THE ASYMPTOTIC NUMBER OF PRIME ALTERNATING LINKS

SÉBASTIEN KUNZ-JACQUES AND GILLES SCHAEFFER

ABSTRACT. The first precise asymptotic result in enumerative knot theory is the determination by Sundberg and Thistlethwaite (*Pac. J. Math.*, 1998) of the growth rate of the number A_n of prime alternating links with n crossings. They found λ and positive constants c_1, c_2 such that

$$c_1\, n^{-7/2}\lambda^n \leq A_n \leq c_2\, n^{-5/2}\lambda^n.$$

In this extended abstract, we prove that the asymptotic behavior of A_n is in fact

$$A_n \underset{n\to\infty}{\sim} c_3\, n^{-7/2}\lambda^n,$$

where c_3 is a constant with an explicit expression.

RÉSUMÉ. Le premier résultat d'énumération asymptotique exacte en théorie des nœuds est dû à Sundberg et Thistlethwaite (*Pac. J. Math.*, 1998), qui ont donné un encadrement du nombre A_n d'entrelacs premiers alternants à n croisements en déterminant λ et des constantes positives c_1 et c_2 telles que

$$c_1\, n^{-7/2}\lambda^n \leq A_n \leq c_2\, n^{-5/2}\lambda^n.$$

Nous montrons ici que le comportement asymptotique de A_n est en fait

$$A_n \underset{n\to\infty}{\sim} c_3\, n^{-7/2}\lambda^n,$$

où c_3 est une constante connue explicitement.

INTRODUCTION

Enumerative questions have been given few answers in the field of knot theory. Until Ernst and Sumners [ES87] in 1987, it was not even known that the number of knots grows exponentially fast. However, such problems seem very appealing:

> When introduced to the subject of knot theory, it is natural to ask how the number of knots and links grows in relation to crossing number.
>
> C. Sundberg & M. Thistlethwaite, *Pac. J. Math. 1998.*

This is the opening sentence of an article [ST98] in which these authors obtain the first exact growth rate for the number of some natural objects of knot theory (to our knowledge), namely *prime alternating links* (a link, unlike a knot, may be composed of several intertwined components; it is alternating if it admits an alternating planar representation — see Section 1 for definitions). The interest in alternating links arises from the Tait conjecture, proven by Menasco and Thistlethwaite [MT93]: two planar diagrams representing the same alternating link are related by a sequence of transformations of a single simple type, *the flype*. This is an important simplification with respect to the general case where the more complex Reidemeister moves are needed.

A natural approach to the enumeration of knots and links proceeds via their planar diagrams: the set of planar diagrams is indeed a classical class of *planar maps*, whose enumeration has been known since Tutte in the 60's [Tut63]. The stumbling block of this method lies however in the difficulty of taking into account equivalence classes of diagrams representing the same knot. The *tour de force* of Sundberg and Thistlethwaite has been to carry this program to a complete success for prime alternating *tangles*, that is, links with one rigid, oriented crossing. Tangles play exactly with respect to links, the role of *rooted* maps with respect to maps: the root gives a canonical way to start a recursive decomposition and allows enumeration. Let us state the fundamental result of [ST98].

Theorem 1 (Sundberg and Thistlethwaite). *Let a_n be the number of prime alternating tangle types with n crossings*[1]. *Then*[2]

$$a_n \sim \frac{3}{4}\sqrt{\frac{c_0}{\pi}}\, n^{-5/2}\lambda^n, \quad \textit{where} \quad \lambda = \frac{101+\sqrt{21001}}{40} \approx 6.15,$$

and

$$c_0 = \frac{5^7(21001 - 61\sqrt{21001})^3}{2(153+27\sqrt{21001})^5} \approx 0.063.$$

This result follows from an expression of the generating function $\sum_n a_n z^n$, that we shall discuss in great details in Section 4. Having completely addressed the case of tangles, Sundberg and Thistlethwaite studied the maximum and minimum numbers of tangles corresponding to a link. As a result they could obtain the growth rate of the number of prime alternating links:

Theorem 2 (Sundberg and Thistlethwaite). *Let A_n be the number of prime alternating link types with n crossings. Then, for $n \geq 3$,*

$$\frac{a_{n-1}}{8(2n-3)} \leq A_n \leq \frac{a_{n-1}}{2}, \quad \textit{and} \quad \lim_{n\to\infty}(A_n)^{1/n} = \lambda.$$

In the present article we provide the exact asymptotic behaviour of A_n:

Theorem 3. *Let A_n be the number of prime alternating link types with n crossings. Then*

$$A_n \sim \frac{a_{n-1}}{4c\,n} \sim \frac{3}{16c}\sqrt{\frac{c_0}{\pi}}\, n^{-7/2}\lambda^n,$$

where

$$c = \frac{1}{2}\left(\frac{371}{\sqrt{21001}} - 1\right) \approx 0.780.$$

Large planar maps have almost surely no symmetries [RW95], so that a large map with n edges has almost surely $4n$ distinct rootings and the asymptotic number of maps is just the number of rooted maps, divided by $4n$. As tangles are to links what rooted maps are to maps, one could expect an easy proof using methods of [RW95].

However the situation for links is much more intricate than expected because of the occurrence of local symmetries. This is illustrated by the fact that the ratio nA_n/a_n tends to a non trivial algebraic constant; unlike maps, links without global symmetries do not have a known number of rootings. Let us now outline our proof.

After definitions (Section 1), the decomposition of alternating links used in [ST98] is rephrased in terms of a tree-like structure (Section 2). Symmetries of links are then discussed and global symmetries are proven exponentially negligible. An almost sure linear lower bound for the number of distinct rootings of a link is also obtained (Section 3). Generating functions are then introduced (Section 4) and used to compute the average number of distinct rootings of a link and to prove that a concentration property holds (Section 5). This allows to conclude a first proof of Theorem 3. An alternative proof is then also discussed, which implies that a random link almost surely contains a large polyhedral component. For a full version of this extended abstract, see our web pages.

The determination of the generating function of the a_n was reproduced by Zinn-Justin and Zuber in [ZJZ] using the terminology of physics and extended in [ZJZ00] to the case of tangles with two colors. The extension to any number of colors might lead the enumeration of alternating *knots*, but seems out of reach at present. As for general knots and links, the classification via Reidemeister moves seems not rigid enough to allow enumeration with similar techniques.

[1] To keep consistency with [ST98], the rigid oriented crossing does not count in the number of crossings.

[2] Throughout this article $f(n) \sim g(n)$ stands for $\lim_{n\to\infty} f(n)/g(n) = 1$.

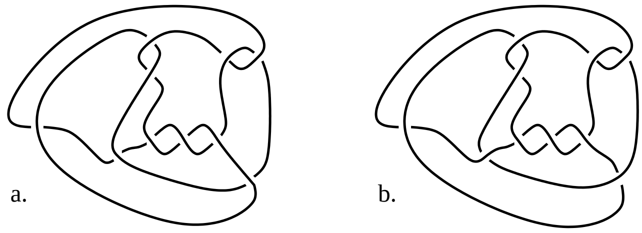

FIGURE 1. A general and an alternating planar link diagram.

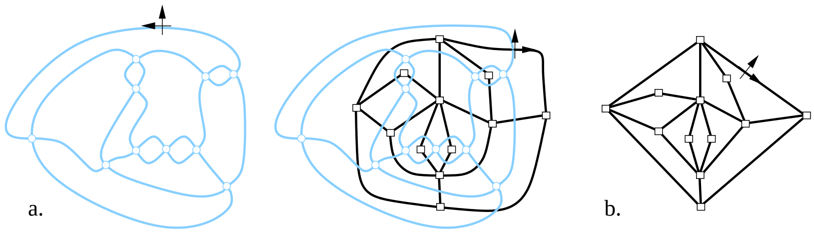

FIGURE 2. Duality: a rooted 4-regular map and the dual quadrangulation.

Acknowledgements. The authors would like to thank Bétréma for bringing the work of Sundberg and Thistlethwaite to their attention.

1. Definitions

Knots and links are topological objects and as such have classical topological definitions. Almost as classical is the possibility to represent knots and links by planar diagrams. Because our treatment is purely combinatorial, we shall give definitions in these terms and rely on existing treatment for the proof that such definitions correspond to topological ones (see in particular [ST98, Con70]).

1.1. Maps and alternating planar link diagrams.

A *planar map* is a two-cell embedding of a connected graph on the sphere, considered up to all[3] homeomorphisms of the sphere. A planar map is 4-*regular* if all its vertices have degree four (see Figure 2a; ignore the small arrows until Section 1.3).

A *planar link diagram* consists of a 4-regular planar map together with a over-undercrossing structure: each vertex is made a *crossing* by distinguishing two opposite *undercrossing* edges from the other two *overcrossing* ones (see Figure 1a). A planar link diagram is *alternating* if each edge is undercrossing at one end and overcrossing at the other (see Figure 1b).

In the rest of the paper we shall concentrate on alternating link diagrams. An immediate but useful remark is that the whole over-undercrossing structure of an alternating link diagram is completely determined by one crossing. Moreover on any 4-regular planar map, the choice of a crossing structure at one vertex can be consistently propagated to the whole map to form an alternating link diagram.

It will be useful also to use the dual representation of 4-regular maps as *quadrangulations*, obtained by putting a vertex in each face and joining them by dual edges. This involutive construction is illustrated by Figure 2.

1.2. Prime diagrams, flypes and links.

The first step in classifying link diagrams is a restriction. A planar link diagram is *prime* if it cannot be separated into two parts, each containing at least one crossing, by a loop intersecting only two edges (see Figure 3). In standard map terminology, a planar link diagram is prime if the underlying 4-regular map is edge-2-connected, or equivalently if the dual

[3]Unlike in [Tut63], homeomorphisms that change the orientation are allowed here.

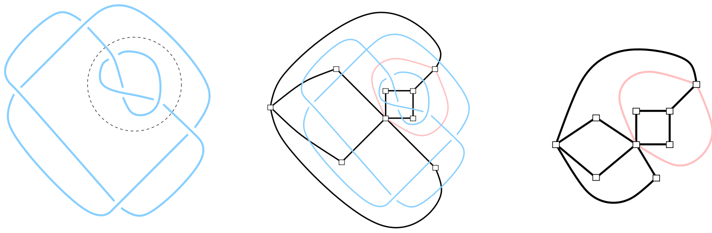

FIGURE 3. Defect of primality, or dually, forbidden multiple edges.

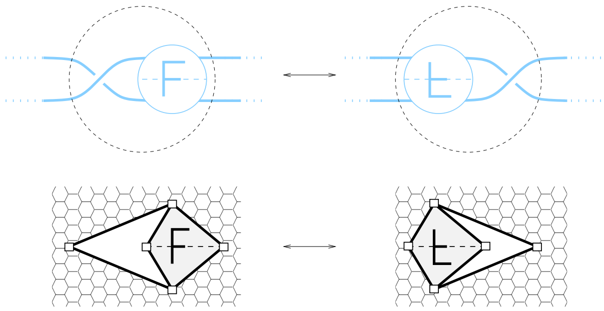

FIGURE 4. The flype equivalence, on diagrams or on quadrangulations.

quadrangulation does not contain multiple edges or loops. *From now on we only consider prime alternating link diagrams and all quadrangulations are implicitly supposed not to contain multiple edges or loops.*

The *flype* is a local transformation of planar link diagrams that takes place inside an imaginary loop intersecting four edges. It is best defined by a simple picture (see Figure 4). Equivalently, in terms of quadrangulations, the flype is a local transformation inside a 4-cycle $\mathcal{C}$ that exchanges a face sharing two edges with $\mathcal{C}$ with the complement sub-quadrangulation, the latter being vertically symmetrized.

Finally a *prime alternating link*[4] is a flype-equivalence class of prime alternating planar link diagrams[5]. It is exactly the Tait conjecture, proven by Menasco and Thistlethwaite [MT93], that these combinatorial prime alternating links are in one-to-one correspondence with topological prime alternating link types.

1.3. **Rooted maps and tangles.** A planar map is *rooted* if a triple of incident vertex, half-edge and face is chosen and called the *root*. The root is pictorially indicated by a couple of orthogonal arrows pointing at the root vertex and root face respectively (see Figure 2a) and we usually[6] draw rooted maps so that the root face is the unbounded face. Remark that the root arrows are naturally transported to the dual quadrangulation (see Figure 2b).

A *tangle diagram* is a rooted 4-regular planar map; as for links, it is endowed with an over-under-crossing structure by the convention that the root edge is undercrossing at the root vertex. The equivalent definition of [ST98] is recovered upon deleting the root vertex and attaching the legs to four named points of a large imaginary circle.

[4]or rather *link type*, but this makes no difference from our combinatorial point of view

[5]Let us recall here that our diagrams are already up to homeomorphisms of the sphere.

[6]There are in general several distinct representations of a map in the plane up to orientation preserving homeomorphisms *of the plane*, depending *e.g.* on the choice of unbounded face.

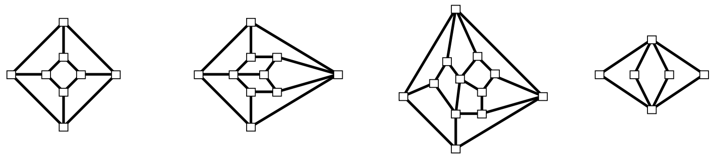

FIGURE 5. Irreducible quadrangulations and an axis with four faces.

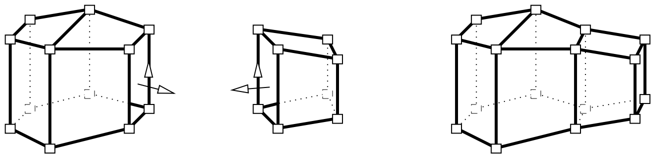

FIGURE 6. A connected sum along roots, represented on spheres.

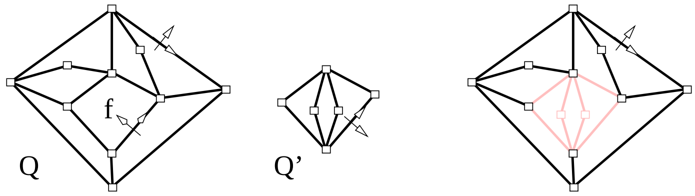

FIGURE 7. The substitution of Q' into Q at f, represented in the plane.

The action of flypes on tangle diagram is restricted: a flype can occur only inside a loop that does not enclose the root vertex. With this restriction, a *tangle* (or tangle type) is a flype-equivalence class of tangle diagrams.

2. The tree-like structure of link diagrams

In this section we rephrase the decomposition used by Sundberg and Thistlethwaite in terms of a tree-like structure. This will allows us in the next section to give a simple description of flype-equivalence. As pointed out by Mullin and Schellenberg, decompositions are easier to describe on the dual quadrangulations than on maps or diagrams.

2.1. **Irreducible quadrangulations, axes and connected sums.**

The following quadrangulations will serve us as atomic components for decompositions:

- An *irreducible* quadrangulation is a quadrangulation in which each 4-cycle is the boundary of a face (see Figure 5, and Figure 2b for a counterexample); it must moreover have at least four faces.
- For $k \geq 3$, the *axis* with k faces is the quadrangulation made of two axial vertices joined by k parallel chains of two edges (see Figure 5).

Now let us define the *connected sum* of two quadrangulations. Let Q_1 and Q_2 be two quadrangulations, f_i be a face of Q_i for $i = 1, 2$, and ϕ be one of the eight possible identifications of the two cycles of edges f_1 and f_2. Removing the face f_i of the sphere underlying Q_i yields a sphere with a hole. The connected sum of these two spheres for $i = 1, 2$ can be done in a unique way with respect to the identification ϕ (see Figure 6).

The resulting quadrangulation is the connected sum of Q_1 and Q_2 according to ϕ. This operation is associative: the sum $Q_1 \oplus_\phi Q_2 \oplus_\psi Q_3$ is well defined provided ϕ and ψ identify distinct faces of Q_2.

Given two rooted quadrangulations, their *connected sum along roots* is the connected sum according to the only identification of their root faces that superpose their root edges and root vertices.

Similarly, given a rooted quadrangulation Q, and a non-root face f of Q, there is a canonical way (*e.g.* bfs-traversal) to assign secondary a root r to Q that points toward f. The *substitution* of a rooted quadrangulation Q' into Q at f is their connected sum along r and the root of Q' (see Figure 7). Remark that the substitution of a 4-cycle (*i.e* the smallest quadrangulation) has no effect.

2.2. **Sum decomposability.** The following lemma is due to Tutte [Tut63] and was clarified by Mullin and Schellenberg [MS68]. It is implicitly used by Sundberg and Thistlethwaite with a distinct terminology.

Lemma 1. *Let Q be a rooted quadrangulation with at least three faces.*

Then there exists a maximal integer $k \geq 3$ such that Q is the substitution in a rooted quadrangulation Q_0 having k non-root faces $f_1,\ldots,f_k$ of some rooted quadrangulations $Q_1,\ldots,Q_k$, where Q_0 is required to be

- *either an irreducible quadrangulation,*
- *or an axis.*

This maximal decomposition is unique and Q_0 is called the core *of Q.*

If the core Q_0 is irreducible then Q is said to be polyhedral*. In such cases, any quadrangulation (even a 4-cycle) can occur as Q_i.*

If the core Q_0 is an axis with axial vertices x_1 and x_2, then Q is called algebraic*. In such cases, any quadrangulation (even a 4-cycle) can occur as Q_i provided its core is not an axis with the same two axial vertices x_1, x_2. (This would contradict maximality of k.)*

Proof. See Mullin and Schellenberg [MS68]. □

Exercise. The right hand side quadrangulation of Figure 7 is made of a cube (like the leftmost quadrangulation of Figure 5) in which three axes and two 4-cycles have been substituted. One of these substitutions is illustrated by Figure 7.

2.3. **The tree-like structure of a quadrangulation.** The recursive application of Lemma 1 yields a decomposition of quadrangulations into tree-like hierarchies. To make this assertion more precise, let us consider trees whose vertices are *nodes* and edges are *connections*. A node is here

- either an irreducible quadrangulation,
- or an axis,

and nodes are connected by face identifications, subject to the constraint inherited from Lemma 1: a connection that involves an irreducible quadrangulation is an identification of two faces f_1 and f_2 by any of the eight possible bijections, but a connection that join two axes must be one of the four bijections that do not identify axial vertices of the two axes. Finally a face must take part in at most one connection. From now on such tree-like structures are simply referred to as *trees*.

The *connected sum* of a tree is obtained by performing a connected sum along each connection, with respect to the identification provided. The result is a quadrangulation and associativity ensures that the order of summation does not matter.

A *free* face of a tree is a face taking part in no connection. Thus, the free faces of a tree are the faces of its connected sum. A tree is *rooted* if one of its nodes is rooted and the root face is free.

Theorem 4. *Rooted trees are in one-to-one correspondence with rooted quadrangulations. Similarly trees are in one-to-one correspondence with quadrangulations.*

The nodes of a tree are exactly the irreducible sub-quadrangulations and maximal sub-axes of the corresponding quadrangulation.

Proof. The first assertion follows by recursive application of Lemma 1. The second is then immediate from the definition of rooted trees. For the third one, the nodes of a tree are by construction sub-quadrangulations of its connected sum. Conversely an irreducible quadrangulation or a maximal axis cannot be broken in pieces by applications of Lemma 1 and thus must appear in the tree. (More formal proofs are easily adapted from [MS68] if needed.) □

3. Flypes, roots and asymmetry

3.1. **Articulated trees.** Flypes, as illustrated by Figure 4, are inherently acting on axes. A non-trivial flype exchanges a face and the complementary sub-quadrangulation inside a 4-cycle, that is to say a free face and one (or a succession of) connected faces. Moreover the symmetry occurring in the sub-quadrangulation simply results in the tree in a local symmetry of the identification used in the connection.

An *articulated tree* is a flype-equivalence class of trees. The restriction of the action of flypes on tangle diagrams translates into the restriction that a flype cannot act directly on the root face, *i.e* the root face cannot be the free face on which a flype is based. With this restriction, a rooted articulated tree is a flype-equivalence class of rooted trees.

Theorem 5. *Rooted articulated trees are in one-to-one correspondence with prime alternating tangles. Similarly, articulated trees endowed with an over-undercrossing structure are in one-to-one correspondence with prime alternating links.*

Remark that in the unrooted case, one of the two over-undercrossing structures need be specified to turn an articulated tree into a prime alternating link, because there is no more a root to do it.

3.2. **Choosing the root of a tree: root candidates.** In order to relate the number of articulated trees to the number of rooted articulated trees, we need to count the number of rootings of an articulated tree, *i.e.* the number of different rooted trees that can be obtained from an articulated tree by choosing a root.

Two kinds of identifications between *a priori* different rootings may arise: local identifications between root placements on the same node due to flypes, and global identifications due to symmetries of the tree. Let us first ignore symmetries. Then an analysis of the flype-equivalence on rooted and unrooted trees leads to a characterisation of the distinct *root candidates* of an articulated tree. (So that, unless there is a symmetry, each root candidate yields a different rooted articulated tree.) The following lemma will allow us to compute their average number in Section 4.

Lemma 2. *The number of root candidates of a tree T is eight times the number of* root face candidates*, which is defined by counting*

- *one root face candidate for each free face on an irreducible node,*
- *ℓ root face candidates for each axis with $\ell \geq 1$ connected faces and at least one free face.*

3.3. **Almost sure asymmetry of articulated trees.** A symmetry of an articulated tree T is given by a permutation of its nodes together with a map isomorphism of every node onto its image, such that the whole transformation yields a tree equivalent to T up to flypes.

An articulated tree is *symmetric* if it has a non-trivial symmetry. Such a symmetry induces a permutation of the nodes of the tree.

Theorem 6. *Symmetric articulated trees are exponentially rare amongst articulated trees.*

Proof (sketch). Building on earlier results of Bender, Gao and Richmond ([BGR92]), Richmond and Wormald proposed in [RW95] an elegant method to prove asymmetry results for a wide variety of families of maps. The proof of Theorem 6 essentially follows the same approach, although details require adaptations due to flypes. Let us recall the main steps of the method.

First, in accordance to [BGR92], given any irreducible quadrangulation I, there exists a constant $\alpha > 0$ such that the proportion of trees of size n which have less than αn leaves bearing copies of I is bounded above by some exponentially decreasing function of n. Such trees can thus be neglected.

Then a nontrivial symmetry s of a tree T must permute the copies of I; moreover if I itself has no symmetry, the action of s on these copies cannot have any fixed point but only cycles of length at least two. Thus one copy of I per cycle is enough to determine the way all copies of I are glued in T: a fraction at least $\alpha n/2$ of the copies have no choice. Among all the possible ways to glue copies of I in a tree, only an exponentially small fraction thus lead to symmetric trees. This is sufficient to conclude. □

3.4. An almost sure lower bound for the number of rootings.

Lemma 3. *The proportion of articulated trees with n free faces but less than $2n$ distinct rootings is exponentially small.*

Proof. Let T be an articulated tree with n free faces and less than $2n$ distinct rootings, and ℓ the number of root candidates of T. If $\ell > 2n$, there exists root candidates which are on different nodes and yield the same rooted tree; the tree must then be symmetric. These trees are exponentially rare according to previous results of this section. If on the contrary $\ell \leq 2n$, by definition of root candidates T has many free faces locally equivalent under the action of flypes; therefore it must have many faces on axes. On each of these axes, removing all the free faces but one (or two on leaves) yields a tree T' of size $\lceil n/2 \rceil$. The lemma is then proven by combining the (much smaller) number of such trees T' with the (not so large) number of ways to reconstruct the original tree. $\square$

4. Generating series and singularity analysis

In this section, the combinatorial decompositions are converted into functional equations, yielding asymptotic enumerating results.

4.1. Generating series in one variable and singular expansions.

Let $W(z) = \sum_{n\geq 1} a_n z^n$ be the generating series of rooted articulated trees with respect to the number of non root free faces (*i.e.* of tangles w.r.t. the number of crossings). Then Lemma 1 translates into the following equations (see also [ST98, ZJZ]), where $U(z)$ and $2D(z)$ are the generating series of polyhedral and algebraic articulated trees respectively,

$$(1) \qquad \left\{ \begin{array}{rclrcl} W(z) &=& z + U(z) + 2D(z), & G(z) &=& U(z) + D(z), \\ D(z) &=& \dfrac{z}{1-z}\dfrac{1}{1-G(z)} + \dfrac{G(z)^2}{1-G(z)} - z, & U(z) &=& P(W(z)), \end{array} \right.$$

where
$$P(y) = \sum_{n\geq 5} p_{n+1} y^n$$
with p_n the number of irreducible quadrangulations with n faces. This latter series was obtained by Tutte [Tut63] and can be rewritten

$$(2) \qquad P(y) = \frac{1}{2(y+2)^3}\left((1-4y)^{3/2} + 2y^2 - 10y - 1\right) - \frac{2}{1+y} - y + 2.$$

In particular this series has a unique dominant singularity in $\sigma = 1/4$. Sundberg and Thistlethwaite used Expression (2) and a system like (1) to prove that $W(z)$ has a unique dominant singularity $1/\lambda$ and has a singular expansion in a slit neighborhood of $1/\lambda$

$$(3) \qquad W(z) = \sigma - a(1-\lambda z) + \sqrt{c_0}(1-\lambda z)^{3/2} + \mathcal{O}\left((1-z)^2\right),$$

for some $a > 0$ and the constants λ and c_0 of Theorem 1. Using transfer theorems of [FO90], this yields the asymptotic behaviour of a_n as in Theorem 1.

4.2. Taking root candidates into account.

The generating functions can be refined to take the number of root candidates into account: let $W(z,v) = \sum_{n,k} a'_{n,k} z^n v^k$, where $a'_{n,k}$ is the number of rooted articulated trees with n non root free faces and k root face candidates (recall that the number of root candidates is eight times the number of root face candidates). Lemma 2 leads to following adaptation of System (1) in which the indeterminate v is used to mark root face candidates

$$(4) \qquad \left\{ \begin{array}{rcl} W(z,v) &=& zv + U(z,v) + 2D(z,v), \qquad G(z,v) \;=\; U(z,v) + D(z,v), \\ D(z,v) &=& \dfrac{zv}{1-z}\dfrac{1}{1-vG(z,v)} + \dfrac{G(z,v)^2}{1-G(z,v)} - zv, \\ U(z,v) &=& P(W(z,v)). \end{array} \right.$$

From this system the asymptotic behaviour of the number of root candidates follows.

Lemma 4. *The average number of root candidates in a rooted tree with n free faces is equivalent to $8cn$ as n goes to infinity, where c is the constant of Theorem 3.*

Proof. Let us use the notation $f'_v(z) = \frac{\partial}{\partial v} f(z,v)|_{v=1}$. Then

$$\begin{aligned} W'_v(z) &= z + U'_v(z) + 2D'_v(z), \qquad G'_v(z) \;=\; U'_v(z) + D'_v(z), \\ D'_v(z) &= \frac{z}{1-z}\left(\frac{1}{1-G(z)} + \frac{G'_v(z)+G(z)}{(1-G(z))^2}\right) + \frac{G'_v(z)}{(1-G(z))^2} - G'_v(z) - z, \\ U'_v(z) &= W'_v(z) P'(W(z)). \end{aligned}$$

As expected this linear system of equations in the series $\{W'_v, U'_v, D'_v, G'_v\}$ is of maximal rank and allows to express these series in terms of the original generating functions. For instance,

$$W'_v(z) = z\left[2(1-z)G(z)(G(z)-1) + (1-2z) - P'(W(z))\right]^{-1}.$$

No new singularity appears on the circle of convergence and using singular expansions of P' and G, a singular expansion of $W'_v(z)$ near $z = 1/\lambda$ is obtained:

$$W'_v(z) = a' - \tfrac{3}{2} c\sqrt{c_0}(1-\lambda z)^{1/2} + O(1-\lambda z).$$

The mean number $\mu_n = cn$ of root face candidates then follows from $[z^n]W'_v(z) = \mu_n a_n$, using transfer theorems and the asymptotic behaviour of a_n. The same asymptotic mean is obtained for polyhedral or algebraic articulated trees, using U'_v or D'_v. ☐

Lemma 5. *The variance of the number of root candidates is equivalent to sn for some $s > 0$. In particular the number of root candidates is concentrated around $8cn$.*

Proof. Similar to the proof of Lemma 4, with more computer algebra. ☐

5. The asymptotic number of links

We are now ready to obtain an equivalent of T_n, the number of (unrooted) articulated trees of size n. Let $T_n = T_n^+ + T_n^0 + T_n^-$ (respectively $a_n = a_n^+ + a_n^0 + a_n^-$) where

- T_n^0 (resp. a_n^0) is the number of (resp. rooted) articulated trees with n non root free faces and between $8cn - n^{3/4}$ and $8cn + n^{3/4}$ root candidates,
- $T_n^{\pm}$ (resp. $a_n^{\pm}$) is the number of (resp. rooted) articulated trees with n non root free faces and more/less than $8cn \pm n^{3/4}$ root candidates,

Chebyshev's inequality applied to the number of root candidates reads

$$\Pr(|r_n - 8cn| > x\sqrt{sn}) = \mathcal{O}\left(1/x^2\right),$$

where the mean $8cn$ and variance sn are given by Lemmas 4–5. With $x = n^{1/4}$ this immediately implies $a_n^0 \sim a_n$, $a_n^+ = \mathcal{O}(a_n/n^{1/2})$, and $a_n^- = \mathcal{O}(a_n/n^{1/2})$.

But according to results of Section 3, except for an exponentially negligible fraction of trees, the number of root candidates is the number of distinct rootings so that $T_n^+ \cdot 8cn = \mathcal{O}(a_n^+)$ and $T_n^+ = \mathcal{O}(a_n/n^{3/2})$. Similarly, using the lower bound of Lemma 3, $T_n^- 2n = \mathcal{O}(a_n^-)$ and $T_n^- = \mathcal{O}(a_n/n^{3/2})$.

On the contrary, $T_n^0 \cdot 8cn \sim a_{n-1}$, so that

$$T_n \sim a_{n-1}/8cn.$$

To conclude the proof of Theorem 3, it remains to deal with the missing over-undercrossing structure. As already mentioned, each articulated tree T may provide two prime alternating links depending on the over-undercrossing structure that is chosen. If these two links are equal, the tree has a non-trivial symmetry; thus with probability 1, they are different. Finally, the number A_n of prime alternating links with n crossing satisfies

$$A_n \sim 2T_n \sim a_{n-1}/4cn.$$

☐

6. The largest irreducible node and an alternative proof

In the complete version of this article, we give a second proof of Theorem 3, that provides a convincing cross-examination of the value of the constant c that plays a central rôle in the main formula. This alternative proof is based on the following result, of independent interest.

Theorem 7. *The proportion of articulated trees of size n that contain an irreducible node of size larger than $\lfloor n/2 \rfloor$ tends to 1 as n tends to infinity.*

Equivalently, in Sundberg and Thistlethwaite's terminology, a random prime alternating link almost surely contains a very large polyhedral template.

Similar results hold for random maps, see *e.g.* [BFSS00]. The proof goes by refining the generating function of rooted articulated trees to take into account the size of the root node. This allows the computation of another constant p such that the limit proportion in Theorem 7 is $pc = 1$.

The alternative proof of Theorem 3 can then be described as follows (up to rather more technical details). The large irreducible node of Theorem 7 is almost surely asymmetric by [RW95]. This allows almost surely a bijective decomposition into two rooted articulated trees, one of which has a large root node. The number of such tuples can then be computed to recover Theorem 3.

The strength of this proof as a cross-examination of c lies in the remark that p and c are computed independently and yet yield the result $pc = 1$, which has a natural interpretation as the probability of an almost sure event.

References

[BFSS00] C. Banderier, P. Flajolet, G. Schaeffer, and M. Soria. Planar maps and Airy phenomena. In U. Montanari, J. Rolim, and E. Welzl, editors, *Automata, Languages, and Programming*, number 1853 in LNCS, pages 388–402, 2000. Proceedings of the 27th ICALP Conference, Geneva, July 2000.

[BGR92] E. A. Bender, Z. Gao, and L. B. Richmond. Submaps of maps. I. General 0-1 laws. *J. Combin. Theory Ser. B*, 55(1):104–117, 1992.

[Con70] J. H. Conway. An enumeration of knots and links, and some of their algebraic properties. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 329–358. Pergamon, Oxford, 1970.

[ES87] C. Ernst and D. W. Sumners. The growth of the number of prime knots. *Math. Proc. Cambridge Philos. Soc.*, 102(2):303–315, 1987.

[FO90] P. Flajolet and A. Odlyzko. Singularity analysis of generating functions. *SIAM J. Discrete Math.*, 3(2):216–240, 1990.

[MS68] R. C. Mullin and P. J. Schellenberg. The enumeration of c-nets via quadrangulations. *J. Combinatorial Theory*, 4:259–276, 1968.

[MT93] W. Menasco and M. Thistlethwaite. The classification of alternating links. *Ann. of Math. (2)*, 138(1):113–171, 1993.

[RW95] L. B. Richmond and N. C. Wormald. Almost all maps are asymmetric. *J. Combin. Theory Ser. B*, 63(1):1–7, 1995.

[ST98] C. Sundberg and M. Thistlethwaite. The rate of growth of the number of prime alternating links and tangles. *Pacific J. Math.*, 182(2):329–358, 1998.

[Tut63] W. T. Tutte. A census of planar maps. *Canad. J. Math.*, 15:249–271, 1963.

[ZJZ] P. Zinn-Justin and J.-B. Zuber. Matrix Integrals and the Counting of Tangles and Links. `arXiv:math-ph/9904019`, to appear in *Discrete Math.*, Proceedings of the 11th international conference on Formal Power Series and Algebraic Combinatorics (FPSAC'99).

[ZJZ00] P. Zinn-Justin and J.-B. Zuber. On the counting of colored tangles. *Journal of Knot Theory and its Ramifications*, 9:1127–1141, 2000. Preprint available at `arXiv:math-ph/0002020`.

Sébastien Kunz-Jacques, LIX, École Polytechnique, F-91128 Palaiseau Cedex.
E-mail address: `kunzjacq@lix.polytechnique.fr`

Gilles Schaeffer, LORIA–CNRS, Campus Sciences, B.P. 101, F-54602 Villers-lès-Nancy.
E-mail address: `Gilles.Schaeffer@loria.fr`