A COMBINATORIAL INTERPRETATION OF SUPER-CATALAN NUMBERS OF ORDER TWO.

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ABSTRACT. The numbers $S(n,m) = (2n)!(2m)!/(n!m!(n+m)!)$ were proved to be positive integers by Ira Gessel. For $m = 1$, the $S(n,1)/2$ are the Catalan numbers, and have dozens of combinatorial interpretations.

We give a combinatorial interpretation for $S(n,2)/2$, the super-Catalan numbers of order two. Indeed these numbers enumerate a family of plane trees that we call *balanced blossom trees.* These trees are related to combinatorial proofs of Tutte's formulas for various families of planar maps.

1. INTRODUCTION

In [2], Ira Gessel discusses, among other nice numbers, the super-Catalan numbers

$$S(n,m) = \frac{(2n)!(2m)!}{n!m!(n+m)!} = \binom{2n}{n}\binom{2m}{m}/\binom{n+m}{n}.$$

These numbers are integer but no combinatorial interpretation of them is known for $m > 2$. For $m = 1$, $S(n,m)/2$ is a Catalan number, with dozens of combinatorial interpretations. For $m = 2$ a combinatorial interpretation was given in terms of couples of paths with matching height [3].

In this note we provide an alternative combinatorial interpretation for the case $m = 2$. It is based on a method that was introduced to give a combinatorial interpretation to formulas of Tutte for planar maps [4] (see also [1] for more recent developments).

2. BALANCED BLOSSOM TREES

Let $\mathcal{C}$ be the familly of planted binary trees. More precisely $\mathcal{C}_n$ is the set of planted plane trees that have:

- n white nodes (or inner vertices) of degree 3,
- and $n + 2$ leaves, including the root vertex.

Figure 1(i) shows a planted binary tree with $n = 8$ nodes, $n - 1 = 7$ inner edges and $n + 2 = 10$ leaves (including the root). The number of planted complete binary trees with n nodes is well known to be the nth Catalan number, $|\mathcal{C}_n| = \frac{1}{n+1}\binom{2n}{n}$.

Let us now define the family $\mathcal{B}$ of blossom trees. Take T_0 a planted binary tree with n nodes. On each of the $n - 1$ inner edges, add a black node, and attach to this node an edge leading to a vertex of degree one. Call the resulting planted tree T. The $n - 1$ new vertices of degree one of T are called *buds*, to distinguish them from the $n + 2$ leaves that were already in T_0. In pictures we represent buds by black arrows. A tree T obtained in this way is a blossom tree of the family $\mathcal{B}_n$ if around each white vertex the neighboring buds all turn in the same direction. An example of blossom tree is shown on Figure 1(ii).

To produce a blossom tree of $\mathcal{B}_n$ from a tree of $\mathcal{C}_n$, two choices of orientation are *a priori* possible around each node. But as soon as the orientation is chosen around one vertex, all other orientations are forced. There are thus exactly two consistent global orientations for each tree T_0 of $\mathcal{C}_n$, and two resulting blossom trees T_+, T_- of $\mathcal{B}_n$. These two trees differ in particular by the orientation at their root. This construction shows that the number of blossom trees in $\mathcal{B}_n$ is

$$|\mathcal{B}_n| = 2|\mathcal{C}_n| = \frac{2}{n+1}\binom{2n}{n}.$$

Take a blossom tree T in $\mathcal{B}_n$. Put your favorite animal (say it is a squirrel) on the left hand side of a leaf ℓ of T and have it turn completely around the tree in counterclockwise direction. Along its turn, the squirrel draws a path with $2n + 1$ steps in his notebook: a down step $(1, -1)$ when a leaf is met, a up step $(1, 1)$

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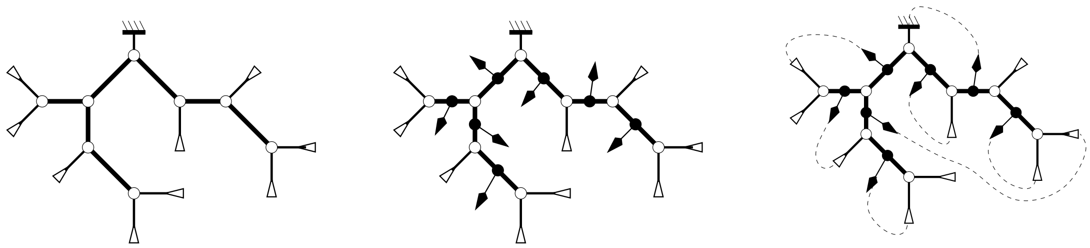

FIGURE 1. (i) A planted binary tree. (ii) A blossom tree. (iii) The matched and free leaves.

when a bud is met. If the path starts at level zero, it ends at level -3 since there are three more leaves than buds. The leaf ℓ is called *single* if the path reaches level -3 only at the last step. A tree from $\mathcal{B}$ is called *balanced* if its root leaf is balanced. Let $\mathcal{B}_n^*$ be the subset of $\mathcal{B}_n$ of balanced blossom trees.

The key observation is that the *cyclic* order in which buds and leaves are met do not depend on the starting leaf. Hence changing the starting leaf ℓ just result in a cyclic shift of the path. Hence the cycle lemma applies [5]. Since the path finishes at level -3, there are exactly 3 single leaves in T. Moreover these 3 leaves remain single upon rerooting T at any leaf. Here is an alternative proof, illustrated by Figure 1(iii). The usual intepretation of Dyck paths as balanced parenthesis words yields a algorithm to find the 3 single leaves: greedily match each bud which is immediately followed by a leaf in counterclockwise order in the infinite face; when all buds are matched, the 3 single leaves remain.

Consider now a blossom tree T with a marked *single* leaf ℓ (which may be the root). Upon exchanging the rôle of the root and the single leaf, it can also be seen as a balanced blossom tree with a marked leaf. Hence the equality

$$|\mathcal{B}_n| \cdot 3 \;=\; |\mathcal{B}_n^*| \cdot (n+2).$$

This proves the following theorem, giving our interpretation of the number $S(n,2)/2$.

Theorem 1. *The number of balanced blossom trees is thus*

$$|\mathcal{B}_n^*| = \frac{3}{n+2} \cdot 2|\mathcal{C}_n| = \frac{6(2n)!}{n!(n+2)!} = S(n,2)/2.$$

3. Comments

Binary trees are encoded by Dyck words, and these Dyck words can be counted using the cycle lemma. Hence our construction might be considered a "second order application" of the cycle lemma...

This result can easily be extended to k-ary trees. For $k = 2p+1$ odd, around nodes of positive orientation place $p+1$ buds on the left and p on right of each incident inner edge. For $k = 2p$ place p buds on each side, so that there is problem of orientation.

Corollary 2. *Let $\mathcal{B}_{k,n}$ be the familly of blossom trees constructed from k-ary trees with n white nodes by adding a black node with $k-1$ buds on each inner edge as described before. Then*

$$\begin{aligned}
|\mathcal{B}_{k,n}^*| &= \frac{k+1}{(k-1)n+2} \cdot \frac{1}{(k-1)n+1}\binom{kn}{n}, \quad \textit{if } k \textit{ is odd, and} \\
|\mathcal{B}_{k,n}^*| &= \frac{(k+1)}{(k-1)n+2} \cdot \frac{2}{(k-1)n+1}\binom{kn}{n}, \quad \textit{if } k \textit{ is even.}
\end{aligned}$$

In particular these numbers are integers.

This suggests to look for a corresponding generalisation of super-Catalan numbers.

References

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