Factorizations of large cycles in the symmetric group

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Abstract

The factorizations of an n-cycle of the symmetric group $\mathfrak{S}_n$ into m permutations with prescribed cycle types $\alpha_1, \ldots, \alpha_m$ describe topological equivalence classes of one pole meromorphic functions on Riemann surfaces. This is one of the motivations for a vast literature on counting such factorizations. Their number, denoted by $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$, is also known as a connection coefficient of the center of the algebra of the symmetric group, whose multiplicative structure it describes.

The relation to Riemann surfaces induces the definition of a *genus* for factorizations. It turns out that this genus is fully determined by the cycle types $\alpha_1, \ldots, \alpha_m$, and that it has a determinant influence on the complexity of computing connection coefficients. In this article, a new formula for $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$ is given, that makes this influence of the genus explicit. Moreover, our formula is cancellation–free, thus contrasting with known formulae in terms of characters of the symmetric group. This feature allows us to derive non trivial asymptotic estimates.

Our results rely on combining classical methods of the theory of characters of the symmetric group with a combinatorial approach that was first introduced in the much simpler case $m = 2$ by Goupil and Schaeffer.

Key words: symmetric group, conjugacy classes, connection coefficients.

1 Introduction

1.1 Definitions and notations

Let us first recall some notations. Let n and k be positive integers; a *partition* of n into k parts is a non increasing k-tuple of positive integers $\beta = (\beta_1, \ldots, \beta_k)$

such that $\beta_1+\cdots+\beta_k = n$. Let $|\beta|$ denote the *weight* n of β, and $\ell(\beta)$ its *length* k. We also write $\beta \vdash n$. The *rank* $r(\beta)$ of β is defined by $r(\beta) + \ell(\beta) = |\beta|$. If b_i is the number of parts of β equal to i, then the exponential notation $\beta = 1^{b_1} \dots n^{b_n}$ will be used as well. The *cycle type* of a permutation $\sigma \in \mathfrak{S}_n$ is the partition of n whose parts are the lengths of the cycles in the representation in disjoint cycles of σ. The permutations of $\mathfrak{S}_n$ of given cycle type β form a conjugacy class denoted by $\mathcal{C}_\beta$.

Let n and m be positive integers, $\alpha_1,\dots,\alpha_m$ and β be $m+1$ partitions of n. For convenience's sake, we shall consistently use the notations ℓ_i and r_i instead of $\ell(\alpha_i)$ and $r(\alpha_i)$. Denote by $c^\beta_{\alpha_1,\dots,\alpha_m}$ the number of m-tuples of permutations in $\mathfrak{S}_n$ of respective cycle type $\alpha_1, \alpha_2, \dots \alpha_m$ whose product is equal to a given permutation of cycle type β. In other terms, for any permutation τ of cycle type β,

$$c^\beta_{\alpha_1,\dots,\alpha_m} \;=\; |\{(\sigma_1,\dots,\sigma_m) \in \mathcal{C}_{\alpha_1} \times \cdots \times \mathcal{C}_{\alpha_m} \mid \sigma_1 \cdots \sigma_m = \tau\}|\,.$$

The formal sums $C_\beta = \sum_{\sigma \in \mathcal{C}_\beta} \sigma$ for $\beta \vdash n$ belong to the group algebra of the symmetric group $\mathfrak{S}_n$ and, more precisely, form a basis of its center $\mathcal{Z}_n$. Thus the (multiplicative) structure constants $c^\beta_{\alpha_1,\dots,\alpha_m}$ can equivalently be defined by the following linearization relations in $\mathcal{Z}_n$:

$$C_{\alpha_1} \cdots C_{\alpha_m} \;=\; \sum_{\beta \vdash n} c^\beta_{\alpha_1,\dots,\alpha_m} C_\beta,$$

from which they got their name.

1.2 Context and motivations

It was already known by Hurwitz that factorizations in the symmetric group have a topological interpretation. We refer to [**?**,**?**,**?**] for a description of this connection and simply indicate – somewhat loosely – that factorizations represent topological equivalence classes of meromorphic functions on Riemann surfaces, up to homeomorphisms of the domain. As only connected surfaces are usually considered, the corresponding factorizations must satisfy the additional *transitivity* condition that their factors should generate a group acting transitively on $\{1,\dots,n\}$. In this case, up to labelling the sheets over a regular point, the correspondence can be made one-to-one. Factorizations are used in practice to conduct the experimental topological classification of functions, and, in this context, enumerative results like ours are used to check for exhausion of the search space [**?**].

A remarkable feature of the above mentionned correspondence is that it defines a genus of transitive factorizations, which turns out to have a simple combinatorial description in terms of the cycle type of factors: the *genus* g of a transitive factorization of a permutation of cycle type β into m permutations of respective cycle types $\alpha_1, \ldots, \alpha_m$ is given by

$$\sum_{i=1}^{m} r_i = \ell(\beta) + n - 2 + 2g.$$

The fact that g has to be a non negative integer so that such factorizations exist is immediate from its topological interpretation, but can also be proved inductively directly from its combinatorial definition (see *e.g.* [?]). The topological intuition suggests that the situation should become more and more involved as the genus increases, and one would expect this to show on enumerative result. As a first result, Hurwitz gave a remarkable formula for the number of genus zero transitive factorizations in transpositions ([?], see also [?,?,?,?] for recent development).

In the general case, the classical expression for the structure constants $c^{\beta}_{\alpha_1,\ldots,\alpha_m}$ is a summation over products of evaluations of irreducible characters of the symmetric group (see Proposition 4). Unfortunately, huge cancellations often occur in these sommations, hiding the influence of the genus and precluding any prediction of the order of magnitude of the constants. Moreover the evaluation of characters is usually done via Murnaghan–Nakayama rule so that these formulae are more properly described as evaluation algorithms, from which infinite families of constants cannot be evaluated simultaneously.

1.3 The particular case of n-cycles

It was observed in [?,?] that the formula expressing structure constants in terms of characters translates into a remarkably simple expression of their generating function in terms of Schur functions. However, extracting coefficients in these generating functions also involves alternating sign summations and, unsurprisingly, are more or less equivalent to the direct computations via characters. Factorizations of an n-cycle are somewhat less involved because the character generating function is simpler, but, even in this case, large cancellations occur.

However, besides Hurwitz' transpositions, a few other families of structure constants are known to be given by simple explicit formulae, the most intriguing of which may be $c^{\alpha}_{(n),(n-1,1)} = 2(n-2)!$ if permutations of cycle type α are odd, and 0 otherwise. Several results of this type have been obtained in the early 80's, by Walkup [?], Boccara [?], Bertram and Wei [?] or Stanley [?], and later by Jackson [?], Goulden [?], Goupil [?] or Jones [?]. Almost all these results concern decompositions of an n-cycle into very restricted families of factors, and they do not rely on any kind of topological intuition.

Observe that factorizations of an n-cycle are necessarily transitive, since the factorized cycle belongs to the group generated by the factors. Hence the genus of such a factorization is well-defined and satisfies:

$$\sum_{i=1}^{m} r_i \;=\; n-1+2g.$$

2 Main theorem

2.1 Two former results

An important progress was made for the study of such factorizations with Goulden and Jackson's formula in the minimal genus case [?,?,?,?]: for all partitions $\alpha_1, \dots, \alpha_m$ of n such that $r_1 + \cdots + r_m = n-1$,

$$c^{(n)}_{\alpha_1,\dots,\alpha_m} \;=\; n^{m-1}\prod_{i=1}^{m}\frac{1}{\ell_i}\binom{\ell_i}{a_{i,1},\dots,a_{i,n}} \;=\; n^{m-1}\prod_{i=1}^{m}\frac{(\ell_i-1)!}{\mathrm{Aut}(\alpha_i)}$$

in which, for any partition $\beta = (\beta_1, \dots, \beta_\ell) = 1^{b_1}\dots n^{b_n}$, $\mathrm{Aut}(\beta) = b_1!\cdots b_n!$ is the number of permutations σ in $\mathfrak{S}_\ell$ such that $\beta_{\sigma(i)} = \beta_i$ for all i in $[\![1,\ell]\!]$ (notation borrowed from [?]). Remark that this formula generalizes the classical Cayley formula $c^{(n)}_{(1^{n-2}2)^{n-1}} = n^{n-2}$.

Then, a detailed case analysis of low genera led Goupil to introduce the following symmetric polynomials for all non negative g and ℓ:

$$S_g(x_1,\dots,x_\ell) \;=\; \sum_{p_1+\ldots+p_\ell=g}\;\prod_{j=1}^{\ell}\frac{1}{x_j}\binom{x_j}{2p_j+1}.$$

They have a simple generating function:

$$\begin{aligned}\sum_{g\geqslant 0} S_g(x_1,\dots,x_\ell)t^{2g} \;&=\; \prod_{j=1}^{\ell}\frac{1}{x_jt}\sum_{p\geqslant 0}\binom{x_j}{2p+1}t^{2p+1}\\ &=\; \prod_{j=1}^{\ell}\frac{(1+t)^{x_j}-(1-t)^{x_j}}{2x_jt}.\end{aligned}$$

Using these symmetric polynomials, Goupil and Schaeffer were able in [?] to extend Goulden and Jackson's result in the special case $m=2$: for all partitions α_1 and α_2 of n,

$$c^{(n)}_{\alpha_1,\alpha_2} \;=\; \frac{n}{2^{2g}}\prod_{i=1}^{2}\frac{(\ell_i-1)!}{\mathrm{Aut}(\alpha_i)}\sum_{g_1+g_2=g}\;\prod_{i=1}^{2}\ell_i^{(2g_i)}S_{g_i}(\alpha_i),$$

where $g = \frac{1}{2}(n-1-r_1-r_2)$ and $x^{(k)}$ is the raising factorial $x(x+1)\dots(x+k-1)$.

2.2 Statement of the main theorem

In view of Goulden and Jackson's and Goupil and Schaeffer's formulae, it is natural to conjecture an immediate extension of the latter to $m \geqslant 2$. However, this extension turns out to be false because of the existence of a factor P trivial for $m \leqslant 2$. In order to define this factor, let us fix some notations. Recall that the *elementary symmetric functions* are defined by $e_\mu = e_{\mu_1} \dots e_{\mu_\ell}$ for any partition $\mu = (\mu_1, \dots, \mu_\ell)$, where for each $k \in \mathbb{N}$:

$$e_k = \sum_{1 \leqslant j_1 < j_2 < \dots < j_k} x_{j_1} x_{j_2} \cdots x_{j_k}.$$

Falling factorial powers are defined by $(x)_k = x(x-1)\cdots(x-k+1)$ for any integer k, and the mapping $\mathcal{D}: \ x^k \mapsto (x)_k$ can be extended multiplicatively to monomials in distinct variables and then linearly to symmetric functions. Finally, for $\mu = 1^{m_1} \dots n^{m_n}$, the partition $3^{m_1} 5^{m_2} \dots (2n+1)^{m_n}$ is denoted by $2\mu + 1$.

With these notations, we define symmetric functions P_g by setting $P_0 = 1$, and for all positive g:

$$P_g = \sum_{\mu \vdash g} \frac{\mathcal{D}\left(e_{2\mu+1}\right)}{\mathrm{Aut}(\mu)}.$$

The main result of the present article can now be stated as follows:

Theorem 1 *Let n be a positive integer, and $\alpha_1, \dots, \alpha_m$ be partitions of n, with, for all i, $\alpha_i = 1^{a_{i,1}} \dots n^{a_{i,n}}$, $\ell(\alpha_i) = \ell_i$ and $r(\alpha_i) = r_i$. Let the genus g be defined by $\sum_i r_i = n - 1 + 2g$. Then:*

$$c^{(n)}_{\alpha_1,\dots,\alpha_m} = \frac{n^{m-1}}{2^{2g}} \cdot \prod_{i=1}^{m} \frac{(\ell_i - 1)!}{\mathrm{Aut}(\alpha_i)} \cdot \sum_{g_0 + \dots + g_m = g} \left(P_{g_0}(\mathbf{r} - 2\mathbf{g}) \prod_{i=1}^{m} \ell_i^{(2g_i)} S_{g_i}(\alpha_i) \right) \quad (1)$$

where $\mathbf{r} - 2\mathbf{g} = (r_1 - 2g_1, \dots, r_m - 2g_m)$.

2.3 A discussion of the main theorem

Let us point out some properties of Formula (1).

- First of all, it is a summation over positive contributions. This is certainly not the case of character theoretic formulae, and a substantial part of our proof is directed towards the construction of a sign reversing involution to eliminate negative contributions. This allows us to derive asymptotic results for different kinds of limit at fixed genus (see below).

- The symmetric polynomial $P_g(x_1,\dots,x_m)$ has degree $3g$, so that the number of terms involved in its summation is a polynomial in m of degree $3g$. First few values of P_g and S_g are given in Appendix.
- The symmetric polynomial $S_g(x_1,\dots,x_\ell)$ has degree $2g$ and the number of terms in its summation is $\binom{\ell+2g-1}{2g}$, *i.e.* a polynomial in ℓ of degree $2g$. Since $S_g(x_1,\dots,x_\ell) = S_g(x_1,\dots,x_\ell,1,\dots,1,0,\dots,0)$, the evaluation $S_g(\alpha_i)$ only depends on the partition $\overline{\alpha_i} = 2^{a_{i,2}}\dots k^{a_{i,k}}$. (In fact it does not depend on parts equal to 2 either.)
- The higher genus correction to Goulden and Jackson's formula

$$\sum_{g_0+\cdots+g_m=g} P_g(\mathbf{r}-2\mathbf{g}) \prod_{i=1}^{m} \ell_i^{(2g_i)} S_{g_i}(\alpha_i)$$

  is a polynomial in the parts $\alpha_{i,j}$ (for fixed g, m, n and ℓ_i). In other words, besides $\mathrm{Aut}(\alpha_\mathrm{i})$, there is no further dependence on the multiplicities $a_{i,j}$, *i.e.* on the fact that some parts may be equal. Moreover, in terms of *reduced* partitions $\overline{\alpha_i} = 2^{a_{i,2}}\dots k^{a_{i,k}}$ (*i.e.* withdrawing parts equal to 1), the correction reads, using the equalities $\ell_i = \left(\sum_{j\neq i} r_j\right) + 1 - 2g$, and $r(\alpha_i) = r(\overline{\alpha_i})$,

$$\sum_{g_0+\cdots+g_m=g} P_g(\mathbf{r}-2\mathbf{g}) \prod_{i=1}^{m} \left(\sum_{j\neq i} r(\overline{\alpha_j}) \ + 1 - 2g\right)^{(2g_i)} S_{g_i}(\overline{\alpha_i}),$$

  and is, for fixed g, m and $\ell(\overline{\alpha_i})$ but independantly of n, a polynomial of total degree $4g$ in the parts of the reduced partitions.

To sum up, Theorem 1 shows that the genus g has a determining influence on the complexity of connection coefficients computing: as mentioned above, the number of terms in the summation is polynomial for fixed g, but increases exponentially with g. Similar phenomena are observed in related results of Goulden *et al.* [**?**,**?**, and references therein]. These authors consider transitive factorizations of a permutation of type β into *transpositions*. Their expression overlaps with ours when $\beta = n$ (our restriction) and $\alpha_i = 1^{n-2}2$ (their restriction).

2.4 Some asymptotic corollaries

Asymptotic results for structure constants were mainly considered in the limit where n is fixed and m goes to infinity. In particular, this implies that the genus goes as well to infinity. While this is natural in the study of random walks in the symmetric group (see [**?**]), it is also of interest to obtain asymptotic results at fixed genus in view of the connection with topological interpretations. In this context, parts of length one are not interesting (since they correspond to regular points as opposed to critical points), so that it is more natural to stress the genus and reduced cycle types in notations: for $\alpha_1, \dots \alpha_m$ being reduced partitions (*i.e.* without parts equal to 1), let

$$f^g_{\alpha_1,\dots,\alpha_m} = \begin{cases} c^{(n)}_{1^{a_1}\alpha_1,\dots,1^{a_m}\alpha_m} & \text{if } n > \max_i(|\alpha_i|), \\ 0 & \text{otherwise,} \end{cases}$$

where $n = \sum_i r_i + 1 - 2g$ and $a_i = n - |\alpha_i|$. In other terms, $f^g_{\alpha_1,\dots,\alpha_m}$ is the number of factorizations of genus g of a maximal cycle into m permutations of respective reduced cycle types $\alpha_1,\dots,\alpha_m$.

Formula 1 is useful to give asymptotic estimate for connection coefficients in the limit where g is fixed and n goes to infinity. As an illustration, we prove in Section 6 the following two corollaries. (Throughout this article, $f(x) \underset{x\to\infty}{\sim} g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$.)

Corollary 2 (Large number of factors) *Let g be a non negative integer, and $\alpha = 2^{a_2} \dots k^{a_k}$ a partition (without parts equal to 1) of rank r and length ℓ. Then there exists a constant $c(g, \alpha)$ (given in the proof) such that, for m going to infinity,*

$$f^g_{(\alpha)^m} \underset{m\to\infty}{\sim} c(g,\alpha) \cdot \frac{(\ell-1)!^m}{\mathrm{Aut}(\alpha)^m} \cdot (mr)^{m\ell-1+3g}.$$

In particular, for $\alpha_1, \dots \alpha_m$ being transpositions, this yields a generalized Cayley formula:

$$f^g_{(2)^m} \underset{m\to\infty}{\sim} \frac{m^{m-1+3g}}{24^g\, g!}.$$

This latter formula nicely extends either to involutions with k cycles:

$$f^g_{(2^k)^m} \underset{m\to\infty}{\sim} \frac{(km)^{km-1+3g}}{k^m 24^g\, g!},$$

or to k-cycles:

$$f^g_{(k)^m} \underset{m\to\infty}{\sim} c(g,k) \cdot ((k-1)m)^{m-1+3g}.$$

Corollary 3 (Large factors) *Let g be a non negative integer, and consider m partitions $\alpha_i = 1^{a_{i,1}} \dots k^{a_{i,k}}$. Let $x \cdot \alpha_i$ denote the partition $x^{a_{i,1}} \dots (kx)^{a_{i,k}}$. For x going to infinity, there exists a constant $c(g; \alpha_1, \dots, \alpha_m)$ (given in the proof) such that:*

$$f^g_{x\cdot\alpha_1,\dots,x\cdot\alpha_m} \underset{x\to\infty}{\sim} c(g;\alpha_1,\dots,\alpha_m) \cdot x^{4g-1+\sum_i \ell_i}.$$

Considering this homothetic limit was suggested to us by Dimitri Zvonkine.

Let us mention that structure constants have an interpretation in terms of some 2-cell embeddings of graphs, called *cacti*, *maps* or more generally *constellations*. This interpretation is intermediate between factorizations and meromorphic functions (see [?]). This allows to work with discrete, combinatorial objets, without completely loosing the topological intuition. In this context, the special case $m = 2$, $\alpha_2 = 2^{n/2}$ was known to Walsh and Lehman [?]. But unlike Goulden and Jackson in [?], and although we also use some graphical interpretations in the course of the proof, we were unable to use constellations. Such a relation would be very interesting to find in so far as it could provide a constructive proof of our result: in the present state, our derivation starts from the (non constructive) character theoretic formula and we are unable to present a reasonable algorithm to list all the factorizations counted by $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$.

The proof relies on the same approach that was successfully applied to the much simpler case $m = 2$ in [?]. In Section 3, the formula given by character theory is interpreted as a weighted sum over some combinatorial objects. This first interpretation is closely related to the one used by Goulden in [?] and similar to that of [?]. Then, in Section 4, a new interpretation in terms of *starry* graphs is developed. These graphs are the key ingredient allowing us to proceed in a formal analogy with the case $m = 2$, although the objects and details are more involved. An involution principle is applied to cancel negative contributions and to obtain a weighted sum involving cyclomatic numbers of graphs (Theorem 15). In Section 5, this weighted sum is finally related to the number of orientations of graphs and explicitly computed.

3 A combinatorial interpretation of the character theoretic formula

Character theory provides an expression for $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$. For any partition β of n, the cardinality z_β of the centralizer of any permutation in $\mathcal{C}_\beta$ is equal to $1^{b_1} b_1!\ 2^{b_2} b_2!\ \ldots\ n^{b_n} b_n!$. For convenience, we consistently use the notation z_i for z_{α_i}. If moreover we denote by χ^γ the irreducible character of $\mathfrak{S}_n$ indexed by the partition γ, its evaluation at the conjugacy class $\mathcal{C}_\beta$ by χ^γ_β, and its degree by f^γ, then Frobenius formula can be expressed as (see *e.g.* [?, p68]):

Proposition 4 *Let $\alpha_1, \ldots, \alpha_m$ and β be partitions of $n \in \mathbb{N}$. Then:*

$$c^{\beta}_{\alpha_1,\ldots,\alpha_m} = \frac{n!^{m-1}}{z_1 \ldots z_m} \sum_{\gamma \vdash n} \frac{\chi^\gamma_{\alpha_1} \cdots \chi^\gamma_{\alpha_m}}{(f^\gamma)^{m-1}} \chi^\gamma_\beta.$$

This formula motivates the search for a convenient expression for evaluations of irreducible characters of $\mathfrak{S}_n$. We first recall a classical rule for computing them using Ferrers diagrams, and then seek for simplifications that occur in the particular case $\beta = (n)$.

3.1 Murnaghan-Nakayama rule

Let α and β be two partitions of a positive integer n. A *rim hook tableau* of type (α, β) is a Ferrers diagram of shape α filled with positive integers such that, for all $i \in [\![1, \ell(\beta)]\!]$,

- the cells filled with integers i to $\ell(\beta)$ form a Ferrers diagram of weight $\beta_i + \cdots + \beta_{\ell(\beta)}$,
- the β_i cells filled with i form a *rim hook*, *i.e.* a connected set that contains no pattern of the type: $\begin{array}{|c|c|}\hline i & i \\ \hline i & i \\ \hline\end{array}$.

This means that a rim hook tableau of type (α, β) can be regarded as a diagram of shape α filled with the rows of β, in such a way that each row of β forms a rim hook of α. The *weight* $W(T)$ of a rim hook tableau T is the number of patterns $\begin{array}{|c|}\hline i \\ \hline i \\ \hline\end{array}$ in it.

Example: A rim hook tableau of type $((5,4,3,1),(6,4,3))$:

$$\begin{array}{|c|c|c|c|c|}\hline 2 \\ \cline{1-1} 2 & 1 & 1 \\ \cline{1-3} 2 & 2 & 1 & 1 \\ \cline{1-4} 3 & 3 & 3 & 1 & 1 \\ \hline\end{array}$$

Its three rim hooks have respective contributions 2, 2 and 0, hence its weight is 4.

$\diamondsuit$

With these definitions, the following rule allows to compute evaluations of irreducible characters of $\mathfrak{S}_n$ (see [**?**]):

Proposition 5 (Murnaghan-Nakayama rule) *Let α and β be two partitions of a positive integer n. Then:*

$$\chi^{\alpha}_{\beta} \;=\; \sum_{T} (-1)^{W(T)}$$

where the sum runs over all rim hook tableaux of type (α, β).

Evaluations of the irreducible characters of $\mathfrak{S}_n$ at the class of the n-cycles can be immediately deduced from this rule, since no diagram can be filled with a single rim hook unless it is itself a hook:

Proposition 6

$$\chi^{\gamma}_{(n)} \;=\; \begin{cases} (-1)^r & \textit{if there exists } r \in [\![0, n-1]\!] \textit{ such that } \gamma = 1^r(n-r), \\ 0 & \textit{otherwise.} \end{cases}$$

This reduces the summation in Proposition 4 to a summation over hook diagrams. Moreover, we obtain from Murnaghan-Nakayama rule the value of f^{γ} when γ is a hook:

$$\forall r \in [\![0, n-1]\!], \quad f^{1^r(n-r)} \;=\; \chi^{1^r(n-r)}_{1^n} \;=\; \binom{n-1}{r}.$$

Hence Proposition 4 can be rewritten into:

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} \;=\; \frac{n!^{m-1}}{z_1 \ldots z_m} \sum_{r=0}^{n-1} \binom{n-1}{r}^{1-m} (-1)^r \chi^{1^r(n-r)}_{\alpha_1} \ldots \chi^{1^r(n-r)}_{\alpha_m}. \qquad (2)$$

3.2 Quasi-painted diagrams

In order to give a combinatorial interpretation of Formula (2), we derive from Murnaghan-Nakayama rule another expression of characters involved in it. Let us define a *painted diagram* of shape α as a Ferrers diagram whose cells are painted in black ($\bullet$) or white ($\circ$), such that each row is monochrome. A *quasi-painted diagram* is such that all cells are black or white but one which contains a cross, and all rows are monochrome but the last one which has the form:

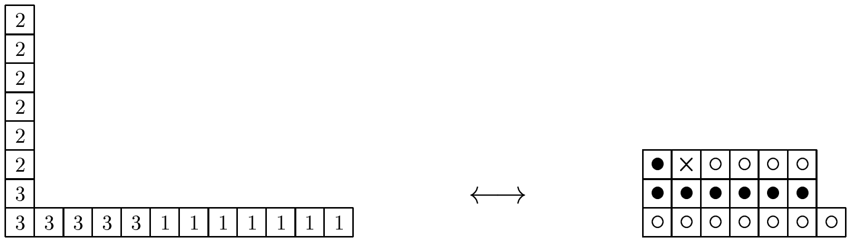
$\bullet$	$\bullet$	$\cdots$	$\bullet$	$\times$	$\circ$	$\cdots$	$\circ$	$\circ$

i.e. $\bullet^p \times \circ^q$ for some $p \geqslant 0$, $q \geqslant 0$. Let $\widehat{\alpha}$ be a quasi-painted diagram of shape α, $|\widehat{\alpha}|_{\bullet}$ denote the number of its black cells, and $\ell_{\bullet}(\widehat{\alpha})$ the number of its black rows (in particular the last row, which contains the cross, does not count).

Proposition 7 *Let $r \in [\![0, n-1]\!]$ and $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vdash n$. Then the rim hook tableaux of type $(1^r(n-r), \alpha)$ are in one-to-one correspondence with the quasi-painted diagrams of shape α with r black cells.*

PROOF. White cells of α are those that fill the horizontal part of the hook $1^r(n-r)$, black ones fill the vertical part, and the crossed cell corresponds to the position of the corner cell of $1^r(n-r)$. $\square$

Example: A rim hook tableau of type $(1^7\,12,(7,6,6))$ and the associated quasi-painting of $(7,6,6)$:



$\diamondsuit$

Hence we get the following reformulation of Murnaghan-Nakayama rule in terms of quasi-paintings, in the special case of $\chi_\alpha^{1^r(n-r)}$:

Proposition 8 *Let $r \in [\![0, n-1]\!]$ and $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vdash n$. Then:*

$$\chi_\alpha^{1^r(n-r)} \;=\; \sum_{\widehat{\alpha}} (-1)^{r-\ell_\bullet(\widehat{\alpha})},$$

where the sum runs over the quasi-paintings $\widehat{\alpha}$ of α with r black cells.

3.3 Painted heightened diagrams

In this section, we derive from Proposition 8 an expression for characters $\chi_\alpha^{1^r(n-r)}$ that involves painted diagrams instead of quasi-painted ones.

Let $\{\bullet,\circ\}^p$ denote the set of words of length p over the alphabet $\{\bullet,\circ\}$, and $\mathfrak{S}(\bullet^p\circ^q)$ the set of words of length $p+q$ with exactly p letters $\bullet$ and q letters $\circ$.

With these notations, the following lemma is straightforward:

Lemma 9 (shuffle lemma) *Let p and q be positive integers. Define:*

$$\begin{aligned} \varphi : \{\bullet,\circ\}^{p+q+1} &\longrightarrow \{\bullet,\circ\}^* \times \{\bullet,\circ\}^* \\ u &\mapsto (v,w) \end{aligned}$$

where $vw = u$ and $|v|$ is maximal with respect to the conditions: $|v|_\bullet \leqslant p$ and $|v|_\circ \leqslant q$.

Then φ maps $\{\bullet,\circ\}^{p+q+1}$ bijectively onto

$$\bigcup_{i=1}^{p} \left(\mathfrak{S}(\bullet^i\circ^q) \times \circ\{\bullet,\circ\}^{p-i}\right) \;\cup\; \bigcup_{j=1}^{q} \left(\mathfrak{S}(\bullet^p\circ^j) \times \bullet\{\bullet,\circ\}^{q-j}\right).$$

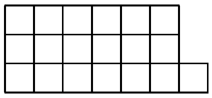
Let $\alpha = (\alpha_1, \ldots, \alpha_\ell)$ be a partition of an integer $n > 1$. For any k in $[\![0, \alpha_\ell - 1]\!]$, its k-th *heightened partition* α^k is the partition of $n-1$ defined as follows:

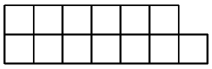
$$\alpha^0 \;=\; (\alpha_1,\ \ldots,\ \alpha_{\ell-1},\ \underbrace{1,\ \ldots,\ 1}_{\alpha_\ell - 1\text{ times}}),$$

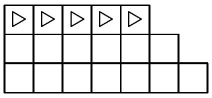
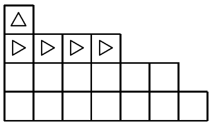
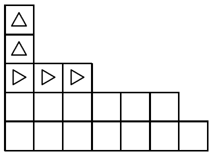
and for all integer $1 \leqslant k < \alpha_\ell$,

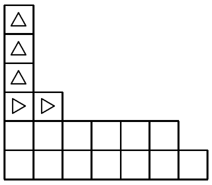
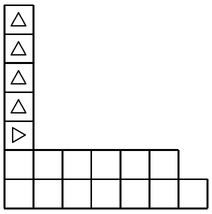
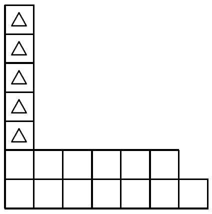
$$\alpha^k \;=\; (\alpha_1,\ \ldots,\ \alpha_{\ell-1},\ k,\ \underbrace{1,\ \ldots,\ 1}_{\alpha_\ell - 1 - k\text{ times}}).$$

The partition $(\alpha_1, \ldots, \alpha_{\ell-1})$ is denoted by α^*, and the partition $1^{\alpha_\ell - 1 - k}k$ is called a *heightened hook* of α; $1^{\alpha_\ell - 1 - k}$ is its *vertical part* and k (if $k \neq 0$) its *horizontal part*.

EXAMPLE: Let us consider the partition $\alpha = (7, 6, 6) =$  .

Then $\alpha^* =$  and the heightenings of α are:

$\alpha^5 =$  , $\alpha^4 =$  , $\alpha^3 =$  ,

$\alpha^2 =$  , $\alpha^1 =$  , $\alpha^0 =$  ,

where cells of the vertical part of the heightened hook are marked with a $\vartriangle$, and those of its horizontal part with a $\vartriangleright$. $\diamondsuit$

With these notations and Lemma 9, Proposition 8 becomes:

Proposition 10 *Let $r \in [\![0, n-1]\!]$ and $\alpha = (\alpha_1, \ldots, \alpha_\ell) \vdash n$. Then:*

$$\chi_\alpha^{1^r(n-r)} \;=\; \sum_{k=0}^{\alpha_\ell - 1} 2^{\ell - \ell(\alpha^k) - 1} \sum_{\widehat{\alpha^k}} (-1)^{r - \ell_\bullet(\widehat{\alpha^*})},$$

where the inner sum runs over all painted diagrams $\widehat{\alpha^k}$ of shape α^k with r black cells.

Remark that any painting $\widehat{\alpha^k}$ induces a painting $\widehat{\alpha^*}$ of α^*; in particular, $\ell_\bullet(\widehat{\alpha^*})$ is well defined and equal to $\ell_\bullet(\widehat{\alpha^k})$.

PROOF. The summation in Proposition 8 can be rewritten as follows:

$$\begin{aligned}\chi_\alpha^{1^r(n-r)} &= \frac{1}{2^{\alpha_\ell}} \sum_{\widehat{\alpha}} \sum_{u \in \{\bullet,\circ\}^{\alpha_\ell}} (-1)^{r-\ell_\bullet(\widehat{\alpha})} \\ &= \frac{1}{2^{\alpha_\ell}} \sum_{\widehat{\alpha}} \sum_{u \in \{\bullet,\circ\}^{\alpha_\ell}} (-1)^{r-\ell_\bullet(\widehat{\alpha^*})}.\end{aligned}$$

Let us define a one-to-one correspondence between couples $(\widehat{\alpha}, u)$ consisting of a quasi-painting of α and a word u in $\{\bullet,\circ\}^{\alpha_\ell}$, and triples $(k, \widehat{\alpha^k}, w')$ consisting of a non negative integer $k < \alpha_\ell$, a painting of α^k and a word of length $k+\delta_k^0$, where δ_k^0 denotes Kronecker's symbol.

For any painting $\widehat{\alpha}$ of α, let p and q be such that its last row is $\bullet^p \times \circ^q$. For any word u, let $(v,w) = \varphi(u)$ defined as in the shuffle lemma, and $k = |w|-1$. Then the corresponding painting $\widehat{\alpha^k}$ of the heightened diagram α^k is defined as follows:

- the painting of rows $\alpha_1, \ldots, \alpha_{\ell-1}$ of α induces a painting of the same rows in α^k.
- the colour of the horizontal part of the heightened hook, if any, *i.e.* if $k \neq 0$, is *not* the first letter of w.
- the colour of the remaining cells is given by the word v.

Moreover, we take w' to be w if $k=0$ and w without its first letter otherwise.

The mapping $\widehat{\alpha} \mapsto (k, \widehat{\alpha^k}, w')$ is one-to-one and preserves the number of black cells, so that:

$$\begin{aligned}\chi_\alpha^{1^r(n-r)} &= \frac{1}{2^{\alpha_\ell}} \sum_{k=0}^{\alpha_\ell-1} \sum_{\widehat{\alpha^k}} \sum_{w' \in \{\bullet,\circ\}^{k+\delta_k^0}} (-1)^{r-\ell_\bullet(\widehat{\alpha^*})} \\ &= \sum_{k=0}^{\alpha_\ell-1} 2^{-\alpha_\ell+k+\delta_k^0} \sum_{\widehat{\alpha^k}} (-1)^{r-\ell_\bullet(\widehat{\alpha^*})},\end{aligned}$$

where the internal sum runs over the painted diagrams of shape α^k with r black cells. Since $1+\ell(\alpha^k)-\ell = \alpha_\ell - k - \delta_k^0$, the proof is complete. $\square$

Example: Consider the painting $\widehat{\alpha} = \begin{array}{|c|c|c|c|c|c|c|c|} \hline \bullet & \times & \circ & \circ & \circ & \circ \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \hline \end{array}$; then $p = 1$ and $q = 4$.

Let $u = \circ\bullet\bullet\circ\circ\bullet \in \{\bullet, \circ\}^6$, then $\varphi(u) = (\circ\bullet, \bullet\circ\circ\bullet)$, $k = 3$, and the corresponding painting of α^3 is $\begin{array}{|c|c|c|c|c|c|c|} \hline \circ \\ \hline \bullet \\ \hline \circ & \circ & \circ \\ \hline \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \hline \circ & \circ & \circ & \circ & \circ & \circ & \circ \\ \hline \end{array}$. $\diamondsuit$

4 A graphical summation

For any m-tuple $\mathfrak{a}$ of partitions $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,\ell_i})$ of n, let $I_{\mathfrak{a}}$ denote the set $[\![0, \alpha_{1,\ell_1} - 1]\!] \times \cdots \times [\![0, \alpha_{m,\ell_m} - 1]\!]$. Then, for any $\boldsymbol{k} = (k_1, \ldots, k_m) \in I_{\mathfrak{a}}$, the m-tuple $(\alpha_1^{k_1}, \ldots, \alpha_m^{k_m})$ is denoted by $\mathfrak{a}^{\boldsymbol{k}}$. Last, let $\ell_i^{\boldsymbol{k}} = \ell(\alpha_i^{k_i})$.

We are now able to derive from Formula (2) a new expression for connection coefficient $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$. Each evaluation of a character uses a summation over painted diagrams with r black cells. Expanding these summations leads to summing over configurations of m painted diagrams with the same number r of black cells:

$$\frac{n^{m-1}}{z_1 \cdots z_m} \sum_{r=0}^{n-1} \sum_{\boldsymbol{k} \in I_{\mathfrak{a}}} \sum_{\widehat{\mathfrak{a}^{\boldsymbol{k}}}} \left(\prod_{i=1}^{m} 2^{\ell_i - \ell_i^{\boldsymbol{k}} - 1} (-1)^{r - \ell_\bullet(\widehat{\alpha_i^*})} \right) (-1)^r [r!(n-1-r)!]^{m-1} \tag{3}$$

where the innermost summation runs over paintings $\widehat{\mathfrak{a}^{\boldsymbol{k}}}$ of $\mathfrak{a}^{\boldsymbol{k}}$ with r black cells in each diagram $\alpha_i^{k_i}$.

4.1 Painted starry graphs

The factor $[r!(n-1-r)!]^{m-1}$ in Expression (3) can easily be interpreted. In order to give a natural description for other factors as well, we introduce the following model:

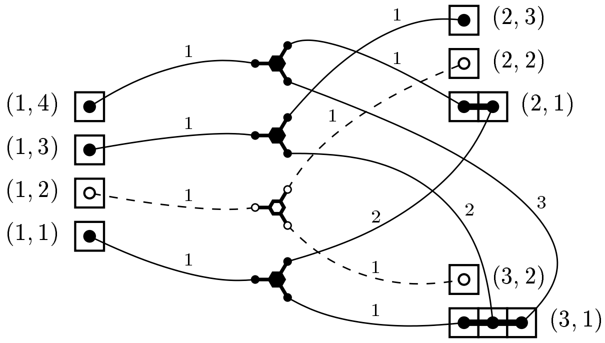
Let $\alpha_1, \ldots, \alpha_m$ be partitions of a positive integer p, with $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,\ell_i})$ for all $i \in [\![1, m]\!]$. A *starry graph* of type $\mathfrak{a} = (\alpha_1, \ldots, \alpha_m)$ is a bipartite graph Γ satisfying the following conditions:

- its two kinds of vertices are, on the one hand, $\ell_1 + \ldots + \ell_m$ *row* vertices, and on the other hand p *star* vertices,

- row vertices correspond bijectively to rows of the diagrams,
- for any $1 \leqslant i \leqslant m$, $1 \leqslant j \leqslant \ell_i$, the row vertex (i,j) has degree α_{ij},
- star vertices are *unlabelled* and have degree m,
- edges incident to row vertex (i,j) are labelled 1 to $\alpha_{i,j}$,
- for any star vertex, and any $i \in [\![1,m]\!]$, there is exactly one index $j \in [\![1,\ell_i]\!]$ such that row vertex (i,j) is adjacent to the given star vertex.

A starry graph $\widehat{\Gamma}$ is *painted* if its vertices are painted black or white in such a way that any two adjacent vertices have the same colour. If the colour of its row vertices is given by the colour of the rows of $\widehat{\mathfrak{a}}$, $\widehat{\Gamma}$ is said to be of type $\widehat{\mathfrak{a}}$. The set of starry graphs of type $\mathfrak{a}$ is denoted by $\mathcal{G}(\mathfrak{a})$ and the set of painted starry graphs whose type is a painting of $\mathfrak{a}$ by $\widehat{\mathcal{G}}(\mathfrak{a})$.

Example: Let us consider the partitions $\alpha_1 = 1^4$, $\alpha_2 = 1^2 2$, $\alpha_3 = 1\,3$. The picture below represents a painted starry graph on $(\alpha_1, \alpha_2, \alpha_3)$.



$\diamondsuit$

Let us consider m painted diagrams with p black and q white cells each. Any starry graph on them has p black and q white *undistinguishable* star vertices. Each black (respectively white) cell of a given diagram is adjacent to a different black (respectively white) star, hence the number of starry graphs on these diagrams is $\frac{1}{p!\,q!}\,(p!\,q!)^m$.

Proposition 11 *Let $\mathfrak{a}$ be a m-tuple of partitions of n, and $\widehat{\mathfrak{a}^{\boldsymbol{k}}}$ a painted heightening of $\mathfrak{a}$ with r black cells in each diagram. Then the number of painted starry graphs of type $\widehat{\mathfrak{a}^{\boldsymbol{k}}}$ is $[r!(n-1-r)!]^{m-1}$.*

This gives an interpretation of factor $[r!(n-1-r)!]^{m-1}$ in Expression (3) in terms of painted starry graphs of type $\widehat{\mathfrak{a}^{\boldsymbol{k}}}$. Let us rewrite Expression (3) in these terms. The summation over r leads to a summation over all paintings, so that we obtain:

Proposition 12

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} \;=\; \frac{n^{m-1}}{z_1\ldots z_m\; 2^{2g}} \sum_{\boldsymbol{k}\in I_{\mathfrak{a}}} \sum_{\widehat{\Gamma}\in\widehat{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}})} 2^{\kappa(\Gamma)-c(\Gamma)}(-1)^{(m-1)\rho(\Gamma)-\sum_i \ell_\bullet(\widehat{\alpha}_i^*)} \qquad (4)$$

in which $\rho(\Gamma)$ denotes the number of black stars of Γ, and $\kappa(\Gamma)$ its cyclomatic number, i.e. *$e(\Gamma)-v(\Gamma)+c(\Gamma)$, with $e(\Gamma)$, $v(\Gamma)$ and $c(\Gamma)$ denoting respectively the numbers of its edges, vertices and connected components.*

PROOF. Remark that $\rho(\Gamma)$ corresponds to r, so that to derive Formula (4) from Formula (3), we only have to prove:

$$\prod_{i=1}^{m} 2^{\ell_i-\ell_i^{\boldsymbol{k}}-1} \;=\; 2^{\kappa(\Gamma)-c(\Gamma)-2g}.$$

According to the definition of g, we have $\sum_{i=1}^m(\ell_i-1)=(m-1)(n-1)-2g$. The number $r(\Gamma)$ of row vertices of Γ is $\sum_{i=1}^m \ell_i^{\boldsymbol{k}}$, the number $s(\Gamma)$ of star vertices is $n-1$ and the number of edges is $m(n-1)$, so that:

$$\begin{aligned}\sum_{i=1}^m \ell_i-\ell_i^{\boldsymbol{k}}-1 \;&=\; [(m-1)(n-1)-2g]-\sum_{i=1}^m \ell_i^{\boldsymbol{k}}\\ &=\; [e(\Gamma)-s(\Gamma)-2g]-r(\Gamma)\\ &=\; \kappa(\Gamma)-c(\Gamma)-2g.\end{aligned}$$

□

4.2 Connected components and evenness

Let Γ be a starry graph of type $\mathfrak{a}^{\boldsymbol{k}}=(\alpha_1^{k_1},\ldots,\alpha_m^{k_m})$, and $\Gamma^{(1)}\sqcup\ldots\sqcup\Gamma^{(c(\Gamma))}$ its splitting in connected components. It induces a splitting of each partition $\alpha_i^{k_i}$.

The following proposition is straightforward from the definition of painted starry graphs:

Proposition 13 *Let $\widehat{\Gamma}$ be a painted starry graph. Then each connected component of Γ is monochrome. In other words, painted starry graphs $\widehat{\Gamma}$ are in bijection with couples made of a (non painted) starry graph Γ and a subset $\mathcal{B}$ of the set $\mathcal{C}(\Gamma)$ of its components – the black ones.*

Hence we can derive from Proposition 13 a new formulation of the innermost summation in Equation (4):

$$\sum_{\Gamma\in\mathcal{G}(\mathfrak{a}^{\boldsymbol{k}})} 2^{\kappa(\Gamma)-c(\Gamma)} \sum_{\mathcal{B}\subset\mathcal{C}(\Gamma)} (-1)^{\varepsilon(\mathcal{B})},$$

in which $\varepsilon(\mathcal{B})$ is defined as follows: let $e(\Gamma^{(c)})$, $s(\Gamma^{(c)})$, $r(\Gamma^{(c)})$ and $h(\Gamma^{(c)})$ denote respectively the numbers of edges, star vertices, row vertices and heightened row vertices in the component $\Gamma^{(c)}$; we associate to each connected component the following parameter:

$$\begin{aligned}\varepsilon(\Gamma^{(c)}) &= (m-1)\, s(\Gamma^{(c)}) - \sum_{i=1}^{m} \ell(\alpha_i^{*(c)}) \\ &= \left[e(\Gamma^{(c)}) - s(\Gamma^{(c)})\right] - \left[r(\Gamma^{(c)}) - h(\Gamma^{(c)})\right] \\ &= \kappa(\Gamma^{(c)}) - 1 + h(\Gamma^{(c)}),\end{aligned}$$

and this notation is extended to subsets of $\mathcal{C}(\Gamma)$:

$$\forall \mathcal{B} \subset \mathcal{C}(\Gamma),\ \varepsilon(\mathcal{B}) = \sum_{c\in\mathcal{B}} \varepsilon(\Gamma^{(c)}).$$

A starry graph is said *totally even* if $\varepsilon(\Gamma^{(c)})$ is even for all $c \in \mathcal{C}(\Gamma)$. We denote by $\mathcal{E}(\mathfrak{a}^{\boldsymbol{k}})$ the set of totally even starry graphs of type $\mathfrak{a}^{\boldsymbol{k}}$.

4.3 A sign-reversing involution

Lemma 14 *Let $\Gamma \in \mathcal{G}(\mathfrak{a}^{\boldsymbol{k}})$. Then:*

$$\sum_{\mathcal{B}\subset\mathcal{C}(\Gamma)} (-1)^{\varepsilon(\mathcal{B})} = \begin{cases} 2^{c(\Gamma)} & \text{if } \Gamma \text{ is totally even,} \\ 0 & \text{otherwise.}\end{cases}$$

PROOF. First case is obvious. In the other case, let c be the lowest index such that $\varepsilon(\Gamma^{(c)})$ is odd. Consider the involution ϑ of the set of subsets $\mathcal{C}(\Gamma)$ mapping $\mathcal{B} \subset \mathcal{C}(\Gamma)$ on the symmetric difference $\mathcal{B}\triangle\{c\}$. Then ϑ is an involution without fixed point, and for all $\mathcal{B} \subset \mathcal{C}(\Gamma)$, $\varepsilon(\mathcal{B}) \not\equiv \varepsilon(\vartheta(\mathcal{B})) \mod 2$, so that the contributions of all subsets cancel two by two. □

Hence the contribution of a starry graph is $2^{\kappa(\Gamma)}$ if it is totally even and 0 otherwise, which proves the following theorem:

Theorem 15 *Let $\mathfrak{a}$ be a m-tuple of partitions of n. Then:*

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{z_1 \ldots z_m\ 2^{2g}} \sum_{\boldsymbol{k}\in I_{\mathfrak{a}}} \sum_{\Gamma\in\mathcal{E}(\mathfrak{a}^{\boldsymbol{k}})} 2^{\kappa(\Gamma)}.$$

Some particular cases can be directly deduced from this first expression of our theorem. Observe that, for any starry graph Γ built on any heightening of $\mathfrak{a}$,

$$\begin{aligned}\sum_{c\in\mathcal{C}(\Gamma)} \varepsilon(\Gamma^{(c)}) &= (m-1)s(\Gamma) - \sum_{i=1}^{m}(\ell_i - 1)\\ &= (m-1)(n-1) - [(m-1)(n-1) - 2g]\\ &= 2g.\end{aligned}$$

Remark that this supplies a proof that $c_{\mathfrak{a}}^{(n)} = 0$ unless g is an integer: if it is not, no totally even starry graph can be built on a heightening of $\mathfrak{a}$, hence the summation is empty.

Let us now assume that $\mathfrak{a}$ is such that g is an integer, and consider a connected starry graph Γ in $\mathcal{G}(\mathfrak{a}^{\boldsymbol{k}})$. Then its cyclomatic number only depends on $\mathfrak{a}^{\boldsymbol{k}}$:

$$\kappa(\Gamma) \;=\; \varepsilon(\Gamma) + 1 - h(\Gamma) \;=\; 2g + 1 + \sum_{i=1}^{m}\left(k_i + \delta_{k_i}^{0} - \alpha_{i,\ell_i}\right).$$

Suppose that $\mathfrak{a}$ is such that any starry graph on any heightening $\mathfrak{a}^{\boldsymbol{k}}$ is connected. According to Theorem 15,

$$c_{\alpha_1,\ldots,\alpha_m}^{(n)} \;=\; \frac{2\,n^{m-1}}{z_1\ldots z_m}\sum_{\boldsymbol{k}\in I_{\mathfrak{a}}} \mathrm{card}\left(\mathcal{G}(\mathfrak{a}^{\boldsymbol{k}})\right)\; 2^{k_i+\delta_{k_i}^{0}-\alpha_{i,\ell_i}}.$$

Now, $\mathrm{card}\left(\mathcal{G}(\mathfrak{a}^{\boldsymbol{k}})\right) = (n-1)!^{m-1}$, and for any integer p, $\sum_{k=0}^{p-1} 2^{k+\delta_k^0} = 2^p$. This proves the following:

Corollary 16 *If $(\alpha_1,\ldots,\alpha_m)$ is such that any starry graph on any of its heightenings is connected, then either $c_{\alpha_1,\ldots,\alpha_m}^{(n)} = 0$ or*

$$c_{\alpha_1,\ldots,\alpha_m}^{(n)} \;=\; \frac{2}{n!}\prod_{i=1}^{m}\mathrm{card}(\mathcal{C}_{\alpha_i}).$$

(Recall that $\mathcal{C}_{\alpha_i}$ is the conjugacy class of permutations of type α_i.)

In particular, this connectedness condition is satisfied if one of the partitions is equal to $(n-1,1)$. Hence this formula generalizes the well-known result that $c_{(n),(n-1,1)}^{\alpha} = 2(n-2)!$ for any odd partition α.

5 Orientability and explicit enumeration

5.1 Evenness function and orientability

Let Γ be a graph, $\mathcal{V}$ its set of vertices. Let us call *evenness function* on Γ any mapping $\varphi: \mathcal{V} \to \{0,1\}$. The cardinality of $\varphi^{-1}(0)$ is called its *weight* and denoted by $w(\varphi)$. An orientation of edges of Γ is said *φ-compatible* if for each vertex v, $\varphi(v)$ and its outdegree have the same evenness. A graph Γ which has such an orientation is said *φ-orientable.*

Proposition 17 *A connected graph Γ is φ-orientable if and only if*

$$\kappa(\Gamma) \not\equiv w(\varphi) \mod 2.$$

In this case, it has exactly $2^{\kappa(\Gamma)}$ φ-compatible orientations.

PROOF. First suppose that $\kappa(\Gamma) = 0$, *i.e.* that Γ is a tree. We prove the result inductively on the number of vertices of Γ. The case where Γ has only one vertex (and hence no edge) is obvious: Γ is φ-orientable if and only if $w(\varphi) = 1$, *i.e.* if $w(\varphi) \not\equiv \kappa(\Gamma) \mod 2$. In this case, there is only possible orientation – the void one. Otherwise, Γ has a leaf ℓ adjacent to a vertex a. Let Γ' be the graph obtained by deleting ℓ in Γ. We define the following evenness function on Γ':

$$\varphi' : \ v \longmapsto \begin{cases} \varphi(v) & \text{if } v \neq a \\ \varphi(a) + \varphi(\ell) + 1 & \text{if } v = a. \end{cases}$$

Γ is φ-orientable if and only if Γ' is φ'-orientable, and each compatible orientation of Γ corresponds to exactly one orientation of Γ'. Moreover, $w(\varphi') \equiv w(\varphi) \mod 2$, hence by induction Γ is φ-orientable if and only if $w(\varphi) \equiv 1 \mod 2$, and has only one compatible orientation. This proves the case $\kappa(\Gamma) = 0$.

Let us prove the proposition inductively on $\kappa(\Gamma)$. Suppose that $\kappa(\Gamma) \geqslant 1$, then Γ has a simple cycle $(v_1, \ldots, v_k)$, with $k \geqslant 3$. Let (v_1, v_2) be an edge of the cycle. Let Γ' be obtained from Γ by deleting (v_1, v_2). Let us consider the φ-compatible orientations of Γ in which (v_1, v_2) is oriented from v_1 to v_2, and denote by φ' be the evenness function on Γ' defined by:

$$\varphi' : \ v \longmapsto \begin{cases} \varphi(v) & \text{if } v \neq v_1 \\ \varphi(v_1) + 1 & \text{if } v = v_1. \end{cases}$$

Then the restrictions to Γ' of these orientations are exactly its φ'-compatible orientations. Since $\kappa(\Gamma') = \kappa(\Gamma) - 1$, by induction $\kappa(\Gamma') \not\equiv w(\varphi') \mod 2$ and the number of these orientations is $2^{\kappa(\Gamma')}$. Since $w(\varphi') \not\equiv w(\varphi) \mod 2$, there exists φ-compatible orientations of Γ in which (v_1, v_2) is oriented from v_1 to v_2 if and only if $\kappa(\Gamma) \not\equiv w(\varphi) \mod 2$. In this case the number of such orientations is $2^{\kappa(\Gamma)-1}$, and Γ has also $2^{\kappa(\Gamma)-1}$ φ-compatible orientations in which (v_1, v_2) is oriented from v_2 to v_1, which concludes the proof. □

5.2 Last enumeration

We use the latter results on starry graphs with a judicious choice of evenness function: let $\mathfrak{a}$ be a m-tuple of partitions of n, $\mathfrak{a}^{\boldsymbol{k}}$ a heightening of $\mathfrak{a}$ and Γ a starry graph in $\mathcal{G}(\mathfrak{a}^{\boldsymbol{k}})$. We define the evenness function φ by $\varphi(v) = 0$ if v is a row vertex taken from a heightened hook and $\varphi(v) = 1$ otherwise. We denote by $\varphi^{(c)}$ the restriction of φ to the connected component $\Gamma^{(c)}$ of Γ.

Proposition 18 Γ *is totally even if and only if it is φ-orientable.*

PROOF. A starry graph Γ is totally even if and only if, for all connected component $\Gamma^{(c)}$, $\varepsilon(\Gamma^{(c)}) \equiv 0 \mod 2$, and it is φ-orientable if and only if, for all $\Gamma^{(c)}$, $\kappa(\Gamma^{(c)}) \not\equiv w(\varphi^{(c)}) \mod 2$. But $w(\varphi^{(c)})$ is by definition the number of rows of the heightened hook that belong to the c-th component, *i.e.* $h(\Gamma^{(c)})$. Since $\varepsilon(\Gamma^{(c)}) = \kappa(\Gamma^{(c)}) - 1 + h(\Gamma^{(c)})$, this ends the proof. □

The set of φ-compatible orientations of graphs in $\mathcal{G}(\mathfrak{a}^{\boldsymbol{k}})$ is denoted by $\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}})$. We can immediately derive the following equality from Lemma 14 and Propositions 17 and 18:

$$\sum_{\Gamma \in \mathcal{E}(\mathfrak{a}^{\boldsymbol{k}})} 2^{\kappa(\Gamma)} = \operatorname{card}\left(\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}})\right).$$

Hence Theorem 15 becomes:

Theorem 19

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{z_1 \ldots z_m \; 2^{2g}} \sum_{\boldsymbol{k} \in I_{\mathfrak{a}}} \operatorname{card}\left(\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}})\right).$$

So we have reduced the problem to counting the elements of $\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}})$. Let us now partition $\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}})$ through the following criteria: the partition $\hat{\mu}$ composed of the outdegrees of the star vertices, and the map ψ that associates to each row vertex α_{ij} its outdegree ψ_{ij}. We consider $\hat{\mu}$ as a partition because the star vertices are not labelled, unlike row vertices. Then:

$$\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}}) = \bigcup_{(\psi,\hat{\mu})} \vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}}, \psi, \hat{\mu}).$$

Proposition 20 *By construction, ψ and $\hat{\mu}$ satisfy the following properties.*

- *Since $\varphi(v) = 1$ on any star vertex v, the partition $\hat{\mu}$ has no even part. Hence $|\hat{\mu}| \geqslant n-1$ and $|\hat{\mu}| \equiv n-1 \mod 2$. Star vertices with outdegree 1 are said* simple, *the others,* i.e. *those whose outdegree is at least 3, are said* complex.
- *For all $i \in [\![1,m]\!]$, exactly $\ell_i - 1$ indices j are such that row vertex (i,j) satisfies $\varphi(i,j) = 1$; hence $\sum_{j \geqslant 1} \psi_{ij} \geqslant \ell_i - 1$ and $\sum_{j\geqslant 1} \psi_{ij} \equiv \ell_i - 1 \mod 2$.*
- *Each edge of the graph contributes exactly once to the outdegree of a vertex, hence: $|\hat{\mu}| + \sum_{1 \leqslant i \leqslant m} \sum_{j \geqslant 1} \psi_{ij} = m(n-1)$.*

For any $i \geqslant 1$, let m_i denote the number of parts of size $2i+1$ in $\hat{\mu}$, and consider the partition $\mu = 1^{m_1} 2^{m_2} \ldots$; let $g_0 = |\mu|$, *i.e.* half the number of extra outgoing edges of star vertices, and for all $i \in [\![1,m]\!]$, let g_i be half the number of extra outgoing edges of row vertices taken from diagram i. In other terms,

$$|\hat{\mu}| = n - 1 + 2g_0 \quad \text{and} \quad \forall i \in [\![1,m]\!], \ \sum_{j \geqslant 1} \psi_{ij} = \ell_i - 1 + 2g_i.$$

Let us first compute card $\left(\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}}, \psi, \hat{\mu})\right)$ for given ψ and μ. We have to choose for each row α_{ij} the position of the ψ_{ij} outgoing edges. These give raise to $\prod_{j \geqslant 1} \binom{\alpha_{ij}}{\psi_{ij}}$ possible choices for each diagram.

We next have to choose which of the other cells are linked to the complex star vertices and how. Using the notations described in the introduction, this gives

$$\frac{\left(\mathcal{D}\left(e_{2\mu+1}\right)\right)\left(n - 1 - \sum_j \psi_{1j}, \ldots, n - 1 - \sum_j \psi_{mj}\right)}{\mathrm{Aut}(\mu)}$$

possible choices, *i.e.*

$$\frac{\left(\mathcal{D}\left(e_{2\mu+1}\right)\right)\left(r_1 - 2g_1, \ldots, r_m - 2g_m\right)}{\mathrm{Aut}(\mu)}.$$

Linking the last incoming cells to the simple star vertices, which are not indexed, yields no further choice, so it remains only to link the outgoing cells to star vertices in such a way that each diagram is adjacent to each star. This gives $\left(\sum_j \psi_{ij}\right)!$ choices for diagram α_i.

Hence card $\left(\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}}, \psi, \hat{\mu})\right)$ is equal to:

$$\prod_{i,j} \binom{\alpha_{ij}}{\psi_{ij}} \cdot \prod_i \left(\sum_j \psi_{ij}\right)! \ \cdot \ \frac{\left(\mathcal{D}\left(e_{2\mu+1}\right)\right)(\mathbf{r} - 2\mathbf{g})}{\mathrm{Aut}(\mu)}.$$

Let us now sum over ψ with a prescribed m-tuple $(g_1,\ldots,g_m)$. Since ψ_{ij} is odd if and only if $j<\ell_i$, we obtain:

$$\frac{\left(\mathcal{D}\left(e_{2\mu+1}\right)\right)(\mathbf{r}-2\mathbf{g})}{\mathrm{Aut}(\mu)}\cdot\prod_{i=1}^{m}\left((\ell_i-1+2g_i)!\sum_{p_1+\cdots+p_{\ell_i}=g_i}\binom{k_i}{2p_{\ell_i}}\prod_{j=1}^{\ell_i-1}\binom{\alpha_{ij}}{2p_j+1}\right).$$

Summing over $(g_1,\ldots,g_m)$ and μ leads to card $\left(\vec{\mathcal{G}}(\mathfrak{a}^{\boldsymbol{k}})\right)$:

$$\sum_{g_0+\cdots+g_m=g}P_{g_0}(\mathbf{r}-2\mathbf{g})\prod_{i=1}^{m}\left((\ell_i-1+2g_i)!\sum_{p_1+\cdots+p_{\ell_i}=g_i}\binom{k_i}{2p_{\ell_i}}\prod_{j=1}^{\ell_i-1}\binom{\alpha_{ij}}{2p_j+1}\right).$$

Following Theorem 19, we now have to sum over $\boldsymbol{k}$, but the identity

$$\sum_{k_i=0}^{\alpha_{i\ell_i}-1}\binom{k_i}{2p_{\ell_i}}=\binom{\alpha_{i\ell_i}}{2p_{\ell_i}+1},$$

yields the following expression for $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$, which is equivalent to Theorem 1:

$$\frac{n^{m-1}}{z_1\ldots z_m\ 2^{2g}}\sum_{g_0+\cdots+g_m=g}P_{g_0}(\mathbf{r}-2\mathbf{g})$$
$$\prod_{i=1}^{m}\left((\ell_i+2g_i-1)!\sum_{p_1+\cdots+p_{\ell_i}=g_i}\prod_{j=1}^{\ell_i}\binom{\alpha_{ij}}{2p_j+1}\right).$$

□

6 Asymptotic results

Formula (1) allows to catch non trivial asymptotic results at fixed genus. We shall here consider two different limits, both with the weight n of involved partitions going to infinity.

6.1 Large number of identical factors

First we let m go to infinity together with n, with identical factors. The simpler particular case is that of transpositions: let $T=1^{n-2}\,2$, and $\alpha_i=T$

for $1 \leqslant i \leqslant m$. Then $m = n-1+2g$ and

$$\begin{aligned}
c^{(n)}_{T^{n-1+2g}} &= \frac{n^{n-2+2g}}{2^{2g}} P_g(1,\ldots,1)\\
&= \frac{n^{n-2+2g}}{2^{2g}} \sum_{\mu\vdash g} \frac{1}{\mathrm{Aut}(\mu)} \binom{n-1+2g}{\ell(\mu)+2g} \binom{\ell(\mu)+2g}{2\mu_1+1,\ldots,2\mu_{\ell(\mu)}+1}\\
&= \frac{n^{n-2+2g}}{2^{2g}} \sum_{\ell=0}^{g} \binom{n-1+2g}{\ell+2g} \sum_{\substack{\mu\vdash g\\ \ell(\mu)=\ell}} \frac{1}{\mathrm{Aut}(\mu)} \binom{\ell+2g}{2\mu_1+1,\ldots,2\mu_\ell+1},
\end{aligned}$$

so that $c^{(n)}_{T^{n-1+2g}}$ is a polynomial of degree $n-2+5g$ in n. Remark the surprising fact that the coefficient of $\binom{n-1+2g}{\ell+2g}$ is the number of set partitions of $\{1,\ldots,\ell+2g\}$ into ℓ subsets of cardinality odd and at least three. It would be interesting to have a combinatorial interpretation of this fact. Observe that this cancellation–free result could have been obtained after some algebra from those of [**?**,**?**], that give the following formulation:

$$c^{(n)}_{T^{n-1+2g}} = \frac{1}{n!} \sum_{i=0}^{n-1} (-1)^i \binom{n-1}{i} \left[\binom{n}{2} - ni\right]^{n-1+2g}. \qquad (5)$$

Remark that to compute large exact values, an efficient approach is to determine the g coefficients of the polynomial using the g first evaluations of Goulden's formula (5), and then to use our expression.

Asymptotically, the dominant contribution is clearly obtained from our expression for $\mu = 1^g$, so that the number of factorizations of genus g into $m = n-1+2g$ transpositions is estimated by

$$c^{(n)}_{T^{n-1+2g}} \underset{n\to\infty}{\sim} \frac{n^{n-2+5g}}{24^g\, g!}, \quad \text{or equivalently} \quad c^{(n)}_{T^m} \underset{m\to\infty}{\sim} \frac{m^{m-1+3g}}{24^g\, g!}.$$

More generally, let us consider factorizations of genus g in m factors of type $1^p\alpha$, where $\alpha = 2^{a_2}\ldots k^{a_k}$ is a partition of an integer n_0 without trivial parts, with length ℓ and rank r. The relation $mr = n_0+p-1+2g$ gives the number of fix points $p = mr - n_0 + 1 - 2g$, so that such factorizations exist as soon as $m > \dfrac{n_0+2g-1}{r}$.

Then, since $S_{g_i}(1^p\alpha) = S_{g_i}(\alpha)$,

$$c^{(n_0+p)}_{(1^p\alpha)^m} = \frac{n^{m-1}}{2^{2g}} \frac{(p+\ell-1)!^m}{(p!\ \mathrm{Aut}(\alpha))^m} \sum_{g_0+\cdots+g_m=g} P_{g_0}(\mathbf{r}-2\mathbf{g}) \prod_{i=1}^{m} (\ell+p)^{(2g_i)} S_{g_i}(\alpha)$$

where $\mathbf{r}-2\mathbf{g} = (r-2g_1,\ldots,r-2g_m)$. Summands corresponding to compositions $(g_1,\ldots,g_m)$ with same underlying partitions ν are equal, and the factors of the product only contribute for $g_i > 0$. Hence:

$$c^{(n_0+p)}_{(1^p\alpha)^m} = \frac{n^{m-1}}{2^{2g}} \frac{(p+\ell-1)!^m}{(p!\ \mathrm{Aut}(\alpha))^m} \sum_{\substack{(g_0,\nu)\vdash g \\ \nu=1^{n_1}\dots g^{n_g}}} \binom{m}{\ell(\nu)} \binom{\ell(\nu)}{n_1,\dots,n_g} P_{g_0}(\mathbf{r}-2\nu) \prod_{i=1}^{\ell(\nu)} (\ell+p)^{(2\nu_i)} S_{\nu_i}(\alpha)$$

where $\mathbf{r}-2\nu$ is the m-uple $(r-2\nu_1,\dots,r-2\nu_{\ell(\nu)},r,\dots,r)$.

Let us first consider $P_{g_0}(\mathbf{r}-2\nu)$. For m going to infinity, the dominating term is the term of degree $3g_0$ of $\mathcal{D}(e_{3g_0})/g_0!$:

$$g_0!\ P_{g_0}(\mathbf{r}-2\nu) \sim r^{3g_0}\binom{m-\ell(\nu)}{3g_0}\binom{3g_0}{3,\dots,3} \sim \frac{(mr)^{3g_0}}{3!^{g_0}}.$$

Indeed it is the contribution of largest degree in m among a finite number of terms.

In the summation at fixed g_0, the product contributes with a polynomial in p whose degree is $2(g-g_0)$ *i.e.* does not depend on ν. Hence the largest contribution is determined by the degree of $\binom{m}{\ell(\nu)}$ and given by $\nu = 1^{g-g_0}$. Therefore, since $n \sim p \sim mr$, we obtain the following equivalent:

$$c^{(n_0+p)}_{(1^p\alpha)^m} \underset{m,p\to\infty}{\sim} \frac{p^{m\ell-1}}{2^{2g}} \frac{(\ell-1)!^m}{\mathrm{Aut}(\alpha)^m} \sum_{g_0+g_1=g} \frac{m^{g_1}}{g_1!} \frac{(mr)^{3g_0}}{6^{g_0} g_0!}\, p^{2g_1} S_1(\alpha)^{g_1}$$

$$\underset{m,p\to\infty}{\sim} (mr)^{m\ell-1+3g} \frac{(\ell-1)!^m}{\mathrm{Aut}(\alpha)^m} \sum_{g_0+g_1=g} \frac{S_1(\alpha)^{g_1}}{r^{g_1}\, 4^g\, 6^{g_0}\, g_0!\, g_1!}.$$

This yields Corollary 2. Observe that $S_1(\alpha) = \frac{1}{6}\sum_i(\alpha_i-1)(\alpha_i-2)$, so the constant $c(g,a)$ can also be expressed as:

$$\frac{1}{24^g} \sum_{g_0+g_1=g} \frac{1}{r^{g_1}\, g_0!\, g_1!} \sum_{i=1}^{\ell} (\alpha_i-1)(\alpha_i-2).$$

The special cases are obtained as follows: for involutions with k cycles, we have $\alpha = 2^k$, $r = \ell = k$, $p = k(m-2)+1-2g$, and $S_1(\alpha) = 0$; thus we obtain:

$$c^{(2k+p)}_{(1^p2^k)^m} \underset{m\to\infty}{\sim} \frac{(km)^{km-1+3g}}{k^m 24^g\, g!}.$$

For k-cycles, the result is straightforward. For instance, in the particular case of 3-cycles, we have $\alpha = 3$, $r = 2$, and $\ell = 1$, hence we obtain more precisely:

$$c^{(3+p)}_{(1^p3)^m} \underset{m\to\infty}{\sim} \frac{(2m)^{m-1+3g}}{24^g} \sum_{g_0=0}^{g} \frac{1}{g_0!(g-g_0)!} \underset{m\to\infty}{\sim} \frac{(2m)^{m-1+3g}}{12^g\, g!}.$$

6.2 Large factors

A second kind of limit was suggested to us by Dimitri Zvonkine: we fix m and let non trivial parts go to infinity homothetically. In order to do that, let us choose our m original partitions $\alpha_i = 1^{a_{i,1}}\ 2^{a_{i,2}} \dots k^{a_{i,k}}$ with respective weights n_i. We shall denote by $x \cdot \alpha_i$ the partition $x^{a_{i,1}} \dots (kx)^{a_{i,k}}$ and consider the number of factorizations of genus g in permutations of respective reduced cycle type $x \cdot \alpha_1, \dots, x \cdot \alpha_m$, when x goes to infinity. According to the genus relation, these permutations must have additional fix points so that the total number n of elements on which they act satisfies the equality:

$$n \ = \ \sum_{i=1}^{m} r(x \cdot \alpha_i) + 1 - 2g,$$

where $r(x \cdot \alpha_i) = xn_i - \ell_i$. Hence, if we denote $n_0 = n_1 + \cdots + n_m$ and $\ell_0 = \ell_1 + \cdots + \ell_m$, n must satisfy:

$$n \ = \ xn_0 - \ell_0 + 1 - 2g.$$

Let us first consider the behaviour of polynomials $S_g(x \cdot \beta)$ for any given partition β and integer g: fix points of $x \cdot \beta$ do not interfere, so that:

$$S_g(x \cdot \beta) \ = \sum_{p_1+\cdots+p_\ell = g} \ \prod_{j=1}^{\ell} \frac{1}{x\beta_j} \binom{x\beta_j}{2p_j+1} \underset{x\to\infty}{\sim} x^{2g}\ s_g(\beta),$$

where

$$s_g(\beta) \ = \ \sum_{p_1+\cdots+p_\ell = g} \ \prod_{j=1}^{\ell} \frac{\beta_j^{2p_j}}{(2p_j+1)!}.$$

Now turn to $P_{g_0}(\mathbf{r} - 2\mathbf{g})$: since, for any $i \in [\![1, m]\!]$, $r(x \cdot \alpha_i) \sim xn_i$, the contribution is again dominated by the largest degree term $\mathcal{D}(e_{3^{g_0}})/g_0!$. Thus:

$$\begin{aligned} g_0!\ P_{g_0}(\mathbf{r} - 2\mathbf{g}) \ &\underset{x\to\infty}{\sim} \ (\mathcal{D}(e_{3^{g_0}}))\,(xn_1, \dots, xn_m) \\ &\underset{x\to\infty}{\sim} \ e_{3^{g_0}}(xn_1, \dots, xn_m) \\ &\underset{x\to\infty}{\sim} \ x^{3g_0}\, e_{3^{g_0}}(n_1, \dots, n_m). \end{aligned}$$

Finally, for any $i \in [\![1, m]\!]$, denote by p_i the number of fix points needed for α_i. Then $p_i = n - xn_i \sim x(n_0 - n_i)$, and

$$\frac{(\ell(x \cdot \alpha_i) + 2g_i - 1)!}{\mathrm{Aut}(x \cdot \alpha_i)} \ = \ \frac{(p_i + \ell_i + 2g_i - 1)!}{p_i!\ \mathrm{Aut}(\alpha_i)} \underset{x\to\infty}{\sim} \frac{p_i^{\ell_i + 2g_i - 1}}{\mathrm{Aut}(\alpha_i)}.$$

Hence:

$$\frac{2^{2g}}{n^{m-1}} \cdot c^{(n)}_{x\cdot\alpha_1,\dots,x\cdot\alpha_m}$$
$$\underset{x\to\infty}{\sim} \sum_{g_0+\cdots+g_m=g} \frac{x^{3g_0}\ e_{3^{g_0}}(n_1,\dots,n_m)}{g_0!} \prod_{i=1}^{m} \frac{[x(n_0-n_i)]^{\ell_i+2g_i-1}}{\mathrm{Aut}(\alpha_i)}\ x^{2g_i}\ s_{g_i}(\alpha_i).$$

This yields Corollary 3, since the dominating term is obtained for $g_0=0$:

$$c^{(n)}_{x\cdot\alpha_1,\dots,x\cdot\alpha_m} \underset{x\to\infty}{\sim} \left(\frac{n_0^{m-1}}{2^{2g}} \sum_{g_1+\cdots+g_m=g} \prod_{i=1}^{m} \frac{(n_0-n_i)^{\ell_i+2g_i-1}\ s_{g_i}(\alpha_i)}{\mathrm{Aut}(\alpha_i)}\right) \cdot\ x^{\ell_0+4g-1}\ .$$

A First values

First values of the polynomials S_g (*i.e.* for small g):

$$\begin{aligned}
S_0(x_1,\dots,x_\ell) &= 1,\\
S_1(x_1,\dots,x_\ell) &= \frac{1}{3!}\sum_{i=1}^{\ell}(x_i-1)_2,\\
S_2(x_1,\dots,x_\ell) &= \frac{1}{5!}\sum_{i=1}^{\ell}(x_i-1)_4\ +\ \frac{1}{(3!)^2}\sum_{1\leqslant i<j\leqslant \ell}(x_i-1)_2(x_j-1)_2.
\end{aligned}$$

First values of P_g:

$$\begin{aligned}
P_0 &= 1,\\
P_1 &= \mathcal{D}(e_3) \ = \sum_{i_1<i_2<i_3} x_{i_1}x_{i_2}x_{i_3},\\
P_2 &= \mathcal{D}(e_5)\ +\ \frac{1}{2}\mathcal{D}(e_{3^2}),\\
&= \sum_{i_1<\cdots<i_5} x_{i_1}\cdots x_{i_5} + \frac{1}{2}\mathcal{D}\left(\left(\sum_{i_1<i_2<i_3} x_{i_1}x_{i_2}x_{i_3}\right)^2\right),\\
&= \sum_{i_1<\cdots<i_5} x_{i_1}\cdots x_{i_5} + 10\sum_{j_1<\cdots<j_6} x_{j_1}\cdots x_{j_6} + 3\sum_{i,\,j_1<\cdots<j_4}(x_i)_2 x_{j_1}\cdots x_{j_4}\\
&\quad + \sum_{i_1<i_2,\,j_1<j_2}(x_{i_1})_2(x_{i_2})_2 x_{j_1}x_{j_2} + \frac{1}{2}\sum_{i_1<i_2<i_3}(x_{i_1})_2(x_{i_2})_2(x_{i_3})_2.
\end{aligned}$$

The *next* case after Goupil and Schaeffer's formula, *i.e.* $m=3$, reads:

$$c^{(n)}_{\alpha_1,\alpha_2,\alpha_3} \ = \ \frac{n^2}{2^{2g}}\prod_{i=1}^{3}\frac{(\ell_i-1)!}{\mathrm{Aut}(\alpha_i)} \ \cdot \sum_{g_0+\cdots+g_3=g}\frac{1}{g_0!}\prod_{i=1}^{3}(r_i-2g_i)_{g_0}\ell_i^{(2g_i)}S_{g_i}(\alpha_i).$$

For $g = 1$, this reduces to:

$$\frac{n^2}{4}\prod_{i=1}^{3}\frac{(\ell_i-1)!}{\mathrm{Aut}(\alpha_i)} \cdot \left(r_1r_2r_3 + \sum_{i=1}^{3}\ell_i(\ell_i+1)S_1(\alpha_i)\right)$$

with $\ell_1 = r_2 + r_3 - 1$, and $r_i = r(\overline{\alpha_i})$ so that the correction is indeed seen to be a polynomial of degree 4 in the parts of $\overline{\alpha_i}$.

For $g = 2$, we obtain, with $\ell_1 = r_2 + r_3 - 3$,

$$\frac{n^2}{16}\prod_{i=1}^{3}\frac{(\ell_i-1)!}{\mathrm{Aut}(\alpha_i)} \cdot \Bigg(\frac{1}{2}(r_1)_2(r_2)_2(r_3)_2 + r_1r_2r_3\sum_{i=1}^{3}\ell_i^{(2)}S_1(\alpha_i)\frac{r_i-2}{r_i}$$
$$+\sum_{1\leqslant i<j\leqslant 3}\ell_i^{(2)}\ell_j^{(2)}S_1(\alpha_i)S_1(\alpha_j) + \sum_{i=1}^{3}\ell_i^{(4)}S_2(\alpha_i)\Bigg).$$

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