quantum field theory. Plane curves are viewed as configurations of loops on a random planar lattices, that are in turn interpreted as a model of 2d quantum gravity with matter. The identification of the universality class of this model yields the conjecture.

Since approximate counting or sampling planar curves with more than a few dozens of intersections is an open problem, direct confrontation with numerical data yields no convincing indication on the correctness of our conjectures. However, our physical approach yields a more general conjecture about connected systems of curves. We take advantage of this to design an original and feasible numerical test, based on recent perfect samplers for large planar maps. The numerical data strongly support our identification with a conformal field theory recently described by Read and Saleur.

1. Introduction.

Our motivation for this work is the enumeration of topological equivalence classes of smooth open and closed curves in the plane (see Figure 1; precise definitions are given in Section 2.1). The problem of characterizing closed curves was considered already by Gauss and has generated many works since then: see [19] and references therein. Our interest here is in the numbers a_p and α_p of such open and closed curves with p self-intersections, and more precisely we shall consider the asymptotic properties of a_p and α_p when p goes to infinity. The numbers a_p were given up to $p = 10$ in [8] and have been recently computed up to $p = 22$ by transfer matrix methods [10]. Asymptotically, as p goes to infinity, one expects the relation $a_p \sim 4\,\alpha_p$ to hold (see below), so that we concentrate on the numbers a_p.

In the present paper we propose a physical reinterpretation of the numbers a_p that leads to the following conjecture, and we present numerical results supporting it.

Conjecture 1. *There exist constants τ and c such that*

$$\alpha_p \underset{p\to\infty}{\sim} \frac{1}{4} a_p \underset{p\to\infty}{\sim} c\,\tau^p \cdot p^{\gamma-2},$$

where

$$\gamma = -\frac{1+\sqrt{13}}{6} \doteq -0.76759... \qquad (1)$$

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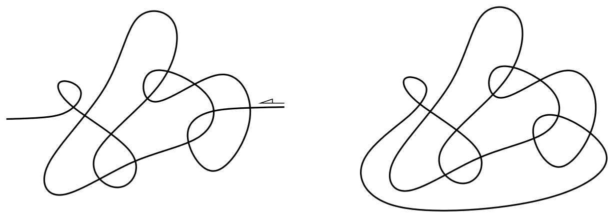

FIGURE 1. An open plane curve and the associated closed curve.

From the data of [10] one has the numerical estimate: $\tau \doteq 11.4$. But the main point in Conjecture 1 lies not so much in the existence of τ as in the explicit value of γ. It should indeed be observed that γ, rather than τ, is the interesting information in the asymptotic of a_p. Observe for instance that the value of γ is left unchanged if one redefines the size of a closed curve as the number $p' = 2p$ of arcs between crossings. More generally, as discussed in Section 8, the exponent γ determines the branching behavior of generic large curves under the uniform distribution, and is *universal* in the sense that the same value is expected in the asymptotic of related families of objects like prime self-intersecting curves or alternating knots.

Conjecture 1 is similar in nature to the conjecture of Di Francesco, Golinelli and Guitter on the asymptotic behavior of the number of plane meanders [6]. The two problems do not fall into the same universality class (in particular the predictions for the exponent γ are different in the two problems). However our approach to design a numerical test is applicable also to the meander problem.

The rest of the paper is organized as follows. Precise definitions are given, and a more general family of drawings is introduced that play an important rôle in the identification of the associated physical model (Section 2). The physical background leading to the conjecture is then discussed (Section 3) and a numerically testable quantity is proposed (Section 4). The sampling method is briefly presented (Section 5) before the analysis of numerical data (Section 6). We conclude with some variants and corollaries of the conjecture (Section 8).

2. Plane curves and colored planar maps

2.1. **Plane curves and doodles.** For p a positive integer, let $\mathcal{A}_p$ be the set of equivalence classes of self-intersecting loops γ in the plane, that is: (i) γ is a smooth mapping $S^1 \to \mathbb{R}^2$; (ii) there are p points of self-intersection, all of which are regular crossings; (iii) two loops γ and γ' are equivalent if there exists an orientation preserving homeomorphism h of the plane such that $\gamma'(S^1) = h(\gamma(S^1))$.

Similarly let A_p be the set of equivalence classes of self-intersecting open curves γ in the plane: (i) γ is a smooth mapping $[0,1] \to \mathbb{R}^2$ and $\gamma(0)$ and $\gamma(1)$ belong to the infinite component of $\mathbb{R}^2 \setminus \gamma((0,1))$; (ii) there are p points of self-intersection, all of which are regular crossings; (iii) two open curves are equivalent if there exists an orientation preserving homeomorphism h of the plane such that $\gamma'([0,1]) = h(\gamma([0,1]))$ and $\gamma'(i) = h(\gamma(i))$ for $i = 0, 1$.

Observe that, unlike closed curves, open curves are oriented from the initial point $\gamma(0)$ to the final point $\gamma(1)$. Moreover a unique closed curve is obtained from an open curve by connecting the final point to the initial one in counterclockwise direction around the curve. These definitions are illustrated by Figure 1.

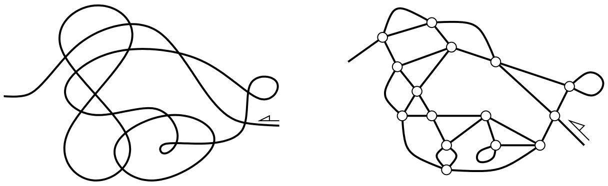

FIGURE 2. A doodle and the associated rooted planar map.

In order to study the families $\mathcal{A}_p$ and A_p and to obtain Conjecture 1 we introduce a more general class of drawings, that we call *doodles*. For given positive integers p and k, let $A_{k,p}$ be the set of equivalence classes of $(k+1)$-uples $\Gamma = (\gamma_0, \gamma_1, \ldots, \gamma_k)$ of curves drawn on the plane such that: (i) the curve γ_0 is an open curve of the plane: γ_0 is a smooth mapping $[0,1] \to \mathbb{R}^2$, and $\gamma_0(0)$ and $\gamma_0(1)$ belong to the infinite component of $\mathbb{R}^2 \setminus (\gamma_0((0,1)) \cup \bigcup_i \gamma_i(S^1))$; (ii) for $i \geq 1$, each γ_i is a loop, that is a smooth map $S^1 \to \mathbb{R}^2$; (iii) there are p points of intersection (including possibly self-intersections) of these curves, all of which are regular crossings; (iv) the union of the curves is connected, (v) two doodles $\Gamma = (\gamma_0, \ldots, \gamma_k)$ and $\Gamma' = (\gamma'_0, \ldots, \gamma'_k)$ are equivalent if there exists an orientation preserving homeomorphism h of the plane such that $\gamma'_0([0,1]) \cup \bigcup_i \gamma'_i(S^1) = h(\gamma_0([0,1]) \cup \bigcup_i \gamma_i(S^1))$ and $\gamma'_0(x) = h(\gamma_0(x))$, for $x = 0, 1$. In other terms, a doodle is made of an open curve intersecting a set of loops, that are considered up to continuous deformations of the plane. An example of doodle is given in Figure 2 (left-hand side).

2.2. **Colored planar maps.** An equivalent presentation of doodles is in terms of rooted planar maps [22, 23]. A *planar map* is a proper embedding of a connected graph in the plane considered up to homeomorphisms of the plane. It is *4-regular* if all vertices have degree four. It is *rooted* if one root edge is marked on the infinite face and oriented in counterclockwise direction. Equivalently the root edge can be cut into an in- and an out-going half-edge (also called *legs*) in the infinite face. There is an immediate one-to-one correspondence between doodles with p crossings, and 4-regular planar maps with p vertices and two legs. This correspondence is illustrated on Figure 2.

We shall consider the number $a_{k,p} = \operatorname{card} A_{k,p}$ of doodles with p crossings and k loops and more specifically we shall consider the asymptotic properties of $a_{k,p}$ as p (and possibly k) goes to infinity. It turns out to be convenient to introduce the generating function $a_p(n)$ as k varies:

$$a_p(n) = \sum_{k=0}^{\infty} a_{k,p}\, n^k \tag{2}$$

The requirement that a doodle is connected implies that it cannot contain more loops than crossings so that $a_p(n)$ is a polynomial in the (formal) variable[1] n. For real valued n, $a_p(n)$ can be understood as a weighted summation over all doodles with p crossings, and, more specifically for n a positive integer, $a_p(n)$ can be interpreted as the number of *colored* doodles in which each loop has been drawn using a color taken from a set of n distinct colors.

[1]The subsequent interpretation in terms of colored doodles and the strong tradition in the physics litterature are our (admittingly poor) excuses for the use of n to denote a formal variable.

On the one hand, in the special case $k = 0$, $a_{0,p} = a_p$ gives by definition the number of open self-intersecting plane curves. Observe that a_p is also given by $n = 0$ since $a_p(0) = a_{0,p}$. On the other hand, the generating functions of the $a_p(n)$ for other values of n, namely $n = 1, 2, -2$, have been computed exactly (see respectively [22, 26, 28]). We elaborate now on the case $n = 1$ since it will play a crucial role in what follows.

The number $a_p(1)$ counts the number of doodles with p crossings irrespective of the number of loops k. In terms of maps, $a_p(1)$ is the number of rooted 4-regular planar maps with p vertices. The number of such planar maps is known [3, 22], from which one can compute the asymptotics:

$$a_p(1) = 2\frac{3^p\,(2p)!}{p!(p+2)!} \underset{p\to\infty}{\sim} \frac{2}{\sqrt{\pi}}\,12^p p^{-5/2}. \tag{3}$$

Observe in this case the power-law exponent $-5/2$, which is *universal* for rooted planar maps in the sense that it is observed for a variety of families of rooted planar maps (see [7]). As opposed to this, the exponential growth factor 12 is specific to the family of rooted 4-regular planar maps.

There is a physical interpretation of the power-law behavior $p^{-5/2}$: it is given by two-dimensional gravity. This explanation begs to be generalized to any n, and we shall explore such a possibility now.

3. Two-dimensional quantum gravity and asymptotic combinatorics

The purpose of this section is to give the rationale behind our conjectures. We place the discussion at a rather informal level that we hope achieves a double purpose: on the one hand it should give an intuition of the path leading to our conjectures to the reader with zero-knowledge in quantum field theory (QFT), and on the other hand it should convince the expert of this (quantum) field. For our defense, let us observe that filling in more details would require a complete course on QFT, with the result of not getting much closer to a mathematical proof.

3.1. **From planar maps to two-dimensional quantum gravity.** The main idea of the physical interpretation of the numbers $a_p(1)$ is to consider planar maps as discretized random surfaces (with the topology of the sphere). As the number of vertices of the map grows large, the details of the discretization can be assimilated to the fluctuations of the *metric* on the sphere. To make this idea more precise, let us describe a way to associate a metric on the sphere to a given 4-regular map m: to each vertex of m associate a unit square and identify the sides of these squares according to the edges of m (arbitrary number of corners of squares get identified); the result is by construction a metric space with the topology of a sphere. Upon taking a 4-regular map uniformly at random in the set of map with p edges, a random metric sphere with area p is obtained.

Now, physics tells us that the metric is the dynamical field of general relativity *i.e.* gravity, and that this type of fluctuations in the metric are characteristic of a quantum theory. In our case it means that, as p becomes large, the discrete nature of the maps can be ignored and there exists a scaling limit, the properties of which are described by two-dimensional euclidian quantum gravity. In particular any parameter of random planar maps that makes sense in the scaling should converge to its continuum analog. A fundamental parameter of this kind turns out to be precisely the number of (unrooted) planar maps: it is expected to scale to the partition function $Z_{\mathrm{g}}(A)$ of two-dimensional quantum gravity with spherical topology at fixed area A, through a relation of the form

$$\frac{1}{p} a_p(1) \underset{p\to\infty}{\sim} Z_{\mathrm{g}}(A), \text{ with } A \text{ proportional to } p. \tag{4}$$

(Here the factor $1/p$ is due to the fact that the partition function does not takes the rooting into account.) The only thing we want to retain from $Z_{\mathrm{g}}(A)$ is that the power law dependance of its large area asymptotic takes the form $A^{-7/2}$, in accordance with Formula (3). (Trying to give here a precise description of the partition function Z_{g} would carry us to far away, and anyway the arguments in this section are non rigourous.)

In the case $n = 1$, this is the whole physical picture: a fluctuating but empty two-dimensional spacetime – there is no matter in it. What happens when $n \neq 1$? As already discussed, an appealing image is to consider that one must "decorate" the planar map by coloring each curve γ_i with n colors. Alternatively, the physicist's view would be to consider that we have put a statistical lattice model (of crossing loops) on a random lattice (the planar map). This perfectly fits with the previous interpretation of the planar map as a fluctuating two-dimensional spacetime: as we learn from physics, in the limit of large size, adding a statistical lattice model amounts to coupling matter to quantum gravity. Matter is described by a quantum field theory (QFT) living on the two-dimensional spacetime. The parameters of the lattice model that survive in the scaling limit are recovered in the critical (long distance) behavior of this QFT, which in turn is described by a conformal field theory (CFT). Then, provided we can find a CFT describing the lattice model corresponding to a given $n \neq 1$, the analog of Relation (4) holds with the partition function $Z_{\mathrm{g+CFT}(n)}(A)$ of this CFT coupled to gravity: in the large size limit,

$$\frac{1}{p} a_p(n) \underset{p\to\infty}{\sim} Z_{\mathrm{g+CFT}(n)}(A). \tag{5}$$

In this picture, the only thing we need to know about the CFT that describes the scaling limit of our model is its *central charge* c, which roughly counts its number of degrees of freedom. Indeed, the study of CFT coupled to gravity was performed in [4, 5, 13], resulting in the following fundamental prediction: the partition function $Z_{\mathrm{g+CFT}}(A)$ of gravity dressed with matter has a power-law dependence on the area of the form $A^{\gamma-3}$ where the critical exponent γ depends only on the central charge of the underlying CFT via (for $c < 1$)

$$\gamma = \frac{c - 1 - \sqrt{(1-c)(25-c)}}{12}. \tag{6}$$

Returning to our asymptotic enumeration problem (not forgetting the extra factor p which comes from the marked edge), we find:

$$a_p(n) \underset{p\to\infty}{\sim} \mathrm{e}^{\sigma\, p \,+\, (\gamma-2)\,\log p \,+\, \kappa} \tag{7}$$

where σ, κ are unspecified "non-universal" parameters, whereas the "universal" exponent γ is given by Eq. (6) with the central charge c of the *a priori* unknown underlying CFT(n). The absence of matter, that is the case $n = 1$, corresponds to a CFT with central charge $c = 0$: one recovers $\gamma - 2 = -5/2$ as expected from Eq. (3). In general, all parameters in Eq. (7) are functions of n; assuming furthermore that their dependence on n is smooth in a neighborhood of $n = 1$, one can recover by Legendre transform of $\sigma(n)$ the asymptotics of $a_{k,p}$ as k and p tend to infinity with the ratio k/p fixed. Observe finally that the knowledge of the CFT could give more informations. For instance, the irrelevant operators of the CFT control subleading corrections to Eq. (7).

3.2. **The identification of two candidate models.** We now come to the issue of the determination of the CFT for a arbitrary n. An observation made in [27], based on a matrix integral formulation, is that this CFT must have an $O(n)$ symmetry (for n positive integer – for generic n this symmetry becomes rather formal and cannot be realized as a unitary action of a compact group on the Hilbert space). There exists a well-known statistical model with $O(n)$ symmetry, a model of (dense) *non-crossing* loops [16], whose continuum limit for $|n| < 2$ is described by a CFT with central charge

$$c_{\mathrm{I}} = 1 - 6(\sqrt{g} - 1/\sqrt{g})^2 \qquad n = -2\cos(\pi g), \quad 0 < g < 1 \tag{8}$$

In [27] it was speculated that there is no phase transition between the model of non-crossing loops, which we call model I, and our model of crossing loops, and therefore the central charge is the same and given by Eq. (8). If this were the case, the study of irrelevant operators of this CFT would allow moreover to predict that subleading corrections to Eq. (7) have power-law behavior for all $|n| < 2$ with exponents depending continuously on n.

However, another scenario is possible. In [17], it was suggested that $O(n)$ models, for $n < 2$, possess in general a low temperature phase with *spontaneous symmetry breaking* of the $O(n)$ symmetry into a subgroup $O(n-1)$. This is a well-known mechanism[2] in QFT (see e.g. [25] chapters 14, 30), which produces Goldstone bosons living on the homogeneous space $O(n)/O(n-1) = S^{n-1}$. In the low energy limit the bosons become free and the central charge is simply the dimension of the target space S^{n-1}:

$$c_{\mathrm{II}} = n - 1 \qquad n < 2 \tag{9}$$

For generic real n ($n < 2$) this is only meaningful in the sense of analytic continuation, but we assume it can be done and call it model II. This CFT possess a marginally irrelevant operator, leading to main corrections to leading behavior (7) of *logarithmic* type *i.e.* in $\frac{1}{\log p}$, $\frac{\log\log p}{(\log p)^2}$ etc.

It was furthermore argued in [17] that the critical phase of model II is generic in the sense that the low-energy CFT is not destroyed by small perturbations – the most relevant $O(n)$-invariant perturbation is the action itself, which corresponds to a marginally irrelevant operator for $n < 2$. On the contrary, the model I of non-crossing loops is unstable to perturbation by crossings; some numerical work on regular lattices (at $n = 0$) [9] tends to suggest that it flows towards model II.

Note that both Conjectures (8) and (9) supply the correct value $c = 0$ for $n = 1$ and $c = 1$ for the limiting case $n = 2$.[3] Of course, in no way do we claim that these are the *only* possible scenarios which fit with known results – one might

[2]The Mermin–Wagner theorem, which forbids spontaneous symmetry breaking of a continuous symmetry in two dimensions, only applies to n integer greater or equal to 2.

[3]Actually, the two resulting $c = 1$ theories are not identical: the one from model I seems to be the wrong one, although this is a subtle point on which we do not dwell here.

have a plateau of non-critical behavior ($c = 0$) around $n = 1$, for instance; or two-dimensional quantum gravity universality arguments might not apply at all in some regions of n – but they seem the most likely candidates and therefore it is important to find a numerically accessible quantity which at least discriminates between the two conjectures.

4. The general conjectures and a testable parameter

The physical reinterpretation of doodles as a model on random planar lattices has led us to postulate that the weighted summation over doodles satisfies

$$a_p(n) \sim c_0(n)\, \tau(n)^p \cdot p^{\gamma(n)-2},$$

with the critical exponent $\gamma(n)$ given in terms of the central charge $c(n)$ by

$$\gamma(n) = \frac{c(n) - 1 - \sqrt{(1-c(n))(25-c(n))}}{12}.$$

Moreover we have presented two concurrent models which fix the value of $c(n)$. Since negative values of n create additional technical difficulties (appearance of complex singularities in the generating function, cf [28]), we formulate the conjectures in the restricted range $0 \le n < 2$:

Conjecture 2 (Model I). *Colored doodles are in the universality class of dense non-crossing loops, so that for* $0 \le n < 2$, $n = -2\cos(\pi g)$, $1/2 \le g < 1$,

$$c(n) = 1 - 6(\sqrt{g} - 1/\sqrt{g})^2.$$

Conjecture 3 (Model II). *Colored doodles are in the universality class of models with spontaneously broken* $O(n)$ *symmetry, so that for* $0 \le n < 2$,

$$c(n) = n - 1.$$

Observe that Conjecture 3 implies Conjecture 1 for $n = 0$, while Conjecture 2 would give $c(0) = 1 - 6(\sqrt{2} - 1/\sqrt{2})^2 = -2$ and $\gamma(0) = -1$. According to the discussion of the previous section, Conjecture 3 appears more convincing. In order to get a numerical confirmation, we look for a way to discriminate between the two.

Since the model at $n = 1$ is much easier to manipulate, we look for such a quantity at $n = 1$. Of course the known value of the exponent $\gamma(1)$ is a natural candidate but as already mentioned both conjectures agree on this: we propose instead the derivative of the exponent at $n = 1$,

$$\gamma' \equiv \frac{d}{dn}_{|n=1} \gamma(n). \tag{10}$$

The reason that it can easily be computed numerically is that it appears in the expansion of the average number of loops $\langle k \rangle_p$ for a uniformly distributed random planar map with p vertices. Indeed one easily finds

$$\langle k \rangle_p = \frac{d}{dn}_{|n=1} \log a_p(n) \underset{p\to\infty}{=} \sigma' p + \gamma' \log p + \kappa' + o(1) \tag{11}$$

Here we have assumed expansion (7) to be uniform with smoothly varying constants $\sigma(n)$, $\gamma(n)$, $\kappa(n)$ in some neighborhood of $n = 1$, and written $\sigma' \equiv \frac{d}{dn}_{|n=1}\sigma(n)$, $\kappa' \equiv \frac{d}{dn}_{|n=1}\kappa(n)$.

The conjectures 2 and 3 provide the following predictions for γ':

$$\gamma' = \begin{cases} \frac{3\sqrt{3}}{4\pi} = 0.413\ldots & \text{in CFT I} \\ \frac{3}{10} = 0.3 & \text{in CFT II} \end{cases} \tag{12}$$

The quantity $\langle k \rangle_p$ is not known theoretically, so that we cannot immediately conclude in either direction. However it is possible to estimate it numerically using random sampling.

5. Sampling random planar maps

In this section we present the algorithm we use to sample a random map from the uniform distribution on rooted 4-regular planar maps with p vertices. The problem of sampling random planar maps with various constraints under the uniform distribution was first approached in mathematical physics using markov chain methods [12, 2]. However these methods require a large and unknown number of iterations, and only approximate the uniform distribution. Another approach was proposed based on the original recursive decompositions of Tutte [22] but has quadratic complexity [1], and is limited as well to p of order a few thousands.

We use here a more efficient method that was proposed in [20, 21] along with a new derivation of Tutte's formulas. The algorithm, which we outline here in the case of 4-regular maps, requires only $O(p)$ operations to generate a map with p vertices and manipulates only integers bounded by $O(p)$. Moreover maps are sampled exactly from the uniform distribution. The only limitation thus lies in the space occupied by the generated map. In practice we were able to generate maps with up to 100 million vertices, with a generation speed of a million vertices per second.

The algorithm relies on a correspondence between rooted 4-regular planar maps and a family of trees that we now define. A *blossom tree* is a planted plane tree such that

- vertices of degree one are of two types: buds and leaves;
- each inner vertex has degree four and is incident to exactly one bud;
- the root is a leaf.

An example of blossom tree is shown on Figure 3. By definition a blossom tree with p inner vertices has $p+2$ leaves (including the root) and p buds. Observe that removing the buds of a blossom tree gives a planted complete binary tree with p inner vertices, and that conversely 3^p blossom trees can be constructed out of given binary tree with p inner vertices. Since the number of binary trees with p inner vertices is well known to be the Catalan number $\frac{1}{p+1}\binom{2p}{p}$, the number of blossom tree is seen to be

$$3^p \cdot \frac{1}{p+1}\binom{2p}{p}.$$

Let us define the *closure* of a blossom tree. An example is shown on Figure 3. Buds and leaves of a blossom tree with p inner vertices form in the infinite face a cyclic sequence with p buds and $p+2$ leaves. In this sequence each pair of consecutive bud and leaf (in counterclockwise order around the infinite face) are merged to form an edge enclosing a finite face containing no unmatched bud or leaf. Matched buds and leaves are eliminated from the sequence of buds and leaves in the infinite face and the matching process can be repeated until there is no more buds available. Two leaves then remain in the infinite face.

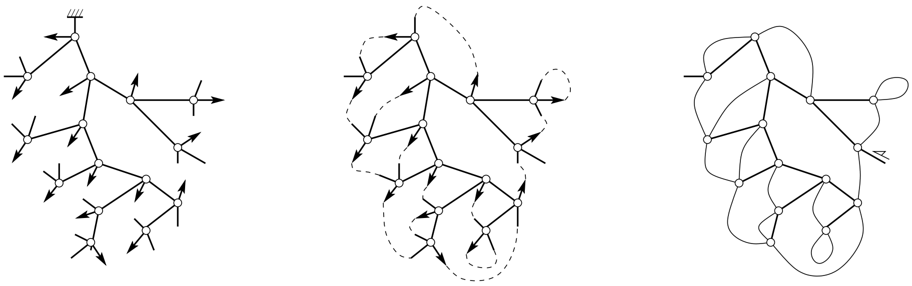

FIGURE 3. A blossom tree and its closure. Buds are represented by arrows. Dashed edges connect pairs of matched buds and leaves.

Proposition 1 ([20]). *Closure defines a $(p+2)$-to-2 correspondence between blossom trees and rooted four-regular planar maps. In particular the number of rooted four-regular planar maps is*

$$\frac{2}{p+2} \cdot \frac{3^p}{p+1} \binom{2p}{p}.$$

This proposition implies that to generate a random map according to the uniform distribution on rooted 4-regular planar maps with p vertices one can generate a blossom tree according to the uniform distribution on blossom tree and apply closure. A synopsis of the sampling algorithms is given below. An implementation is available on the web page of G.S.

Random sampling of a rooted 4-regular maps with p vertices.

Step 1. Generate a random complete binary tree T_1 according to the uniform distribution on complete binary trees with p inner vertices. (This is done in linear time using *e.g.* prefix codes [24].)

Step 2. Convert T_1 into a random blossom tree T_2 from the uniform distribution on blossom trees with p inner vertices: independantly add a bud on each vertex in a uniformly chosen position among the three possibilities.

Step 3. Use a stack (a.k.a. a last-in-first-out waiting line) to realise the closure of T_2 in linear time: Perform a counterclockwise traversal of the infinite face until p buds and leaves have been matched; when a bud is met, put b into the stack; when a leaf ℓ is met and the stack is non empty, remove the last bud entered in the stack and match it with ℓ.

Step 4. Choose uniformly the root between the two remaining leaves.

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6. SIMULATION RESULTS

The algorithm described in the previous section allows to generate random rooted 4-regular planar maps with p vertices and two legs, with uniform probability. One can compute various quantities related to the map thus generated and then average over a sample of maps, as always in Montecarlo simulations. Here the main quantity of interest is the number of loops of the map. If we generate N maps of size p so that the ith map has $k_{p,i}$ loops, $1 \le i \le N$, then $\frac{1}{N}\sum_{i=1}^{N} k_{p,i}$ has an expectation value of $\langle k \rangle_p$ and a variance of $\frac{1}{N}\langle\langle k^2 \rangle\rangle_p$, where $\langle k \rangle_p = \frac{d}{dn}_{|n=1} \log a_p(n)$ and $\langle\langle k^2 \rangle\rangle_p =$

TABLE 1. Numerical values k_ℓ of the average number of loops of maps with $p = 2^\ell$ vertices. The error (standard deviation) on the last digit is given in parentheses.

ℓ	1	2	3	4	5	6
k_ℓ	0.1111(0)	0.3228(0)	0.6605(0)	1.2120(0)	2.1640(1)	3.8970(1)
ℓ	7	8	9	10	11	12
k_ℓ	7.1764(1)	13.5372(1)	26.0524(2)	50.8704(2)	100.2890(3)	198.9060(6)
ℓ	13	14	15	16	17	18
k_ℓ	395.916(1)	789.716(2)	1577.089(4)	3151.607(7)	6300.44(1)	12597.83(2)
ℓ	19	20	21	22	23	24
k_ℓ	25192.45(3)	50381.35(5)	100759.0(1)	201514.3(2)	403023.8(4)	806043.2(7)

$\frac{d^2}{dn^2}_{|n=1} \log a_p(n)$ (the latter can of course itself be estimated as the expectation value of $\frac{1}{N-1}\sum_{i=1}^N k_{p,i}^2 - \frac{1}{N(N-1)}(\sum_{i=1}^n k_{p,i})^2$). According to Eq. (7), both $\langle k\rangle_p$ and $\langle\langle k^2\rangle\rangle_p$ are of order p for p large. However, we are interested in corrections to the leading behavior of $\langle k\rangle_p$ which are of order $\log p$, cf Eq. (11), so that we need to keep the *absolute* error small. This implies that the size of the sample N should scale like p, or that the computation time grows quadratically as a function of p.

In practice we have produced data for $p = 2^\ell$ with $\ell \leq 24$, the sample size being of the order of up to 10^7. To ensure a good sampling we used the "Mersenne twister" pseudo-random generator [15], which is both fast and unbiased. The last few values of ℓ are only given to show where the statistical error begins to grow large due to limited memory and computation time. Let us call k_ℓ the numerical value found for $\langle k\rangle_{p=2^\ell}$. The results obtained are shown on table 1.

First, as a rough check of the asymptotic behavior, let us define $u_\ell = 2k_\ell - k_{\ell+1}$. If expansion (11) is correct, then u_ℓ must display an affine behavior as a function of ℓ: $u_\ell = (\ell-1)\gamma' \log 2 + \kappa' + O(1/\ell)$. Indeed, as one can see on Fig. 4, this is the case.

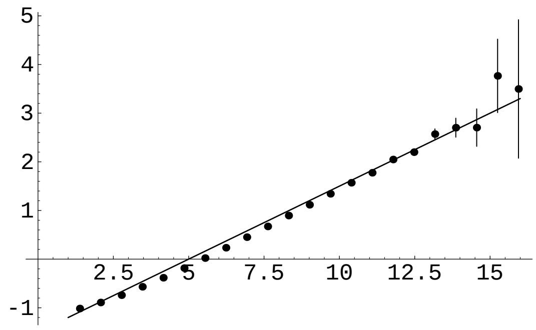

FIGURE 4. The set of points $u_\ell = 2k_\ell - k_{\ell+1}$ as a function of $\log p = \log 2 \cdot \ell$ with their error bars, as well as a proposed asymptote of slope 0.3.

By comparison with the proposed asymptote it seems clear that γ' is close to 0.3. To make this statement more precise, one can try to fit the set of the k_ℓ to

TABLE 2. Fits for the k_ℓ. χ^2 is the minimized weighted sum of squared errors.

ℓ_{min}	2	3	4	5	6	7
σ'	0.04804410	0.04804398	0.04804388	0.04804382	0.04804377	0.04804374
γ'	0.2952	0.3018	0.3071	0.3113	0.3148	0.3175
κ'	-0.364	-0.408	-0.445	-0.475	-0.501	-0.522
χ^2	18273	8067.63	3414.53	1384.07	522.297	187.471
ℓ_{min}	8	9	10	11	12	13
σ'	0.04804371	0.04804370	0.04804369	0.04804368	0.04804368	0.04804367
γ'	0.3196	0.3213	0.3226	0.3236	0.3246	0.3266
κ'	-0.539	-0.553	-0.563	-0.572	-0.582	-0.600
χ^2	64.4297	24.3678	12.7841	9.30634	8.00342	6.30457
ℓ_{min}	14	15	16	17	18	19
σ'	0.04804366	0.04804365	0.04804364	0.04804364	0.04804363	0.04804365
γ'	0.3289	0.3340	0.3440	0.3392	0.3700	0.3129
κ'	-0.624	-0.680	-0.795	-0.737	-1.13	-0.373
χ^2	5.69736	4.72577	3.66152	3.58534	2.8422	2.24532

$\sigma' p + \gamma' \log p + \kappa'$, where ℓ ranges from $\ell = \ell_{min}$ to $\ell = \ell_{max} = 24$ and ℓ_{min} is varied. The results are reported on Tab. 2. Unfortunately the confidence level remains fairly low until ℓ_{min} becomes so high that statistical error is huge, which tends to indicate strong corrections to the proposed fit.

It is important to understand that if Conjecture 2 were true, then the corrections to asymptotic behavior would be power-law – starting with $p^{-1/2}$. This means that the procedure used in Tab. 2 should converge quickly to the correct values of σ', γ', κ' (to check this we have performed a similar analysis with a model of non-crossing loops on random planar maps and obtained fast convergence with high accuracy – 2 digits on γ'). Here the range of values of γ' seems to be 0.29–0.34, far from the value predicted by Conjecture 2. It is therefore our view that the numerical data render Conjecture 2 extremely unlikely.

On the other hand, the value 0.3 predicted by Conjecture 3 remains possible. The fluctuations observed even for very high p would be caused by the logarithmic corrections present in model II due to the marginally irrelevant operator, as mentioned in Section 3.2. This operator is expected to induce a correction in $1/\log p$ (which is in principle computable exactly using quantum field theory techniques, since it is universal; progress on this will be reported elsewhere), plus higher corrections, all of which remain significant in our range of data. This would also explain why it is so hard to extract useful information from the first few (exact) values of $a_p(n)$ given in [10, 11].

In conclusion, and in view of the theoretical as well as numerical evidence, our belief is that Conjecture 3 is indeed correct.

7. Variants and corollaries

First observe that planar maps have in general no symmetries. More precisely the fraction of planar maps with p edges that have a non trivial automorphism group goes to zero exponentially fast under very mild assumption on the family considered [18]. If this (very plausible) property holds then a typical closed curve will be obtained by closing d different open curves, where d is the degree of the outer face. But the average degree of faces in any fixed 4-regular planar map is four. Thus the relation $a_p \sim 4\alpha_p$.

Second let us give a property illustrating the importance of the critical exponent γ as opposed to the actual value of τ. A closed plane curve C is said to be α-*separable*, for $0 < \alpha \leq 1$ a constant, if there exist two simple points x and y of C such that $\Gamma \setminus \{x, y\}$ is not connected and both connected components contain at least p^α crossings. The pair (x, y) is called a cut of C. In other terms, C is α-separable if it is obtained by gluing the endpoints of two big enough open plane curves (up to homeomorphisms of the sphere).

Corollary 1. *Assume Conjecture 1 is valid, and consider a uniform random closed plane curve Γ_p with p crossings. The probability that Γ_p is 1-separable decays at least like $p^\gamma \doteq p^{-0.77}$. More generally, if $\alpha > 1/(1-\gamma) = (7-\sqrt{13})/6 \doteq 0.56$, the probability that Γ_p is α-separable goes to zero as p goes to infinity.*

For comparison, $\gamma = -1/2$ and $1/(1-\gamma) = 2/3$ for doodles, which are thus easier to separate.

Indeed let us compute the expected number of inequivalent cuts of a closed plane curve with p crossings. When considered up to homeomorphisms of the sphere, close plane curves with a marked cut are in one-to-one correspondence with pairs of open plane curves. Hence, with a factor p for the choice of infinite face,

$$p \cdot \sum_{p'=q}^{p-q} \frac{a_{p'} a_{p-p'}}{\alpha_p} < cst \cdot p \cdot \sum_{p'=q}^{p-q} \frac{(p')^{\gamma-2}(p-p')^{\gamma-2}}{p^{\gamma-2}} = O(pq^{\gamma-1}). \tag{13}$$

In particular if $q \gg p^{1/(1-\gamma)}$ this expectation goes to zero as p goes to infinity.

It is typical that in the computation of probabilistic quantities, like in Equation (13), the exponential growth factors cancel, leading to behaviors that are driven by polynomial exponents. This explains the interest of in these *critical exponents* and gives probabilistic meaning to their apparent *universality*. As a final illustration of this point let us present two variants of Conjecture 1: (Definitions of prime self-intersecting curves and alternating knots can be found in [10, 14].)

Conjecture 4. *The number α'_p of closed prime self-intersecting curves with p crossings and the number α''_p of prime alternating knots with p crossings lie in the same universality class as closed self-intersecting curves: there are constants τ', τ'', c', c'' such that*

$$\alpha'_p \sim c' \, \tau'^p \cdot p^{\gamma-2}, \qquad \alpha''_p \sim c'' \, \tau''^p \cdot p^{\gamma-3},$$

where γ is given in Conjecture 1.

Observe that knot diagrams are naturally considered up to homeomorphisms of the sphere [10, 14], while we have considered plane curves up to homeomorphisms of the plane. This explains the discrepancy of a factor p in Conjecture 4 for α''_p, since one of the $p+2$ faces of a spherical diagram must be selected to puncture the sphere and put the diagram in the plane.

8. Conclusion

We have given arguments supporting Conjecture 1 for the asymptotic number of plane curves with a large number of self-intersections, as well as the more general Conjecture 3. The numerical results provided in Section 6 support Conjecture 1 only indirectly since they are related to another specialization of Conjecture 3 (derivative at $n = 1$ versus $n = 0$). However the alternative proposal is not compatible with either of these new numerical results (as is the case of Conjecture 2) or earlier ones.

Our method to test the conjecture could be applied to other models like open curves with endpoints that are not constrained to stay in the infinite face, or the meanders studied by Di Francesco et al.

Acknowledgements. P.Z.-J. would like to thank J. Jacobsen for pointing out Refs. [9] and [17] to him.

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