A bijective census of nonseparable planar maps

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Abstract

Bijections are obtained between nonseparable planar maps and two different kinds of trees : description trees and skew ternary trees. A combinatorial relation between the latters and ternary trees allows bijective enumeration and random generation of nonseparable planar maps. The involved bijections take account of the usual combinatorial parameters and give a bijective proof of formulae established by Brown and Tutte. These results, combined with the recent bijections of [DG] and [GW96], give a purely combinatorial enumeration of two-stack-sortable permutations.

1 Introduction

Nonseparable planar maps have been enumerated by W.G Brown and W.T. Tutte [Tut63, Bro63, BT64], using a recursive decomposition and the so-called quadratic method for solving the occurring equations. They have obtained formulae for the number of these maps by number of edges or by number of vertices and faces. The quadratic method is a powerful method for enumeration of maps (see [GJ83] for a survey) but gives very few insights into involved combinatorial structures: in particular, very simple parameterizations are obtained which have no known combinatorial interpretation. In the case of nonseparable planar maps, W.G. Brown and W.T. Tutte derived the following formulae:

$$|\mathcal{NS}_n| = \frac{2}{n+1} \cdot \frac{1}{3n+1}\binom{3n+1}{n}, \tag{1}$$

and

$$|\mathcal{NS}_{i,j}| = \frac{(2i+j+1)!(2j+i+1)!}{(i+1)!(j+1)!(2i+1)!(2j+1)!}, \tag{2}$$

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enumerating nonseparable planar maps with $n+1$ edges or with $i+2$ vertices and $j+2$ faces.

The interest in enumerating nonseparable planar maps has been renewed when J. West [Wes90, Wes91] observed that two-stack-sortable permutations are enumerated by the same sequence of numbers as nonseparable planar maps. One-stack-sortable permutations were first introduced and enumerated by D. Knuth [Knu73]. J. West's conjecture was first proven by D. Zeilberger [Zei92] and I.M. Gessel [Ges]. A combinatorial proof was then given by S. Dulucq, S. Gire and O. Guibert [DG] by constructing eight bijections between different classes of permutations with forbidden sequences, eventually going from two-stack-sortable permutations to permutations encoding nonseparable planar maps. A simpler bijection was given more recently by I.P. Goulden and J. West in [GW96]. Although these bijections explain the combinatorial connection between two-stack-sortable permutations and nonseparable planar maps, they do not give a full combinatorial proof of the enumeration formulae: it still relies on W.G. Brown and W.T. Tutte's original derivation by the quadratic method. Our aim in the present paper is to obtain formulae (1) and (2) in a purely combinatorial way. We use the notion of skew ternary tree introduced by A. Del Lungo, F. Del Ristoro and J.G. Penaud, who conjectured that these trees are in bijection with nonseparable planar maps. In [DLDRP97], these authors gave independently a proof of formula (1), which is different from ours and do not allow deterministic random generation. Our approach can be adapted to several other families of trees (see [CJS97]).

We now give some combinatorial definitions in order to be able to state the main results of the paper.

1.1 Three combinatorial objects

A *plane rooted tree* consists of a mapping τ from a set V of vertices into the set of (possibly empty) sequences (or ordered lists) of distinct elements of V and of an element r of V called the *root*, satisfying the following condition:

- For any v in V, there exists a unique sequence $(v_1, \ldots, v_p)$ $(p \geqslant 1)$ such that $v_1 = r$, $v_p = v$ and $v_{i+1} \in \tau(v_i)$ for $i = 1, \ldots, p-1$.

For any vertex v the elements of $\tau(v)$ are called the *sons* of v and v is their *father*. If v has no son it is a *leaf*, otherwise it is an *internal node*. An internal node with p sons is also said to have p *subtrees* which are rooted on the sons. In the sequel, trees are drawn such that the root is at the top of the picture.

1.1.1 Description trees

A *description tree* is a planar rooted tree whose vertices are labelled according to the following rules:

- The root is not labelled.
- Leaves are labelled 1, except when the tree is reduced to one leaf (which, being the root, is not labelled).

- Each non-root internal node is labelled with an integer between 1 and the sum of labels of its sons.

The word *description* refers to a natural encoding of a decomposition of nonseparable maps by such trees (cf. section 2.1). The set of all description trees with n non-root vertices is denoted by $\mathcal{DT}_n$ (for $n \geqslant 0$) and is partitioned into the subsets $\mathcal{DT}_{i,j}$ of trees with $i+1$ leaves and $j+1$ internal nodes (for $i+j+1=n$, $i \geqslant 0$, $j \geqslant 0$ and $n \geqslant 1$). We also consider the set $\mathcal{DT}^*$ of trees obtained from description trees by labelling also the root according to the same rule as for internal nodes, the root-leaf tree admitting a unique labelling by 1. For a tree B in $\mathcal{DT}_{i,j}$ let $s(B)$ be the sum of labels of the root's sons. The tree B is then the underlying description tree of $s(B)$ distinct trees of $\mathcal{DT}^*_{i,j}$.

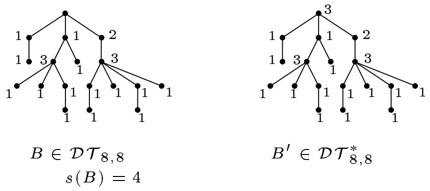


Figure 1: Description trees

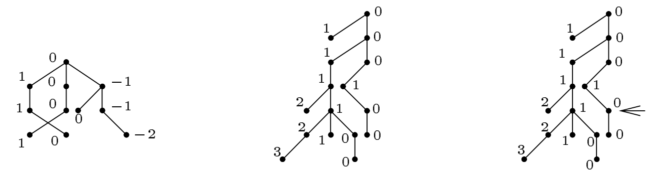
1.1.2 Skew ternary trees

A *ternary tree* is a rooted tree whose nodes have at most one son of each of the following type: *left*, *middle* and *right* (in the notations used for plane trees, $\tau(v)$ is no more a sequence but a triple of elements of $\{\emptyset\} \cup V$ satisfying the same condition). Vertices of ternary trees are naturally labelled by the shift index: the root is labelled 0 and the non-root vertices take the label of their father, to which is added respectively $+1$, $+0$ or -1 when they are left, middle or right sons. A ternary tree whose labels are all nonnegative is said to be *skew*.

The set of all ternary trees with n vertices is denoted by $\mathcal{T}_n$ (for $n \geqslant 0$) and is partitioned into the subsets $\mathcal{T}_{i,j}$ of ternary trees with $i+1$ even labels and j odd ones (for $i+j+1=n$, $i \geqslant 0$, $j \geqslant 0$ and $n \geqslant 1$). The corresponding sets of skew ternary trees are denoted by $\mathcal{ST}_n$ and $\mathcal{ST}_{i,j}$. For a tree A in $\mathcal{ST}_{i,j}$ let $z(A)$ denote the number of vertices with label 0 in A. There are then $z(A)$ ways to mark a vertex with label 0 of the tree A. More generally, we consider skew ternary trees with a marked vertex with label k, and we represent these with an arrow directed towards the marked vertex. The set of marked skew ternary trees is denoted by $\mathcal{ST}^*$ and is partitioned into $\mathcal{ST}^{*k}$ for $k \geqslant 0$, the subsets of those trees whose marked vertex has label k.

As for the set $\mathcal{T}$, it is partitioned into the subsets $\mathcal{T}^{\perp k}$ of ternary trees with minimum label $-k$ for $k \geqslant 0$.

One can notice that $\mathcal{ST}_{i,j} = \mathcal{T}^{\perp 0}_{i,j}$.



$T \in \mathcal{T}^{\perp 2}_{5,5}$ $\qquad$ $A \in \mathcal{ST}_{8,7}$ $\qquad$ $A \in \mathcal{ST}^{*0}_{8,7}$

Figure 2: Ternary and skew ternary trees

1.1.3 Nonseparable planar maps

A *planar map* is an embedding of a connected graph in the sphere, partitioning its surface into simply connected regions called faces. The maps we consider are rooted by orienting a *root edge* away from a *root vertex*. The face on the right of the root edge is called the *exterior face* and the map is represented in the plane by a stereographic projection in which the exterior face is mapped onto the infinite face. Two rooted planar maps are considered to be the same if there exists an homeomorphism of the sphere mapping one onto the other. A useful equivalent combinatorial representation can be given in terms of permutations (see for instance [Cor75]). Let $\mathcal{M}_n$ denote the set of planar maps with $n+1$ edges for $n \geqslant 0$. A planar map M with only one face can be viewed as a plane rooted tree whose root is the root vertex of M and whose first son of the root is the end of the root edge of M. The *degree* of a face or a vertex is its number of incident edges. A *corner* is a neighborhood of a vertex which is incident to two half-edges and included in a face. A vertex of degree d defines d corners.

A vertex v is called a *cut-vertex* if there is a partition of the edges of the map into two non-void classes such that v is the only vertex to be incident with edges of the two classes. A *nonseparable planar map* is a planar map without cut-vertex nor loop-edge. Let $\mathcal{NS}_n$ denote the set of nonseparable planar maps with $n+1$ edges and $\mathcal{NS}_{i,j}$ the subset of maps having $i+2$ vertices and $j+2$ faces. Euler's formula for maps gives the relation $i+j+1=n$ for i and j nonnegative. For a map M in $\mathcal{N}S_{i,j}$ let $e(M)$, $v(M)+1$ and $f(M)+1$ denote the number of edges, vertices and faces and $r(M)$ the number of non-root vertices which are incident with the exterior face.

A *bi-rooted* map is a (rooted) nonseparable planar map with a marked non-root vertex on the exterior face. For a (rooted) map M the number of bi-rooted maps having M as underlying map is $r(M)$. The set of bi-rooted nonseparable planar maps is denoted by $\mathcal{NS}^*$. For a bi-rooted map M' whose marked vertex is the pth after the root vertex on the exterior face, let $d(M') = p$.

1.2 A catalog of small objects

Table (ii) in the appendix contains all the previously defined objects for small values of n. It suggests the existence of bijections between them.

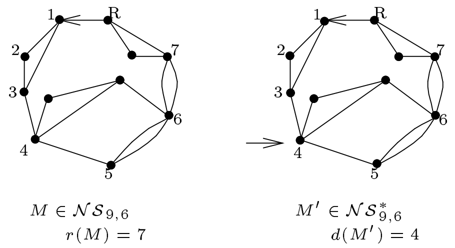


Figure 3: Nonseparable planar maps

1.3 Outline and main results

In this paper we adopt a combinatorial approach to answer the problem of enumerating nonseparable planar maps and hence two-stack-sortable permutations. In section 2, description trees are shown to encode naturally the decomposition of nonseparable planar maps, thus giving a recursive bijection between these objects. The same approach is then successfully applied to skew ternary trees which admit a similar but more intricate decomposition, also encoded by description trees:

Theorem 1 *There exists a bijection between nonseparable planar maps with $n+1$ edges, $i+2$ vertices, $j+2$ faces and exterior face degree $k+1$ and description trees with $n+1$ vertices, $i+1$ of which are leaves, $j+1$ internal nodes and with labels of the root's sons summing to k. The same description trees are in bijection with skew ternary trees with n vertices, $i+1$ of which have an even label, j an odd one and containing k labels 0. In other terms:*

$$\begin{array}{rcccc} & \mathcal{NS}_{i,j} & \simeq & \mathcal{DT}_{i,j} & \simeq & \mathcal{ST}_{i,j} \\ \textit{and} & \mathcal{NS}^*_{i,j} & \simeq & \mathcal{DT}^*_{i,j} & \simeq & \mathcal{ST}^{*0}_{i,j}. \end{array}$$

These bijections carry all the usual combinatorial parameters as summarized in table i. A straightforward factorization of the decomposition trees used by I.P. Goulden and J. West in [GW96] allows us to reduce them to our description trees. Their bijection can therefore be composed with ours to get a direct fully combinatorial enumeration of two-stack-sortable permutations (denoted in the table by $\mathcal{TSS}$).

In section 3, we establish the following theorem which is the key of our combinatorial proof.

Theorem 2 *There exists a bijection between skew ternary trees with n vertices having a marked vertex with label k and ternary trees with n vertices and minimum label $-k$ or $-(k+1)$. Moreover, the repartition of the parity of labels is conserved or switched according to the parity of k. More precisely:*

$$\begin{array}{rccc} & \mathcal{ST}^{*2k}_{i,j} & \simeq & \mathcal{T}^{\perp 2k}_{j,i} \cup \mathcal{T}^{\perp 2k+1}_{j,i}, \\ \textit{and} & \mathcal{ST}^{*2k+1}_{i,j} & \simeq & \mathcal{T}^{\perp 2k+1}_{i,j} \cup \mathcal{T}^{\perp 2k+2}_{i,j}. \end{array}$$

By considering together all skew ternary trees with a marked even label (or all with an odd one) we get the following bijections:

$$\begin{array}{rccc} & \mathcal{ST}^{*\mathrm{even}}_{i,j} & \simeq & \mathcal{T}_{j,i}, \\ \text{and} & \mathcal{ST}_{i,j} \cup \mathcal{ST}^{*\mathrm{odd}}_{i,j} & \simeq & \mathcal{T}_{i,j}. \end{array}$$

$\mathcal{NS}$	edges $(n+1)$	vertices $(i+2)$	faces $(j+2)$	cut-vertices in decomposition (m)	exterior edges $(k+1)$
$\mathcal{DT}$	vertices $(n+1)$	leaves $(i+1)$	nodes $(j+1)$	sons of root $(m+1)$	sum of the labels of the root's sons (k)
$\mathcal{ST}$	vertices (n)	even labels $(i+1)$	odd labels (j)	first zeros $(m+1)$	zeros (k)
$\mathcal{TSS}$	permutation on$\{1,\ldots,n\}$	descents (i)	ascents (j)	cf [GW96]	right-left max (k)

Table i: A summary of the combinatorial parameters

If any vertex is allowed to be marked the resulting bijection is:

$$\begin{aligned} \mathcal{ST}_{i,j} \cup \mathcal{ST}^*_{i,j} &\simeq \mathcal{T}_{i,j} \oplus \mathcal{T}_{j,i}, \\ \text{hence} \quad \mathcal{ST}_n \cup \mathcal{ST}^*_n &\simeq \mathcal{T}_n \oplus \mathcal{T}_n. \end{aligned}$$

Since n distinct vertices can be marked, j of which have odd labels, these bijections imply the following equality:

$$\begin{aligned} (j+1) \cdot |\mathcal{ST}_{i,j}| &= |\mathcal{T}_{i,j}|, \\ \text{and} \quad (n+1) \cdot |\mathcal{ST}_n| &= 2 \cdot |\mathcal{T}_n|. \end{aligned}$$

As recalled in section 4, ternary trees deserve a classical encoding in terms of Lukasievicz words (see [Lot84] chapter 10) which is easily extended to take account of the parity of the vertices, following a method of L. Chottin [Cho73]. These encodings together with the previous bijections give combinatorial proofs of the formulae of W.G. Brown and W.T. Tutte:

Corollary 1 *The number of nonseparable planar maps with $n+1$ edges is $\frac{2}{n+1}$ times the number of ternary trees with n vertices:*

$$|\mathcal{NS}_n| = \frac{2}{n+1}|\mathcal{T}_n| = \frac{2}{n+1} \cdot \frac{1}{3n+1}\binom{3n+1}{n},$$

and the number of nonseparable planar maps with $i+2$ vertices and $j+2$ faces is $\frac{1}{j+1}$ times the number of ternary trees in $\mathcal{T}_{i,j}$:

$$|\mathcal{NS}_{i,j}| = \frac{1}{j+1}|\mathcal{T}_{i,j}| = \frac{(2i+j+1)!(2j+i+1)!}{(i+1)!(j+1)!(2i+1)!(2j+1)!}.$$

These results also stand for two-stack-sortable permutations on $\{1,\ldots,n\}$ with i descents and j ascents. In particular the distribution is symmetric in terms of descents and ascents, a result which is not clear on permutations.

The previous bijections immediately yield the following corollary:

Corollary 2 *Random generation of skew ternary trees with n vertices can be achieved in linear time. Random generation of nonseparable planar maps with n edges (or i vertices and j faces) or of two-stack sortable permutations on $\{1,\ldots,n\}$ can be achieved in quadratic time in the worst case.*

Indeed theorem 1 reduces random generation of nonseparable planar maps to random generation of skew ternary trees, via description trees. This reduction requires a recursive decomposition, hence is quadratic in the worst case (but the average complexity is at most $O(n^{3/2})$ because the average value of labels in description trees is constant and the height of a ternary tree is $O(\sqrt{n})$). Then theorem 2 reduces random generation of skew ternary trees to random generation of ternary trees which is linear (using for instance Lukasievicz words). Moreover this reduction is linear. It should be noted that the manipulated numbers never exceed the size of the generated objects.

Random generation algorithms are also known for planar maps ([Den94], [Sch97]) but not for 3-connected planar maps.

2 Decomposition of nonseparable planar maps and skew ternary trees

The two following sections are devoted to two decompositions which together prove theorem 1.

2.1 Nonseparable planar maps and description trees

We recall here a decomposition of nonseparable planar maps due to W.G. Brown [Bro63] and we give an encoding in terms of description trees. This encoding can be viewed as a modification of the encoding used by I.P. Goulden and J. West [GW96].

Lemma 1 (Brown) *The deletion of the root edge leads to the decomposition*

$$\begin{array}{rcl} \mathcal{NS} & \xrightarrow{\sim} & (\mathcal{NS}^*)^+, \\ M & \mapsto & (M_1^*,\ldots,M_{m+1}^*), \end{array}$$

where $e(M) = 1 + \sum e(M_i^*)$, $v(M) = \sum v(M_i^*)$, $f(M) = 1 + \sum f(M_i^*)$, $r(M) \leqslant \sum r(M_i^*)$ and $r(M) = \sum d(M_i^*)$.

Proof : The deletion of the root edge of M gives a map M' with m cut-vertices ($m \geqslant 0$). Let x_{m+1} be the root vertex and x_0 the head of the root edge of M. A cut-vertex of M' is incident to the two faces which are incident to the root edge of M and the m cut-vertices form a subsequence $(x_1,\ldots,x_m)$ of the sequence of vertices

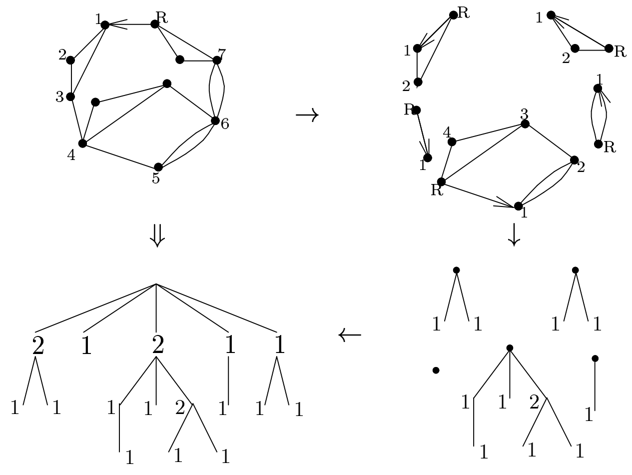


Figure 4: Decomposition of a nonseparable planar map and its recursive encoding into a description tree.

incident to the exterior face. All cut-vertices can be split into two vertices, leaving $m+1$ nonseparable components between successive cut-vertices of the sequence. The component delimited by (x_i, x_{i+1}) is rooted on the exterior edge of M, pointing away from x_i, and has a marked vertex x_{i+1}. It is therefore a bi-rooted map of $\mathcal{NS}^*$. This decomposition is clearly reversible and carries the combinatorial parameters as stated. □

Corollary 3 *A nonseparable planar map M of $\mathcal{NS}_{i,j}$ such that $r(M) = k$ is encoded by a description tree of $\mathcal{DT}_{i,j}$ such that the sum of labels of its root's sons equals k.*

This encoding is obtained recursively: the map M splits into $m+1$ bi-rooted maps $M_1^*, \ldots, M_{m+1}^*$. The root of its description tree has then $m+1$ sons and each subtree is obtained from one of the bi-rooted maps:

- either M_q^* is the single edge map and the qth subtree is a leaf with label 1,
- or M_q^* is obtained from a map M_q by marking the pth vertex on the exterior face (i.e. p edges stay outside in M). The qth subtree is then the description tree of M_q in which the root is labelled p. By construction, $1 \leqslant p \leqslant r(M_q)$ and therefore k is within 1 and the sum of labels of the sons.

2.2 Skew ternary trees and description trees

The aim of this section is to give a decomposition of skew ternary trees similar to the decomposition of nonseparable planar maps. This is done by recursively encoding skew ternary trees into description trees. The parameter $z(T)$ plays the same role for ternary trees as $r(M)$ for maps:

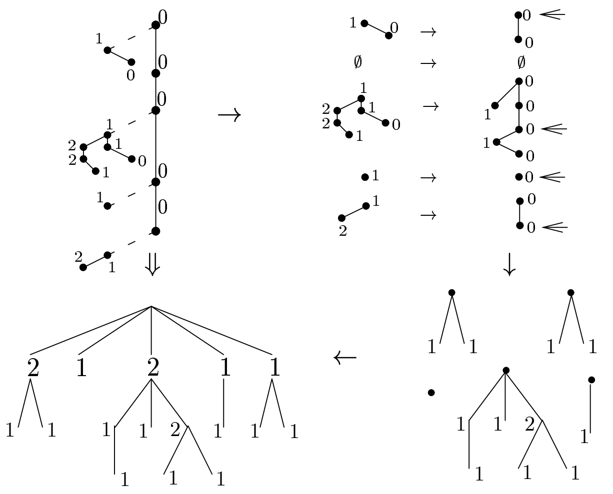


Figure 5: Decomposition of a skew ternary tree, transformation of the 1-rooted subtrees and recursive encoding into a description tree.

Lemma 2 *The sequence of middle sons starting from the root can be broken to get the decomposition:*

$$\begin{array}{rcl} \mathcal{ST} & \xrightarrow{\sim} & \left(\{\emptyset\} \cup \mathcal{ST}^{\surd 1}\right)^{+}, \\ T & \mapsto & (T_1^{\surd 1}, \ldots, T_{m+1}^{\surd 1}), \end{array}$$

where $z(T) = \sum \left(1 + z(T_i^{\surd 1})\right)$ *and* $\mathcal{ST}^{\surd 1}$ *denote the set of skew ternary trees with root label* 1 *(i.e.* 1*-rooted skew ternary trees). Here* $m+1$ *denote the number of vertices in the sequence of middle sons with label* 0 *starting from the root.*

Proof : This decomposition is straightforward: a skew ternary tree is 0-rooted and following the sequence of middle sons from the root vertex yields a sequence of vertices with label 0, each having a (possibly empty) left 1-rooted skew subtree. □

However to get a recursive decomposition we need to make a skew ternary tree with a marked vertex with label 0 from each of the 1-rooted subtrees of the sequence. This is achieved by the following lemma, which is a consequence of the forthcoming theorem 3:

Lemma 3 *There is a bijection between* 1*-rooted skew ternary trees and skew ternary trees with a marked vertex with label* 0*:*

$$\begin{array}{rcl} \mathcal{ST}_{i,j}^{\surd 1} & \xrightarrow{\sim} & \mathcal{ST}_{i+1,j-1}^{*0}, \\ T & \mapsto & T', \end{array}$$

where $z(T') > z(T)$ *and the marked zero of* T' *is the* $(z(T)+1)$*th in the postfix order.*

Proof : this lemma is proved as corollary 5 in section 3.3.

Using this lemma, the recursive decomposition of skew ternary trees is complete and can be encoded into description trees:

Corollary 4 *A skew ternary tree T of $\mathcal{ST}_{i,j}$ with k zeros is encoded by a description tree of $\mathcal{DT}_{i,j}$ such that the sum of labels of its root's sons equals k.*

This encoding is obtained recursively: the skew ternary tree T split into $m+1$ possibly empty 1-rooted skew ternary trees $T_1^{\surd 1}, \dots, T_{m+1}^{\surd 1}$. The root of its description tree has then $m+1$ sons and each subtree is obtained from one of the 1-rooted tree:

- either $T_q^{\surd 1}$ is empty and the qth subtree is a leaf with label 1,
- or $T_q^{\surd 1}$ is mapped by applying the lemma onto a skew ternary tree T_q^{*0} having its $(z(T_q^{\surd 1})+1)$th zero marked. The qth subtree is then the description tree of T_q in which the root is labelled $z(T_q^{\surd 1})+1$.

3 Ternary and marked skew ternary trees

The aim of this section is to prove theorem 2. In order to clarify the exposition of the proof, we consider more general labellings of ternary trees: a ternary tree is k-rooted when its root has label k instead of 0. Labels of other vertices are deduced as before from the root's, according to their shift index. A k-rooted skew ternary tree is then a k-rooted ternary tree with minimum label 1 or 0. This definition extends the definition of (0-rooted) skew ternary tree. Let $\mathcal{ST}^{\surd k}$ denote the set of k-rooted skew ternary trees. Unless explicitly stated otherwise, skew ternary trees are still to be considered as 0-rooted.

Since k-rooted skew ternary trees contain a vertex with label 1 or 0, changing the root label into 0 and adjusting others labels gives:

$$\begin{aligned}\mathcal{ST}_{i,j}^{\surd 2k+1} &\simeq \mathcal{T}_{j-1,i+1}^{\perp 2k} \cup \mathcal{T}_{j-1,i+1}^{\perp 2k+1},\\ \mathcal{ST}_{i,j}^{\surd 2k+2} &\simeq \mathcal{T}_{i,j}^{\perp 2k+1} \cup \mathcal{T}_{i,j}^{\perp 2k+2}.\end{aligned}$$

Theorem 2 can therefore be restated with this notation:

Theorem 3 *For any nonnegative integers i, j and k,*

$$\begin{aligned} \mathcal{ST}_{i,j}^{*2k} &\simeq \mathcal{ST}_{i-1,j+1}^{\surd 2k+1}\\ \textit{and}\quad \mathcal{ST}_{i,j}^{*2k+1} &\simeq \mathcal{ST}_{i,j}^{\surd 2k+2}.\end{aligned}$$

The proof is made by constructing a mapping from skew ternary trees with a marked vertex with label k to k-rooted skew ternary trees and showing it is bijective. It goes in three steps: the bijection is first constructed for a restricted class of trees, namely skew ternary trees with a marked leaf. The fundamental construction is described there. In the second part, the mapping is extended to skew ternary trees with a marked internal node through a decomposition of these trees. The reverse construction is then given.

3.1 Preliminary construction: reversing a branch

A *branch* is a tree whose internal nodes all have only one son. It is encoded either by the sequence of edges' types ($[m, l, r, l, l, m, r, l, l, l]$ for instance) or by the sequence of vertices' labels ($[0, 0, 1, 0, 1, 2, 2, 1, 2, 3, 4]$ for the same example).

The involution ϕ consists in reversing the sequences of labels:

$$\phi : [l_0, l_2, \dots, l_p] \to [l_p, \dots, l_1, l_0].$$

In terms of edges, the involution $l \leftrightarrow r$ and $m \to m$ need moreover to be applied to each letter.

Let T be a ternary tree with a marked leaf: the tree T splits into a *main branch* B joining the root to the marked leaf and a family of (possibly empty) lateral subtrees associated with the edges of B. The tree T can therefore be encoded by a sequence of triples (edge's type, leftmost subtree, rightmost subtree) where the subtrees are the two (possibly empty) other subtrees of the top-vertex of the considered edge.

The involution ϕ is extended to sequences of triples in the following way: reverse the sequence and apply to each triple the involution $(l, S_1, S_2) \leftrightarrow (r, S_1, S_2)$ and $(m, S_1, S_2) \to (m, S_2, S_1)$.

$$\begin{aligned} T &= [(m,s_1,\emptyset),(l,s_2,\emptyset),(r,s_3,s_4),(l,s_5,\emptyset),(l,s_6,s_7),(m,s_8,s_9),(r,s_{10},s_{11}),\\ &\quad (l,s_{12},s_{13}),(l,s_{14},s_{15}),(l,s_{16},s_{17})],\\ \phi(T) &= [(r,s_{16},s_{17}),(r,s_{14},s_{15}),(r,s_{12},s_{13}),(l,s_{10},s_{11}),(m,s_9,s_8),(r,s_6,s_7),\\ &\quad (r,s_5,\emptyset),(l,s_3,s_4),(r,s_2,\emptyset),(m,\emptyset,s_1)]. \end{aligned}$$

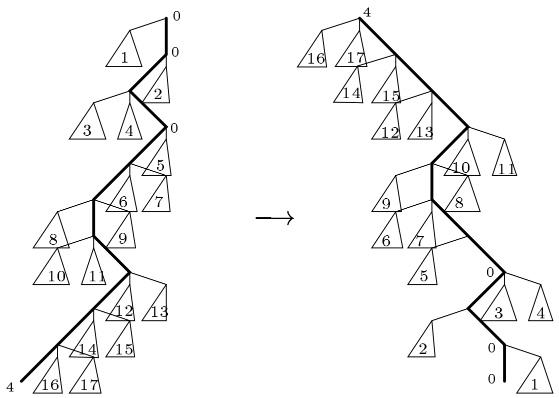


Figure 6: An illustration of the involution ϕ on sequences of triples and its interpretation in terms of trees.

Now let A be a skew ternary tree with a marked leaf with label k, and $\mathcal{S}$ be associated sequence of triples. Let us consider the tree which is encoded by $\phi(\mathcal{S})$ as k-rooted, so that the labels on the main branch are not modified by ϕ. This tree has a marked leaf with label 0 at the end of its main branch, let $\psi(A)$ be the k-rooted tree obtained by deleting it. Although $\psi(A)$ has no marked leaf, we still call the main branch of $\psi(A)$ the shortened image of the main branch of A. Since the deleted leaf was the image of the root of A which has label 0, its father has a label 1 or 0. We call this vertex the *pivot* of $\psi(A)$, it ends its main branch and has no son with label 0. Note that if A is the root-leaf tree, there is no pivot since $\psi(A)$ is reduced to an empty tree.

The lateral subtrees of A are moved around the main branch by ψ: if a subtree was hanging on the left of the main branch of A, it is hanging on the right of the main branch of $\psi(A)$. This move implies that its labels are decreased by 2. Conversely a right-hanging subtree of A becomes a left-hanging subtree of $\psi(A)$ and its labels are increased by 2. Since A is a skew tree, all its labels are nonnegative and the resulting labels of $\psi(A)$ are:

- greater or equal to 2 in left-hanging subtrees,
- nonnegative on the main branch,
- greater or equal to -2 in right-hanging subtrees.

The mapping ψ is therefore a mapping from skew ternary trees with a marked leaf with label k into a restricted class of k-rooted ternary trees. We now prove a more precise result:

Lemma 4 *The mapping ψ is injective and its image consists of all k-rooted ternary trees with minimum label between -2 and 1 (plus the empty tree).*

Proof : We already know that the trees in the image are k-rooted ternary trees, containing a label 1 or 0 and whose labels are greater than or equal to -2. Conversely, let A' be a k-rooted tree with minimum label within -2 and 1. Since A' has no marked leaf, the key point to find A such that $\psi(A) = A'$ is the determination of its main branch and more precisely of its pivot. As noticed before, the pivot should have label 1 or 0 and no son with label 0. It should moreover define a main branch splitting the tree in left-hanging and right-hanging subtrees which satisfy the labelling conditions given before.

Let us consider the following searching algorithm which determines a vertex v_0:

- let v be the first vertex in postfix order labelled with 1 or 0,
- if the branch joining the root to v has no negative label, v_0 is v,
- otherwise v_0 is the father of the first reached vertex with label -1 when descending the branch from the root to v.

This construction yields a vertex v_0 and a branch. By construction, left-hanging trees are before v_0 in postfix order and have therefore labels greater or equal to 2. The same argument implies that v_0 has no son with label 0. Hence a marked leaf with label 0 can be added to complete the main branch of A'. Then A' splits in a sequence of triples and the involution ϕ can be applied. The occurring tree is easily seen to be skew with a marked leaf with label k and the construction is clearly the reverse of ψ. $\square$

A second construction will be needed in the next section. We define below a mapping δ on p-rooted ternary trees with nonnegative labels. Let A be such a tree and consider its root as the first vertex of a sequence of middle sons with label p, sided with left and right (possibly empty) subtrees, respectively $(p+1)$- and $(p-1)$-rooted. The image $\delta(A)$ is then the $(p+2)$-rooted ternary tree which is built around a sequence of vertices of the same length, starting at the root and sided on the left with the right subtrees of A and on the right with its left subtrees. The labels in subtrees are therefore increased by 4 on the left and conserved on the right.

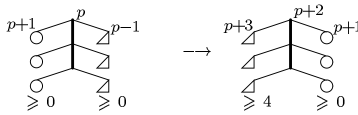


Figure 7: A tree and its image by δ.

3.2 Mapping marked skew ternary trees onto ternary trees

We now consider skew ternary trees having a marked vertex (internal node or leaf) with label k and we construct the mapping

$$\Psi : \mathcal{ST}^{*k} \longrightarrow \mathcal{ST}^{\surd k+1}.$$

Let A be a tree of $\mathcal{ST}^{*k}$ and v its marked vertex. Vertex v is the father of three (possibly empty) subtrees A_1, A_2 and A_3 with respective root labels $k+1$, k and $k-1$. The tree A' obtained by the deletion of these three subtrees has a marked leaf and is therefore decomposed into its main branch, left-hanging and right-hanging subtrees.

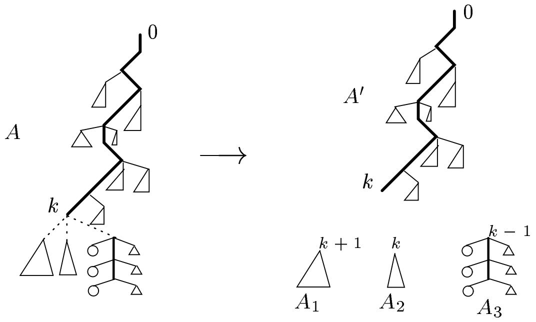


Figure 8: The different components of a skew ternary tree having a marked vertex with label k.

The set $\mathcal{ST}^{*k}$ is partitioned into two disjoint subsets and Ψ is defined in each case:

implicit case: there is a vertex with label 1 or 0 in A_1, A_2 or in a left-hanging subtree of A'. The tree $\Psi(A)$ is then constructed as follows: the tree $\delta(A_3)$ is $(k+1)$-rooted, so we can add a vertex with label $k+1$ as middle son of the last vertex in its middle sons sequence. This vertex gets as left, middle and right sons the subtrees $\psi(A')$, A_1 and A_2 respectively. The labels in A_1 and A_2 need no modification whereas those of $\psi(A')$ have to be increased by 2 so that the root label agrees with its father's label. All the labels in $\Psi(A)$ are nonnegative and this tree *implicitly* contains a vertex with

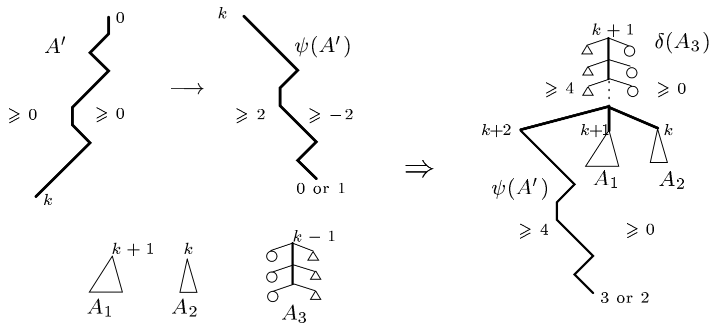


Figure 9: The transformation in the implicit case.

label 1 or 0. Indeed, A_1, A_2 and the left-hanging subtrees of A' are moved without any modification of their labels and one of them contains such a vertex by hypothesis.

explicit case: all labels in A_1, A_2 and the left-hanging subtrees of A' are greater or equal to 2. The tree $\Psi(A)$ is then constructed as follows: the tree $\delta(A_3)$ is $(k+1)$-rooted, so we can add a vertex with label $k+1$ as middle son of the last vertex in its middle sons sequence. This vertex gets as left, middle and right sons the subtrees A_2, A_1 and $\psi(A')$ respectively. The labels in A_1 and $\psi(A')$ need no modification whereas those of A_2 have to be increased by 2 so that the root label agrees with its father's label. All the labels in $\Psi(A)$ are nonnegative and this tree *explicitly* contains

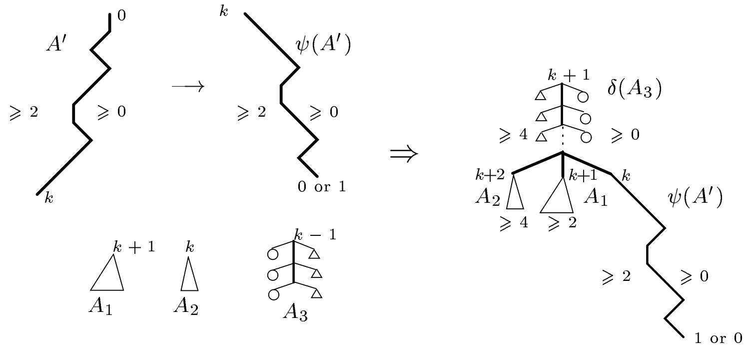


Figure 10: The transformation in the explicit case.

a vertex with label 1 or 0. Indeed, the labels of $\psi(A')$ where not modified and the pivot of $\psi(A')$ is labelled 1 or 0.

Remark: in the two cases, the number of vertices is invariant since one vertex with label $k+1$ is added and one with label 0 is deleted (in $\psi(A')$). Moreover, with the exception of this vertex, labels are modified by even values: the repartition of the parity of labels is therefore invariant except for a label 0 becoming $k+1$. This remark explains that the statements of theorems 2 and 4 contain two cases depending upon the parity of k.

3.3 Reverse construction

We now show that the previously defined mapping Ψ is a bijection and we state a more refined version of the $k=0$ case as corollary 5.

Let T be a $(k+1)$-rooted skew ternary tree (recall it contains a label 1). The following algorithm builds a skew ternary tree with a marked label k: it starts with reconstructing A_3 by descending along middle sons of the root, until it recognize an implicit or an explicit case.

1. **Initialization:** at the beginning A_3 is an empty tree.
2. The root of the tree T has three (possibly empty) subtrees T_1, T_2 and T_3.
3. If $k=0$ or if there is a vertex with label 2 or 3 in T_1 then we are in the implicit case (cf. figure 9): T_1 is $k+2$-rooted and let T_1' be the k-rooted copy of T_1. The tree T_1' is empty or contains a label 1 or 0 and has only labels that are greater or equal to -2. It is therefore the image by ψ of a skew ternary tree A' with a k-labelled leaf. The subtrees T_2 and T_3 are identified with the subtrees A_1 and A_2 and a skew ternary tree A is easily reconstructed such that $\Psi(A)=T$.
4. Otherwise all labels of T_1 are greater or equal to 4 (or T_1 is empty).
   (a) If the labels of T_2 are greater or equal to 2 (or T_2 is empty), there is a label 1 in T_3 (because there is a 1 in T and the root label is greater or equal to 2). We are therefore in the explicit case (cf. figure 10): T_3 is the image of a tree A' by ψ. The tree T_2 is identified as A_1 and the k-rooted copy of T_1 as A_2. A skew ternary tree A is then easily reconstructed such that $\Psi(A)=T$.
   (b) Otherwise the subtree T_2 contains a label 1. Since it is also $k+1$-rooted we can go back to step 2 and apply the algorithm to T_2 after having added T_3 and T_1 to the bottom of A_3.

This algorithm ends for all $k+1$-rooted skew ternary trees and it is easily seen that it reverses Ψ. Hence theorem 2 is proven. □

The case $k=0$ can be refined to get the following corollary which is a restatement of the end of lemma 3.

Corollary 5 *Let A be a skew ternary tree whose $p+1$th zero (in postfix order) is marked then the tree $\Psi(A)$ has p zeros.*

Proof : since $k=0$, it is an implicit case and moreover the subtree A_3 is empty. The possible zeros of $\Psi(A)$ are therefore in the image of A_1 or A_2 or in a right-hanging subtree of $\psi(A')$. These zeros are the images of the zeros of A_1, A_2 or of the left-hanging subtrees of A' which are exactly the zeros before the marked vertex of A in postfix order. $\square$

4 Coding ternary trees

The preceding bijections already allow us to substitute Lagrange inversion for ternary trees to the quadratic method used by W.G. Brown and W.T. Tutte. However, in order to have a complete bijective proof and to allow random generation, it is useful to encode ternary trees by words. This is done using the Lukacievicz language and Raney's theorem (see [Lot84], chapter 11) and yields the following result:

Theorem 4 (Dvoretski, Motzkin, Raney) *There exists a bijection between ternary trees with n internal vertices and conjugation classes of words with n letters x_2 and $2n+1$ letters x_0.*

Hence we recover the well-known number of ternary trees with n vertices:

$$|\mathcal{T}_n| = \frac{1}{3n+1}\binom{3n+1}{n}.$$

This proves the first formula of corollary 1 and allows deterministic uniform random generation of skew ternary trees with n vertices in linear complexity.

In order to take into account the parity of the vertices, this encoding can refined: vertices of a ternary tree are colored in two colors by their parity and Lukacievicz words can be colored as well. Then the previous theorem can be extended using a method due to Chottin ([Cho73] or [Lot84] ex.11.3.4):

Theorem 5 (Chottin) *The set of ternary trees with $i+1$ even and j odd vertices can be partitioned into disjoint subsets of size $2(i+1)$ such that these subsets are in bijection with pairs of conjugation classes of words with respectively $i+1$ letters x_2, $2j+1$ letter x_0 and j letters y_2, $2i+2$ letters y_0.*

Hence we get

$$\begin{aligned} |\mathcal{T}_{i,j}| &= 2(i+1)\frac{1}{(i+2j+2)}\binom{i+2j+2}{i+1}\frac{1}{(j+2i+2)}\binom{j+2i+2}{j} \\ &= \frac{(i+2j+1)!(j+2i+1)!}{(i+1)!j!(2i+1)!(2j+1)!}. \end{aligned}$$

This proves the second formula of corollary 1 and allows deterministic uniform random generation of skew ternary trees with $i+1$ even and j odd vertices in linear complexity.

Acknowledgments

We wish to thank Robert Cori for suggesting this problem and patiently encouraging us.

Appendix

n	i	j	$\mathcal{DT}_n$	$\mathcal{ST}_n$	$\mathcal{NS}_n$
0					
1	0	0			
2	0	1			
	1	0			
3	0	2			
	1	1			
	1	1			
	1	1			
	1	1			
	2	0			

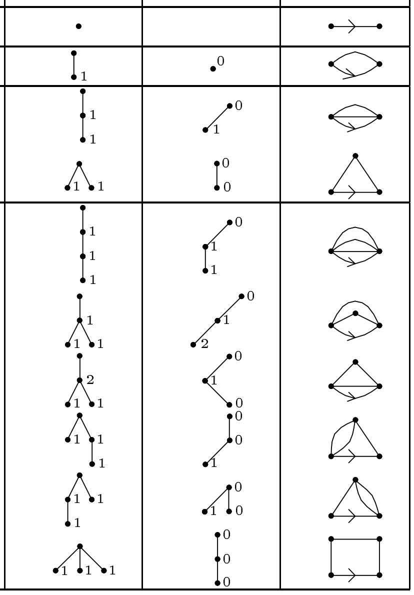


Table ii: All small objects.

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