Optimal Coding and Sampling of Triangulations

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Abstract. We present a bijection between the set of plane triangulations (aka. maximal planar graphs) and a simply defined subset of plane trees with two leaves per inner node. The construction takes advantage of the minimal realizer (or Schnyder tree decomposition) of a plane triangulation.
This yields a simple interpretation of the formula for the number of plane triangulations with n vertices. Moreover the construction is simple enough to induce a linear random sampling algorithm, and an explicit information theory optimal encoding.

1 Introduction

This paper addresses three problems on finite *triangulations*, or *maximal planar graphs*: coding, counting, and sampling. The results are obtained as consequences of a new bijection, between triangulations endowed with their minimal realizer and trees in the simple class of plane trees with two leaves per inner node.

Coding The coding problem was first raised in algorithmic geometry: find an encoding of triangulated geometries which is as compact as possible. As demonstrated by previous work, a very effective "structure driven" approach consists in distinguishing the encoding of the combinatorial structure, – that is, the triangulation – from the geometry – that is, vertex coordinates (see [26] for a survey and [16] for an opposite "coordinate driven" approach). Three main properties of the combinatorial code are then desirable: *compacity*, that is minimization of the bit length of code words, *linear complexity* of the complete coding and decoding procedure, and *locality*, that is the possibility to navigate efficiently (and to code the coordinates by small increments).

For the fundamental class $\mathcal{T}_n$ of triangulations of a sphere with $2n$ triangles, several codes of linear complexity were proposed, with various bit length $\alpha n(1+o(1))$: from $\alpha = 4$ in [6, 11, 18], to $\alpha = 3.67$ in [21, 29], and recently $\alpha = 3.37$ bits in [7]. The information theory bound on α is $\alpha_0 = \frac{1}{n}\log|T_n| \sim \frac{256}{27} \approx 3.245$ (see below). In some sense the compacity problem was given an optimal solution for general recursive classes of planar maps by Lu *et al.* [19, 22]. For a fixed class, say triangulations, this algorithm does not use the knowledge of α_0, as expected for a generic algorithm, and instead relies on a cycle separator algorithm and, at bottom levels of recursion, on an exponential optimal coding algorithm. This leads to an algorithm difficult to implement with low complexity constants. Moreover the implicit nature of the representation makes it unlikely that locality constraints can be dealt with in this framework: known methods to achieve locality require the code to be based on a spanning tree of the graph.

Counting The exact enumeration problem for triangulations was solved by Tutte in the sixties [31]. The number of rooted triangulations with $2n$ triangles, $3n$ edges and $n+2$ vertices is

$$T_n = \frac{2\,(4n-3)!}{n!(3n-1)!}. \tag{1}$$

(This formula gives the previous constant $\alpha_0 = \frac{256}{27}$.) More generally Tutte was interested in *planar maps*: embedded planar multigraphs considered up to homeomorphisms of the sphere. He obtained several elegant formulas akin to (1) for the number of planar maps with n edges and for several subclasses (bipartite maps, 2-connected maps, 4-regular maps). It later turned out that constraints of this kind lead systematically to explicit enumeration results for subclasses of maps (in the form of algebraic generating functions, see [5] and references therein). A natural question in this context is to find simple combinatorial proofs explaining these results, as opposed to the technical computational proofs *à la Tutte*. This was done in a very general setting for maps without restrictions on multiple edges and loops [9, 27]. Two main ingredients are at the heart of this approach: dual breath-first search to go from maps to trees, and a closure operation for the inverse mapping. When loops are forbidden, the first ingredient is no longer suited, but it was shown that it can be replaced by bipolar orientations [24, 28]. When multiple edges are forbidden as well, the situation appears completely different and neither of the previous methods directly apply.

It should be stressed that planar graphs have in general non-unique embeddings: a given planar graph may underlie many planar maps. This explains that, as opposed to the situation for maps, no exact formula is known for the number of planar graphs with n vertices (even the asymptotic growth factor is not known, see [7, 23]). However according to Whitney's theorem, 3-connected planar graphs have an essentially unique embedding. In particular the class of triangulations is equivalent to the class of maximal planar graphs (a graph is maximal planar if no edge can be added without losing planarity).

Sampling A perfect (resp. approximate) random sampling algorithm outputs a random triangulation chosen in $\mathcal{T}_n$ under the uniform distribution (resp. under an approximation thereof): the probability to output a specific rooted triangulation T with $2n$ vertices is (resp. is close to) $1/T_n$. Safe for an exponentially small fraction of them, triangulations have a trivial automorphism group [25], so that as far as polynomial parameters are concerned, the uniform distribution on rooted or unrooted triangulations are indistinguishable.

This question was first considered by physicists willing to test experimentally properties of two dimensional quantum gravity: it turns out that the proper discretization of a typical quantum universe is precisely obtained by sampling from the uniform distribution on rooted triangulations [4]. Several approximate sampling algorithms were thus developed by physicists for planar maps, including for triangulations [3]. Most of them are based on Markov chains, the mixing times of which are not known (see however [17] for a related study). A recursive perfect sampler was also developed for cubic maps, but has at least quadratic complexity [1]. More efficient and perfect samplers were recently developed for a dozen of classes of planar maps [5, 29]. These algorithms are linear for triangular maps (with multiple edges allowed) but have average complexity $O(n^{5/3})$ for the class of triangulations.

Most random sampling algorithms are usually either based on Markov chains, or on enumerative properties. On the one hand, an algorithm of the first type perform a random walk on the set of configurations until it has (approximately) forgotten its start point. This is a very versatile method that requires little knowledge of the structures. It can even allow for perfect sampling in some restricted cases [32]. However in most cases it yields only approximate samplers of at least quadratic complexities. On the other hand, algorithms of the second type take advantage of exact counting results to construct directly a configuration from the uniform distribution [15]. As a result these perfect samplers often operate in linear time with little more than the amount of random bits required by information theory bounds to generate a configuration [2, 13]. It is very desirable to obtain such an algorithm when the combinatorial class to be sampled displays simple enumerative properties, like Formula (1) for triangulations.

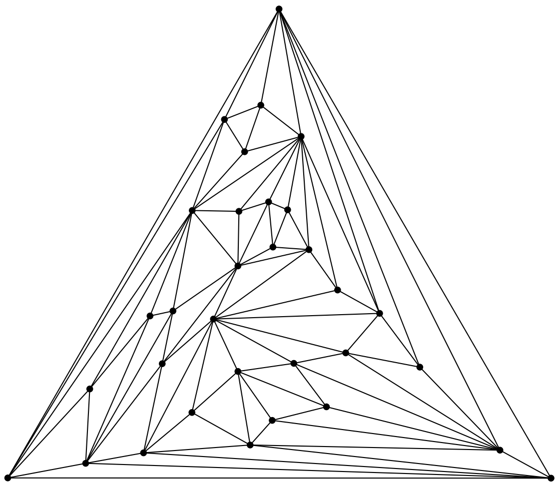


Fig. 1. A random triangulation with 60 triangles.

New results The central result of this paper is a one-to-one correspondence between the triangulations of $\mathcal{T}_n$ and the *balanced trees* of a new simple family $\mathcal{B}_n$ of plane trees. We give a linear *closure algorithm* that constructs a triangulation out of a balanced tree, and conversely, a linear *opening algorithm* that recovers a balanced tree as a special depth-first search spanning tree of a triangulation endowed with its minimal realizer. Realizers, or Schnyder tree decompositions, where introduced by Schnyder [30] to compute graph embeddings and have proved a fundamental tool in the study of planar graphs [8, 10, 14, 20]. The role played in this paper by minimal realizers of triangulations is akin to the role of breadth-first search spanning trees in planar maps [9, 27, 29], and of minimal bipolar orientations in 2-connected maps [24, 28], however the closure algorithm is very different from the closure used in the latter works. Our bijection allows us to address the three previously discussed problems.

From the coding point of view, our encoding in terms of trees preserves the entropy and satisfies linearity: each triangulation is encoded by one of the $\binom{4n}{n}$ bit strings of length $4n$ with sum of bits equal to n. The techniques of [18] to ensure locality apply to this $4n$ bit encoding. Optimal compacity can then be reached still within linear time, using for instance [7, Lemma 7].

From the exact enumerative point of view, the outcome of this work is a bijective derivation of Formula (1), giving it a simple interpretation in terms of trees. As far as we know, this is the first such bijective construction for a natural family of 3-connected planar graphs.

As far as random sampling is concerned, we obtain a linear time algorithm to sample random triangulations according to the (perfect) uniform distribution. In practice the speed we reach is about 100,000 vertices per second on a standard PC and triangulations with millions of vertices can be generated.

2 A one-to-one correspondence

Let us first recall some definitions, illustrated by Figure 2.

Definition 1. *A* planar map *is an embedding of a connected planar graph in the oriented sphere. It is* rooted *if one of its edges it distinguished and oriented; this determines a root edge, a root vertex (its origin) and a root face (to its right), which is usually chosen as infinite face for drawing in the plane.*

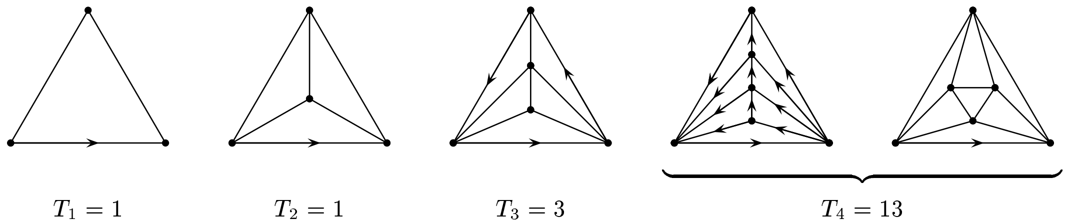


Fig. 2. The smallest triangulations with their inequivalent rootings.

A triangular map *is a rooted planar map with all faces of degree 3. It is a* triangulation *if moreover it has no loop or multiple edge. A triangular map of size* n *has* $2n$ *triangular faces,* $3n$ *edges and* $n+2$ *vertices; the three vertices incident to the root face are called* external*, as opposed to the* $n-1$ internal *other ones. The set of triangulations of size* n *is denoted by* $\mathcal{T}_n$.

2.1 From trees to triangulations

In view of Formula (1), it seems natural to ask for a bijection between triangulations and some kind of quaternary trees: indeed the number of such trees with n nodes is well known to be $\frac{(4n)!}{n!(3n+1)!}$. It proves however more interesting to consider the following less classical family of plane trees, illustrated by Fig. 3:

Definition 2. *Let* $\mathcal{B}_n$ *be the set of plane trees with* n *nodes each carrying two leaves and rooted on one of these leaves.*

It will prove useful to make a distinction between *nodes* (vertices of degree at least 2) and *leaves* (vertices of degree 1), and between *inner edges* (connecting two nodes) and *external edges* (connecting a node to a leaf).

The partial closure We introduce here a partial closure operation that merges leaves to nodes in order to create triangular faces.

Let B be a tree of $\mathcal{B}_n$. The border of the infinite face consists of inner and external edges. An *admissible triple* is a sequence (e_1, e_2, e_3) of two successive inner edges followed by an external one in counterclockwise direction around the infinite face. An admissible triple is thus formed of three edges $e_1 = (v, v')$, $e_2 = (v', v'')$ and $e_3 = (v'', \ell)$ such that v, v' and v'' are nodes and ℓ is a leaf. The *local closure* of such an admissible triple (e_1, e_2, e_3) consists in merging the leaf ℓ with the node v so as to create a bounded face of degree 3. The external edge $e_3 = (v'', \ell)$ then becomes an internal edge (v'', v). For instance the first three edges after the root around the infinite face of the tree of Figure 4(a) form an admissible triple, and the local closure of this triple produces the

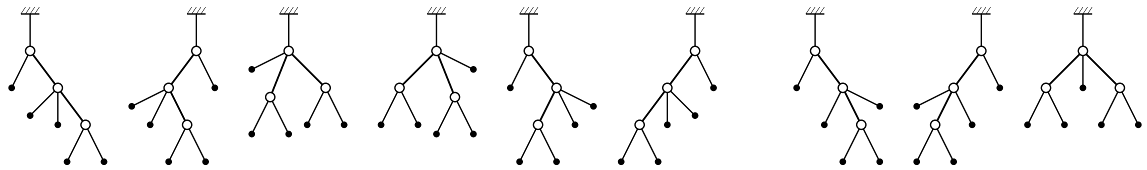
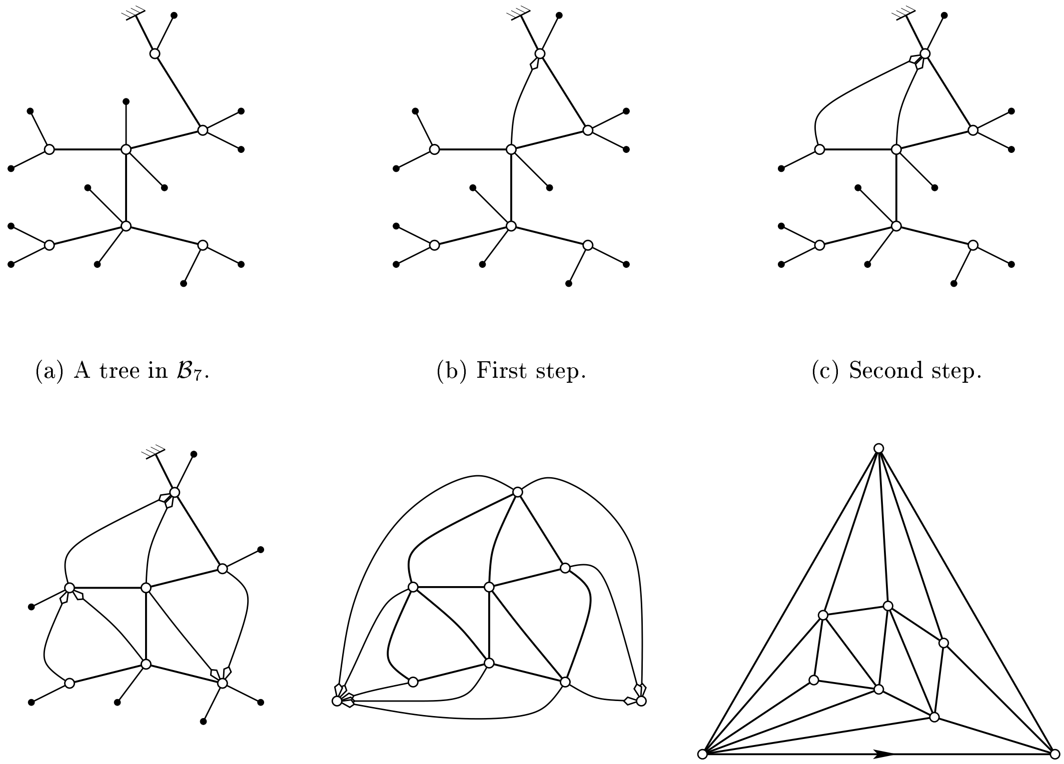


Fig. 3. The 9 elements of the set $\mathcal{B}_3$.



(a) A tree in $\mathcal{B}_7$. (b) First step. (c) Second step.

(d) Partial closure. (e) Two more vertices. (f) Complete closure.

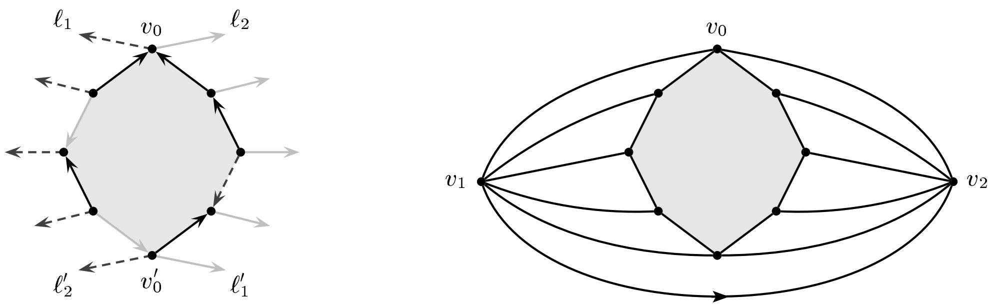
Fig. 4. Complete closure construction on an element of $\mathcal{B}_7$.

planar map of Figure 4(b). In turn, the first three edges of this map form a new admissible triple, and its local closure yields the map of Figure 4(c).

The *partial closure* $\tilde{B}$ of a tree B is the result of the greedy recursive application of local closure to all admissible triples available. The partial closure of the tree of Figure 4(a) is shown on Figure 4(d). At a given step of the construction, there are usually several admissible triples, but their local closures are independent so that the order in which they are performed is irrelevant and the final map $\tilde{B}$ is uniquely defined.

In the tree B there are two more external edges than sides of inner edges in the infinite face, and this property is preserved by local closures. When the partial closure ends, there is no more admissible triple but some leaves remain unmatched. Hence in the infinite face of $\tilde{B}$ no two inner edges can be consecutive: each inner edge lies between two external edges, as illustrated by Figures 4(d) and 5(a) (ignore orientations and colors for the time being). More precisely the external edges and sides of inner edges alternate except at two special nodes: these two nodes v_0 and v_0' each carry two external edges with leaves ℓ_1, ℓ_2 and ℓ_1', ℓ_2' such that ℓ_1 (resp. ℓ_1') follows ℓ_2 (resp. ℓ_2') in the infinite face.

Observe that the partial closure of a tree is defined regardless of which of its leaves is the root. A tree B of $\mathcal{B}_n$ is said to be *balanced* if its root leaf is one of the two leaves ℓ_1 or ℓ_1' of its partial closure $\tilde{B}$. Let $\mathcal{B}_n^*$ be the subset of balanced trees of $\mathcal{B}_n$. The fourth, sixth and eighth trees in Figure 3 are balanced. The following immediate property shall be useful later on.



(a) After partial closure. (b) After complete closure.

Fig. 5. Generic situation.

Property 1. Let B be a balanced tree; then each local closure is performed in clockwise direction around B.

The complete closure Let B be a balanced tree of $\mathcal{B}_n^*$, and call v_0 and v'_0 the two special nodes of $\tilde{B}$ that carry the leaves ℓ_1, ℓ_2, ℓ'_1 and ℓ'_2. The *complete closure* of B is obtained from its partial closure as follows (see Figures 4 and 5(b)):

1. merge ℓ_1, ℓ'_2 and all leaves in between at a new vertex v_1;
2. merge ℓ'_1, ℓ_2 and all leaves in between at a new vertex v_2;
3. add a root edge (v_1, v_2).

The result of this complete closure is clearly a triangular map, which we denote $\bar{B}$. Apart from the orientation of the root, the complete closure is more generally well defined for any tree of $\mathcal{B}_n$ and does not depend on which of the $2n$ leaves is the root of the tree. Since $2n$ rooted trees correspond to a given unrooted one B (or n in the exceptional case where B has a global symmetry), in general $2n$ trees have the same image. This image is a triangular map with a marked (non oriented) edge (v_1, v_2). However the use of the subset of balanced trees allows us to state a plain one-to-one correspondence rather than a "$2n$-to-2" one. We shall prove the following theorem in Section 3:

Theorem 1. *The complete closure is a one-to-one correspondence between the set $\mathcal{B}_n^*$ of balanced plane trees with n nodes with two leaves per node, and the set $\mathcal{T}_n$ of rooted triangulations of size n.*

Although the constructions are formally unrelated, the terminology we used here is reminiscent from [9, 24, 27], where bijections were proposed between some trees and planar maps with multiple edges.

2.2 From triangulations to trees

Minimal realizer We shall use the following notion, due to Schnyder [30].

Definition 3. *Let T be a triangulation, with root edge (v_1, v_2), and with v_0 its third external vertex. A* realizer *of T is a coloration of its internal edges in three colors c_0, c_1 and c_2 satisfying the following conditions:*

Fig. 6. Local property of a realizer.

- *for each $i \in \{0, 1, 2\}$, edges of color c_i form a spanning oriented tree of $T \setminus \{v_{i+1}, v_{i+2}\}$ rooted on v_i; this induces an orientation of edges of color c_i toward v_i, such that each vertex has exactly one outgoing edge of color c_i;*
- *around each internal vertex, outgoing edges of each color always appear in the cyclic order shown on Figure 6, and entering edges of color c_i appear between outgoing edges of the two other colors.*

From now on, this second condition is referred to as Schnyder condition*.*

Realizers of triangulations satisfy a number of nice properties [12, 14, 30], among which we shall use the following ones:

Proposition 1. – *Every triangulation has a realizer.*
- *The set of realizers of a triangulation can be endowed with an order for which there is a unique minimal (resp. maximal) element.*
- *The minimal realizer of a triangulation T is the unique realizer of T that contains no direct cycle.*
- *The minimal realizer of a triangulation can be computed in linear time.*

Depth-first search opening Let T be a triangulation, endowed with its minimal realizer. Let (v_1, v_2) be its root edge, v_0 be the other external vertex. We construct a spanning tree of $T \setminus \{v_1, v_2\}$ using a right-to-left depth-first search traversal of T, modified to accept edges only if they are oriented toward the root:

1. delete (v_1, v_2), and detach (v_0, v_1) and (v_0, v_2) from v_1 and v_2 to form two new leaves ℓ_1, ℓ_2 attached to v_0;
2. set $v \leftarrow v_0$ and $e \leftarrow (v_0, \ell_2)$, and mark v and e;
3. as long as $e \neq (v_0, \ell_1)$, repeat:
   (a) $e' \leftarrow (v, u)$, the edge after e around v *in clockwise direction*;
   (b) *special orientation test:* if e' is oriented $v \to u$ and is not marked, mark e' and detach it from u to produce a leaf attached to v;
   (c) otherwise, if u is marked and e' is not, set $e \leftarrow e'$;
   (d) otherwise, mark both u and e' if necessary and set $e \leftarrow e'$ and $v \leftarrow u$.

Step 3c prevents the opening algorithm from forming a cycle of marked edges and ensures that it eventually terminates. Let $S(T)$ be the visited tree, containing all marked edges. Without Step 3b, the opening algorithm would be a standard right-to-left depth-first search, and $S(T)$ would be the corresponding spanning tree. We shall prove the following proposition:

Proposition 2. *For any triangulation T, the tree $S(T)$ is a spanning tree of $T \setminus \{v_1, v_2\}$. Moreover it is the unique balanced tree with complete closure T.*

Because of the minimal orientation of T (without counterclockwise circuit), we shall see that the condition of Step 3c is in fact never satisfied. This line of the algorithm could thus as well be ignored: it was included only to make clear the fact that the algorithm terminates.

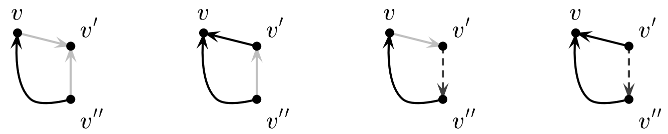


Fig. 7. The different cases of closure of a leaf.

3 Proofs

3.1 The closure produces a triangulation

The closure construction adds edges to a planar map and only creates triangular faces. It is thus clear that the resulting map is a triangular map with external vertices v_0, v_1 and v_2, and with exactly two more vertices than B has nodes. Let us show that $\bar{B}$ is indeed a triangulation, *i.e.* has no multiple edge.

Let B be a balanced tree of $\mathcal{B}_n^*$. By definition the root leaf ℓ_1 of B is immediately followed around v_0 in clockwise direction by a second leaf ℓ_2. Set ℓ_1 in color c_1, ℓ_2 in color c_2, and other edges incident to v_0 in color c_0. Upon orienting all inner edges of B toward v_0 and all external edges toward their leaf, all vertices but v_0 have three outgoing edges. Since the tree B is acyclic, its orientation induces a unique coloration of edges satisfying the Schnyder condition (Figure 6) at all vertices but v_0.

Lemma 1. *The orientation and coloration of edges still satisfy the Schnyder condition on each node but v_0 after the partial closure of B.*

Proof. This lemma is checked iteratively, by observing that each face created during the partial closure falls into one of the four types indicated on Figure 7 (up to cyclic permutation of colors). Indeed, consider an admissible triple (e_1, e_2, e_3). Assuming that the external edge e_3 to be closed is of color c_0, only two colors are possible for e_2 in view of the Schnyder condition at v''. In each case again, only two colors are possible for e_1. Finally in all four cases, the merging of ℓ into v does not contradict the Schnyder condition at v.

Property 2. If a (triangular) face of $\tilde{B}$ is oriented so that its sides form a circuit, then this circuit is necessarily oriented in the clockwise direction. More generally, each circuit in $\tilde{B}$ is created by the closure of a (last) leaf, the orientation of which imposes on the circuit to be clockwise.

Lemma 2. *After the complete closure, the Schnyder condition is satisfied at each internal vertex, and, apart from the three edges of the root face, each external vertex v_i is incident only to entering edges of color c_i.*

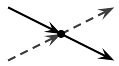
Proof. As illustrated by Figure 5(a), the Schnyder condition on nodes along the border of the partial closure implies that all external edges between ℓ_1 and ℓ_2' (resp. ℓ_2 and ℓ_1') are of color c_1 (resp. c_2). This is readily checked iteratively by a case analysis akin to the previous one.

The following proposition can be seen as an independant result on realizers.

Proposition 3. *A triangular map endowed with a colored 3-orientation satisfying the Schnyder condition on its inner vertices and the special condition on its three external vertices is in fact a triangulation endowed with a realizer.*

Proof. Let us first consider the color c_0 of the external vertex v_0. By Schnyder condition each inner vertex has exactly one outgoing edge of color c_0. In particular any cycle of edges of color c_0 is in fact a circuit. Moreover from each inner vertex originates a unique oriented path of color c_0, ending either in v_0 or on a circuit of color c_0.

Now consider two paths with distinct colors, say c_0 and c_1. In view of the Schnyder coloring, a crossing between these two paths is necessarily of the type:



Hence two such paths can only cross once. Here crossing is taken in the (weak) sense having a vertex in common, even if this is just the origin of the path.

As a consequence monochrome circuits are vertex disjoint, and thus ordered by inclusion with respect to the external face. Consider a vertex v on an innermost circuit C. The Schnyder condition at v provides an edge e going out of v into the inner region delimited by C. Since this region contains no cycle the oriented path extending e has to cross C a second time, in contradiction with the previous discussion. This excludes monochrome circuits and proves that, for each $i = 0, 1, 2$, edges of color c_i form an oriented tree rooted at v_i. In particular multiple edges are excluded, and the coloring satisfies the definition of a realizer.

Combining Proposition 3 and Property 2, we obtain the following corollary that concludes the first part of the proof.

Corollary 1. *Upon keeping colors, the closure maps a balanced trees B of $\mathcal{B}_n^*$ on a triangulation with $n + 2$ vertices endowed with its minimal realizer.*

3.2 The depth-first search opening is inverse to closure

The following Lemmas 3-6 imply Proposition 2, and, together with Corollary 1, conclude the proof of Theorem 1.

Lemma 3. *The depth-first search opening visits all vertices of $T \setminus \{v_1, v_2\}$.*

Proof. Assume that the inner vertex v is not visited by the opening algorithm, that is to say, v does not belong to $S(T)$. By definition of realizers, there is a unique oriented path P of color c_0 starting in v and ending in v_0. Let t be the last vertex on P that does not belong to $S(T)$, and $u \in S(T)$ the next vertex on P. Then the edge between t and u is oriented toward u and should have been included in the tree when u was visited.

Lemma 4. *The conditions of Step 3c are never satisfied.*

Proof. Consider the first time the conditions of Step 3c are satisfied. Up to that point an oriented tree S was constructed that contains v and u but not the edge (v, u). Since the unmarked edge (v, u) was not considered by Step 3b, it is oriented from u toward v.

Let E be the set of edges that were already cut by Step 3b. Then S is the initial part of the right-to-left depth-first search tree of $T \setminus E$. In particular since the edge (u, v) is probed from v, the vertex u is an ancestor of v in the tree. But then the tree contains an oriented path from v to u, that forms a couterclockwise circuit with (u, v). This contradicts the minimality of the orientation.

Lemma 5. *Edges that are cut by the opening algorithm lie on the left-hand side of the tree, as in Property 1. Hence the complete closure of $S(T)$ is T.*

Proof. As already observed, as the algorithm proceeds the tree which is constructed can be thought of as the right-to-left depth-first search tree of a submap of T. In particular when the algorithm probes an edge $e' = (v, u)$, this edge lies on the left-hand side of the tree, as in Property 1.

To check that the complete closure of $S(T)$ is T, it is sufficient to check that a cut edge would be properly replaced by local application of the closure algorithm. Since cut edges are bordered on one side by the infinite face and the final tree is a spanning tree, then the other face is bounded, that is triangular. Hence when $e' = (v, u)$ is cut, the vertex u lie two corners away from v along the infinite face in clockwise direction, as specified for admissible triples.

Lemma 6. *At most one spanning tree of $T \setminus \{v_1, v_2\}$ satisfies Property 1.*

Proof. Assume there are two such trees S and S'. Consider a left-to-right depth-first search traversal of both trees in parallel. Let (u, v) be the first met edge that belongs to one of them – say S – and not to the other one. Let V be the set of vertices that belong to the subtree of S rooted in v. As the tree S' is also a spanning tree, there exists in S' an oriented path from u to v_0; let (s, t) be the first oriented edge on this path such that t does not belong to V. The orientation of (s, t) forbids to this edge to belong to the tree S; it corresponds thus in that tree to the closure of a leaf of the vertex s. According to Property 1, this closure is performed clockwise around S. This contradicts the fact that the edge (u, v) has been visited before (s, t) in the depth-first search traversal. There is therefore only one covering tree of T that satisfies Property 1.

4 Applications

4.1 An explicit optimal code for triangulations

As a first byproduct of Theorem 1, we obtain a code of triangulations in $\mathcal{T}_n$ by balanced trees in $\mathcal{B}_n$. Since a triangulation can be endowed with its minimal realizer in linear time (Proposition 1), the tree code can be obtained in linear time. Another fundamental feature of our code is that the tree code is a spanning tree of the original triangulation, making locality amenable to the techniques of [18]. Elements of $\mathcal{B}_n$ can themselves be coded by bit strings of length $4n - 2$ and weight $n - 1$ using a trivial variant of the usual prefix code for trees.

Theorem 2. *A tree B of $\mathcal{B}_n$ can be linearly represented by the word $s(B)$ that is obtained by writing* 1 *for "down" steps along inner edges, and* 0 *for leaves and for "up" steps along inner edges, during left-to-right depth-first search traversal.*

Hence a code for triangulations which is a subset of the set S of bit strings with length $4n - 2$ and weight $n - 1$. According to [7, Lem. 7] it can be given in linear time a representation as a bit string of length $\log |S| + o(n) \sim \log \binom{4n}{n} \sim \frac{256}{27} n$.

4.2 A bijective proof of Formula (1)

Proposition 4. *The set $\mathcal{B}_n$ has cardinality $\frac{2}{4n-2} \cdot \binom{4n-2}{n-1}$.*

Proof. As for classical prefix code of trees, the code words corresponding to trees of $\mathcal{B}_n$ can be easily characterized: they are the bit strings of length $4n-2$ with weight $n-1$ such that any proper prefix satisfies $3|u|_1 - |u|_0 > -2$. Now the number of such bit strings is readily obtained by the cycle lemma: in each cyclic class of words with length $4n - 2$ and weight $n - 1$, exactly 2 elements among $4n - 2$ are codes words (or 1 among $2n - 1$ for symmetric classes).

Now as seen in Section 2.1, any tree in $\mathcal{B}_n$ has two particular leaves among its $2n$ ones, and it is balanced if and only if one of these two is its root. Hence the ratio of balanced trees in $\mathcal{B}_n$ is $\frac{2}{2n}$. From Theorem 1 we obtain:

Corollary 2. *The number of triangulations with* $2n$ *triangles,* $3n$ *edges and* $n+2$ *vertices is* $\frac{2}{2n} \cdot \frac{2}{4n-2} \cdot \binom{4n-2}{n-1}$*, which is exactly Formula (1).*

4.3 Linear time perfect random sampling of triangulations

The closure construction provides a sampling algorithm with linear complexity:

1. generate a random bit string of length $4n-2$ and weight $n-1$;
2. choose randomly one of its two cyclic shift that code an element of $\mathcal{B}_n$;
3. decode this word to construct the corresponding tree;
4. construct its partial closure by turning around the tree; using a stack, this can be done in at most two complete turns, hence in linear time;
5. complete the closure and choose a random orientation for the edge (v_1, v_2).

Theorem 3. *This algorithm produces in linear time a random triangulation uniformly chosen in* $\mathcal{T}_n$*.*

Observe that Steps 1–3 correspond to a special case of the classical algorithm described *e.g.* in [2] for sampling trees.

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