A bijection for triangulations of a polygon with interior points and multiple edges

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Abstract

Loopless triangulations of a polygon with k vertices in $k+2n$ triangles (with interior points and possibly multiple edges) were enumerated by Mullin in 1965, using generating functions and calculations with the quadratic method.

In this article we propose a simple bijective interpretation of Mullin's formula. The argument rests on the method of *conjugacy classes of trees*, a variation of the cycle lemma designed for planar maps. In the much easier case of loopless triangulations of the sphere ($k=3$), we recover and prove correct an unpublished construction of the second author.

Key words: Enumeration, Planar maps, Trees, Bijections, Tutte.
1991 MSC: 05C30

1 Introduction

In 1965, R.C. Mullin published the following formula for the number $T^*_{k,n}$ of planar loopless triangulations of a rooted k-gon into $k+2n$ triangles (see below for precise definitions):

$$T^*_{k,n} = \frac{2^{n+2}(2k+3n-1)!(2k-3)!}{(n+1)!(2k+2n)!(k-2)!^2} \tag{1}$$

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for all $k \geq 2$ and $n \geq 0$ (see [1] or [2, p. 145]), which extends the well-known formula for the number of triangulations of a k-gon without interior points:

$$T^*_{k,-1} \;=\; \frac{(2k-4)!}{(k-1)!(k-2)!} \tag{2}$$

for all $k \geq 3$. By duality this formula also accounts for the number of rooted non-separable planar maps with a root vertex of degree k and $k+2n$ vertices of degree three.

In his work, R.C. Mullin was closely following the seminal steps of W.T. Tutte in his *census* papers [3–5]. In particular Formula (1) extends Tutte's formula

$$T_n \;=\; T^*_{3,n-2} \;=\; \frac{2^{n+1}(3n)!}{n!(2n+2)!} \tag{3}$$

for the number T_n of rooted loopless triangulations of the sphere with $2n$ triangles (or non-separable cubic maps with $2n$ vertices). The proof itself relies, following Tutte, on a recursive decomposition of triangulations that yields a recurrence for their number. Encoding the latter into generating functions then allows for a solution through the quadratic method and a few pages of calculus.

Ever since their discovery, efforts have been made to find derivations reflecting the elegant and simple product form of this and other formulas of Tutte for planar maps. The first bijective results were for general planar maps [6]. A simpler and more versatile construction, the *conjugation of trees*, was proposed in the second author's PhD thesis [7,8]. It led to the proof of a new formula for planar constellations, generalizing results of Tutte and Hurwitz [9]. This method was recently further extended to include refined enumerations according to degree distributions [10–12].

However these extensions do not apply to families of loopless or non-separable maps. The first bijective results in this context were recursive constructions for the family of all non-separable maps [13,14]. A simpler direct bijection was later given for this family using an adaptation of the conjugation of tree principle [8]. As for loopless triangulations, a similar construction was outlined in [8] for the case $k = 3$, that is Formula (3), but it could not be extended to fit the two parameter formula (1) for $T^*_{k,n}$.

In this article we introduce a slight variation of the family of triangulations under consideration, the cardinality of which is easily deduced from $T^*_{k,n}$. In view of this new family $\mathcal{T}_{k,n}$, which is defined below, Mullin's formula reads

$$T_{k,n} \;=\; |\mathcal{T}_{k,n}| \;=\; \frac{2^{n+2}}{2k+2n}\binom{2k-2}{k}\binom{2k+3n}{n+1}. \tag{4}$$

The purpose of the present article is to provide a bijective construction of Formula (4). A main ingredient in our construction is again the *conjugation of trees* principle, and this confirms the adequacy of this approach to the bijective enumeration of planar maps. However the bijection involves two new ingredients with respect to the treatment of Tutte's formulas. On the one hand, a *special* vertex is introduced in the construction, that allows to account for parameter k of Mullin's formulas. On the other hand, as opposed to the case of constellations [9], the inverse construction does not rely on breadth-first search. Instead, in order to deal with non-separability, one has to resort to more difficult recursive arguments.

The triangulations we consider here have no loop but may have multiple edges. Although the number of triangulations without multiple edges has a simple expression, it is not easily given by restriction of the present construction. The conjugation of tree principle can be applied as well to triangulations without multiple edge, but involves yet another kind of inverse construction, so as to take into account 3-connectivity [15].

The rest of the article is organized as follows: after Formula (4) for the cardinality of $\mathcal{T}_{k,n}$ has been proved equivalent to Formula (1) for $T^*_{k,n}$, we exhibit a simple family $\mathcal{E}_{k,n}$ of trees (balanced blossom trees) that are clearly enumerated by Formula (1), and we define a mapping φ from $\mathcal{E}_{k,n}$ that we claim onto $\mathcal{T}_{k,n}$ (Sections 2 and 3). This first part is rather simple and hopefully gives a convincing bijective interpretation of Formula (1). For the yet unconvinced and conscientious reader comes then the hardest part, as often with bijections, namely the proof that the image of the mapping φ is indeed $\mathcal{T}_{k,n}$ and that it is one-to-one (Section 4).

It is worth indicating here that the proof was considerably simplified with respect to a preliminary version that was presented at the International Conference on Formal Power Series and Algebraic Combinatorics, in Melbourne, july 2002.

2 The enumerative formula for rooted loopless triangulations

2.1 Definitions on planar maps

Let us make more precise the definitions of the objects under consideration. A (planar) *map* is a two-cell embedding of a connected planar graph into the oriented sphere considered up to orientation preserving homeomorphisms of the sphere. Multiple edges are allowed. The *degree* of a vertex or a face is the number of (sides of) edges incident to that vertex or face. A face is a k-*gon*

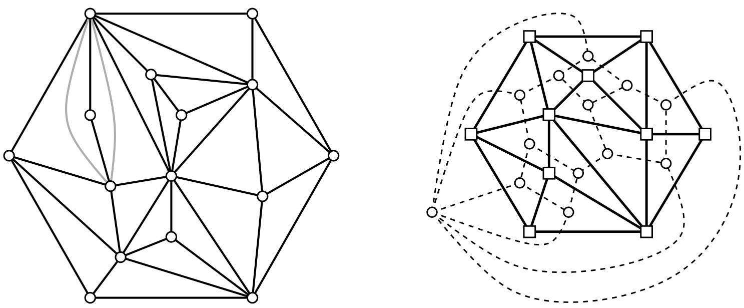

Fig. 1. A triangulation of an hexagon with a double edge; another one and its dual.

if it has degree k and it is incident to k distinct vertices. A *cut-vertex* is a vertex whose deletion disconnects the map. A planar map is *non-separable* if it contains no cut-vertex and no loop.

A map is *rooted* if one edge, called the *root*, is chosen and oriented. The startpoint of the root and the face on its right hand side are called respectively root vertex and root face. Unless explicitely mentioned, the root face is taken as infinite face when representing maps in the plane. The *dual* M^* of a map M is obtained from M by putting a vertex in each face of M and an edge of M^* across each edge of M. If M is rooted, the root edge of M^* is the dual of the root edge of M, oriented in such a way that the root vertex of M^* is the dual of the root face of M. This construction is clearly involutive on unrooted maps (see Figure 1).

2.2 Rooted triangulations

A *triangulation* is a planar map such that each face has degree three. We will only consider loopless triangulations, hence faces are "real" triangles, in the sense that they are 3-gons. However they are only "topological" triangles, in the sense that multiple edges are allowed. In particular these triangulations do not necessarily admit a representation with straight edges.

A *triangulation of a rooted k-gon* is a planar map without loops such that the root face is a k-gon while all other faces have degree three. A *rooted triangulation of a k-gon* is the same thing except that the distinguished k-gon need not be the root face. A triangulation of a k-gon has $k + 2n$ triangles for some integer $n \geq -1$, and hence $2k + 3n$ edges and $k + n + 1$ vertices (k exterior and $n + 1$ interior ones). Let $\mathcal{T}_{k,n}$ be the set of rooted triangulations of a k-gon into $k + 2n$ triangles, and let $T_{k,n} = |\mathcal{T}_{k,n}|$. Then

$$k\, T_{k,n} \;=\; 2(2k + 3n)\, T^*_{k,n},$$

as immediately follows upon considering doubly rooted triangulations with one root on the polygon and the other anywhere: these can be viewed either as rooted loopless triangulations of a k-gon in which an edge of the k-gon is distinguished (and oriented so that the k-gon is on its right hand side), or as loopless triangulations of a rooted k-gon in which an edge is distinguished and oriented.

Hence Mullin's formula (Formula (1)) becomes

$$T_{k,n} \;=\; 2^{n+3}\frac{(2k+3n)!(2k-3)!}{k\,(n+1)!(2k+2n)!(k-2)!^2},$$

and can be rewritten as previously claimed:

$$T_{k,n} \;=\; \frac{2^{n+2}}{2k+2n}\binom{2k-2}{k}\binom{2k+3n}{n+1}.$$

This formula holds for any $k \geq 2$ and any $n \geq -1$: it specializes correctly for $k \geq 3$, $n = -1$, according to Formula (2); the degenerate case $k = 2$ and $n = -1$ yields $T_{2,-1} = 1$ and accounts for a special vertex with a single loop. Observe also that $2nT_n = T_{3,n-2}$: indeed a map in $\mathcal{T}_{3,n-2}$ can be viewed as a rooted loopless triangulation with $2n$ triangles among which one is distinguished (the 3-gon).

2.3 Dual family

A *cubic map* is a map with all vertices of degree three, and a *near-cubic map* is a map with all vertices of degree three, except maybe one. Let $\mathcal{C}_n$ and $\mathcal{C}_{k,n}$ be respectively the set of non-separable cubic maps with $2n$ vertices and the set of non-separable near-cubic maps with a special vertex of degree k and $k+2n$ vertices of degree three. They are respectively the dual sets of $\mathcal{T}_n$ and $\mathcal{T}_{k,n}$, since a loop is mapped by duality onto a separating edge.

3 The constructive census of triangulations

In this section we construct a set of simple objects counted by $T_{k,n}$ and a transformation of these objects that we claim is a bijection onto $\mathcal{T}_{k,n}$.

3.1 Terminology for trees

All the trees we are interested in are planted plane trees. In the context of planar maps, it is convenient to define a plane tree as a planar map with

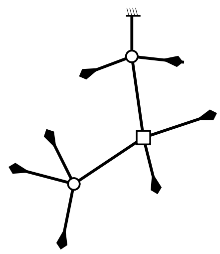

Fig. 2. A plane tree of $\mathcal{A}_{3,1}$.

only one face. This is equivalent to the classical recursive definitions. *Planted* means that one vertex of degree one is distinguished and called the root.

We shall consider an enriched terminology for trees, with two kinds of vertices of degree one (*buds* and *leaves*), three kinds of vertices of larger degree (*generic*, *pathological* and *special*), and four kinds of edges (*generic links*, *special links*, *inner edges* and *stems*). Buds and leaves shall always be incident to stems (as opposed to links or inner edges). In pictures, buds are represented by arrows, links by dashed lines, and generic and pathological vertices by circles and the special vertex by a square. The root of a planted tree shall always be a leaf (that is, not a bud). This terminology reflects the very different roles played in our contructions by otherwise similar items.

3.2 Planted plane trees

The first remark is that the following binomial coefficient, taken from Formula (4),

$$A_{k,n} = \binom{2k+3n}{n+1} = \frac{1}{2k+3n+1}\binom{2k+3n+1}{1, n+1, 2k+2n-1}$$

is the number of planted plane trees with (see also Figure 2)

- one special vertex of degree $2k-2$,
- $n+1$ generic vertices, of degree four,
- $2k+2n$ leaves (including the root) and their $2k+2n$ stems,
- and $n+1$ inner edges connecting the generic and special vertices.

This is nothing but the classical formula for planted plane trees with given numbers of vertices of each degree [2, p. 113]. Let us call the family of these trees $\mathcal{A}_{k,n}$.

Formula (4) now reads

$$T_{k,n} \;=\; \frac{2}{2k+2n}\, 2^{n+1} \binom{2k-2}{k} A_{k,n}. \tag{5}$$

(a) Generic vertices. (b) Special vertex, with $k = 2$.

Fig. 3. From trees to blossom trees.

Observe in this formula the numbers of leaves $(2k + 2n)$, of generic vertices $(n + 1)$, and of edges incident to the special vertex $(2k - 2)$.

3.3 Blossom trees

The interpretation of the formula continues with the factor

$$B_{k,n} \;=\; 2^{n+1}\binom{2k-2}{k}\, A_{k,n}.$$

Since a tree A of $\mathcal{A}_{k,n}$ has $n+1$ generic vertices of degree four, the factor 2^{n+1} can be interpreted as the number of ways to select two opposite corners on each generic vertex, while the binomial factor appears as the number of ways to select $k - 2$ of the $2k - 2$ edges incident to the special vertex.

Given such a selection, let us apply the transformation of Figure 3.(a) to generic vertices and, that of Figure 3.(b) to the special vertex. Each generic vertex is expanded into two generic vertices of degree four joined by a generic link, each one carrying a bud. Each selected edge around the special vertex is transformed to make room for a special link and two buds attached to a pathological vertex of degree four. In these constructions, buds always immediately precede links in counterclockwise direction around new vertices.

The set $\mathcal{B}_{k,n}$ of trees that are constructed in this manner from trees of $\mathcal{A}_{k,n}$ is of course of cardinality $B_{k,n}$. We call them *blossom trees.* By construction blossom trees are exactly the planted plane trees with (see also Figure 4)

- one special vertex incident to $k - 2$ special links and k edges;
- $k-2$ pathological vertices of degree four, incident to the $k-2$ special links, and each carrying two buds right before the link in counterclockwise order;
- $2n+2$ generic vertices of degree four, organized in $n+1$ pairs connected by generic links, each vertex carrying one bud right before the link in counterclockwise order;
- $2k + 2n$ leaves, $2k + 2n - 2$ buds, and their $4k + 4n - 2$ stems;
- $n + 1$ inner edges connecting some generic, pathological or special vertices.

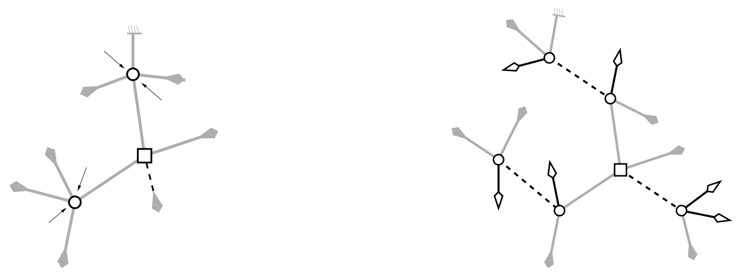

Fig. 4. A selection on the tree of Figure 2, and the resulting blossom tree of $\mathcal{B}_{3,1}$.

Formula (4) now reads

$$T_{k,n} = \frac{2}{2k+2n} B_{k,n}, \tag{6}$$

making it inviting to distinguish two leaves among the $2k + 2n$.

3.4 Balanced blossom trees

The *partial closure* of a blossom tree B consists in the following greedy procedure (see Figure 5). Start with $B^{(0)} = B$, $i = 1$.

(1) Find a bud b_i and a leaf ℓ_i such that, walking from b_i to ℓ_i around the infinite face of $B^{(i-1)}$ in counterclockwise direction, no other bud or leaf is met.
(2) Fuse b_i, ℓ_i and their stems into an edge m_i so as to create a bounded face f_i enclosing the previous walk. In particular this new bounded face f_i contains no bud or leaf.
(3) Call the resulting map $B^{(i)}$ and, if it still contains buds, increment i and return to Step (1).

Observe that the latter loop continues until there is no more free bud. The operation in Step (2) is called the *matching* of b and ℓ, and the resulting edge is called a *matching edge*.

The result of this partial closure is a planar map $\bar{B} = B^{(2k+2n-2)}$ with $k + 2n$ vertices of degree four, one special vertex of degree $2k - 2$, and two remaining leaves that we call *free* in the infinite face. This map $\bar{B}$ is independent of the exact order in which buds and leaves have been matched, (exactly like in a balanced parenthesis word, there is only a partial order of inclusion of pairs, and a greedy algorithm performing the matching has a freedom in the order it deals with incomparable pairs).

A blossom tree is called *balanced* if its root is one of the two leaves that remain free throughout partial closure. Let $\mathcal{E}_{k,n}$ be the subset of balanced trees in $\mathcal{B}_{k,n}$. Two blossom trees are called *conjugated* if they can be obtained one from another simply by changing the root leaf. The resulting conjugacy classes of

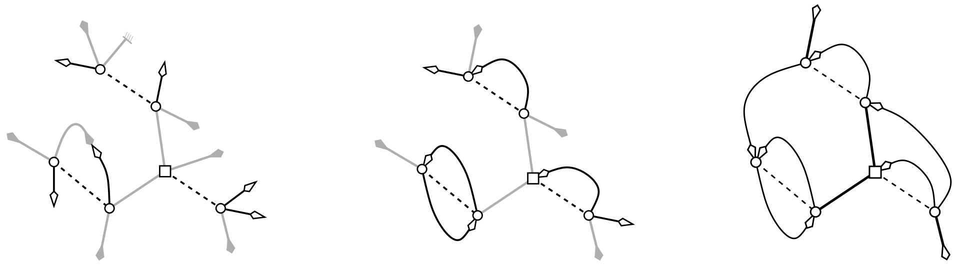

Fig. 5. The partial closure of the unbalanced blossom tree of Figure 4.

$\mathcal{B}_{k,n}$ are naturally associated with unplanted trees. Matchings between buds and leaves only depend on the conjugacy class of the blossom tree, hence we can also consider the partial closure of an unplanted tree.

Now consider a blossom tree B with root leaf r and let ℓ be one of the two leaves of B that remain free throughout partial closure. Taking now ℓ as root of B, a balanced blossom tree with a secondary distinguished leaf r is obtained. This yields[1]:

$$2\ B_{k,n} \ = \ (2k+2n)\, E_{k,n}$$

where $E_{k,n}$ denote the number of balanced blossom trees. This relation and Formula (6) allow to rewrite finally Formula (4) as

$$T_{k,n} \ = \ E_{k,n},$$

and we are led to seek a bijection between triangulations and balanced blossom trees.

3.5 The case of $\mathcal{T}_n$

A similar (but much simpler) construction provides an interpretation of Tutte's enumerative formula for the set $\mathcal{T}_n$ of loopless triangulations with $2n$ triangles, that can be rewritten in the following way:

$$T_n \ = \ \frac{2}{2n+2}\, 2^n\, \frac{1}{2n+1}\binom{3n}{n}. \tag{7}$$

The coefficient $\frac{1}{2n+1}\binom{3n}{n}$ is the number of planted plane ternary trees with n internal nodes, that is trees with n generic vertices of degree four, $n-1$ inner edges and $2n+2$ stems and leaves (including the root). The blossom

[1] Observe that this relation is the translation for conjugacy classes of trees of the *cycle lemma* for conjugacy classes of Łukasiewicz words. This lemma, initially due to Dvoretzki and Motzkin, underlies Raney's combinatorial proof of the Lagrange inversion formula [16, Chap. 11]. This analogy motivates our choice of terminology.

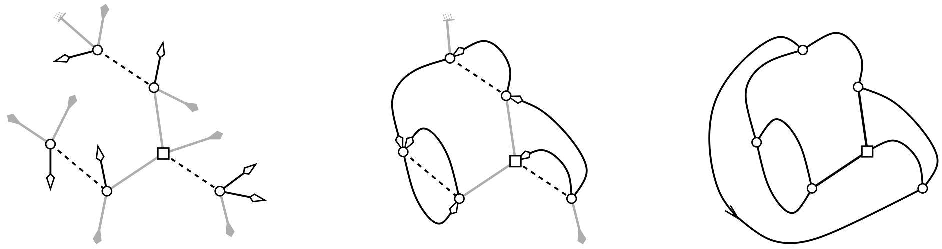

Fig. 6. The partial and complete closure of a balanced conjugate of the blossom tree of Figure 4.

trees obtained from these trees by the transformation of Figure 3.(a) have $2n$ generic vertices with their n links and $2n$ buds, $2n+2$ leaves, $4n+2$ stems and $n-1$ inner edges. Let $\mathcal{B}_n$ be the set of these blossom trees without special vertex. After the partial closure of any of these trees, two leaves remain unmatched, so the ratio of balanced blossom trees in $\mathcal{B}_n$ is $\frac{2}{2n+2}$. Hence the corresponding subset $\mathcal{E}_n$ has cardinality

$$E_n \;=\; \frac{2}{2n+2} \;\cdot\; 2^n \;\cdot\; \frac{1}{2n+1}\binom{3n}{n} \;=\; T_n.$$

Let $\mathcal{E}$ denote the set of all balanced blossom trees (with or without special vertex).

3.6 The complete closure

In fact the bijection was already almost completely described. Let us define the *complete closure* φ as a mapping defined on the set $\mathcal{E}$. An example is shown in Figure 6. Given B a tree in $\mathcal{E}$,

(1) Construct the partial closure $\bar{B}$ of B,
(2) Remove all the links and call $\widehat{B}$ the result,
(3) Fuse the two remaining stems of $\widehat{B}$ into a root edge oriented away from the root of B, and call $\varphi(B)$ the resulting rooted planar map.

Our main result, to be proved in the rest of the paper, is the following theorem.

Theorem 1 *The complete closure φ is a bijection from the set $\mathcal{E}_{k,n}$ (resp. $\mathcal{E}_n$) of balanced blossom trees onto the set $\mathcal{C}_{k,n}$ (resp. $\mathcal{C}_n$) of non-separable (near-)cubic maps and by duality onto the set $\mathcal{T}_{k,n}$ (resp. $\mathcal{T}_n$) of loopless triangulations.*

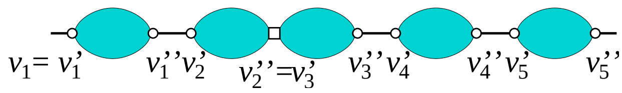

Fig. 7. The chain organisation of a map of $\widehat{C}$.

4 A recursive proof

The very last step of the closure (Step (3) in Section 3.6) is clearly invertible: given a non-separable (near-)cubic map C with root edge r and root vertex v_1, cut r into two stems and reroot the resulting map on the leaf attached to v_1. Let $\widehat{\mathcal{C}}$ denote the set of maps obtained in this way from the set $\mathcal{C}$ of non-separable cubic and near-cubic maps.

In order to prove Theorem 1, we exihibit first a recursive bijective decomposition of maps of $\widehat{\mathcal{C}}$ into smaller maps of the same type. Then we present a related decomposition of balanced blossom trees. These two decompositions are clearly isomorphic, and this proves the existence of a bijection between the two sets of objects. The proof that the closure realizes this bijection is then immediate by observing that the closure transforms the rules of decomposition of trees into the rules of decomposition of maps.

4.1 The decomposition of maps

We shall use the following property of maps of $\widehat{\mathcal{C}}$, which is an immediate consequence of the non-separability of maps of $\mathcal{C}$. An illustration of this lemma is given by Figure 7.

Lemma 2 *The cut vertices of a map $\widehat{C}$ of $\widehat{\mathcal{C}}$ are organised in a chain. There is a unique sequence $v_1', v_1'', v_2', v_2'', \ldots, v_k''$ such that: $v_1' = v_1$, and v_k'' is incident to the second stem of $\widehat{C}$; v_i' and v_i'' are distinct vertices that belong to a same non-separable component of $\widehat{C}$; (v_i'', v_{i+1}') forms a separating edge unless $v_i'' = v_{i+1}'$ is the special vertex.*

Given a map $\widehat{C}$ of $\widehat{\mathcal{C}}$, let v_1 and v_2 denote respectively the root vertex and the vertex carrying the second stem of $\widehat{C}$. If $\widehat{C}$ has a special vertex, call it v. As shown on Figure 8, let us define two oriented paths P_1 and P_2 that both turn in counterclockwise direction around $\widehat{C}$. The *left path* P_1 starts from v_1, and ends at the first vertex it reaches among v_2 and v. Similarly the *right path* P_2 starts from v_2, and stops at v_1 or v. For $i = 1, 2$, define $p_i(\widehat{C})$ to be the number of distinct bounded faces sharing an edge with P_i. The ith bounded face to be met for the first time along P_2 is said to have *index* i. By definition the index of a bounded face incident to P_2 is an integer between 1 and $p_2(\widehat{C})$.

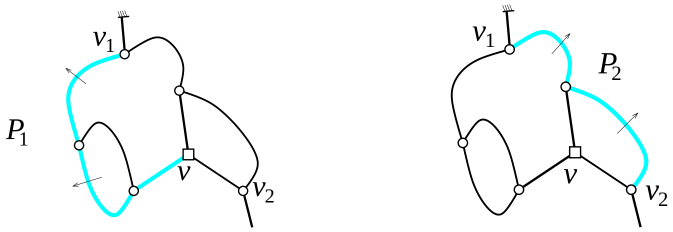

Fig. 8. The paths P_1 and P_2 around $\widehat{C}$. Small arrows indicate first incidences of bounded faces with the paths: $p_1(\widehat{C}) = 2$ and $p_2(\widehat{C}) = 2$.

The parameters p_1 and p_2, together with the number of generic vertices and the degree of the special vertex, will serve to check the isomorphism of the decomposition of maps with the decomposition of trees.

Let c_0 be the degenerate map formed of a stem with two leaves, and c_+ be the degenerate map formed of a special vertex with one stem. Assume by convention that these maps belong to $\widehat{\mathcal{C}}$. Define moreover $\widehat{\mathcal{C}}_0$, the subset of maps without a special vertex, and $\widehat{\mathcal{C}}_+$, the subset of maps with a special vertex. The set $\widehat{\mathcal{C}}$ is now divided into four subsets, and the recursive decomposition is defined separately for a map $\widehat{C}$ of each subset. The decomposition rules are also given in Figure 9.

- $\widehat{C} = c_0$ *or* $\widehat{C} = c_+$*:* base cases.
- $\widehat{C} \in \mathcal{C}_a$*: There is a special vertex v and it belongs to P_1 and P_2.* This case is shown as Case a. on Figure 9. Since v is incident twice to the infinite face, it is a separating vertex of $\widehat{C}$. By Lemma 2, the map $\widehat{C}$ is cut at v into two submaps C_1 and C_2 respectively containing v_1 and v_2. Similar decompositions are applied to C_1 and C_2. Let us describe the decomposition on C_1.

  Observe that $P_1 \subset C_1$, and denote by P_2' the oriented path going from v to v_1 in counterclockwise direction around C_1. By Lemma 2, the separating edges of C_1 form a chain. They are moreover exactly given as the intersection of P_1 and P_2'. Let e be the edge of $P_1 \cap P_2'$ that is the last on P_1 (and the first on P_2'). There are two subcases, as shown on Figure 9, Case a.:
  - *Case a.(i): e is not incident to v.* Let v' be the endpoint of e towards v. Remove v' and transform the 3 incident edges into stems. Let $\varphi_1(C_1)$ be the component containing v, rooted on the right stem, and $\varphi_2(C_1)$ be the other component.
  - *Case a.(ii): e is incident to v.* Then v has degree one in C_1. Let $\varphi_1(C_1) = c_t$, and $\varphi_2(C_1)$ be the map of $\widehat{\mathcal{C}}$ that is obtained by transforming e into a stem and v into a leaf.

  Now set $\Phi_a(\widehat{C}) = (\varphi_1(C_1), \varphi_1(C_2), \varphi_2(C_1), \varphi_2(C_2))$. The mapping Φ_a is a bijection between $\mathcal{C}_a$ and $(\widehat{\mathcal{C}}_0)^2 \times (\{c_+\} \cup \mathcal{C}_a \cup \mathcal{C}_b)^2$.
- $\widehat{C} \in \mathcal{C}_b$*: There is a special vertex v, it belongs to P_2, but not to P_1.* Let $\Phi_b(\widehat{C})$ be obtained by rerooting $\widehat{C}$ on its second stem.

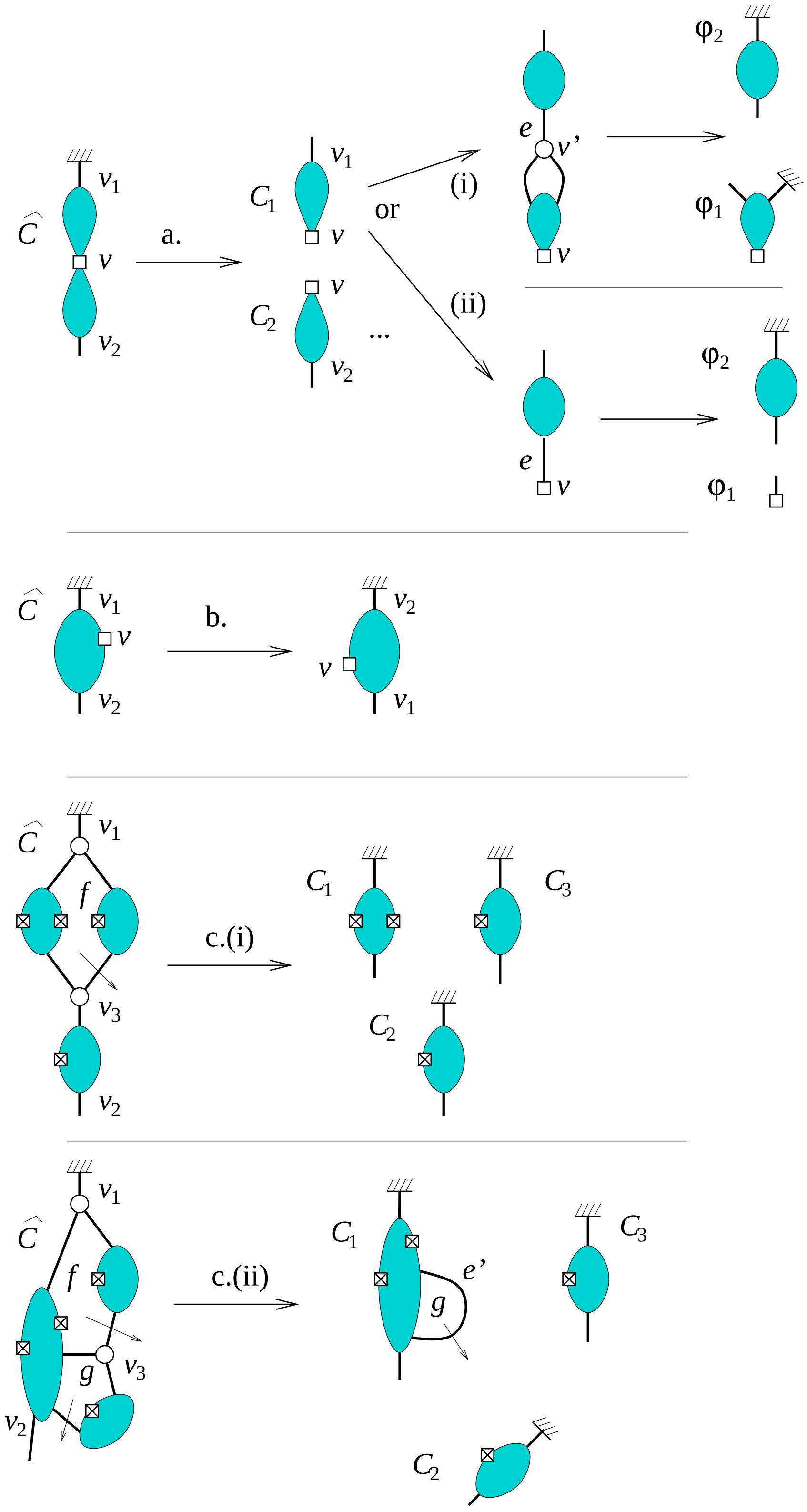

Fig. 9. The decomposition of maps.

- $\widehat{C} \in \mathcal{C}_c$: *There is no special vertex on P_2.* Let f be the bounded face incident to v_1, and v_3 be the first vertex incident to f on P_2. There are two cases:
  - *Case c.(i): v_3 is a separating vertex of $\widehat{\mathcal{C}}$.* Upon deleting v_1 and v_3, and replacing the incident edges by stems, three maps are obtained. They are given names C_1, C_2 and C_3 and rooted as shown on Figure 9, Case c.(i). Set $\Phi_c(\widehat{C}) = (0, C_1, C_2, C_3)$.
  - *Case c.(ii): v_3 is not a separating vertex of $\widehat{\mathcal{C}}$.* Let g be the second bounded face incident to v_3. The face g is incident to the path P_2 at least at the edge before v_3 on P_2. Let e be the first edge of P_2 that is incident to g. Upon deleting v_1 and v_3, and cutting e, three components are obtained. They are named C_1, C_2, and C_3 and rooted as shown on Figure 9, Case c.(ii). The component C_1 contains four stems: form a new edge e' by fusing the stem from v_3 and the stem from e. Observe that e' is then still the first edge of P_2 incident to g in C_1. Let ℓ be the index of the face g in C_1: $1 \leq \ell \leq p_2(C)$. Set $\Phi_c(\widehat{C}) = (\ell, C_1, C_2, C_3)$.

  In Figure 9, Case c., positions where a special vertex can possibly appear are indicated by a crossed box. The mapping Φ_c is a bijection from $\mathcal{C}_c$ onto the restriction of the set $\{(\ell, C_1, C_2, C_3) \mid C_i \in \widehat{\mathcal{C}}, 0 \leq \ell \leq p_2(C_1)\}$ to elements such that: at most one C_i has a special vertex, and if the special vertex is in C_2 or C_3 then it is not on their right path.

Finally observe that the parameters p_1, p_2, the number of generic vertices, and the degree of the special vertex are parameters that can be traced easily through the decomposition.

4.2 The decomposition of trees

In order to describe the parallel decomposition of trees, we need a few notations and three lemmas.

Consider a blossom tree T and an edge e of T. A matching edge $e' = (b, \ell)$, with b the bud and ℓ the leaf, is called *parallel* to e if its endpoints belong to distinct components of $T \setminus e$. In other terms, a matching edge is parallel to e if the unique simple cycle formed by (b, ℓ) and the tree contains e.

The following lemma is immediate upon counting leaves and buds in each subtree.

Lemma 3 *Take T a blossom tree, e a link of T, and b a bud incident to e. Then the matching edge (b, ℓ) is parallel to e.*

Lemma 4 *Take T a blossom tree, and let $\bar{T}$ be its partial closure, as defined in Section 3.4. Let e be an inner edge of T. Then*

- *The two subtrees on each side of e each have one more leaf than buds.*
- *If the inner edge e is a separating edge, then these two subtrees are balanced and each contains one free leaf of T. In particular both subtrees are incident to the infinite face, and so is e.*

The next lemma provides us with a first analogy with non-separable maps.

Lemma 5 *Take T a blossom tree, and let $\bar{T}$ its partial closure. The cut vertices of $\bar{T}$ are organised in a chain as in Lemma 2.*

PROOF. According to Lemma 3, all matching edges and links belong to cycles and are thus not separating edges. Consider a decomposition of $\bar{T}$ into two components C_1 and C_2 incident only at a cut vertex v'. Assume that C_1 and C_2 are not reduced to a stem. The three types of vertices are successively dealt with.

- *v' is a generic vertex.* Let (b, e, e_1, e_2) be the edges incident to v' in counterclockwise order, starting with the bud b, and the generic link e. According to Lemma 3, b and e belong to a simple cycle, hence to a same component, say C_1. The other bud incident to e creates a matching edge parallel to e, and thus a cycle using e_1 or e_2. The other inner edge (in fact e_2) is then a separating inner edge, and according to Lemma 4, both C_1 and C_2 contain a free leaf.
- *v' is a pathological vertex.* Let (b_1, b_2, e, e_1) be the edges incident to v' in counterclockwise order, starting with the two buds b_1, b_2 and the special link e. According to Lemma 3, b_1 (resp. b_2) and e belong to a simple cycle, and thus to the same component. Hence e_1 is a separating inner edge, and both C_1 and C_2 contain a free leaf.
- *v' is the special vertex v.* The partition C_1 and C_2 cuts the counterclockwise cyclic sequence of subtrees around v into two sequences of subtrees: $T'_1, \ldots, T'_{k_1}$ in C_1, and $T''_1, \ldots, T''_{k_2}$ in C_2. At least one of the two components, say C_2, contains a free leaf of T. Then C_2 is incident to the infinite face. This implies that no matching edge can arrive to T'_1 from a bud in another subtree (it would come from C_2 or enclose C_2 in a bounded face).

  If T'_1 is attached to v by a special link e then let T'_0 be the subtree attached to the pathological vertex incident to e. Otherwise let $T'_0 = T'_1$. In both cases T'_0 has one more leaf than buds. Since no matching edge arrives to T'_0, it thus has a free leaf. Hence both C_1 and C_2 contain a single leaf.

The previous case analysis shows that a simple path from one free leaf to the other must use all separating vertices and edges. This yields the chain structure, and concludes the proof of Lemma 5.

In view of Lemma 5, the paths P_1 and P_2 can be defined for $\bar{T}$ as in the

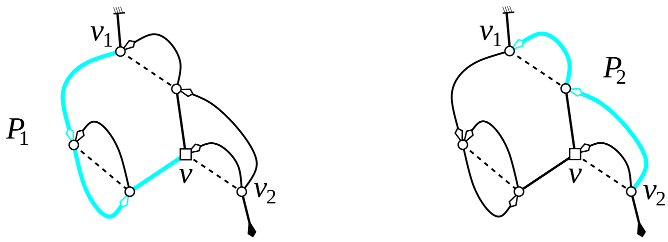

Fig. 10. The paths P_1 and P_2 around $\bar{T}$. Matching edges are easily counted along the paths: $p_1(T) = 2$ and $p_2(T) = 2$.

previous section for $\widehat{C}$. The parameter $p_1(T)$ and $p_2(T)$ are then defined by counting matching edges on P_1 and P_2 respectively. The ith matching edge along P_2 is said to have *index* i, so that the index of a matching edge on P_2 is between 1 and $p_2(T)$.

Let t_0 be the tree reduced to a stem with two free leaves, and t_+ be the tree reduced to a special vertex with one free leaf. Assume by convention that these two trees are balanced blossom trees. Recall that $\mathcal{E}$ denote the set of balanced blossom trees. Let moreover $\mathcal{E}_0$ denote the subset of balanced blossom trees without special vertex, and $\mathcal{E}_+$ the subset of balanced blossom trees with a special vertex, so that $\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}_+$. The set $\mathcal{E}$ is partitioned into four subsets, and the recursive decomposition is defined separately for a tree T of each subset. The decomposition rules are also given in Figure 11.

- *$T = t_0$ or $T = t_+$:* base cases.
- *$T \in \mathcal{E}_a$: There is a special vertex v, and it belongs to P_1 and P_2.* This case is shown as Case a. in Figure 11. Since v is incident twice to the infinite face, it is a separating vertex of $\bar{T}$. By Lemma 5, the map $\bar{T}$ is cut at v into two submaps $\bar{T}_1$ and $\bar{T}_2$ respectively containing v_1 and v_2. This decomposition reflects a decomposition of T into two subtrees T_1 and T_2 at v. Similar decompositions are applied to T_1 and T_2. Let us describe the decomposition on T_1. There are two subcases, as shown on Figure 11, Case a.:
  - *Case a.(i): v has degree at least 2 in T_1.* Reconsider the notation introduced in the proof of Lemma 5 for the case of a special separating vertex: T_1 is decomposed into a sequence of subtrees $T'_1, \ldots, T'_{k_1}$, $k_1 \geq 2$. Assume first that T'_1 is attached to v by an inner edge. No matching edge enters in T'_1 and this subtree has one more leaf than buds. It thus contains a free leaf, and no matching edge leaves it towards another subtree. Therefore the subtrees $T'_2, \ldots, T'_{k_1}$ span a component of $\bar{T}$ that is separable at v but contains no free leaf. This contradicts Lemma 5, thus proving that T'_1 is attached by a special link.

    Remove this special link and the incident pathological vertex, so as to detach a blossom tree $\varphi_2(T_1) = T'_0$, and to create two new free leaves in the component $\varphi_1(T_2)$ of T_1 spanned by $T'_2, \ldots, T'_{k_1}$. Both $\varphi_1(T_1)$ and $\varphi_2(T_2)$ are balanced blossom trees (rooting the former on the second free leaf).

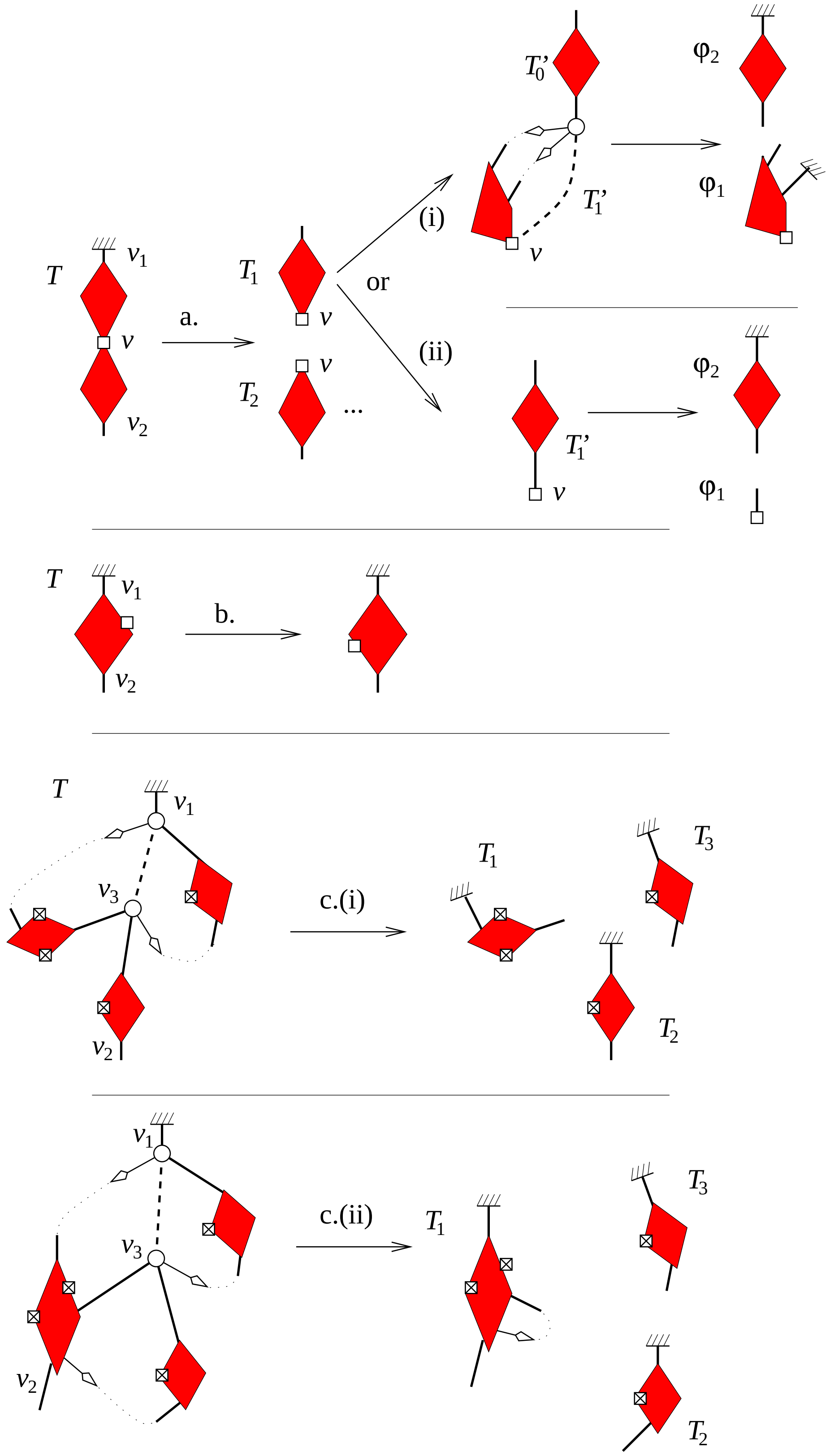

Fig. 11. The decomposition of trees.

Observe, for comparison with the case of maps, that the deleted inner edge is the last separating edge on P_1, since the closure of $\varphi_1(T_1)$ is non-separable.

- *Case a.(ii): v has degree 1 in T_1.* Let $\varphi_1(T_1) = t_+$, and $\varphi_2(T_1)$ be the balanced blossom tree that is obtained by transforming e into a stem and v into a leaf.

Now set $\Phi_a(T) = (\varphi_1(T_1), \varphi_1(T_2), \varphi_2(T_1), \varphi_2(T_2))$. The mapping Φ_a is a bijection between $\mathcal{E}_a$ and $(\mathcal{E}_0)^2 \times (\{t_+\} \cup \mathcal{E}_a \cup \mathcal{E}_b)^2$.

- $T \in \mathcal{E}_b$: *There is a special vertex v, that belongs to P_2, but not to P_1.* Let $\Phi_b(T)$ be obtained by rerooting the tree on the second free leaf.
- $T \in \mathcal{E}_c$: *There is no special vertex on P_2.* In particular the rightmost son of v_1 is neither the special vertex, nor a bud (T is balanced), so that v_1 is a generic vertex. For the root leaf r of T to remain free, the cyclic order around v_1 must be (r, b, e, e_1) with b the bud and e the generic link. Let v_3 be the generic vertex at the other end of e. Upon deleting v_1 and v_3, the tree T is decomposed into three subtrees T_1, T_2 and T_3, named and rooted as indicated by Figure 11, Cases c.(i) and c.(ii). For T to be balanced, its root leaf r must remain free, so that no matching edge can leave T_3: this subtree is balanced. Since T_3 has one more leaf than buds (by construction of blossom trees), and has one leaf matched by the bud of v_3, it can accept no other entering matching edge. Hence no matching edge can leave T_2, and T_2 is balanced. Since T_2 also has one more leaf than buds, two cases remain:
  - *Case c.(i): The second free leaf of T is in T_2.* In other terms the second free leaf of T_2 remains free, and no matching edge can leave T_1: this subtree is balanced as well. Observe, for comparison with the decomposition of maps, that v_3 is separating in $\bar{T}$. Set $\Phi_c(T) = (0, T_1, T_2, T_3)$.
  - *Case c.(ii): The second free leaf of T is not in T_2.* In this case the second free leaf of T_2 is matched by a bud of T_1. Hence the choice of root for T_1 ensures that it is balanced, and that it has a matching edge distinguished that is incident to the infinite face on the right. Let ℓ be the index of this matching edge in T_1, so that $1 \leq \ell \leq p_2(T_1)$. Set $\Phi_c(T) = (\ell, T_1, T_2, T_3)$.

  In Figure 11, positions where a special vertex can possibly appear are indicated by a crossed box. The mapping Φ_c is a bijection from $\mathcal{E}_c$ onto the restriction of the set $\{(\ell, T_1, T_2, T_3) \mid T_i \in \mathcal{E}, 0 \leq \ell \leq p_2(T_1)\}$ to elements such that at most one T_i has a special vertex, and if the special vertex is in T_2 or T_3 then it is not on their right path.

Finally observe that the parameters p_1, p_2, the number of generic vertices, and the degree of the special vertex are parameters that can be easily traced through the decomposition.

The comparison of the decomposition of maps and the decomposition of trees reveals that the two are isomorphic. In particular these decompositions allow to define recursively a bijection between maps and trees that transports the number of generic vertices, the degree of the special vertex if any, and the parameters p_1 and p_2. To conclude the proof of Theorem 1, one checks that the closure as defined in Section 3 transforms the rules of Figure 11 into the rules of Figure 9.

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