A note on
Bipartite Eulerian Planar Maps

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Abstract

Liskovets and Walsh recently counted bipartite eulerian planar maps, that is embeddings of planar bipartite graphs with all vertices of even degrees. In this note we present a simple one-to-one correspondence between the latter and the 4-eulerian maps enumerated by Bousquet-Mélou and Schaeffer (*Adv. Appl. Math.*, 2000). A bijective enumeration then follows.

1 The number of rooted bipartite eulerian planar maps

A *planar map* is a proper embedding of a connected graph in the plane, considered up to continuous deformations of the plane. We consider in this note the set $\mathcal{B}$ of *rooted bipartite eulerian* planar maps: (i) a *root* edge is chosen on the border of the infinite face and oriented counterclockwise, (ii) the vertices are colored in black or white so that each edge joins a black vertex to a white one, (iii) the degree of each vertex is even. Observe that a bipartite eulerian map contains no loop nor bridge but may contain multiple edges.

Duality is an involution on the set of all planar maps in which the roles of vertices and faces are exchanged: to construct the dual M^* of a map M, put a vertex of M^* in each face of M and draw an edge of M^* across each edge of M. An easy classical remark is that duality bijectively maps bipartite maps onto bicolor maps (*i.e.* maps in which faces can be bicolored), that are exactly eulerian maps. As a consequence, the set of bipartite eulerian planar map is self dual, that is stable under duality. Hence alternative characterizations of $\mathcal{B}$ are as the set of rooted *bipartite bicolor* planar maps, or the set of rooted *eulerian dualy eulerian* planar maps.

Liskovets and Walsh [4] recently obtained the following result.

Theorem 1 (Liskovets – Walsh, 2002) *The number of rooted bipartite eulerian planar maps with n edges is*

$$|\mathcal{B}_n| \;=\; \frac{4\cdot 3^{n-1}(3n)!}{(2n+2)!\,n!}. \tag{1}$$

The enumerative theory of planar maps dates back to the sixties with Tutte's seminal papers [6, 7, 8], in which he obtained a number of similar elegant formulas. However, as far as we know, Formula (1) and the formula of Bousquet-Mélou and Schaeffer for *planar constellations* [1] are the first new *simple product form* formulas to appear in planar maps enumeration since Walsh and Lehman's work in the seventies [9].

Tutte used recursive decompositions to obtain functional equations for generating functions, and solved them with the quadratic method. Alternative proofs were given by physicists using matrix integrals [2]. As opposed to these, Bousquet-Mélou and Schaeffer's enumeration of constellations is bijective: the main ingredient is the *conjugation of trees* principle introduced by Schaeffer [5] to give bijective proofs of Tutte's formulas.

Liskovets and Walsh apply Lagrange inversion to derive Formula (1) from a closed form of the generating function of rooted bipartite eulerian planar maps that was obtained by the physicists' matrix integral methodology [3].

In this note we give a bijective proof of Formula (1), based on two ingredients. The first is the *edge-map* or *radial* construction. The second is again *conjugation of trees*, and is a special case of the construction of Bousquet-Mélou and Schaeffer [1]. Taken together, they allow to give a simple *construction* of Formula (1).

2 The edge-map construction

A map is *bi-quartic* if it is bipartite and all its vertices have degree four (compare to Tutte's bi-cubic maps). The following *edge-map construction* (see Figure 1) is a variant of Tutte's derivation of maps.

Theorem 2 *There is a one-to-one correspondence between rooted bipartite eulerian planar maps with n edges and rooted bi-quartic planar maps with n vertices.*

Proof. Let M be a rooted bipartite eulerian map, with vertices of colors black and white (filled or empty squares in Figure 1), and faces of colors green (filled down triangles) and white (empty up triangles). Define its edge-map M^e as follows.

- Place a vertex of M^e (red circle) on each edge of M.
- Add an edge (x, y) of M^e for each corner xy of M, that is, if the two edges x and y of M are successively incident around the same vertex of M.

An edge of M is incident to four corners, hence all vertices of M^e have degree four. Since M is bipartite bicolor, an edge of M joins vertices and separates faces of distinct colors. Thus, around each vertex of M^e the four faces have different types that form either a cycle

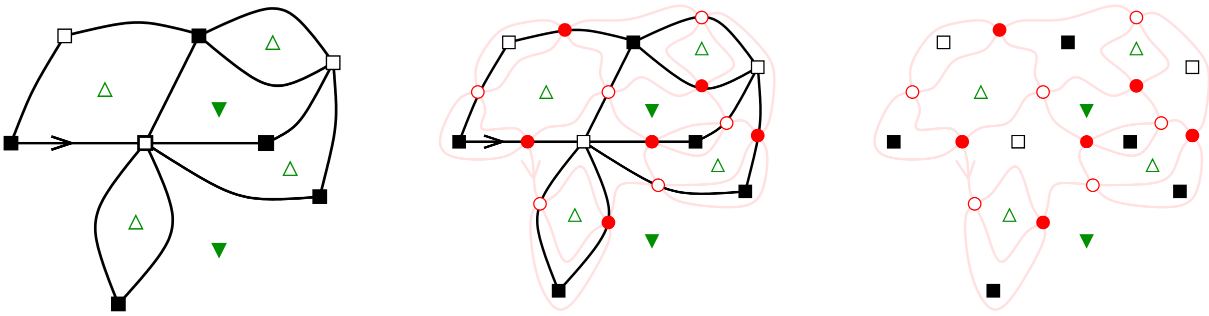

Figure 1: A bipartite eulerian map and its (bi-quartic) edge-map, with the four colors of faces.

filled square - empty triangle - empty square - filled triangle (filled red circles) or a cycle *filled square - filled triangle - empty square - empty triangle* (empty red circles). Adjacent vertices of M^e have opposite cycle orientations so that the map M^e is bi-quartic.

Conversely consider a bi-quartic map E. First E is quartic, hence eulerian, and thus, bicolor. This allows to recover a 2-coloring of its faces in colors *square* and *triangle*. Upon placing vertices in square-faces of E and joining them by edges through vertices of E, a planar map M is recovered that clearly satisfies $M^e = E$.

Observe now that the degree of a vertex of M is equal to the degree of the corresponding square-face of E, and that, similarly the degree of a face of M is the degree of a corresponding triangle-face of E.

Since E is bipartite, its dual E^* is eulerian or, in other terms, the faces of E have all even degree. Thus M and M^* are eulerian, or by duality, M^* and M are bipartite. Thus all colors can be recovered and M is a bipartite eulerian map. $\square$

3 Conjugation of trees

Formula (1) of Liskovets and Walsh is now recovered from the following theorem.

Theorem 3 (Bousquet-Mélou – Schaeffer, 2000) *Rooted bi-quartic planar maps with n vertices are in one-to-one correspondence with balanced blossom trees with $n-1$ nodes and their number is*

$$|\mathcal{B}_n| \;=\; \frac{4}{2n+2}\cdot\frac{3^{n-1}}{2n+1}\binom{3n}{n}. \qquad (2)$$

This result is a corollary of a more general theorem on the enumeration of planar constellations. Indeed bi-quartic maps exactly correspond to maximal 4-eulerian planar maps, so Theorem 3 follows upon taking $m = 4$, $d_1 = n$, and $d_i = 0$ for $i \geqslant 2$ in Theorem 2.3 and 5.2 of [1].

Since the case of bi-quartic planar maps is much simpler than the general case of constellations, we offer in Appendix a short presentation of the construction of Formula (2) using the *conjugation of trees* principle.

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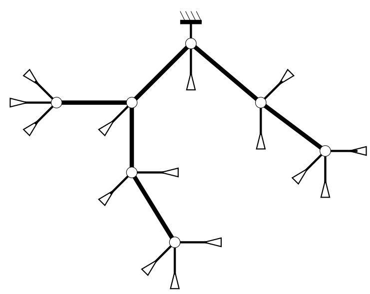
Appendix. A pictorial census of bi-quartic maps

The "construction" of Formula (2) is performed in five elementary steps.

1– There are

$$\frac{1}{2n+1}\binom{3n}{n}$$

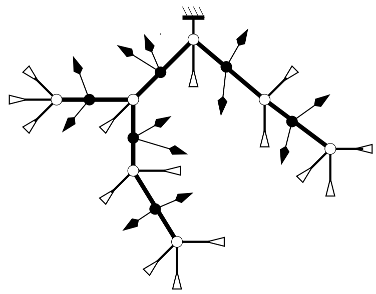
planted ternary trees with n nodes. They have $2n+2$ leaves (root included), and $n-1$ internal edges.



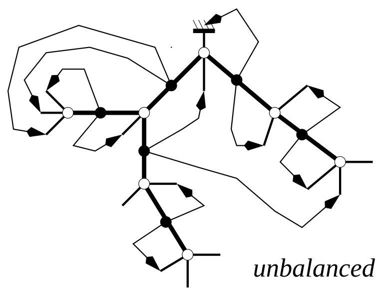
2– Consider all ways to attach two buds in the middle of each inner edge of a planted ternary tree. This can be done in three ways for each inner edge, yielding

$$\frac{3^{n-1}}{2n+1}\binom{3n}{n}$$

planted *blossom* trees with n white and $n-1$ black nodes.



3– Match iteratively pairs of closest buds and leaves in counterclockwise direction around the tree. Each pair (bud, leaf) will be fused into a new edge.
There are $2n-1$ buds and $2n+2$ leaves, so four leaves remain unmatched.

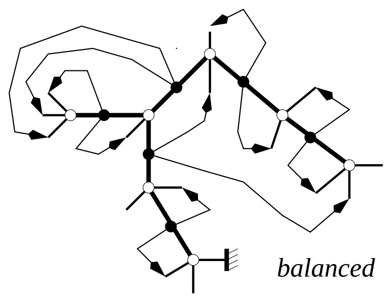


4– A blossom tree is *balanced* if its root remains unmatched.

Among the $2n+2$ possible rootings of an unplanted tree, 4 yield a balanced tree. Hence

$$\frac{4}{2n+2}\cdot\frac{3^{n-1}}{2n+1}\binom{3n}{n}$$

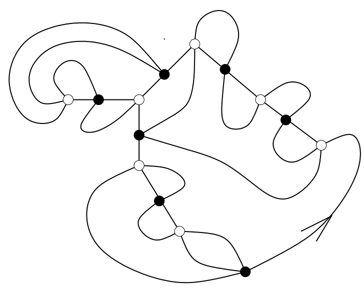
is the number of balanced blossom trees with n white nodes.



5– Upon adding a black root vertex connected to the four free leaves, this *closure* is one-to-one between
- balanced blossom trees (n white nodes),
- and bi-quartic maps ($2n$ vertices).

Therefore the number of bi-quartic maps with $2n$ vertices is

$$\frac{4}{2n+2}\cdot\frac{3^{n-1}}{2n+1}\binom{3n}{n}.$$



This is also the number of bipartite bicolor maps with n edges.